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Generalised Supplementarity and new rule for Empty Diagrams to Make the ZX-Calculus More Expressive

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Abstract. The ZX-Calculus is a powerful diagrammatic language for quantum mechanics and quantum information processing. The completeness of the $\frac{\pi}{4}$ -fragment is a main open problem in categorical quantum mechanics, a program initiated by Abramsky and Coecke. It has recently been proven that this fragment, also called *Clifford+T quantum mechanics*, was not complete, and hence a new rule called *supplementarity* was introduced to palliate it. The completeness of the $\frac{\pi}{4}$ -fragment is a crucial question, for it is the “easiest” approximately universal fragment, whereas the $\frac{\pi}{2}$ -fragment, the *stabiliser quantum mechanics*, is known to be complete but is not approximately universal.

In this paper, we will show that, on the one hand, the $\frac{\pi}{4}$ -fragment is still not complete despite the new supplementarity rule, and that, on the other hand, this supplementarity rule can be generalised to any natural number n . To prove the incompleteness of the $\frac{\pi}{4}$ -fragment, we can show an equality over scalars that is not derivable from the set of axioms, and that will lead to a substitute for the inverse rule, and the obsolescence of the zero rule. We can also show that the generalised supplementarity for any n not a power of 2 is not derivable from the current set of axioms of the ZX-Calculus, but that the rule for some n can be derived from some others. The hierarchy of these rules will be discussed.

1 Introduction

The ZX-Calculus is a powerful diagrammatic language for reasoning in quantum mechanics introduced by Coecke and Duncan [1]. Every diagram is composed of three kinds of vertices: red and green dots which are parametrised by an angle, and a yellow box; and each diagram represents a matrix thanks to the so-called standard interpretation. Moreover, any quantum transformation can be expressed using ZX-diagrams, meaning they are *universal*.

Unlike quantum circuits, the ZX-Calculus comes with a set of equalities between diagrams that preserve the matrix that is represented. Hence, using locally a succession of these equalities, one can prove that two diagrams represent the same matrix, because the language is *sound* i.e. all the equalities do indeed preserve the matrix.

The converse of soundness is called *completeness*. Here, it amounts to being able to transform any diagram into another one as long as both represent the same matrix. It has been proven that the ZX-Calculus is in general not complete [6]. Yet, some restrictions have been proven to be complete. The $\frac{\pi}{2}$ -fragment – the language restricted to angles that are multiples of $\frac{\pi}{2}$, which represents the *stabiliser quantum mechanics* – is complete [2]. The π -fragment – representing the *real stabiliser quantum mechanics* – is also complete [5].

A fragment is *approximately universal* when any quantum transformation can be approached with arbitrarily great precision using only the angles in the fragment. Sadly, the $\frac{\pi}{2}$ -fragment is not approximately universal, but the $\frac{\pi}{4}$ -fragment is. It is called the *Clifford+T quantum mechanics*. Completeness for this fragment is still an open question, one of the main ones in the fields of categorical quantum mechanics – even though a partial answer has been given for the fragment composed of path diagrams involving angles multiple of $\frac{\pi}{4}$ [7].

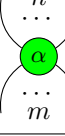






In this paper, we show that in the non-scalar-free version of the ZX-Calculus, the $\frac{\pi}{4}$ -fragment is not complete, showing that a scalar equality is derivable using matrices, but not diagrammatically. We propose to replace the “inverse rule” by this equality, and show that it can prove the former one as well as a third one: the “zero rule”.

We also show that an infinite number of fragments are also incomplete, by proving that a generalised form of the “supplementarity rule” cannot be derived in them. Some of them, though, can be deduced from the rule parametrised with other numbers. We show that the rule parametrised with prime numbers is enough to deduce it for any number, and show that this set – for odd prime numbers – is minimal. We also give some other ways of deducing the rule from other ones – not parametrised with prime numbers.

2 ZX-Calculus

2.1 Diagrams and standard interpretation

A ZX-diagram $D : k \rightarrow l$ with k inputs and l outputs is generated by:

$R_Z^{(n,m)}(\alpha) : n \rightarrow m$		$R_X^{(n,m)}(\alpha) : n \rightarrow m$	
$H : 1 \rightarrow 1$		$e : 0 \rightarrow 0$	
$\mathbb{I} : 1 \rightarrow 1$		$\sigma : 2 \rightarrow 2$	
$\epsilon : 2 \rightarrow 0$		$\eta : 0 \rightarrow 2$	

where $n, m \in \mathbb{N}$ and $\alpha \in \mathbb{R}$

and the two compositions:

- Spacial Composition: for any $D_1 : a \rightarrow b$ and $D_2 : c \rightarrow d$, $D_1 \otimes D_2 : a + c \rightarrow b + d$ consists in placing D_1 and D_2 side by side, D_2 on the right of D_1 .
- Sequential Composition: for any $D_1 : a \rightarrow b$ and $D_2 : b \rightarrow c$, $D_2 \circ D_1 : a \rightarrow c$ consists in placing D_1 on the top of D_2 , connecting the outputs of D_1 to the inputs of D_2 .

The standard interpretation of the stabiliser ZX-diagrams associates to any diagram $D : n \rightarrow m$ a linear map $\llbracket D \rrbracket : \mathbb{C}^{2^n} \rightarrow \mathbb{C}^{2^m}$ inductively defined as follows:

$$\llbracket D_1 \otimes D_2 \rrbracket := \llbracket D_1 \rrbracket \otimes \llbracket D_2 \rrbracket \quad \llbracket D_2 \circ D_1 \rrbracket := \llbracket D_2 \rrbracket \circ \llbracket D_1 \rrbracket \quad \llbracket \square \rrbracket := (1) \quad \llbracket \mathbb{I} \rrbracket := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\llbracket \text{H} \rrbracket := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \llbracket \sigma \rrbracket := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \llbracket \epsilon \rrbracket := (1 \ 0 \ 0 \ 1) \quad \llbracket \eta \rrbracket := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\llbracket \text{Green Circle} \rrbracket := (1 + e^{i\alpha}) \quad \llbracket \text{Green Circle with } \alpha \rrbracket := 2^m \begin{matrix} \overbrace{\left(\begin{matrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & e^{i\alpha} \end{matrix} \right)}^{2^n} \\ (n + m > 0) \end{matrix}$$

$$\text{For any } n, m \geq 0 \text{ and } \alpha \in \mathbb{R}, \quad \llbracket \text{Red Circle with } \alpha \rrbracket = \llbracket \text{H} \rrbracket^{\otimes m} \circ \llbracket \text{Green Circle with } \alpha \rrbracket \circ \llbracket \text{H} \rrbracket^{\otimes n}$$

(where $M^{\otimes 0} = (1)$ and $M^{\otimes k} = M \otimes M^{\otimes k-1}$ for any $k \in \mathbb{N}^*$).

To simplify, the red and green nodes will be represented empty when holding a 0 angle:



Also in order to make the diagrams a little less heavy, when n copies of the same subdiagram occur, we will use the notation $(\cdot)^{\otimes n}$.

ZX-Diagrams are universal:

$$\forall A \in \mathbb{C}^{2^n} \times \mathbb{C}^{2^m}, \exists D, \llbracket D \rrbracket = A$$

This implies dealing with an uncountable set of angles, so it is generally preferred to work with *approximate* universality – the ability to approximate any linear map with arbitrary accuracy – in which only a finite set of angles is involved. The $\frac{\pi}{4}$ -fragment – ZX-diagrams where all angles are multiples of $\frac{\pi}{4}$ – is one such approximately universal fragment, whereas $\frac{\pi}{2}$ -fragment is not.

2.2 Calculus

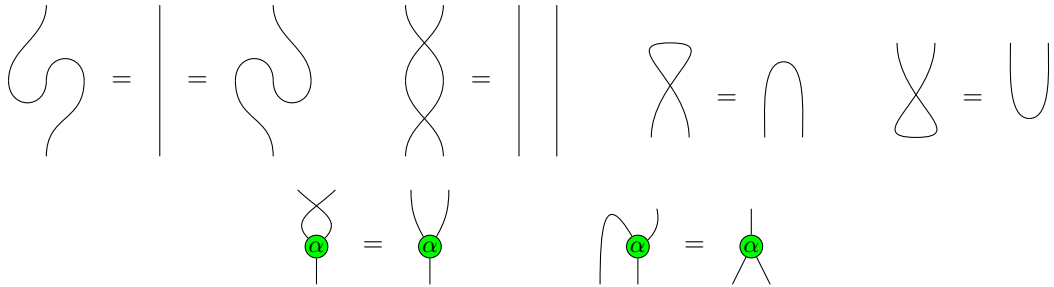
The diagrammatic representation of a matrix is not unique in the ZX-Calculus. Hence, there exists a set of axiomatic equalities between diagrams, summed up in figure 1.

When we can show that a diagram D_1 is equal to another one, D_2 , using a succession of equalities of this set, we write $ZX \vdash D_1 = D_2$. Given that the rules are sound, this means that $\llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket$. The rules can obviously be applied to any subdiagram, meaning, for any diagram D :

$$(ZX \vdash D_1 = D_2) \Rightarrow \begin{cases} (ZX \vdash D_1 \circ D = D_2 \circ D) \wedge (ZX \vdash D \circ D_1 = D \circ D_2) \\ (ZX \vdash D_1 \otimes D = D_2 \otimes D) \wedge (ZX \vdash D \otimes D_1 = D \otimes D_2) \end{cases}$$

The notion of completeness here will take into account the scalars – diagrams with 0 input and 0 output, hence representing a 1×1 matrix –, while in some versions of the ZX-Calculus, the global phase or even all the scalars are ignored.

Only Topology Matters is a paradigm stating that any wire of a ZX-diagram can be bent at will, without changing its semantics:



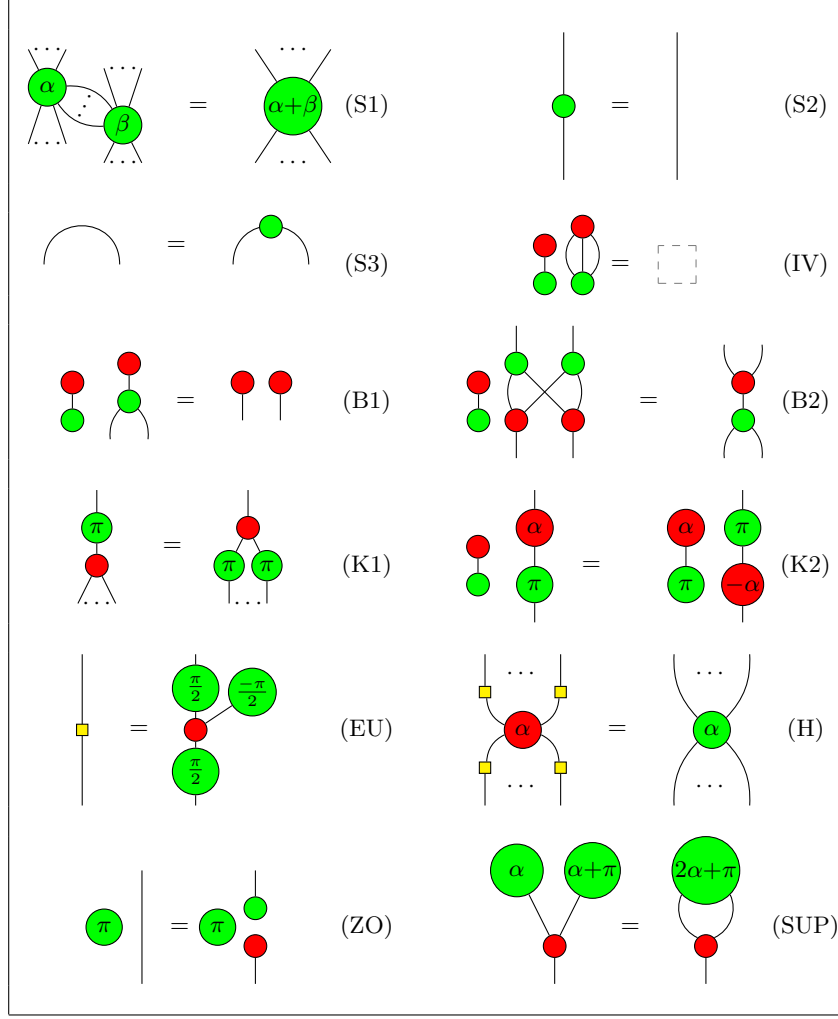


Fig. 1. Rules for the ZX-calculus with scalars, augmented with the supplementarity rule [3]. All of these rules also hold when flipped upside-down, or with the colours red and green swapped. The right-hand side of (IV) is an empty diagram. (\dots) denote zero or more wires, while $(\cdot \cdot)$ denote one or more wires. Note that it has been proven that (K1) could be derived from the other rules [4], and so will be put aside in the sets of rules that will appear in the following.

3 The $\frac{\pi}{4}$ -fragment is not complete


In the figure 1, we can see that the rule (IV) gives a relation of inverse – in the sense of \otimes – between two diagrams, because the empty diagram is instinctively the neutral element for \otimes . We can easily calculate that the two subdiagrams represent

$$\left[\begin{array}{c} \bullet \\ \bullet \end{array} \right] = \sqrt{2} \quad \text{and} \quad \left[\begin{array}{c} \bullet \\ \bullet \end{array} \right] = \frac{1}{\sqrt{2}}$$

Hence, we can create any power – negative or positive integer – of $\sqrt{2}$, by placing the right amount of one of these two diagrams next to each other.

We can represent $\frac{1}{\sqrt{2}}$ in other ways. For instance, we can create a diagram that represents $\cos(\alpha)$ for any $\alpha \in \mathbb{R}$:

$$\left[\begin{array}{c} \bullet \\ \bullet \end{array} \right] = (1 e^{i\alpha}) \circ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + e^{-i\alpha} \\ 1 - e^{-i\alpha} \end{pmatrix} \circ \frac{1}{\sqrt{2}} = \frac{e^{i\alpha} + e^{-i\alpha}}{2} = \cos(\alpha)$$

and in the precise case of $\alpha = \frac{\pi}{4}$, this cos equals $\frac{1}{\sqrt{2}}$. So, simplifying with , we should find:

$$\begin{array}{c} \textcircled{\frac{\pi}{4}} \\ \textcircled{-\frac{\pi}{4}} \end{array} = \boxed{} \quad (\text{E})$$

But this equality is not derivable from the rules of the ZX-Calculus expressed in figure 1.

Indeed, let us define the quantities I_G and I_R :

Definition 1. Let $D = (Q, V_R, V_G, E, E_H)$ be a diagram (of the ZX-Calculus) where Q is the set of inputs/outputs, V_R and V_G the sets of red and green vertices, E a list of elements of $(Q \cup V_R \cup V_G) \times (Q \cup V_R \cup V_G)$ – the links between 2 nodes –, and E_H a list of elements of $(Q \cup V_R \cup V_G) \times (Q \cup V_R \cup V_G)$ – the links between 2 nodes with a Hadamard.

The quantity $I_G(D)$ (resp. $I_R(D)$) is defined as:

$$I_G(D) = \left(\left(\sum_{g \in V_G} \delta(g) \right) + |E_H| \right) \bmod 2 \quad \left(\text{resp. } I_R(D) = \left(\left(\sum_{r \in V_R} \delta(r) \right) + |E_H| \right) \bmod 2 \right)$$

with $\delta(v)$ the degree of vertex v .

In other words, $I_G(D)$ is the parity of the degree of green dots plus the number of Hadamard boxes in the diagram D .

Proposition 1.

Both $I_G(D)$ and $I_R(D)$ are invariants of any non-null diagram – any diagram representing a non-null matrix – of the ZX with regards to the rules in figure 1.

Proof. One just has to check if it is true for all the rules in figure 1 but (ZO) (which deals with null diagrams).

Corollary 1. Since the equality (E) does not deal with null diagrams, and does not hold for the invariants I_G and I_R , $ZX \not\vdash (\text{E})$.

Hence, we propose to add this equality (E) to the set of rules. Now, does this new rule imply another one from the set? The best candidate would precisely be (IV) because it is the only rule – but (E) – that explicitly includes an empty diagram.

First note that the Hopf law

$$\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \bullet \end{array} \quad (\text{HL})$$

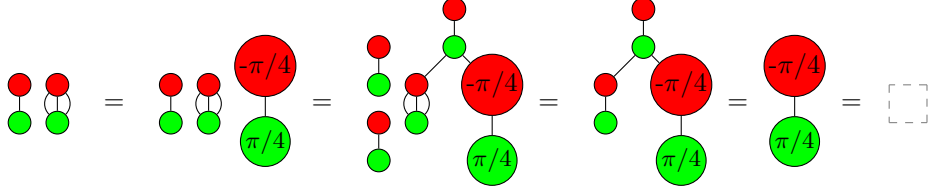
is derivable from (S1), (S2), (S2), (B1) and (B2)– not (IV). So if (IV) is not in the set anymore, the Hopf law is still derivable:

$$\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \bullet \end{array}$$

Proposition 2. (IV) is derivable from the new rule (E)

$$\{(\text{E})\} \cup ZX \setminus \{(\text{IV})\} \vdash (\text{IV})$$

Proof. Using (E), (B1), (HL), (S2), (S1):



We can actually even go a bit further. Indeed, in equation (E), not only does the empty diagram appear, but the invariants I_G and I_R are not respected. This is also the case with rule (ZO). We can then hope that this rule is derivable from the other ones.

Proposition 3. (ZO) is now derivable from the other rules of the ZX-Calculus:

$$\{(E)\} \cup ZX \setminus \{(IV), (ZO)\} \vdash (ZO)$$

Proof. In appendix at page 15.

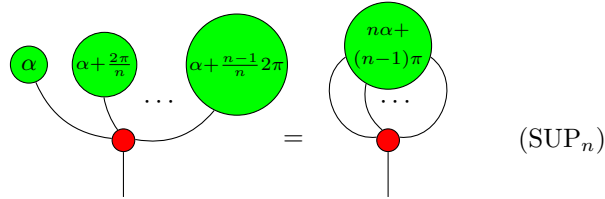
Hence $\{(E)\} \cup ZX \setminus \{(IV), (ZO)\} \vdash (IV), (ZO)$ and the new set of rules is represented in figure 3 in appendix.

Remark 1. In this set of rules, we can still prove that (SUP) is not derivable from the other rules, using the interpretation $\llbracket \cdot \rrbracket_{k,l}^\sharp$ defined in [3] with $k = 3$ and $l = 8$.

4 Generalised Supplementarity

We can show that the supplementarity rule (SUP) can be generalised to any natural number n . This generalisation for a number n will be written (SUP_n) , and the one in figures 1 and 3 will then be written (SUP_2) .

In (SUP_2) , two branches containing α appear, and to each one of them is added either the angle 0 or the angle π , which are the arguments of the two second roots of unity (1 and -1). In the general case, with n branches displaying the n^{th} roots of unity ($\frac{2k\pi}{n}$ for $k \in \llbracket 0; n-1 \rrbracket$), the rule becomes:



Notice that there are n green dots, in the left diagram, and n parallel wires in the right diagram.

In order to prove the soundness of (SUP_n) , let us first define the equality:

$$\alpha \quad \alpha + \frac{2\pi}{n} \quad \dots \quad \alpha + \frac{n-1}{n} 2\pi = \begin{matrix} n\alpha + \\ (n-1)\pi \end{matrix} \quad (1)$$

Lemma 1. (SUP_n) is sound $\forall \alpha \in \mathbb{R} \iff (1)$ is sound $\forall \alpha \in \mathbb{R}$.

Proof. Notice that $\left(\begin{array}{c} | \\ \bullet \\ | \end{array} , \begin{array}{c} | \\ \pi \\ | \end{array} \right)$ form a basis of \mathbb{C}^2 . Hence, using (K1) and (B1):

$$\begin{aligned}
(\text{SUP}_n) \text{ is sound } &\Leftrightarrow \begin{array}{c} \alpha \quad \alpha + \frac{2\pi}{n} \quad \dots \quad \alpha + \frac{n-1}{n} 2\pi \\ \bullet \quad \bullet \quad \dots \quad \bullet \\ | \quad | \quad \dots \quad | \\ \bullet \quad \bullet \quad \dots \quad \bullet \\ | \quad | \quad \dots \quad | \\ \bullet \quad \bullet \quad \dots \quad \bullet \\ | \quad | \quad \dots \quad | \\ \bullet \quad \bullet \quad \dots \quad \bullet \\ | \quad | \quad \dots \quad | \\ \bullet \quad \bullet \quad \dots \quad \bullet \end{array} = \begin{array}{c} n\alpha + \\ (n-1)\pi \\ \bullet \quad \bullet \quad \dots \quad \bullet \\ | \quad | \quad \dots \quad | \\ \bullet \quad \bullet \quad \dots \quad \bullet \\ | \quad | \quad \dots \quad | \\ \bullet \quad \bullet \quad \dots \quad \bullet \end{array} \text{ is sound } \forall \alpha \in \mathbb{R}, \forall k \in \{0, 1\} \\
\forall \alpha \in \mathbb{R} & \\
&\Leftrightarrow \begin{array}{c} \alpha \quad \alpha + \frac{2\pi}{n} \quad \dots \quad \alpha + \frac{n-1}{n} 2\pi \\ \bullet \quad \bullet \quad \dots \quad \bullet \\ | \quad | \quad \dots \quad | \\ \bullet \quad \bullet \quad \dots \quad \bullet \\ | \quad | \quad \dots \quad | \\ \bullet \quad \bullet \quad \dots \quad \bullet \\ | \quad | \quad \dots \quad | \\ \bullet \quad \bullet \quad \dots \quad \bullet \\ | \quad | \quad \dots \quad | \\ \bullet \quad \bullet \quad \dots \quad \bullet \end{array} = \begin{array}{c} n\alpha + \\ (n-1)\pi \\ \bullet \quad \bullet \quad \dots \quad \bullet \\ | \quad | \quad \dots \quad | \\ \bullet \quad \bullet \quad \dots \quad \bullet \\ | \quad | \quad \dots \quad | \\ \bullet \quad \bullet \quad \dots \quad \bullet \end{array} \text{ is sound } \forall \alpha \in \mathbb{R}, \forall k \in \{0, 1\} \\
&\Leftrightarrow \begin{array}{c} \alpha + k\pi \quad \alpha + k\pi + \frac{2\pi}{n} \quad \dots \quad \alpha + k\pi + \frac{n-1}{n} 2\pi \\ \bullet \quad \bullet \quad \dots \quad \bullet \\ | \quad | \quad \dots \quad | \\ \bullet \quad \bullet \quad \dots \quad \bullet \\ | \quad | \quad \dots \quad | \\ \bullet \quad \bullet \quad \dots \quad \bullet \end{array} = \begin{array}{c} n(\alpha + k\pi) + \\ (n-1)\pi \\ \bullet \quad \bullet \quad \dots \quad \bullet \\ | \quad | \quad \dots \quad | \\ \bullet \quad \bullet \quad \dots \quad \bullet \end{array} \text{ is sound } \forall \alpha \in \mathbb{R}, \forall k \in \{0, 1\} \\
&\Leftrightarrow (1) \text{ is sound } \forall \alpha \in \mathbb{R}
\end{aligned}$$

Proposition 4. (SUP_n) is sound.

Proof. According to the previous lemma, (SUP_n) is sound $\forall \alpha \in \mathbb{R}$ if and only if (1) is sound $\forall \alpha \in \mathbb{R}$. The standard interpretation of the right part of the equality yields $1 + e^{i(n\alpha + (n-1)\pi)}$, and the interpretation of the left part amounts to:

$$\prod_{k=0}^{n-1} (1 + e^{i(\alpha + \frac{2k\pi}{n})}) = e^{in(\alpha + \pi)} \prod_{k=0}^{n-1} (e^{-i(\alpha + \pi)} - e^{\frac{2ik\pi}{n}}) = e^{in(\alpha + \pi)} (e^{-in(\alpha + \pi)} - 1) = 1 + e^{i(n\alpha + (n-1)\pi)}$$

Hence (1) is sound which implies (SUP_n) is sound.

4.1 The set of Rules (SUP_n) for n Prime

Let \mathbb{P} be the set of prime numbers.

Proposition 5. The set of equalities with all n odd prime $-\{(\text{SUP}_n) \mid n \in \mathbb{P}, n \geq 3\}$ - is minimal. The equality (SUP_n) for any $n \in \mathbb{N}^*$ can be derived from $\{(\text{SUP}_n) \mid n \in \mathbb{P}\}$.

To prove this proposition, we will need the lemmas:

Lemma 2. Using (B1), (IV), (K1), (S1):

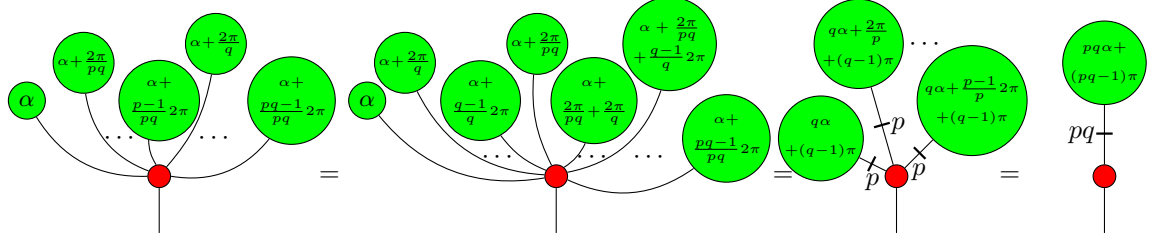
$$\begin{array}{c} \pi \quad \pi \\ \bullet \quad \bullet \\ | \quad | \\ \alpha \quad \beta \end{array} = \begin{array}{c} \bullet \\ | \\ \bullet \\ | \quad | \\ \alpha \quad \beta \end{array} = \begin{array}{c} \pi \\ \bullet \\ | \quad | \\ \alpha \quad \beta \end{array} = \begin{array}{c} \pi \\ \bullet \\ | \\ \alpha + \beta \end{array}$$

Lemma 3. Using (S1), (K2), 2, (IV), (S3), (K1) and the 2π -periodicity of the green dot:

$$\begin{array}{c} \bullet \\ | \\ \alpha \end{array} = \begin{array}{c} \pi \\ \bullet \\ | \\ \alpha \end{array} = \begin{array}{c} \pi \quad \pi \\ \bullet \quad \bullet \\ | \quad | \\ -\alpha \quad \alpha \end{array} = \begin{array}{c} \pi \\ \bullet \\ | \\ \alpha \end{array} = \begin{array}{c} \bullet \\ | \\ \pi \\ | \\ \alpha \end{array} = \begin{array}{c} \bullet \\ | \\ \pi \\ | \quad | \\ \alpha \quad \pi \end{array} = \begin{array}{c} \bullet \\ | \\ \pi \\ | \\ \alpha \end{array} = \begin{array}{c} \bullet \\ | \\ \pi \\ | \\ \alpha \end{array}$$

Proof (Proposition 5).

Derivability: If n is not prime, its supplementarity can be derived. Indeed, suppose n can be decomposed in two numbers p and q ($n = pq$), for which we know the supplementarity rule.



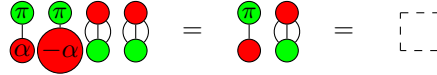
with p -ticked edge representing p parallel wires.

If p or q are not prime numbers, then we can inductively derive their supplementarity rule by decomposing them. Finally:

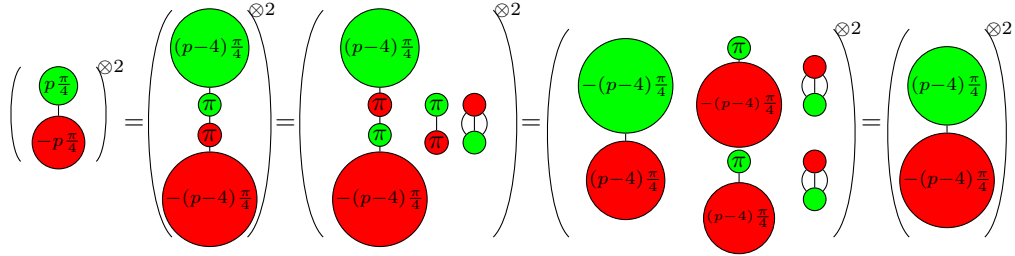
$$\forall n \in \mathbb{N}^*, \quad ZX, (\text{SUP}_p)_{p \in \mathbb{P}} \vdash (\text{SUP}_n)$$

Necessity: Let $p \in \mathbb{P}$ and $p \geq 3$. Let us consider the interpretation $[\![\cdot]\!]_{1, p-1}^{\otimes 2}$ which amounts to multiplying all the angles of a diagram by p , and duplicating the result.

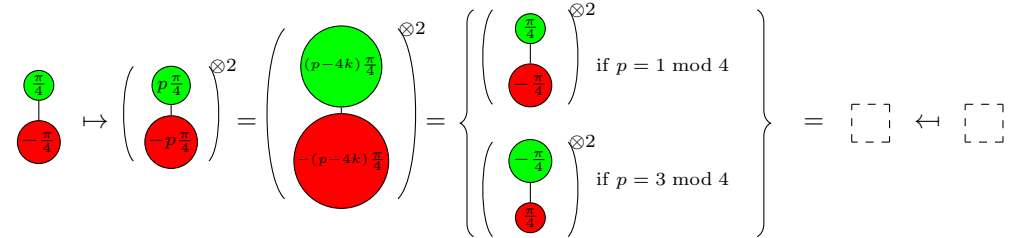
- Since p is odd, all the rules of the ZX hold [3].
- The rule (E) also holds thanks to the duplication. First notice that, using lemmas 2 and 3:



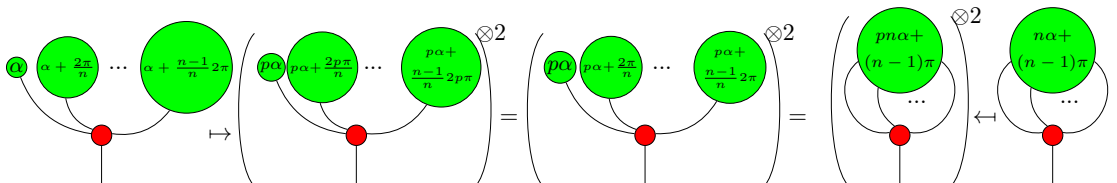
We can show that, using (S1), (K2), (H), and the previous result:



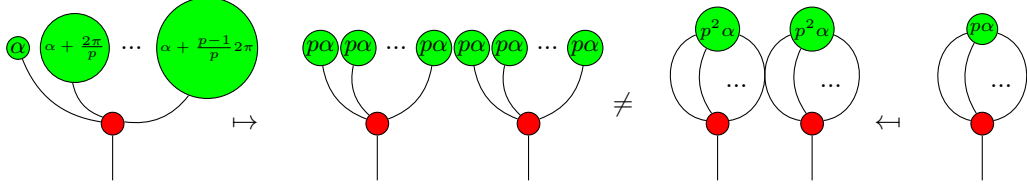
Finally, using the previous result k times:



- The rule (SUP_n) when $n \in \mathbb{P}$, $n \neq p$ holds, since $p \wedge n = 1$:



– The rule (SUP_p) does not hold:



We can show that the two interpretations are different for any angle of the $\frac{\pi}{2p}$ -fragment. Using the same reasoning as for the proof of soundness, an angle α verifying the equality would verify:

$$\begin{cases} 1 + e^{ip^2\alpha} = (1 + e^{ip\alpha})^p \\ 1 - e^{ip^2\alpha} = (1 - e^{ip\alpha})^p \end{cases}$$

By taking the square of the modulus of the two equations, we get:

$$\begin{cases} 2 + 2 \cos(p^2\alpha) = (2 + 2 \cos(p\alpha))^p \\ 2 - 2 \cos(p^2\alpha) = (2 - 2 \cos(p\alpha))^p \end{cases} \Rightarrow 4 = (4 - 4 \cos^2(p\alpha))^p$$

by adding the two lines. So an angle α verifying the two initial equations would also necessarily be of the form $\alpha = \pm \frac{1}{p} \arccos \sqrt{1 - 4^{\frac{1}{p}-1}} + \frac{2k\pi}{p}$, and none of the angles of the $\frac{\pi}{2p}$ -fragment are. More straightforwardly, if $\alpha = \frac{k\pi}{2p}$, then

$$(4 - 4 \cos^2(p\alpha))^p = \left(4 - 4 \cos^2\left(\frac{k\pi}{2}\right)\right)^p = \begin{cases} 0 & \text{if } k = 0 \pmod{2} \\ 4^p & \text{if } k = 1 \pmod{2} \end{cases} \neq 4$$

Every rule but the p -supplementarity (with $p \in \mathbb{P}$ and $p \geq 3$) holds with this interpretation, so it cannot be derived from the others:

$$\forall p \in \mathbb{P}, p \geq 3, \quad ZX, (\text{SUP}_n)_{n \in \mathbb{P}, n \neq p} \not\vdash (\text{SUP}_p)$$

Remark 2. It is important to notice that we have not proven that (SUP_2) could not be derived from the rest. Indeed, the family of interpretations used above only works when p is odd.

Remark 3. We can also notice that all the rules (SUP_n) respect the quantity I_G , so that the rule (E) is still necessary, even if we add this new set of rules.

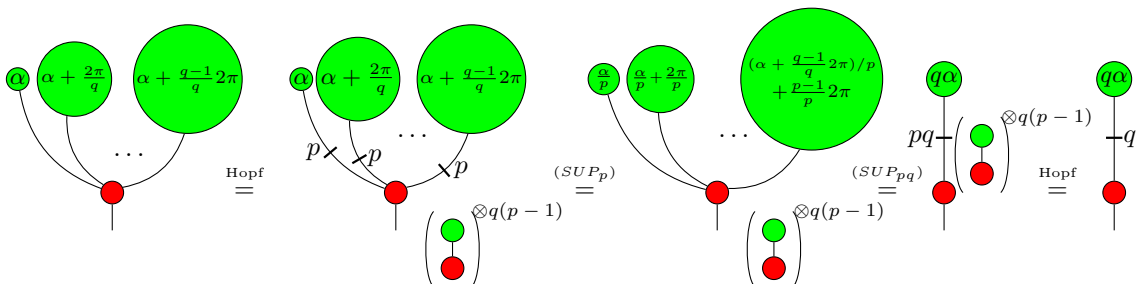
4.2 Discussion on the Supplementarity's Derivability Structure

Let p and q be two natural numbers. We have previously shown that $ZX \cup \{(\text{SUP}_p), (\text{SUP}_q)\} \vdash (\text{SUP}_{pq})$, meaning that we can deduce the supplementarity equality of a number from the equalities of the numbers it is a multiple of. Now, can we deduce this same equality from the equality of some of its multiples? The first result is when p is odd:

Proposition 6.

$$\forall p, q \in \mathbb{N}^* \quad (p = 1 \pmod{2}) \Rightarrow \{(\text{HL}), (\text{IV}), (\text{SUP}_p), (\text{SUP}_{pq})\} \vdash (\text{SUP}_q)$$

Proof. With (HL) the Hopf Law and (IV) the inverse rule, which are both derivable from any of the three sets of rules given:



With p -ticked edge representing p parallel wires. Those are created – always by multiples of two – using the Hopf Law. The 3rd diagram is obtained by rearranging the branches so that we can use (SUP_{pq}) .

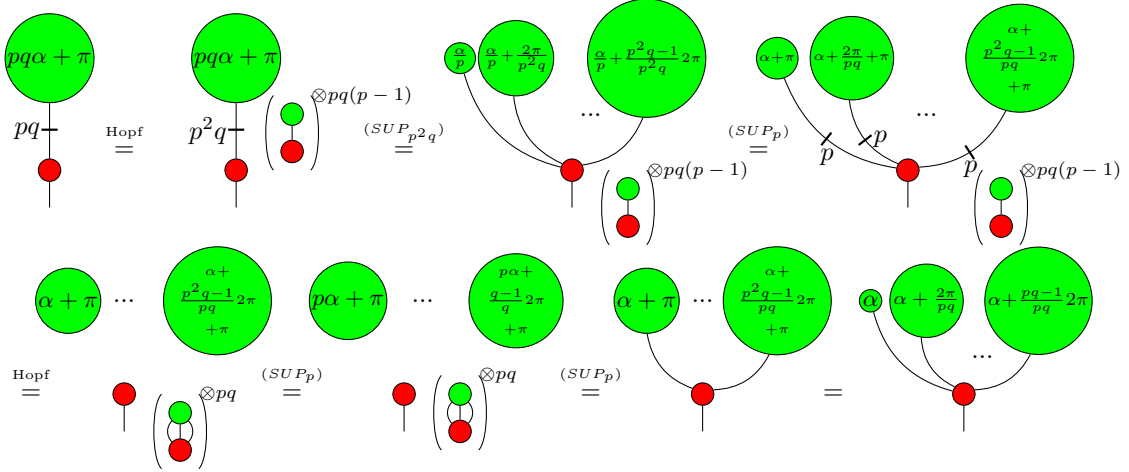
There exists another – weaker – derivation when p is even.

Proposition 7.

$$\forall p, q \in \mathbb{N}^* \quad \{(\text{HL}), (\text{IV}), (\text{SUP}_p), (\text{SUP}_{p^2q})\} \vdash (\text{SUP}_{pq})$$

Proof. If p is odd, then the previous proposition implies the wanted result.

Now, if $p = 0 \pmod 2$:



For the 5th equality, we use the equality 1. The last equality is just a rearranging of the branches.

To sum up, with the rules of the ZX-Calculus depicted in figure 3:

$$\begin{aligned} ZX \cup \{(\text{SUP}_p), (\text{SUP}_q)\} &\vdash (\text{SUP}_{pq}) \\ ZX \cup \{(\text{SUP}_p), (\text{SUP}_{p^2q})\} &\vdash (\text{SUP}_{pq}) \\ (p = 1 \pmod 2) &\Rightarrow ZX \cup \{(\text{SUP}_p), (\text{SUP}_{pq})\} \vdash (\text{SUP}_q) \end{aligned}$$

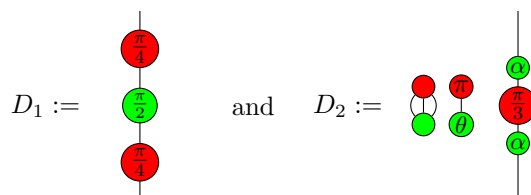
After all this, we propose to add the generalisation of the supplementarity rule to the set of rules of the ZX-Calculus, and to restrict to the set necessary when dealing with particular fragments. The new set of rules of the ZX-Calculus is shown in figure 2.

4.3 The General ZX-Calculus is still not complete

The argument given by Schröder de Witt and Zamdzhiev [6] to show the incompleteness of the general ZX-Calculus is not valid any more – when multiplying the angles by any integer, there is at least one supplementarity that does not hold. But we can patch the demonstration to make it valid again.

Proposition 8. *The general ZX-Calculus is incomplete with the set of rules in figure 2.*

Proof. Consider the following diagrams:



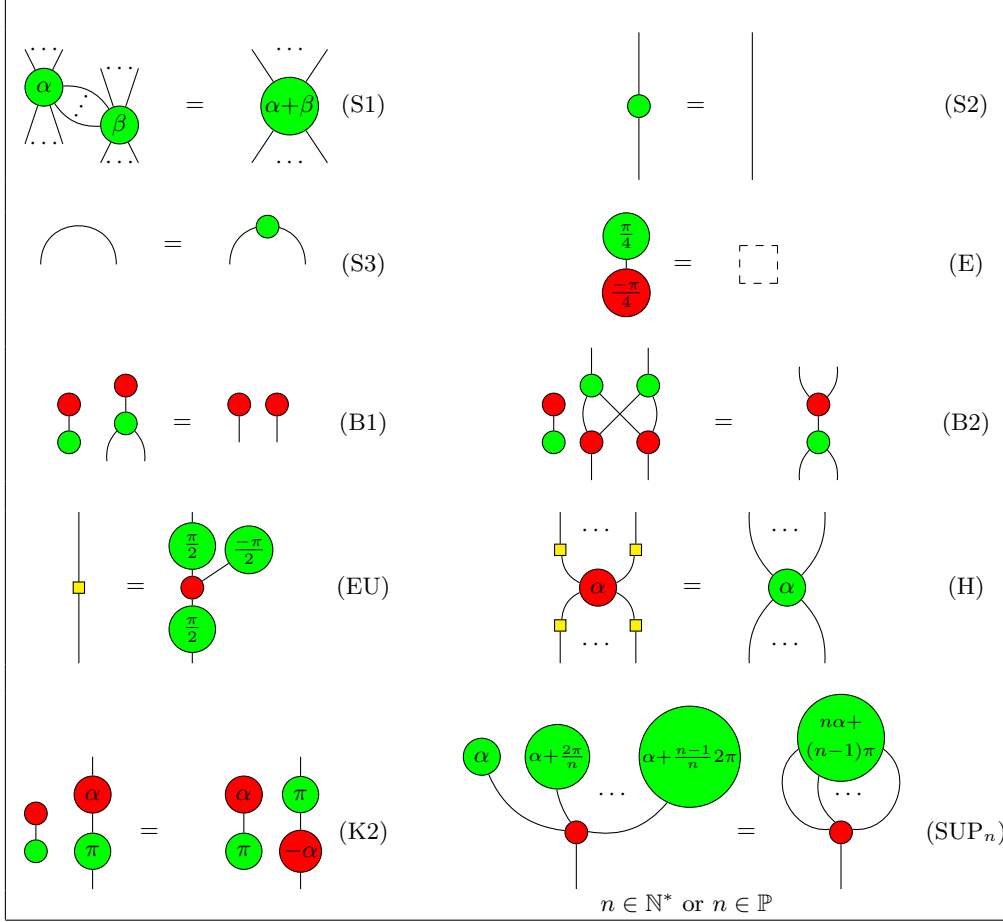


Fig. 2. New set of rules for the ZX-calculus with scalars. All of these rules also hold when flipped upside-down, or with the colours red and green swapped. The right-hand side of (IV) is an empty diagram. (\dots) denote zero or more wires, while $(\cdot\cdot)$ denote one or more wires.

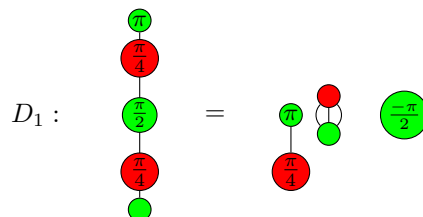
with $\alpha = \arccos\left(\sqrt{\frac{1}{3}}\right)$ and $\theta = \arccos\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{6}\right)$.
Then we can compute $\llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket$.

α is not a rational multiple of π :

One can check that $e^{i\alpha}$ is a root of the polynomial $3X^4 + 2X^2 + 3$ which is irreducible in \mathbb{Z} (since 30203 is a prime number, thanks to Cohn's irreducibility criterion, $3X^4 + 2X^2 + 3$ is irreducible in \mathbb{Z}). The polynomial is not cyclotomic because its coefficient of higher degree is not 1, hence $e^{i\alpha}$ is not a root of unity, i.e. α is not a rational multiple of π .

No more than four values for α are possible when decomposing D_1 in the form D_2 :

When applying the π green state on top and the 0 green state at the bottom of both D_1 and D_2 , we end up with:



And, using (SUP₂):

$$D_2 : \begin{array}{c} \text{red } \pi \\ \text{green } \theta \\ \text{red } \frac{\pi}{3} \\ \text{green } \alpha \\ \text{green } \alpha \\ \text{green } \pi \end{array} = \begin{array}{c} \text{red } \pi \\ \text{green } \theta \\ \text{green } \alpha + \pi \\ \text{red } \frac{\pi}{3} \\ \text{green } \alpha \end{array} = \begin{array}{c} \text{red } \pi \\ \text{green } \theta \\ \text{green } 2\alpha + \pi \\ \text{red } \frac{\pi}{3} \end{array} = \begin{array}{c} \text{red } \pi \\ \text{green } \theta \\ \text{red } \frac{\pi}{3} \\ \text{green } 2\alpha + \pi \\ \text{red } \pi \\ \text{green } \theta \\ \text{green } \pi \end{array}$$

Their interpretations are equal, which means:

$$e^{i\frac{\pi}{4}} \sqrt{2} e^{-i\frac{\pi}{4}} = \frac{1}{2} e^{i\theta} (1 + e^{i\frac{\pi}{3}}) (1 + e^{i(2\alpha+\pi)})$$

$$\text{i.e. } \frac{\sqrt{2}}{2} = e^{i(\theta + \frac{\pi}{6} + \alpha + \frac{\pi}{2})} \cos\left(\frac{\pi}{6}\right) \cos\left(\alpha + \frac{\pi}{2}\right)$$

So using the modulus, $|\cos(\alpha + \frac{\pi}{2})| = \sqrt{\frac{2}{3}}$, thus $\alpha = \pm \frac{\pi}{2} \pm \arccos\left(\sqrt{\frac{2}{3}}\right)$.

Now assume $ZX \vdash D_1 = D_2$, then there exists a finite sequence of rules of the ZX: R_1, \dots, R_n that when correctly applied transform D_2 into D_1 . We will write $D_1 = R_1 \dots R_n D_2$, no matter where the rules are applied.

Now, let

$$q = \max\{p \mid (\text{SUP}_p) \in \{R_1, \dots, R_n\}\}$$

$$S = \{k(q+4)! + 1 \mid k \in \mathbb{N}\}$$

It is easy to show that, $\forall q' \in S$ and for $[\cdot]_{q'}$ the interpretation that multiplies the angles by q' , $\forall i, ZX \vdash [R_i]_{q'}$ and $[D_1]_{q'} = D_1$.

Then, $D_1 = R_1 \dots R_n D_2 \Rightarrow [D_1]_{q'} = [R_1]_{q'} \dots [R_n]_{q'} [D_2]_{q'}$.

Since $\forall i, ZX \vdash [R_i]_{q'}$ and $[D_1]_{q'} = D_1$, we can prove $ZX \vdash D_1 = [D_2]_{q'}$. D_1 has a finite number of decompositions in the form D_2 , but $\{[D_2]_{q'} \mid q' \in S\}$ is infinite – since α is an irrational multiple of π – and all these diagrams are decompositions of D_1 in the form D_2 , hence we end up with a contradiction.

So $ZX \not\vdash D_1 = D_2$, which proves the incompleteness.

5 Application to Trigonometry

First of all, we need a way to create the scalars $\cos(\alpha)$ and $\sin(\alpha)$. To do so, we may notice that:

$$\left[\begin{array}{c} \text{red } \pi \\ \text{green } -\alpha \\ \text{red } 2\alpha \\ \text{green } \frac{\pi}{2} \end{array} \right] = \sqrt{2} e^{-i\alpha} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \circ \sqrt{2} e^{i\alpha} \begin{pmatrix} \cos(\alpha) & \\ & -i \sin(\alpha) \end{pmatrix} \times \frac{1}{2} = \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix}$$

Using the 0-red state in the first case, and the π -red-state in the second case, we can define:

Definition 2. The scalars $\cos(\alpha)$ and $\sin(\alpha)$ are defined as follows:

$$\cos(\alpha) = \left[\begin{array}{c} \text{red } 2\alpha \\ \text{green } \pi \\ \text{green } -\alpha \\ \text{green } \pi \\ \text{green } \pi \end{array} \right] \quad \sin(\alpha) = \left[\begin{array}{c} \text{red } 2\alpha - \pi \\ \text{green } \pi \\ \text{green } \frac{\pi}{2} - \alpha \\ \text{green } \pi \\ \text{green } \pi \end{array} \right]$$

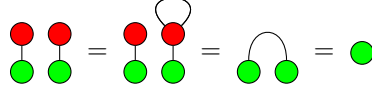
We also define the trivial scalars:

Definition 3.

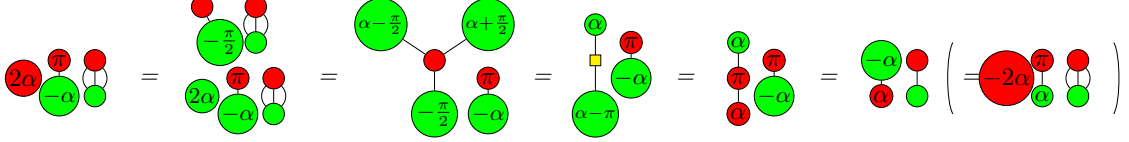
$$1 = \left[\begin{array}{c} - \\ - \\ - \\ - \end{array} \right] \quad 0 = \left[\begin{array}{c} \text{red } \pi \end{array} \right] \quad -1 = \left[\begin{array}{c} \text{red } \pi \\ \text{green } \pi \end{array} \right] \quad \text{so} \quad (-1)^k = \left[\begin{array}{c} \text{red } \pi \\ \text{green } k\pi \end{array} \right]$$

The following lemmas will be used to prove the upcoming propositions:

Lemma 4. Using (S1), (S3) and (B1):



Lemma 5. Using 3, (IV), (SUP₂), (EU), (S1) and (K2):



5.1 Results that do not use the Generalised Supplementarity

Proposition 9. Using the previous definitions, we can derive $\cos(k\pi) = (-1)^k$, $\sin(k\pi) = 0$, $\cos(\pi/2 + k\pi) = 0$ and $\sin(\pi/2 + k\pi) = (-1)^k$.

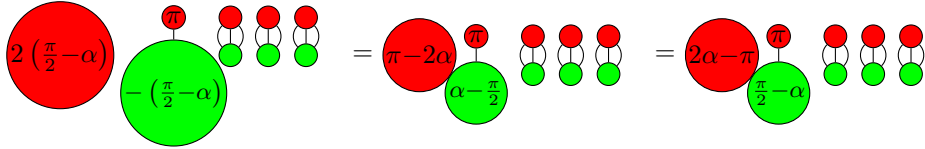
Proof. Obvious when using 4 and (IV).

Proposition 10. Using the previous definition, we can show that the equality $\cos(-\alpha) = \cos(\alpha)$ is derivable in the ZX-Calculus.

Proof. The result is obvious when using the lemma 5.

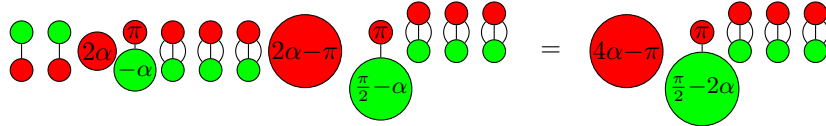
Proposition 11. Using the previous definition, we can show that the equality $\cos(\frac{\pi}{2} - \alpha) = \sin(\alpha)$ is derivable in the ZX-Calculus.

Proof. Replacing α with $(\frac{\pi}{2} - \alpha)$ and using the previous result:



Proposition 12. With the previous definition, the equality $2 \sin(\alpha) \cos(\alpha) = \sin(2\alpha)$ is derivable in the ZX-Calculus.

Proof. Using (SUP₂), (IV) and lemma 2:

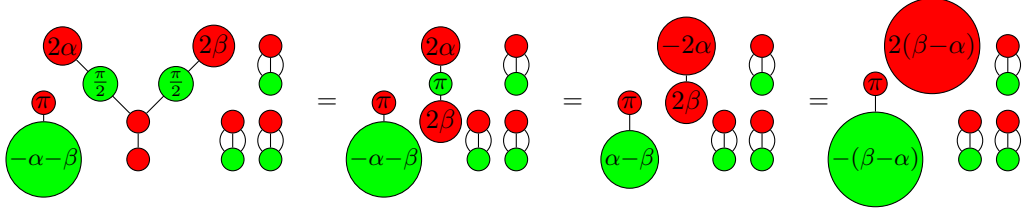


Another interesting vector can be defined this way:

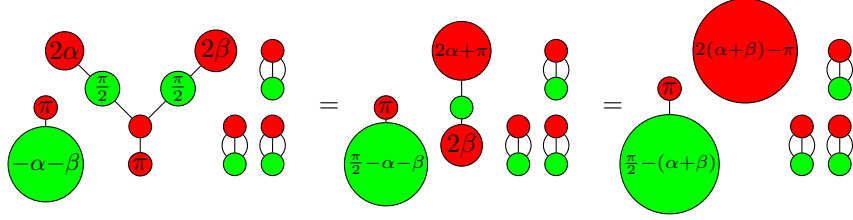
$$\begin{pmatrix} \cos(\alpha) \cos(\beta) + \sin(\alpha) \sin(\beta) \\ \cos(\alpha) \sin(\beta) + \sin(\alpha) \cos(\beta) \end{pmatrix} = \left\| \begin{array}{c} \text{Diagram with nodes } 2\alpha, \frac{\pi}{2}, 2\beta, \pi, -\alpha-\beta \end{array} \right\| \left(= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \circ \begin{pmatrix} \cos(\alpha) \cos(\beta) \\ \cos(\alpha) \sin(\beta) \\ \sin(\alpha) \cos(\beta) \\ \sin(\alpha) \sin(\beta) \end{pmatrix} \right)$$

Proposition 13. Using the previous diagram and the definitions of \cos and \sin , we can derive the equalities $\cos(\alpha) \cos(\beta) + \sin(\alpha) \sin(\beta) = \cos(\beta - \alpha)$ and $\cos(\alpha) \sin(\beta) + \sin(\alpha) \cos(\beta) = \sin(\alpha + \beta)$

Proof. Using (S2), (S1), (K2) and lemma 2:



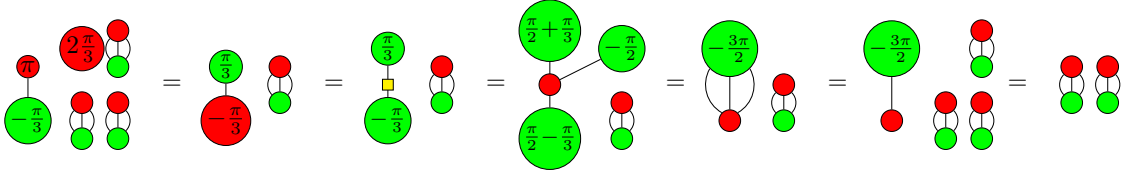
and



5.2 Results that use the Generalised Supplementarity

Proposition 14. *With the previous definition of \cos , we can derive the equality $\cos(\frac{\pi}{3}) = \frac{1}{2}$.*

Proof. Using the lemma 5, (H), (EU), (SUP₃), (HL), (IV):



This last equality is generalisable. Indeed, using only derivable equalities, we can show from $\cos(\frac{\pi}{3}) = \frac{1}{2}$ that $\prod_{k=1}^3 2 \cos\left(\frac{2k-1}{4 \times 3} \pi\right) = \sqrt{2}$.

Proposition 15. *The equality $\prod_{k=1}^n 2 \cos\left(\frac{2k-1}{4n} \pi\right) = \sqrt{2}$ can be derived with the latest version of the ZX-Calculus (figure 2).*

Proof. In appendix at page 15.

Proposition 16. *This equality cannot be derived in the ZX-Calculus without the generalised supplementarity and the new rule (E).*

Proof. If n is an odd prime number, then the interpretation $\llbracket \cdot \rrbracket_{1, n-1}^{\# \otimes 2}$ – used to show the necessity of (SUP _{n}) – holds for all the rules of the ZX without the generalised supplementarity, but does not hold for the equality parametrised by n . Moreover, (E) is used to show the equality for all $n \geq 2$.

6 Conclusion

In this paper, we have proved that the $\frac{\pi}{4}$ -fragment was not complete when taking the scalars into account, showing that a semantically true equality could not be derived in the ZX-Calculus. We have added this equality to the set of rules, which relieved us from 2 other axioms. We have also shown that a generalised version of the supplementarity rule was semantically correct and that it was not derivable in an infinite number of fragments.

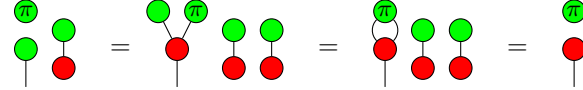
With the most complete version of the ZX-Calculus in figure 2, we have shown that the rule (E) was necessary – not derivable from the other rules – and that the rules (SUP _{p}) with p an odd prime are also necessary. (SUP₂) seems to be also necessary, but no proof has been found yet to show it – the interpretation used in [3] does not work any more when the generalised version of the supplementarity rule is present.

References

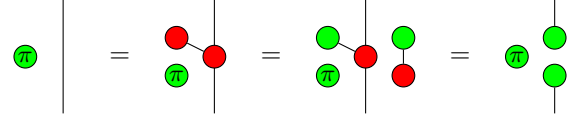
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2. Miriam Backens (2014): *The ZX-calculus is complete for stabilizer quantum mechanics*. *New Journal of Physics* 16(9), p. 093021, doi:[10.1088/1367-2630/16/9/093021](https://doi.org/10.1088/1367-2630/16/9/093021).
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6. Christian Schröder de Witt & Vladimir Zamdzhiev (2014): *The ZX-Calculus is incomplete for quantum mechanics*. *Electronic Proceedings in Theoretical Computer Science* 270(1), pp. 285-292. <https://arxiv.org/abs/1404.3633>.
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7 Appendix

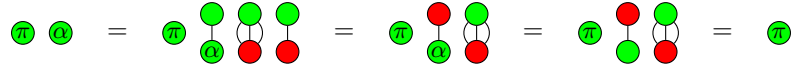
Proof (Proposition 3). Using (B1), (SUP), (HL):



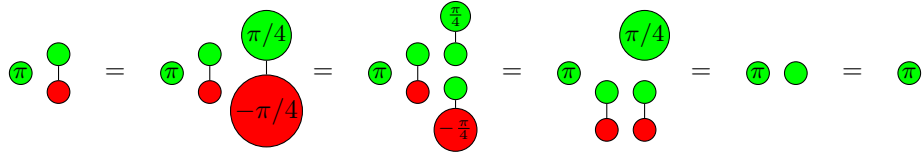
Using (S2), (S1), the previous result, (B1):



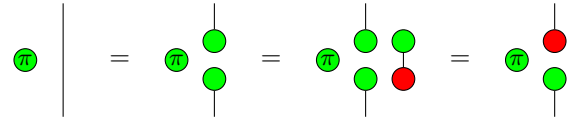
Using (IV), (S1), 3 and the previous results:



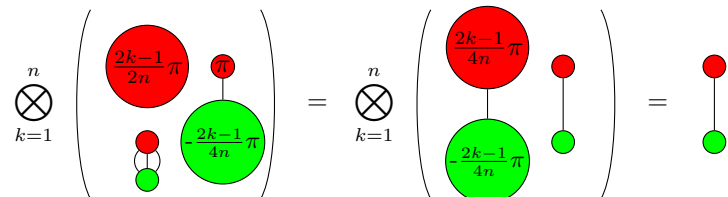
Notice that so far we have not used (E). It becomes necessary in the following, since I_G does not hold there. Using (E), the previous results, (S1) and lemma 4:



Finally:



Proof (Proposition 15). Using the lemma 5, the equation can be expressed as:



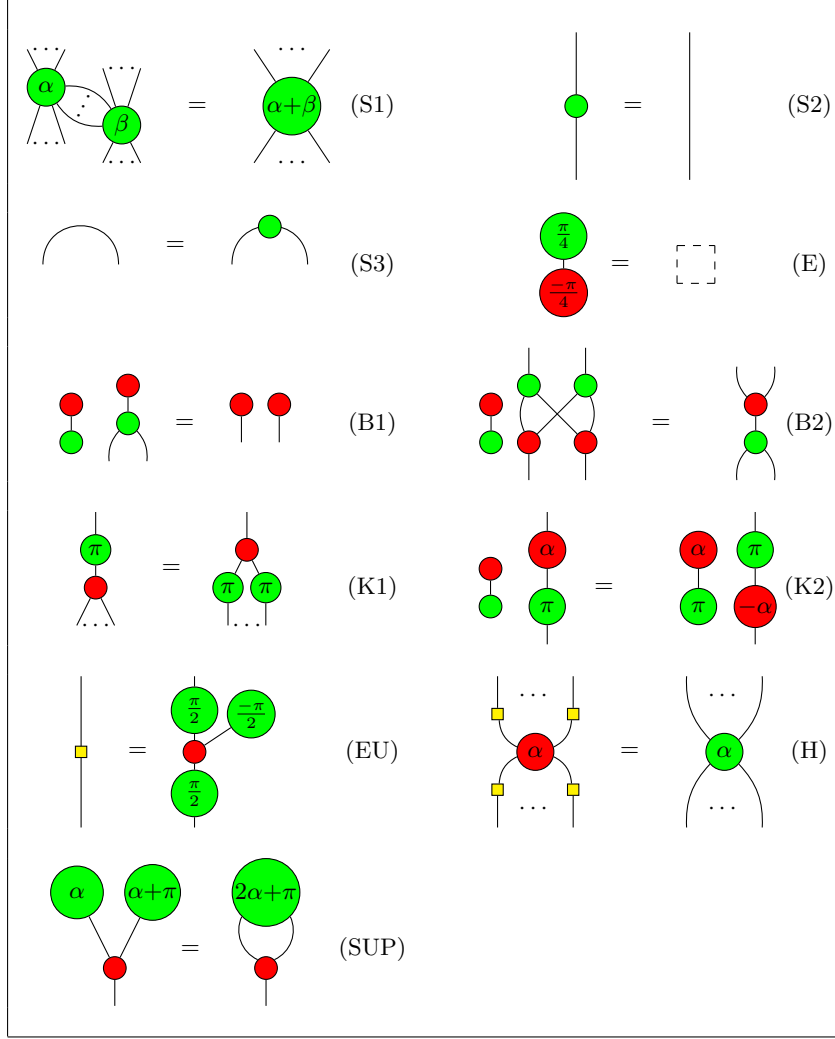


Fig. 3. Intermediate set of rules for the ZX-calculus with scalars. All of these rules also hold when flipped upside-down, or with the colours red and green swapped. The right-hand side of (IV) is an empty diagram. (\dots) denote zero or more wires, while $(\cdot\cdot)$ denote one or more wires.

First note that when applied to $n = 1$, the equality amounts to the rule (E). We can notice that when n is even, proving this equality amounts to showing it for $\frac{n}{2}$:

$$\begin{aligned}
 \bigotimes_{k=1}^n \left(\begin{array}{c} \text{Red circle: } \frac{2k-1}{2n}\pi \\ \text{Green circle: } -\frac{2k-1}{4n}\pi \end{array} \right) &= \bigotimes_{k=1}^{\frac{n}{2}} \left(\begin{array}{c} \text{Red circle: } -\frac{2k-1}{2n}\pi \\ \text{Green circle: } \frac{2k-1}{4n}\pi \end{array} \right) \bigotimes_{l=1}^{\frac{n}{2}} \left(\begin{array}{c} \text{Red circle: } \pi - \frac{2l-1}{2n}\pi \\ \text{Green circle: } \frac{2l-1}{4n}\pi - \frac{\pi}{2} \end{array} \right) \\
 &= \bigotimes_{k=1}^{\frac{n}{2}} \left(\begin{array}{c} \text{Red circle: } \pi - \frac{2k-1}{n}\pi \\ \text{Green circle: } \frac{2k-1}{2n}\pi - \frac{\pi}{2} \end{array} \right) \bigotimes_{l=1}^{\frac{n}{2}} \left(\begin{array}{c} \text{Red circle: } \frac{2l-1}{2(n/2)}\pi \\ \text{Green circle: } -\frac{2l-1}{4(n/2)}\pi \end{array} \right)
 \end{aligned}$$

Hence, any of these equalities can be reduced to the case where n is odd. Let us consider the original equality with n replaced by $(2m + 1)$, and let us also assume that this m is even. Then:

$$\begin{aligned}
\frac{\pi}{2} &= (2m+1)\frac{\pi}{2} = \bigotimes_{k=-m}^m \left(\frac{\pi}{2} + \frac{2k\pi}{2m+1} \right) \\
&= \bigotimes_{k=-m}^{-(\frac{m}{2}+1)} \left(-\frac{\pi}{2} - \frac{2k\pi}{2m+1} \right) \bigotimes_{k=-\frac{m}{2}}^{\frac{m}{2}} \left(\frac{\pi}{2} + \frac{2k\pi}{2m+1} \right) \bigotimes_{k=\frac{m}{2}+1}^m \left(\frac{3\pi}{2} - \frac{2k\pi}{2m+1} \right) \\
&= \bigotimes_{k=1}^{\frac{m}{2}} \left(\frac{4k-1}{2(2m+1)}\pi \right) \bigotimes_{k=1}^{m+1} \left(-\frac{4k-1}{2(2m+1)}\pi \right) \bigotimes_{k=1}^{\frac{m}{2}} \left(\frac{4k-3}{2(2m+1)}\pi \right) \bigotimes_{k=\frac{m}{2}+1}^m \left(\frac{4k-1}{2(2m+1)}\pi \right) \\
&= \bigotimes_{k=1}^{2m+1} \left(\frac{2k-1}{2(2m+1)}\pi \right) \bigotimes_{k=\frac{m/2+1}^{m/2}} \left(-\sum_{k=1}^{\frac{m}{2}} \frac{4k-1}{2(2m+1)}\pi + \sum_{k=\frac{m}{2}+1}^m \frac{8m-4k+5}{2(2m+1)}\pi \right)
\end{aligned}$$

Since:

$$\begin{aligned}
-\sum_{k=1}^{\frac{m}{2}} \frac{4k-1}{2(2m+1)}\pi + \sum_{k=\frac{m}{2}+1}^m \frac{8m-4k+5}{2(2m+1)}\pi &= \frac{m\pi}{4(2m+1)} - \frac{4\pi}{2(2m+1)} \left(\sum_{k=1}^m k \right) + \frac{m\pi}{4} + \frac{2m+3m^2}{2(2m+1)}\pi \\
&= \frac{m\pi + m(2m+1) + 4m + 6m^2 - 4m(m+1)}{4(2m+1)} = \frac{m\pi}{2}
\end{aligned}$$

$$\begin{aligned}
\frac{\pi}{4} \frac{\pi}{4} \frac{\pi}{4} &= \frac{\pi}{4} \frac{\pi}{4} \frac{\pi}{4} = \frac{\pi}{4} \frac{\pi}{4} \frac{\pi}{4} = \frac{\pi}{4} \frac{\pi}{4} \frac{\pi}{4} = \bigotimes_{k=1}^{2m+1} \left(\frac{2k-1}{2(2m+1)}\pi \right)
\end{aligned}$$

And finally:

$$\bigotimes_{k=1}^{2m+1} \left(\frac{2k-1}{2(2m+1)}\pi \right) = \bigotimes_{k=1}^{2m+1} \left(\frac{2k-1}{2(2m+1)}\pi \right) \bigotimes_{k=1}^{2m+1} \left(-\sum_{k=1}^{\frac{2m+1}{4}} \frac{2k-1}{4(2m+1)}\pi \right) = \frac{\pi}{4} \frac{\pi}{4} \frac{\pi}{4} \frac{\pi}{4} = \frac{\pi}{4} \frac{\pi}{4} = \frac{\pi}{4}$$

because

$$-\sum_{k=1}^{\frac{2m+1}{4}} \frac{2k-1}{4(2m+1)}\pi = \frac{2m+1}{4(2m+1)} - \frac{\pi}{2(2m+1)} \sum_{k=1}^{\frac{2m+1}{4}} k = -\frac{2m+1}{4}\pi$$

and since m is even:

$$\frac{\pi}{4} - \frac{m\pi}{2} - \frac{2m+1}{4}\pi = -m\pi = 0 \pmod{2\pi}$$

The reasoning is the same – up to the indexing – when m is odd.