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Cloaking via change of variables for second order quasi-linear elliptic differential equations

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Abstract

The present paper introduces and treats the cloaking via change of variables in the framework of quasi-linear elliptic partial differential operators belonging to the class of Leray-Lions (cf. [14]). We show how a regular near-cloak can be obtained using an admissible (nonsingular) change of variables and we prove that the singular change-of-variable-based scheme achieve perfect cloaking in any dimension $d \geq 2$. We thus generalize previous results (cf. [7], [11]) obtained in the context of electric impedance tomography formulated by a linear differential operator in divergence form.

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1 Introduction

Since ages, the electromagnetic invisibility has long been fascinating men and has had a big interest in the fantasy literature and researchers. From Plato who evoked the invisibility in the myth of Gyges (the myth of the magic ring making any wearer invisible) to the contemporary character of Harry Potter, she was associated with the magic (wizardry, witchery), the supernatural. Its irruption into the scientific and technical sphere could only arouse curiosity or surprise a wide audience.

However just in 2006, Simultaneously Ulf Leonhardt([13]) and John Pendry ([19]) proposed similar concepts to achieve perfect invisibility. The initial idea can be however simple: If one follows the Snell-Descartes Laws, one can also distorts the light path and arranged that in order to avoid a particular area, that area and all objects that can contain become de facto invisible! Of course, this physical idea could germinate in a context where various ingredients seemed to make it realistic.

The key point discovered by Leonhardt and Pendry in 2006 is that the form of Maxwell equations that describe the propagation of electromagnetic waves are invariant under (conformal)-coordinates changes. Therefore the path of the light may be decomposed: In one hand, in a set of intrinsic equations valid whatever the environment and which do not depend on the geometry (the new medium dielectric permittivity $\varepsilon(x)$ and the new medium magnetic permeability $\mu(x)$ are depending only on the transformation and not on the geometry). In the

other hand in a set of equations that characterize the environment and depend on geometry. It is thus possible to associate the change of geometry, that is to say the transformations of the space or coordinates in optical properties and to translate deformation of space in effects equivalent materials: This is the main idea of the transformational Optic, a new way of designing optical devices (the conformal transformations, applicable to two-dimensional case, have been long used by opticians and are special cases for which the properties of material are isotropic). So just, imagine a clever but simple distortion of the space, and apply the principle of transformational optic in order to obtain the optical characteristics of an invisibility Cape/cloak.

Of course, the question that immediately arises is whether we have materials at our disposal, which exhibit suitable characteristics. Let us say immediately that these materials do not exist in nature. One of the most active areas of modern optics is precisely the conception of metamaterials, that is to say structures in which the basic cells, more or less similar, are of sizes of the order of magnitude equal or below the wavelength and contain constituted substructures of conventional materials. The basic cells and their orderly arrangement give the total structure of new optical properties very different from those of basic materials and impossible to find in the range of natural homogeneous materials.

For an exhaustive knowledge/history of the electromagnetic invisibility, see(A. Nicolet [17])

The mechanism, using the change of variables, for obtaining electromagnetic invisibility/cloaking was similar to that of Greenleaf, Lassas and Uhlmann (cf [7]) obtained earlier in 2003 for constructing conductivity ensuring a cloaking in the case of electric impedance tomography. Later Kohn, Shen, Vogelius and Weinstein (cf [11]) in 2008 and that of Kohn, Onofrei, Vogelius and Weinstein (cf [12]) in 2010 use the same mechanism to generalize and improve the results of Greenleaf et al not only for electric impedance tomography but also for the Helmholtz equation. Another adaptations use the change of variables based cloaking to elastic sensing in 2006 ([16]) or to acoustic in 2007 ([3])

Let us remark here that the Dirichlet-to-Neumann-Map and the solution of the Calderón problem ([4] and [2]) play a crucial role for previous research on the cloaking.

the linear partial differential equations in divergence form formalising the context of **Electric Impedance Tomography** are generalized in nonlinear partial differential equations in the framework of the weighted p-Laplace equations (cf [4] and [10]).

In relation with this nonlinear generalization and for the investigation of the Calderón problem, interesting results are obtained (cf [4] and [9]).

The p-Laplace equation has applications in e.g. image processing, fluid mechanics, plastic moulding and modelling of sand-pile.

In contrast to previous investigations on the cloaking, the weighted p-Laplace equations are however, for every $p \neq 2$, not invariant under admissible (conformal) change of variables (i.e. coordinates change). Since the p-Laplace operators belong to the class of Leray-Lions (cf. [14]), we will prove in this paper the following useful results :

An admissible (conformal) coordinates change, transforms a quasi-linear elliptic differential equation in a quasi-linear elliptic differential equation given by operators in the class of Leray-Lions ([14]). We can hence obtain the method of coordinates change for the investigation of cloaking in this nonlinear context formalized by Leray-Lions.

The paper is organized as follows: We begin, in section 2, by introducing frequently used notions needed for the determination of the nonlinear variational Dirichlet problem, introduced in section 3, for second order quasi-linear elliptic differential equations in the class of Leray-Lions, ([9]) and generalizing the weighted p-Laplace equations ([10]).

In section 3, we recall, on Lipschitz domain Ω in \mathbb{R}^d with $d \geq 2$, functions from $\Omega \times \mathbb{R}^d$ to \mathbb{R}^d satisfying structural assumptions defining second order quasi-linear elliptic differential operators/equation belonging to the class of Leray-Lions. We then use ([9]) to recall the solution of the Dirichlet problem given a boundary function in **the Sobolev-Slobodečki**

space $W^{1-\frac{1}{p},p}(\partial\Omega)$ and to recall the Dirichlet-to-Neumann Map.

In section 4, we define admissible change of variables and we prove that it transforms a second order quasi-linear elliptic differential equation in the class of Leray-Lions to a equation of the same art. Moreover this admissible change of variables leaves invariant the Dirichlet-to-Neumann-Map.

In section 5, we adopt the framework of second order quasi-linear elliptic differential equations to define the notion of cloaking. Further, we consider, for Ω the ball B_2 of center $x = 0$ and radius 2, the admissible (nonsingular) change of variables F_ρ for $0 < \rho < 1$ used in ([7] and [11]), and define an associated regular near cloak. We then prove that the unit ball B_1 is almost invisible (in the sence of item2 of theorem 2) as ρ is sufficiently small. Further we prove in theorem 3 that the regular near cloak does not focus on the radial case

For the singular change of variables F , used in previous works(see among others ([7], [11], [12])), we define an associated singular cloak (given on the shell $B_2 \setminus B_1$ by a transformation by F to a given second order quasi-linear elliptic differential operator and on B_1 by an arbitrary second order quasi-linear elliptic differential operator)and we show that it gives a perfect invisibility for the ball B_1 . We prove, in theorem 4, that the potentiel outside the cloaked region with Dirichlet data $\varphi \in W^{1-\frac{1}{p},p}(\partial\Omega)$ is the same obtained for $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$ in (cf. [11] pages 17 and 19) in the linear case of second order differential operator in divergence form modelling the electric impedance tomographie.

The last section is devoted to the application of our results to the linear case treated in ([7], [11])and we prove that our theorem 5 is a generalisation, at the framework of equations in the class of Leray-Lions, of theorem 3 in (cf. [11]) A direct application as for the Laplacian (2-Laplacian) to the p-laplacian for $p \neq 2$ will be treated in a next work...

Because of the nonlinear character of our work, we have developed other methods not related to previous results such as from [5] or from [11]in the linear case.

2 Preliminaries

Recall on Sobolev spaces and related fields

We assume hier that Ω is a bounded Lipschitz domain in \mathbb{R}^d with $d \geq 2$. For $p \in [1, +\infty]$, denote by $L^p(\Omega)$ and $W^{1,p}(\Omega)$ the usual Lebesgue and first Sobolev spaces. We denote by $C^{0,1}(\bar{\Omega})$ the space of all Lipschitz continuous functions on the closure $\bar{\Omega}$ of Ω . The boundary $\partial\Omega$ is equipped with the $(d-1)$ dimensional Hausdorff measure dH when we work with the Lebesgue space $L^p(\partial\Omega)$. Moreover, $C(\partial\Omega)$ denotes the set of all real valued continuous functions on $\partial\Omega$.

2.1 Trace on the boundary

Since Ω is a Lipschitz domain, by J. Nečas ([18] Theorem 4.2, 4.6 and 3.8) the mapping $u \mapsto u|_{\partial\Omega}$ from $C^{0,1}(\bar{\Omega})$ to $C^{0,1}(\partial\Omega)$ has a unique continuous extension mapping

$$T_r : W^{1,p}(\Omega) \longrightarrow L^{p^*}(\partial\Omega)$$

called **trace operator** with $p^* = \frac{p(d-1)}{d-p}$ if $1 \leq p < d$, $p^* \geq 1$ if $p = d$ and $p^* = \infty$. For convenience, it is written $u|_{\partial\Omega}$ instead of $T_r(u)$ for $u \in W^{1,p}(\Omega)$ even if u does not belong to $C(\bar{\Omega})$ and we call $u|_{\partial\Omega}$ or $T_r(u)$ the trace of u .

2.2 Sobolev-Slobodečki space on the boundary

It is well known (cf. J. Nečas [18] section 3.8) that

- $\ker(T_r) = \{u \in W^{1,p}(\Omega) | T_r u = 0\}$ the kernel of T_r coincide with the Sobolev space $W_0^{1,p}(\Omega)$.
- $Rg(T_r) = \{T_r u | u \in W^{1,p}(\Omega)\}$ the range of T_r coincide with **the Sobolev-Slobodečki space** $W^{1-\frac{1}{p},p}(\partial\Omega)$ (cf. [8]) defined as the linear subspace of all $\varphi \in L^p(\partial\Omega)$ with finite semi-norm

$$[\varphi]_p^p = \int_{\partial\Omega} \int_{\partial\Omega} \frac{|\varphi(x) - \varphi(y)|^p}{|x - y|^{d-2+p}} dx dy$$

and equipped with the norm

$$\|\varphi\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} = \|\varphi\|_{L^p(\partial\Omega)} + [\varphi]_p$$

for every $\varphi \in W^{1-\frac{1}{p},p}(\partial\Omega)$

Remark 2.1 *We have the following important results:*

- *The trace operator T_r has a linear bounded right inverse*

$$Z : W^{1-\frac{1}{p},p}(\partial\Omega) \rightarrow W^{1,p}(\Omega)$$

(cf.[18] theorem 5.7)

- *Another crucial propriety of a Lipschitz domain (cf.D. Hauer[9] Lemma 2.1) Ω is that: for $1 < p < \infty$ and for $1 \leq q < \infty$ the space $C^\infty(\bar{\Omega})$ lies dense in $W^{1,p}(\Omega)$ and the set $\{v \mapsto v|_{\partial\Omega}, v \in C^\infty(\bar{\Omega})\}$ is dense in $W^{1-\frac{1}{p},p}(\partial\Omega)$ and in $L^q(\partial\Omega)$.*

3 Second order quasi-linear elliptic differential equations/operators

throughout this paper, we assume that Ω is a bounded domain in \mathbb{R}^d with a Lipschitz boundary $\partial\Omega$, $d \geq 2$ and $1 < p < \infty$

Definition 3.1 *A function $a : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfies a structural assumptions, needed for the definition of a second order quasi-linear elliptic operator if :*

1. *a is a Caratheodory function satisfying the following assumptions:*

For some constants $0 < \alpha \leq \beta < \infty$

2. *$\langle a(x, \xi), \xi \rangle \geq \alpha |\xi|^p$ for a.e $x \in \Omega$ and every $\xi \in \mathbb{R}^d$*
3. *$|a(x, \xi)| \leq \beta |\xi|^{p-1}$ for a.e $x \in \Omega$ and every $\xi \in \mathbb{R}^d$*
4. *$\langle (a(x, \xi_1) - a(x, \xi_2)), (\xi_1 - \xi_2) \rangle > 0$ with $\xi_1 \neq \xi_2$ and for a.e $x \in \Omega$*
5. *$a(x, \lambda\xi) = \lambda |\lambda|^{p-2} a(x, \xi) \quad \forall \lambda \in \mathbb{R}, \lambda \neq 0$ and for a.e $x \in \Omega$*

Where \langle, \rangle is the scalar product in \mathbb{R}^d and $|\xi| = \langle \xi, \xi \rangle^{\frac{1}{2}}$.

Remark 3.2 *1. Under the previous assumptions, the Caratheodory function a defines a second order quasi-linear elliptic operator A , belonging to the class of Leray-Lions operators (cf.[14]) as follows:*

$$Au := -\operatorname{div}(a(x, \nabla u)) \quad \text{in } D'(\Omega) \quad \text{for every } u \in W_{Loc}^{1,p}(\Omega).$$

Hier is $\nabla u = \nabla_x u$ is the gradient of u relatively to the variable x .

2. To the quasi-linear elliptic operator A is developed by (cf.[10]) a nonlinear potential theory.
3. In a very recent work by Daniel Hauer in (cf.[9]) a Dirichlet-to-Neumann map associated with the second order quasi-linear elliptic operator A , was defined and Follow-up of applications to elliptic and parabolic problems.
4. By taking $a(x, \xi) = \sigma(x)|\xi|^{p-2}\xi$ for $\xi \in \mathbb{R}^d$ and σ is a continuous and strongly elliptic matrix on Ω in the sense that for some constants $0 < m < M < \infty$:
 $m|\xi|^2 \leq \sigma(x)\xi, \xi \leq M|\xi|^2$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^d$.
 It is well known that the function a is a function satisfying the previous structural assumptions and the associated operator A is called the weighted p -Laplace operator (cf.[10])(resp.the classical p -Laplace Δ_p obtained by $\sigma(x) = Id_{\mathbb{R}^d}$) and is a prototype of quasi-linear elliptic differential operators belonging to the class of Leray-Lions operators.
5. If $p = 2$ then the weighted 2-Laplace operator is well known as a differential operator in divergence form known in physics as the PDE in electrostatics which is later known as a mathematical formalism for the Electric Impedance Tomography as follows:

$$\nabla \cdot (\sigma \nabla u) = \sum_{i,j} \frac{\partial}{\partial x_i} (\sigma_{ij}(x)) \frac{\partial u}{\partial x_j} \text{ in } \Omega$$

The electrical conductivity $\sigma(x) = \sigma_{ij}(x)$ is, as in remark 4, a strongly elliptic matrix-valued function on Ω . It relates voltage u and the associated electric fields ∇u to the resulting current $\sigma \nabla u$.

In what follows in this section, we will review (cf.D.Hauer [9]) some basic facts about quasi-linear elliptic equations given by

$$-\operatorname{div}(a(x, \nabla u)) = 0$$

Where a satisfies a structural assumptions.

Let $\varphi \in W^{1-\frac{1}{p}, p}(\partial\Omega)$ (the Sobolev-Slobodečki space on the boundary $\partial\Omega$).

We consider the following nonlinear variational Dirichlet problem:

$$\begin{cases} -\operatorname{div}(a(x, \nabla u)) = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega \end{cases} \quad (1)$$

Definition 3.3 1. We call $u \in W_{Loc}^{1,p}(\Omega)$ a weak solution of

$$-\operatorname{div}(a(x, \nabla u)) = 0 \text{ in } \Omega, \quad (2)$$

if u satisfies the following integral equation:

$$\int_{\Omega} (a(x, \nabla u)) \nabla v dx = 0 \text{ for all } v \in W_0^{1,p}(\Omega), \quad (3)$$

2. A weak solution will be called a -harmonic function on Ω .

For later use, we need the following compactness results concerning weak solutions of (2), which is an immediate consequence of (D. Hauer [9] Lemma 2.3) and (L.Boccardo and F. Murat [1] Theorem 2.1 and Remark 2.1).

Lemma 3.4 *If (u_n) is a bounded sequence in $W^{1,p}(\Omega)$ of weak solution of (2), then there is subsequence u_{k_n} of u_n and a weak solution $u \in W^{1,p}(\Omega)$ of (2) such that (u_{k_n}) converges to u weakly in $W^{1,p}(\Omega)$, strongly in L^p , ∇u_{k_n} converges to ∇u strongly in $(L^q(\Omega))^d$ for every $q < p$ (thus a.e in Ω) and $a(x, \nabla u_{k_n})$ converges to $a(x, \nabla u)$ a.e in Ω and weakly in $L^{p'}(\Omega)$.*

Proposition 3.5 *Let $\varphi \in W^{1-\frac{1}{p},p}(\partial\Omega)$, then there exists a unique weak solution of (2) which is solution of the problem (1).*

Proof

Let $\Phi \in W^{1,p}(\Omega)$ be such that $\Phi|_{\partial\Omega} = \varphi$, it is well known that the operator $v \rightarrow A(v) = -\text{div}(a(x, \nabla v + \nabla\Phi))$ satisfies the assumptions of the Minty-Browder theorem [J.L.Lions [15], thm 2.1]. Hence the equation (2) admits a weak solution $u \in W^{1,p}(\Omega)$ satisfying $u - \Phi \in W_0^{1,p}(\Omega)$. We have then

$$\int_{\Omega} a(x, \nabla u) \nabla v dx = 0 \quad \forall v \in W_0^{1,p}(\Omega)$$

Let u_1 be another weak solution with $(u_1 - \Phi) \in W_0^{1,p}(\Omega)$ then $u - u_1 \in W_0^{1,p}(\Omega)$ and

$$\int_{\Omega} a(x, \nabla u_1) (\nabla u - \nabla u_1) dx = \int_{\Omega} a(x, \nabla u) (\nabla u - \nabla u_1) dx = 0$$

and thus

$$\int_{\Omega} (a(x, \nabla u) - a(x, \nabla u_1)) (\nabla u - \nabla u_1) dx = 0.$$

By the strict monotonicity of the function a (assumption 4) we obtain $\nabla u = \nabla u_1$ and by the Poincaré Inequality we get $u = u_1$.

We arrive to the following definition

Definition 3.6 *For a given boundary value $\varphi \in W^{1-\frac{1}{p},p}(\partial\Omega)$, we call a function $u \in W^{1,p}$ a $W^{1,p}$ -solution of the Dirichlet problem (1) on Ω if $(u - Z\varphi) \in W_0^{1,p}(\Omega)$ and u is a weak solution of (2). Z is the linear bounded right inverse of the trace function operator on the boundary $\partial\Omega$ (Remark 2.1).*

In what follows, we will denote for every $\varphi \in W^{1-\frac{1}{p},p}(\partial\Omega)$ by $P\varphi = P^a\varphi$ the $W^{1,p}$ -solution of the Dirichlet problem (1) associated with the function a on Ω .

Remark 3.7 • *for every $\lambda \in \mathbb{R}$ and $\varphi \in W^{1-\frac{1}{p},p}(\partial\Omega)$, we have*

$$P\lambda\varphi = \lambda P\varphi \quad \text{and} \quad |P\varphi| \leq P|\varphi|$$

- *By (cf. [10] Theorem 3.70) we will choose $P\varphi$ a continuous function on Ω .*
- *For useful Properties of the application $P: W^{1-\frac{1}{p},p}(\partial\Omega) \rightarrow W^{1,p}(\Omega)$ (cf D. Hauer [9] lemma 2.5).*

Lemma 3.8 *Let D be a measurable subset in Ω , a and b are two Caratheodory functions respectively from $(\Omega \setminus D) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and from $D \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfying a structural assumptions given at the beginning of this section. Let c be the function from $\Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ given by*

$$c(x, \xi) := \begin{cases} b(x, \xi) & \text{for } x \in D \text{ and } \xi \in \mathbb{R}^d, \\ a(x, \xi) & \text{for } x \in \Omega \setminus D \text{ and } \xi \in \mathbb{R}^d \end{cases} \quad (4)$$

Then c is a function from $\Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfying a structural assumptions.

3.1 Dirichlet-to-Neumann Map (DNP) associated with a quasi-linear elliptic operator

If for a given boundary value $\varphi \in W^{1-\frac{1}{p},p}(\partial\Omega)$, $P\varphi$ and $a(x, \nabla P\varphi)$ are smooth enough up to the boundary $\partial\Omega$ and that ν denote the outward pointing unit normal vector on $\partial\Omega$, the co-normal derivative of $P\varphi$ (associated with the quasi-linear elliptic operator A defined by the function a as at the beginning of this section), on $\partial\Omega$ is formally defined by the dot product $a(x, \nabla P\varphi) \cdot \nu$ on $\partial\Omega$.

It is well known, (e.g. D. Hauer [9]), that the Dirichlet-to-Neumann operator Λ associated with A assign to each Dirichlet boundary data φ the corresponding co-normal derivative of $P\varphi$. We formally set:

$$\Lambda\varphi = a(x, \nabla P\varphi) \cdot \nu$$

Multiplying the equality by some function $\psi \in C^\infty(\overline{\Omega})$ with respect to the inner product on $L^2(\partial\Omega)$ and applying Green's formula yields

$$\int_{\partial\Omega} (\Lambda\varphi)\psi|_{\partial\Omega} dH = \int_{\Omega} a(x, \nabla P\varphi)\nabla\psi dx = \int_{\Omega} a(x, \nabla P\varphi)\nabla Z\psi dx$$

H is the $d - 1$ dimensional Hausdorff measure on the boundary $\partial\Omega$. If in addition $\Lambda\varphi \in L^{p'}(\partial\Omega)$, then by an approximation argument using Remark 2.1 item 2, we can conclude that

$$\int_{\partial\Omega} \Lambda\varphi\psi dH = \int_{\Omega} a(x, \nabla P\varphi)\nabla Z\psi dx \quad \text{for every } \psi \in W^{1-\frac{1}{p},p}(\partial\Omega)$$

If φ and ψ belong to $W^{1-\frac{1}{p},p}(\partial\Omega)$, the integral on the right-hand side of this equation exists, we easily see that the functional $\psi \mapsto \int_{\Omega} a(x, \nabla P\varphi)\nabla Z\psi dx$ belong to the dual space $W^{-(1-\frac{1}{p}),p'}(\partial\Omega)$ of $W^{1-\frac{1}{p},p}(\partial\Omega)$. This justifies why the following definition makes sense and is consistent to the case of smooth functions.

Definition 3.9 We call the mapping $\Lambda : W^{1-\frac{1}{p},p}(\partial\Omega) \longrightarrow W^{-(1-\frac{1}{p}),p'}(\partial\Omega)$, defined by

$$\langle \Lambda\varphi, \psi \rangle = \int_{\Omega} a(x, \nabla P\varphi)\nabla Z\psi dx$$

for every $\varphi, \psi \in W^{1-\frac{1}{p},p}(\partial\Omega)$, the **Dirichlet-to-Neumann map associated with the second order quasi-linear differential operator A** defined by the Function a (see Remark 3.1 1.).

We recall hier the following interesting results (cf. D.Hauer[9] Proposition 3.3).

Proposition 3.10 The mapping $\Lambda : W^{1-\frac{1}{p},p}(\partial\Omega) \longrightarrow W^{-(1-\frac{1}{p}),p'}(\partial\Omega)$ has the following properties:

1. $\langle \Lambda\varphi, \psi \rangle = \int_{\Omega} a(x, \nabla P\varphi)\nabla Z\psi dx$ for every ψ, φ

2. Λ is continuous and monotone, that is for every φ_1, φ_2

$$\langle \Lambda\varphi_1 - \Lambda\varphi_2, \varphi_1 - \varphi_2 \rangle \geq 0$$

3. There exists $C_1 > 0$ such that

$$\|\Lambda\varphi\|_{W^{-(1-\frac{1}{p}),p'}(\partial\Omega)} \leq C_1 \|\varphi\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}^{p-1} \quad \text{for every } \varphi \in W^{1-\frac{1}{p},p}(\partial\Omega).$$

4. There exists a constant $C_2 > 0$ such that

$$\langle \Lambda\varphi, \varphi \rangle \geq C_2 \|\varphi\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}^P \quad \text{for every } \varphi \in W_m^{1-\frac{1}{p},p}(\partial\Omega)$$

Hier $W_m^{1-\frac{1}{p},p}(\partial\Omega)$ is the subspace of all $\varphi \in W^{1-\frac{1}{p},p}(\partial\Omega)$ satisfying the so-called compatibility condition $\int_{\partial\Omega} \varphi dx = 0$ equipped with the induced norm and $W_m^{-(1-\frac{1}{p}),p'}(\partial\Omega)$ is the dual space of $W_m^{1-\frac{1}{p},p}(\partial\Omega)$

4 Change of variables

Along this entire section we consider a Lipschitz domain Ω and a function a satisfying a structural assumptions (cf. section 3).

Let U be an open subset in Ω and G a change of variables given by $y = G(x)$ (where G is an invertible, differentiable and oriented preserving) from U to $G(U)$ and DG (resp. DG^T) the (resp. transposed of the) Jacobian matrix of G with elements $a_{ij} = \frac{\partial y_i}{\partial x_j}$.

We set for u and v in $W^{1,p}(U)$ $I := \int_U a(x, \nabla_x u) \nabla_x v dx$. By the change of variables $y = G(x)$ we get:

$$I = \int_{G(U)} \frac{1}{\det DG(G^{-1}(y))} DG(G^{-1}(y)) a(G^{-1}(y), (DG(G^{-1}(y)))^T \nabla_y \tilde{u}(y)) \nabla_y \tilde{v}(y) dy$$

Where $\tilde{u}(y) = u(G^{-1}(y))$ and $\tilde{v}(y) = v(G^{-1}(y))$ If we set

$$a_G(y, \xi) := \frac{1}{\det DG(G^{-1}(y))} DG(G^{-1}(y)) a(G^{-1}(y), (DG(G^{-1}(y)))^T \xi) \quad \text{for } y \in G(U)$$

then:

$$\int_U a(x, \nabla_x u) \nabla_x v dx = \int_{G(U)} a_G(y, \nabla_y \tilde{u}) \nabla_y \tilde{v} dy$$

Proposition 4.1 *Let U be an open subset in Ω and G a differentiable function from U to $G(U)$. We have for every $x \in U$ and $i \in \{1, \dots, d\}$, the eigenvalues $\lambda_i(x), \beta_i(x)$ of the symmetric matrices $(DG(x))^T DG(x)$ and $DG(x)(DG(x))^T$ are nonnegative and hence for every $\theta \in \mathbb{R}^d$ and $x \in U$:*

$$(\min(\lambda_i(x), i \in \{1, \dots, d\}))^{\frac{1}{2}} |\theta| \leq |DG(x)\theta| \leq (\max(\lambda_i(x), i \in \{1, \dots, d\}))^{\frac{1}{2}} |\theta|$$

and

$$(\min(\beta_i(x), i \in \{1, \dots, d\}))^{\frac{1}{2}} |\theta| \leq |(DG(x))^T \theta| \leq (\max(\beta_i(x), i \in \{1, \dots, d\}))^{\frac{1}{2}} |\theta|$$

Proof

Let for $i \in \{1, \dots, d\}$, $\lambda_i(x), \beta_i(x)$ the eigenvalues and u_i, v_i the orthonormal bases of eigenvectors respectively for the symmetric matrices $(DG(x))^T DG(x)$ and $DG(x)(DG(x))^T$. Then

for every $\theta \in \mathbb{R}^d$, there exist real numbers θ_i, μ_i such that $\theta = \sum_{i=1}^n \theta_i u_i = \sum_{i=1}^n \mu_i v_i$. We thus have:

$$|DG(x)\theta|^2 = \langle (DG(x))^T DG(x)\theta, \theta \rangle = \langle \sum_{i=1}^n \theta_i \lambda_i(x) u_i, \sum_{i=1}^n \theta_i u_i \rangle = \sum_{i=1}^n \theta_i^2 \lambda_i(x)$$

and

$$|DG(x)^T \theta|^2 = \langle DG(x)(DG(x))^T \theta, \theta \rangle = \langle \sum_{i=1}^n \mu_i \beta_i(x) v_i, \sum_{i=1}^n \mu_i v_i \rangle = \sum_{i=1}^n \mu_i^2 \beta_i(x)$$

the sought inequalities are then easy to obtain.

Corollary 4.2 *If the Jacobian matrix of the change of variables G is symmetric then for every $\theta \in \mathbb{R}^d$ and $x \in U$, we have*

$$(\min(|\sigma_i(x)|, i \in \{1, \dots, d\}))|\theta| \leq |DG(x)\theta| \leq (\max(|\sigma_i(x)|, i \in \{1, \dots, d\}))|\theta|$$

where $\sigma_i(x), i \in \{1, \dots, d\}$ are the eigenvalues of the symmetric matrix $DG(x)$.

Proposition 4.3 *Let U be an open subset in Ω and G a change of variables from U to $G(U)$. Assume that there exist c_1, c_2 positive constants such that*

$c_1 < \det DG(G^{-1}(y)) < c_2$ for every $y \in G(U)$, then the function $a_G^{G(U)}$ given by $a_G^{G(U)}(y, \xi) = a_G(y, \xi)I_{G(U)}(y)$ satisfies on $G(U) \times \mathbb{R}^d$ a structural assumptions as by a

Proof

We set $x = G^{-1}(y)$. Let for $i \in \{1, \dots, d\}$, $\lambda_i(y), \beta_i(y)$ the eigenvalues and u_i, v_i the orthonormal bases of eigenvectors respectively for the symmetric matrices $(DG(x))^T DG(x)$ and $DG(x)(DG(x))^T$. From the hypothesis on the determinant of the matrix $(DG(x))$ and since the real numbers $\lambda_i(x), \beta_i(x)$ are nonnegative, we easily obtain that the positive real numbers $\alpha_1 = \min(\lambda_i(x), i \in \{1, \dots, d\}), x \in U$, $\alpha_2 = \max(\lambda_i(x), i \in \{1, \dots, d\}), x \in U$, $\delta_1 = \min(\beta_i(x), i \in \{1, \dots, d\}), x \in U$ and $\delta_2 = \max(\beta_i(x), i \in \{1, \dots, d\}), x \in U$ are positive. By the previous corollary we have;

$$\alpha_1|\theta| \leq |DG(x)\theta| \leq \alpha_2|\theta|$$

and

$$\delta_1|\theta| \leq |(DG(x))^T \theta| \leq \delta_2|\theta|$$

for every $\theta \in \mathbb{R}^d$ and every $x \in U$.

For α and β the constants given by the assumptions 2 and 3 on the function a , the previous inequalities and the hypothesis, we have

$$\langle a_G(y, \xi), \xi \rangle \geq \frac{\alpha}{\det DG(x)} |(DG(x))^T \xi|^p \geq \frac{\alpha(\alpha_1)^p}{\det DG(x)} |\xi|^p \geq \frac{\alpha(\alpha_1)^p}{c_2} |\xi|^p$$

for every $y \in G(U)$. It follows that the function $a_G^{G(U)}$ satisfies on $G(U)$ the assumption 2 with the constant $\frac{\alpha(\alpha_1)^p}{c_2}$.

Moreover, since

$$|a_G(y, \xi)| = \frac{1}{\det DG(x)} |(DG(x))a(x, (DG(x))^T \xi)|$$

we have

$$|a_G(y, \xi)| \leq \frac{\alpha_2}{c_1} |a(x, (DG(x))^T \xi)| \leq \frac{\beta \alpha_2}{c_1} |(DG(x))^T \xi|^{p-1} \leq \frac{\beta \alpha_2 (\delta_2)^{p-1}}{c_1} |\xi|^{p-1}$$

we hence obtain that the function $a_G^{G(U)}$ satisfies on $G(U)$ the assumption 3 with the constant $\beta = \frac{(\alpha_2)^p}{c_1}$.

The other assumptions 4 and 5 and that the function $a_G^{G(U)}$ is a Caratheodory function from $G(U) \times \mathbb{R}^d$ to \mathbb{R}^d are easy verified.

Definition 4.4 (admissible change of variables)

We will say that a piece-wise smooth function $G(y = G(x))$ from Ω to Ω is an admissible change of variables, if G is invertible, $G(x) = x$ at the boundary $\partial\Omega$, there exists a subset E such that E as well $G(E)$ are closed and of Lebesgue measure zero and G and G^{-1} are continuously differentiable respectively on $\Omega \setminus E$ and on $\Omega \setminus G(E)$. Moreover there exist two positive constants c_1 and c_2 such that $c_1 < \det G(x) < c_2$ for every $x \in \Omega \setminus E$

Lemma 4.5 A function G from Ω to Ω is an admissible change of variables if and only if G^{-1} is an admissible change of variables.

Definition 4.6 Let a be a function from $\Omega \times \mathbb{R}^d$ to \mathbb{R}^d satisfying a structural assumptions and G an admissible change of variables. Then the transformed a_G of a by G is defined as follows:

$$\begin{aligned} a_G(y, \xi) &:= \frac{1}{\det DG(x)} DG(x) a(x, (DG(x))^T \xi) && \text{if } y \notin G(E), \\ &:= 0 && \text{if } y \in G(E) \end{aligned} \quad (5)$$

Where $y = G(x)$ and E is the negligible closed subset in the definition of the admissible change of variables.

Theorem 1

Let G be an admissible change of variables on Ω and a_G the transformed of the Caratheodory function a by G then

1. a_G is a function on $\Omega \times \mathbb{R}^d$ satisfying a structural assumptions as a in section 3.
2. a function u is an a -harmonic function on Ω if and only if $\tilde{u} = u \circ G^{-1}$ is an a_G -harmonic function on Ω . The notion of a -harmonic function is referred hier to the definition 3.2 2.

Proof

We have, by the change of variables $y = G(x)$, the following relations between the Caratheodory Functions a and a_G :

$$\int_{\Omega \setminus E} a(x, \nabla_x u) \nabla_x v dx = \int_{\Omega \setminus G(E)} a_G(y, \nabla_y \tilde{u}) \nabla_y \tilde{v} dy$$

Since E and $G(E)$ are negligible we have

$$\int_{\Omega} a(x, \nabla_x u) \nabla_x v dx = \int_{\Omega} a_G(y, \nabla_y \tilde{u}) \nabla_y \tilde{v} dy$$

The function a_G satisfies bei Proposition 4.4 a same structural assumptions on Ω as the function a and because G is an admissible change of variables ($G(x) = x$ on the boundary), we have $v \in W_0^{1,p}(\Omega)$ if and only if $\tilde{v} \in W_0^{1,p}(\Omega)$. It is then easy to see that u is an a -harmonic function on Ω if and only if \tilde{u} is an a_G -harmonic on Ω

Remark 4.7 • the function a_G is called the push-forward of a by the change of variables G (cf.[10] or [11]).

- For the weighted p -Laplace given by $a(x, \xi) = \sigma(x)|\xi|^{p-2}\xi$ we have:

$$a_G(y, \xi) = \frac{1}{\det DG(G^{-1}(y))} DG(G^{-1}(y))\sigma(G^{-1}(y))(DG(G^{-1}(y)))^T |(DG(G^{-1}(y)))^T \xi|^{p-2}\xi$$

This shows that the transformation by G of the weighted p -Laplace operator has not the same form of a p -Laplace

We denote in what follows by $P^b\varphi$, for every $\varphi \in W^{1-\frac{1}{p},p}(\partial\Omega)$, the solution of the Dirichlet problem (1) with boundary value φ defined in section 3 for every function b satisfying a structural assumptions in section 3,

Corollary 4.8 *We have:*

$$P^{\tilde{a}}\varphi = P^{a_G}\varphi \text{ for all } \varphi \in W^{1-\frac{1}{p},p}(\partial\Omega) \quad \text{where} \quad P^{\tilde{a}}\varphi = P^a\varphi \circ G^{-1}.$$

Proof

Since $P\varphi = P^a\varphi$, for every $\varphi \in W^{1-\frac{1}{p},p}(\partial\Omega)$, then it is an a -harmonic solution, it follows that $\tilde{P}\varphi$ is, for every $\varphi \in W^{1-\frac{1}{p},p}(\partial\Omega)$ a solution of the Dirichlet problem (1) defined in section 3 for the Caratheodory Function a_G . Since a_G is by Proposition 4.4 a function from $\Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfying a structural assumptions (section 3), we hence obtain the corollary.

Proposition 4.9 *Let G be an admissible change of variables on Ω , and Λ_G is the Dirichlet-to-Neumann map associated with a_G then $\Lambda = \Lambda_G$*

Proof

We have for every φ, ψ in $W^{1-\frac{1}{p},p}(\partial\Omega)$ and by the definition 3.8 of the Dirichlet-to-Neumann Map associated with the Caratheodory function a

$$\langle \Lambda\varphi, \psi \rangle = \int_{\Omega} a(x, \nabla_x P^a\varphi) \nabla_x Z\psi dx$$

Since $P^a\varphi$ is a solution of the Dirichlet Problem (1) and $(Z\psi - \tilde{Z}\psi) \in W_0^{1,p}(\Omega)$, from the previous corollary 4.4 and from the definition of the Dirichlet-to-Neumann Map associated with the Caratheodory function a_G , we have:

$$\begin{aligned} \langle \Lambda\varphi, \psi \rangle &= \int_{\Omega} a(x, \nabla_x P^a\varphi) \nabla_x Z\psi dx \\ &= \int_{\Omega} a_G(y, \nabla_y P^{\tilde{a}}\varphi) \nabla_y \tilde{Z}\psi dy \\ &= \int_{\Omega} a_G(y, \nabla_y P^{a_G}\varphi) \nabla_y Z\psi dy \\ &= \langle \Lambda_G(\varphi), \psi \rangle \end{aligned} \tag{6}$$

Thus for every φ , $\Lambda(\varphi)$ and $\Lambda_G(\varphi)$ determine identical linear form on $W^{1-\frac{1}{p},p}(\partial\Omega)$. It follows that

$$\Lambda(\varphi) = \Lambda_G(\varphi) \quad \text{for every } \varphi \in W^{1-\frac{1}{p},p}(\partial\Omega)$$

and hence

$$\Lambda = \Lambda_G \quad \text{on } W^{1-\frac{1}{p},p}(\partial\Omega)$$

5 Cloaking via change of variables

Definition 5.1 (cf. [11, 12]) *We say a region of space is cloaked for a particular class of measurements if its contents - and even the existence of the cloak - are invisible using such measurements.*

Remark 5.2 *Let Ω be a relativ compact open set in \mathbb{R}^d such that the Dirichlet problem associated with a given partial differential operator is solvable for a class R of resolutive functions defined at the boundary (in a local setting) or at the complement (in a nonlocal setting) of Ω . We can then consider, for the definition of the cloaking, the class of solutions $S(f)_{f \in R}$ of the partial differential operator as the particular class of measurements.*

We introduce in what follows a definition for the cloaking in the framework of second order quasi-linear elliptic differential equations.

Definition 5.3 *Let a be a fixed function satisfying, as in the beginning of section 3, a structural assumptions and a measurable subset D with $\bar{D} \subset \Omega \subset \bar{\Omega} \subset \Omega'$, Ω and Ω' are open subset. Let a_c be function from $(\Omega' \setminus D) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfying locally a structural assumptions.*

We say that a_c cloaks the region D if its every extention a_D across D in the following form

$$a_D(x, \xi) := \begin{cases} b(x, \xi) & \text{for } x \in D \text{ and } \xi \in \mathbb{R}^d, \\ a_c(x, \xi) & \text{for } x \in \Omega \setminus D \text{ and } \xi \in \mathbb{R}^d \end{cases} \quad (7)$$

produce on Ω the same Dirichlet-to-Neumann-map as a , regardless of the choice of the Caratheodory function b satisfying on $\Omega \times \mathbb{R}^d$ a structural assumptions.

We now explain how change-of-variables-based cloaking for quasi-linear elliptic operators work. Our demarche is much more general but for simplicity we focus on the radial case: $\Omega = B_2$ where B_2 is the ball of center $x = 0$ and radius 2 and the region D to be cloaked is B_1 , the concentric ball of radius 1.

We start by explaining how B_1 can be nearly-cloaked using the following admissible change of variables: Fixing a small parameter $\rho > 0$, consider the piecewise-smooth change of variables from B_2 to B_2 :

$$F_\rho(x) = \begin{cases} \frac{x}{\rho} & \text{if } |x| \leq \rho, \\ \left(\frac{2-2\rho}{2-\rho} + \frac{1}{2-\rho}|x| \right) \frac{x}{|x|} & \text{if } \rho \leq |x| \leq 2, \end{cases} \quad (8)$$

F_ρ expands the ball $B_\rho = \{x \in \mathbb{R}^d / |x| \leq \rho\}$ to B_1 , while mapping the full domain B_2 to itself.

Lemma 5.4 *We have for every $x \in B_2 \setminus B_\rho$*

$$DF_\rho(x) = \left(\frac{1}{2-\rho} + \frac{1}{|x|} \frac{2-2\rho}{2-\rho} \right) I - \frac{2-2\rho}{2-\rho} \frac{1}{|x|} \hat{x} \hat{x}^T$$

$$\det DF_\rho(x) = \left(\frac{1}{2-\rho} \right) \left(\frac{1}{2-\rho} + \frac{1}{|x|} \frac{2-2\rho}{2-\rho} \right)^{d-1} = \frac{(|x| + 2(1-\rho))^{d-1}}{(2-\rho)^d |x|^{d-1}}$$

where I is the identity matrix and $\hat{x} = \frac{x}{|x|}$

Proof

An easy calculation gives the Jacobian matrix $DF_\rho(x)$ which implies that it is symmetric, \hat{x} is an eigenvector with eigenvalues $\frac{1}{2-\rho}$, and $\hat{x}^\perp = \{y/y \perp \hat{x}\}$ is a $d-1$ dimensional eigenspace with eigenvalue $\left(\frac{1}{2-\rho} + \frac{1}{|x|} \frac{2-2\rho}{2-\rho} \right)$.

Moreover for every $0 < \rho < 1$ the key properties of F_ρ are

- F_ρ is continuous and piecewise-smooth
- F_ρ and F_ρ^{-1} are admissible change of variables on B_2 (in the sense of definition 4.5).

5.1 The regular near cloak associated with F_ρ

Let b be an arbitrary Caratheodory function from $B_2 \times \mathbb{R}^d$ to \mathbb{R}^d satisfying structural assumptions of section 3.

Given a function a satisfying the same assumptions as b , we define the “**regular near cloak**” associated with F_ρ as follows:

$$a_b^\rho(y, \xi) = \begin{cases} b(y, \xi) & \text{for } y \in B_1 \text{ and } \xi \in \mathbb{R}^d, \\ a_{F_\rho}(y, \xi) & \text{for } y \in B_2 \setminus B_1 \text{ and } \xi \in \mathbb{R}^d \end{cases} \quad (9)$$

Lemma 5.5 *For every $0 < \rho < 1$, a_b^ρ is, by lemma 3.7 and theorem 1, a function from $B_2 \times \mathbb{R}^d$ to \mathbb{R}^d satisfying structural assumptions of section 3 and $u \in W^{1,p}$ is an a_b^ρ -harmonic solution if and only if $u \circ F_\rho$ is an $(a_b^\rho)_{F_\rho^{-1}}$ -harmonic solution for the following Caratheodory function:*

$$(a_b^\rho)_{F_\rho^{-1}}(x, \xi) = \begin{cases} b_{F_\rho^{-1}}(x, \xi) & \text{for } x \in B_\rho \text{ and } \xi \in \mathbb{R}^d, \\ a(x, \xi) & \text{for } x \in B_2 \setminus B_\rho \text{ and } \xi \in \mathbb{R}^d \end{cases} \quad (10)$$

Which gives that an a_b^ρ -harmonic function is an a -harmonic function outside the closed ball \overline{B}_ρ for every $0 < \rho < 1$.

In what follows, let $\varphi \in W^{1-\frac{1}{p},p}(\partial B_2)$ and $P_\rho \varphi$ the $W^{1,p}$ solution of the following quasi-linear elliptic equation

$$\begin{cases} -\operatorname{div}(a_b^\rho(x, \nabla_x u)) & = 0 & \text{in } B_2, \\ u & = \varphi & \text{on } \partial B_2. \end{cases} \quad (11)$$

We recall hier the following

Lemma 5.6 *Let K be a non-empty compact subset of \mathbb{R}^d and U an open set containing K , then there exists a function $\theta \in C_c^\infty(\mathbb{R}^d)$ such that $0 \leq \theta \leq 1$, $\theta = 1$ on K and $\theta = 0$ outside U*

Proposition 5.7 *There exist constants C_1 and C_2 such that for every $\varphi \in W^{1-\frac{1}{p},p}(\partial B_2)$ and $0 < \rho < 1$, we have*

- $$\left(\int_{B_2 \setminus B_\rho} |\nabla_x (P_\rho \varphi \circ F_\rho(x))|^p dx \right)^{\frac{1}{p}} \leq C_1 \|\varphi\|_{W^{1-\frac{1}{p},p}(\partial B_2)}$$

- $$\left(\int_{B_1} |\nabla_x P_\rho \varphi(x)|^p dx \right)^{\frac{1}{p}} \leq C_2 \|\varphi\|_{W^{1-\frac{1}{p},p}(\partial B_2)}$$

Proof

Let $0 < \rho_0 < 1$ and θ the function associated by the previous lemma with $K = \overline{B}_{\rho_0}$ and $U = B_1$. Let $v = (1 - \theta)P\varphi$ and $w := P_\rho \varphi \circ F_\rho - (1 - \theta)P\varphi$ on B_2 , Since F_ρ is an admissible change of variables, then $w \in W^{1,p}(B_2)$ and $\operatorname{Tr}(w) = 0$, where $\operatorname{Tr}(u)$ is the Trace on the boundary ∂B_2 of any function $u \in W^{1,p}(B_2)$, hence $w \in W_0^{1,p}(B_2)$. Let

$$I = \int_{B_2} (a_b^\rho)_{F_\rho^{-1}}(x, \nabla_x P_\rho \varphi \circ F_\rho(x)) (\nabla_x P_\rho \varphi \circ F_\rho(x)) dx$$

By the Lemma 5.5 and since $v = 0$ on B_{ρ_0} we get that for every $0 \leq \rho \leq \rho_0$

$$I = \int_{B_\rho} b_{F_\rho^{-1}}(x, \nabla_x P_\rho \varphi \circ F_\rho(x)) (\nabla_x P_\rho \varphi \circ F_\rho(x)) dx + \int_{B_2 \setminus B_\rho} a(x, \nabla_x P_\rho \varphi \circ F_\rho(x)) \nabla_x P_\rho \varphi \circ F_\rho(x) dx$$

and hence

$$I \leq \int_{B_2 \setminus B_\rho} a(x, \nabla_x P_\rho \varphi \circ F_\rho(x)) \nabla_x v dx$$

for α and β the constants given by the structural assumptions 2 and 3 on the function a and by the Hoelder inequality we have for p' such that $\frac{1}{p} + \frac{1}{p'} = 1$

$$\alpha \int_{B_2 \setminus B_\rho} |\nabla_x P_\rho \varphi \circ F_\rho(x)|^p dx \leq \beta \left(\int_{B_2 \setminus B_\rho} (|\nabla_x P_\rho \varphi \circ F_\rho(x)|^{p-1})^{p'} dx \right)^{\frac{1}{p'}} \left(\int_{B_2 \setminus B_\rho} |\nabla_x v(x)|^p dx \right)^{\frac{1}{p}}$$

Which gives:

$$\left(\int_{B_2 \setminus B_\rho} |\nabla P_\rho \varphi \circ F_\rho|^p dx \right)^{\frac{1}{p}} \leq \frac{\beta}{\alpha} \left(\int_{B_2} |\nabla_x v|^p dx \right)^{\frac{1}{p}} = \frac{\beta}{\alpha} \|\nabla_x v\|_p$$

It follows, since $\theta \in C_c^\infty(\mathbb{R}^d)$ that

$$\|\nabla_x v\|_p \leq (\|\nabla_x \theta\|_\infty + 1) \|P\varphi\|_{W^{1,p}(B_2)}$$

An easy calculation yields $\|P\varphi\|_{W^{1,p}(B_2)} \leq (1 + 2K\frac{\beta}{\alpha}) \|Z\varphi\|_{W^{1,p}(B_2)}$ where K is the Poincaré constant on B_2 . For $\|Z\|$ the norm of the linear bounded right inverse of the trace, we get

$$\|Z\varphi\|_{W^{1,p}(B_2)} \leq \|Z\| \|\varphi\|_{W^{1-\frac{1}{p},p}(\partial B_2)}$$

putting $C_1 = (1 + 2K(\frac{\beta}{\alpha}))\frac{\beta}{\alpha}(\|\nabla_x \theta\|_\infty + 1) \|Z\|$ we obtain the result sought.

Further bei the change of variable we have

$$\int_{B_\rho} b_{F_\rho^{-1}}(x, \nabla_x P_\rho \varphi \circ F_\rho(x)) (\nabla_x P_\rho \varphi \circ F_\rho(x)) dx = \int_{B_1} b(y, \nabla_y P_\rho \varphi(x)) (\nabla_y P_\rho \varphi(y)) dy$$

hence

$$\int_{B_1} b(y, \nabla_y P_\rho \varphi(y)) (\nabla_y P_\rho \varphi(y)) dy \leq \int_{B_2 \setminus B_\rho} a(x, \nabla_x P_\rho \varphi \circ F_\rho(x)) \nabla_x v dx$$

thus for α_b the constant corresponding to the assumption 2 on the function b we have

$$\begin{aligned} \alpha_b \int_{B_1} |\nabla_x P_\rho \varphi(x)|^p dx &\leq \beta \left(\int_{B_2 \setminus B_\rho} (|\nabla_x P_\rho \varphi \circ F_\rho(x)|^{p-1})^{p'} dx \right)^{\frac{1}{p'}} \left(\int_{B_2 \setminus B_\rho} |\nabla_x v(x)|^p dx \right)^{\frac{1}{p}} \\ &\leq \beta \left(\int_{B_2 \setminus B_\rho} |\nabla_x P_\rho \varphi \circ F_\rho(x)|^p dx \right)^{\frac{1}{p'}} \|\nabla v\|_p \\ &\leq \beta \left(\frac{\beta}{\alpha} \|\nabla v\|_p \right)^{\frac{p}{p'}} \|\nabla v\|_p \end{aligned} \tag{12}$$

it follows that

$$\left(\int_{B_1} |\nabla_x P_\rho \varphi(x)|^p dx \right)^{\frac{1}{p}} \leq \left(\frac{\beta}{\alpha_b} \right)^{\frac{1}{p}} \left(\frac{\beta}{\alpha} \right)^{\frac{1}{p'}} \|\nabla v\|_p = \left(\frac{\alpha}{\alpha_b} \right)^{\frac{1}{p}} \left(\frac{\beta}{\alpha} \right) \|\nabla v\|_p$$

Again putting,

$$C_2 = \left(\frac{\alpha}{\alpha_b} \right)^{\frac{1}{p}} (1 + 2K\frac{\beta}{\alpha}) \frac{\beta}{\alpha} (\|\nabla \theta\|_\infty + 1) \|Z\|$$

we obtain the sought result.

Corollary 5.8 *We have*

$$\left(\int_{B_2} |\nabla_x(P_\rho\varphi \circ F_\rho(x))|^p dx \right)^{\frac{1}{p}} \leq (C_1 + C_2) \|\varphi\|_{W^{1-\frac{1}{p},p}(\partial B_2)}$$

for every $\varphi \in W^{1-\frac{1}{p},p}(\partial B_2)$, every $0 < \rho < 1$ and every $p \leq d$.

Proof

We have Since $F_\rho(x) = \frac{x}{\rho}$ on B_ρ , by a change of variables we easy obtain

$$\int_{B_\rho} |\nabla_x(P_\rho\varphi \circ F_\rho(x))|^p dx \leq \rho^{d-p} \int_{B_1} |\nabla_y(P_\rho\varphi)|^p dy$$

hence

$$\left(\int_{B_\rho} |\nabla_x(P_\rho\varphi \circ F_\rho(x))|^p dx \right)^{\frac{1}{p}} \leq C_2 \|\varphi\|_{W^{1-\frac{1}{p},p}(\partial B_2)}$$

.

Proposition 5.9 *There exists a constant C such that*

$$\|P_\rho\varphi \circ F_\rho\|_{W^{1,p}(B_2)} \leq C \|\varphi\|_{W^{1-\frac{1}{p},p}(\partial B_2)}$$

for every $\varphi \in W^{1-\frac{1}{p},p}(\partial B_2)$, every $0 < \rho < 1$ and every $p \leq d$.

Proof

Since $F_\rho(x) = x$ for $x \in \partial B_2$, then $(P_\rho\varphi \circ F_\rho - P\varphi) \in W_0^{1,p}(B_2)$, by the Poincaré inequality we have $\|P_\rho\varphi \circ F_\rho - P\varphi\|_p \leq K \|\nabla(P_\rho\varphi \circ F_\rho - P\varphi)\|_p$ therefore

$$\|P_\rho\varphi \circ F_\rho\|_p \leq K \|\nabla P_\rho\varphi \circ F_\rho\|_p + \|P\varphi\|_p + K \|\nabla P\varphi\|_p \leq K \|\nabla P_\rho\varphi \circ F_\rho\|_p + (1+K) \|P\varphi\|_{W^{1,p}(B_2)}$$

thus

$$\|P_\rho\varphi \circ F_\rho\|_{W^{1,p}(B_2)} \leq ((C_1 + C_2)(K + 1) + (K + 1)(1 + 2K\frac{\beta}{\alpha} \|Z\|)) \|\varphi\|_{W^{1-\frac{1}{p},p}(\partial B_2)}$$

and it follows that

$$C = (2 + \frac{\alpha}{\alpha_b})^{\frac{1}{p}} (K + 1) (1 + 2K\frac{\beta}{\alpha}) \frac{\beta}{\alpha} (\|\nabla\theta\|_\infty + 1) \|Z\|$$

satisfies the sought inequality.

Proposition 5.10 *Let $p \leq d$ and $u_\rho(x) := P_\rho\varphi \circ F_\rho(x)$ for every $x \in B_2$. Then u_ρ is, for every $0 < \rho < 1$, an a -harmonic function outside the closed ball \bar{B}_ρ , which converge as $\rho \mapsto 0$ to $P\varphi$, the a -harmonic function solution of the problem (1) with Dirichlet data φ . Moreover u_ρ converge to $P\varphi$ as ρ tends to zero, weakly in $W^{1,p}(B_2)$ and strongly in L^p , $\nabla P_\rho\varphi$ converge to $\nabla P\varphi$ a.e on Ω and $a(x, \nabla P_\rho\varphi)$ converge to $a(x, \nabla P\varphi)$ a.e in B_2 and weakly in $L^{p'}(\Omega)$*

Proof

By the Lemma (5.5) and the transformed of a_b^p by the admissible change of variables F_ρ^{-1} , u_ρ is for every $0 < \rho < 1$ an a -harmonic function outside the closed ball \bar{B}_ρ . By the proposition 5.9, the theorem of Banach-Alaoglu, the lemma 3.4, a diagonal procedure and the trace theorem, we obtain that u_ρ converge to $P\varphi$ as ρ tends to zero, weakly in $W^{1,p}(B_2)$ and strongly in L^p and ∇u_ρ converge to $\nabla P\varphi$ a.e on B_2 and $a(x, \nabla u_\rho)$ converge to $a(x, \nabla P\varphi)$ a.e in B_2 and weakly in $L^{p'}(B_2)$

Corollary 5.11 *We have $\lim_{\rho \rightarrow 0} P_\rho \varphi(y) = P\varphi \circ F^{-1}(y)$ for every $1 < |y| \leq 2$ where $F(x) = \lim_{\rho \rightarrow 0} F_\rho(x) = (1 + \frac{1}{2}|x|)\frac{x}{|x|}$ for $x \neq 0$.*

Corollary 5.12 *For every $0 < \rho < 1$ the function $P_\rho \varphi \circ F_\rho$ is, locally uniformly bounded on $B_2 \setminus \overline{B_\rho}$.*

Proof

By Remark 3.7 item 1 and the Harnack inequality (cf. [10])

Proposition 5.13 *For every $\varphi \in W^{1-\frac{1}{p},p}(\partial B_2)$ and function $f \in L^p(B_2)$ we have*

$$\lim_{\rho \rightarrow 0} \int_{B_2 \setminus B_\rho} |(a(x, \nabla_x P_\rho \varphi \circ F_\rho) - a(x, \nabla_x P\varphi))f(x)| dx = 0$$

The convergence does not depend on the Caratheodory function b

Proof

We have for every measurable subset E of B_2

$$\begin{aligned} |\int_E I_{B_2 \setminus B_\rho} a(x, \nabla_x P_\rho \varphi \circ F_\rho) f(x) dx| &\leq \int_E I_{B_2 \setminus B_\rho} |a(x, \nabla_x P_\rho \varphi \circ F_\rho) f(x)| dx \\ &\leq \int_E I_{B_2 \setminus B_\rho} \beta |\nabla_x P_\rho \varphi \circ F_\rho|^{p-1} |f(x)| dx \\ &\leq \beta \left(\int_{B_2 \setminus B_\rho} |\nabla_x P_\rho \varphi \circ F_\rho|^p dy \right)^{\frac{1}{p'}} \left(\int_E |f(x)|^p dx \right)^{\frac{1}{p}} \\ &\leq \beta (C_1 \|\varphi\|_{W^{1-\frac{1}{p},p}(\partial B_2)})^{p-1} \left(\int_E |f(x)|^p dx \right)^{\frac{1}{p}}. \end{aligned} \tag{13}$$

Where C_1 is the constant given in Proposition 5.6. We easy see that the set of functions $\{a(x, \nabla_x P_\rho \varphi \circ F_\rho) I_{B_2 \setminus B_\rho}, 0 < \rho < 1\}$ is uniformly integrable relatively to the parameter ρ and the caratheodory function b . By the Proposition 5.10 and the Vitali Convergence Theorem we get the result sought.

We finish this subsection with the two following nearly cloaking theorems:

Theorem 2

For every $\psi \in W^{1-\frac{1}{p},p}(\partial B_2)$, let $f = (1 - \theta)P\psi$ where θ is a $C_c^\infty(\mathbb{R}^d)$ function from B_2 to \mathbb{R} associated, according to the Lemma 5.6, with the compact set $K = \overline{B_{\rho_0}}$ for a fixed $0 < \rho_0 < 1$ and the open set $U = B_{2\rho_0}$. For every $0 < \rho < \rho_0$, let Λ_b^ρ , $\Lambda_b^\rho(F_\rho^{-1})$ and Λ are the Dirichlet-to-Neumann Maps associated respectively with a_b^ρ , $(a_b^\rho)_{F_\rho^{-1}}$ and a . Then

1. for every $\varphi, \psi \in W^{1-\frac{1}{p},p}(\partial B_2)$ and $0 < \rho < \rho_0$ we have

$$\langle \Lambda_b^\rho \varphi, \psi \rangle - \langle \Lambda \varphi, \psi \rangle = \int_{B_2 \setminus B_r} (a(x, \nabla_x P_\rho \varphi \circ F_\rho) - a(x, \nabla_x P\varphi)) \nabla_x f dx$$

for every r with $\rho < r < \rho_0$

2. for every $\varphi, \psi \in W^{1-\frac{1}{p},p}(\partial B_2)$

$$\lim_{\rho \rightarrow 0} |\langle \Lambda_b^\rho \varphi, \psi \rangle - \langle \Lambda \varphi, \psi \rangle| = 0$$

Proof

Let $\psi, \varphi \in W^{1-\frac{1}{p}, p}(\partial B_2)$. Since the functions f and $P_\rho \varphi \circ F_\rho$ have the function ψ as trace on the Boundary of B_2 , then by propositions 3.9, 4.7 and the lemma 5.5 we have

$$\langle \Lambda_b^\rho \varphi, \psi \rangle = \langle (\Lambda_b^\rho(F_\rho^{-1}))\varphi, \psi \rangle .$$

Let

$$I := \langle \Lambda_b^\rho \varphi, \psi \rangle - \langle \Lambda \varphi, \psi \rangle .$$

By the choose of the function f , we have $\nabla f = 0$ on B_ρ and then

$$\begin{aligned} I &= \int_{B_2} (a_b^\rho)_{F_\rho^{-1}}(x, \nabla_x P_\rho \varphi \circ F_\rho) \nabla_x P_\rho \psi \circ F_\rho dx - \int_{B_2} a(\cdot, \nabla P \varphi) \nabla P \psi dx \\ &= \int_{B_2} (a_b^\rho)_{F_\rho^{-1}}(x, \nabla_x P_\rho \varphi \circ F_\rho) \nabla f dx - \int_{B_2} a(\cdot, \nabla P \varphi) \nabla f dx \\ &= \int_{B_2 \setminus B_\rho} (a(x, \nabla_x P_\rho \varphi \circ F_\rho) - a(x, \nabla P \varphi)) \nabla_x f dx \end{aligned} \quad (14)$$

Hence

$$\langle \Lambda_b^\rho \varphi, \psi \rangle - \langle \Lambda \varphi, \psi \rangle = \int_{B_2 \setminus B_\rho} (a(x, \nabla_x P_\rho \varphi \circ F_\rho) - a(x, \nabla P \varphi)) \nabla_x f dx$$

Since $\nabla f = 0$ on B_{ρ_0} we get the sought result.

By proposition 5.13, we then obtain the validity of item 2.

Remark 5.14 *The ball B_1 is almost invisible as ρ is sufficiently small.*

We have focused on the radial symmetric setting. However our arguments did not use this symmetry in any essential way. Indeed, the Rademacher theorem, an approximation result for $L^1(\Omega)$ functions by smooth functions and the same previous arguments proves:

Theorem 3

Let $G : B_2 \rightarrow \Omega$ be a Lipschitz continuous map with a Lipschitz continuous inverse, and let $D = G(B_1)$ then $H_\rho = G \circ F_\rho \circ G^{-1} : \Omega \rightarrow \Omega$ is peacewise Lipschitz; moreover

- H_ρ expands $G(B_\rho)$ to D and
- $H_\rho(x) = x$ at the boundary $\partial \Omega$

in the same way as in the beginning of this subsection, for given function a and b from $\Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ we define on $\Omega \times \mathbb{R}^d$ the **regular near cloak** associated with H_ρ as follows :

$$a_b^\rho(y, \xi) = \begin{cases} b(y, \xi) & \text{for } y \in D \text{ and } \xi \in \mathbb{R}^d, \\ a_{H_\rho}(y, \xi) & \text{for } y \in \Omega \setminus D \text{ and } \xi \in \mathbb{R}^d \end{cases} \quad (15)$$

Then for every $\varphi, \psi \in W^{1-\frac{1}{p}, p}(\partial B_2)$

$$\lim_{\rho \rightarrow 0} | \langle \Lambda_b^\rho \varphi, \psi \rangle - \langle \Lambda \varphi, \psi \rangle | = 0$$

where Λ_b^ρ and Λ are the Dirichlet to Neumann maps associated respectively with a_b^ρ and a .

Remark 5.15 *The subset $D \subset \Omega$ is almost invisible as ρ is sufficiently small.*

5.2 Singular Cloak

In what follows, let $F(x) = \lim_{\rho \rightarrow 0} F_\rho(x) = (1 + \frac{1}{2}|x|) \frac{x}{|x|}$ for $x \neq 0$.

If we set $y = F(x)$, then $F^{-1}(y) = 2(|y| - 1) \frac{y}{|y|}$ for $|y| > 1$. By the lemma 5.2 we have:

•

$$DF = \left(\frac{1}{2} + \frac{1}{|x|}\right)I - \frac{1}{|x|}\hat{x}\hat{x}^T$$

where $\hat{x} = \frac{x}{|x|}$ For $x \neq 0$

• For $x \neq 0$

$$\det(DF)(x) = \frac{1}{2} \left(\frac{1}{2} + \frac{1}{|x|}\right)^{d-1} = \frac{(|x| + 2)^{d-1}}{2^d |x|^{d-1}}$$

We hence obtain from the previous calculus

Remark 5.16 $(DF)^{-1}$ is uniformly bounded on B_2 but F is not an admissible change of variables and as by previous papers (cf. [11], [12]) it will be called a singular change of variables.

Let b be an arbitrary function from $B_2 \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfying a structural assumption as the function a

Definition 5.17 (Singular cloak associated with F and a)

We define the singular cloak associated with the singular change of variables F , the Function, denoted by a_b^F , from $B_2 \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ given as follows

$$a_b^F(y, \xi) = \begin{cases} b(y, \xi) & \text{for } y \in \overline{B_1} \text{ and } \xi \in \mathbb{R}^d, \\ a_F(y, \xi) & \text{for } y \in B_2 \setminus B_1 \text{ and } \xi \in \mathbb{R}^d \end{cases} \quad (16)$$

where

$$a_F(y, \xi) := \frac{1}{\det DF(x)} DF(x)a(x, (DF(x))\xi) \quad \text{for every } x > 0 \quad \text{and } y = F(x)$$

The following theorems give, for the singular change of variables F , that the associate singular cloak makes the ball B_1 perfectly invisible in the sence of the definitions 5.1 and 5.3.

Theorem 4

Let $\varphi \in W^{1-\frac{1}{p}, p}(\partial B_2)$ and u a locally bounded function from B_2 to $\overline{\mathbb{R}}$ which is a $W_{loc}^{1,p}(\overline{B_2} \setminus \partial B_1)$ solution of the following ‘‘Dirichlet problem’’

$$\begin{cases} \int_{\Omega} (a_b^F(y, \nabla_y u)) \nabla_y v dx = 0 & \text{for all } v \in W_0^{1,p}(B_2), \\ u = \varphi & \text{on } \partial B_2 \end{cases} \quad (17)$$

then there exist a constant C such that u has the following form

$$\begin{cases} u(y) = P\varphi(F^{-1}(y)) & \text{for almost every } y \text{ such that } 1 < |y| \leq 2 \\ u(y) = C & \text{for almost every } y \in B_1 \end{cases} \quad (18)$$

Proof

Let $\varepsilon > 0$ and B_ε the ball with center $x = 0$ and radius ε . We have for $x \in B_2$, $|x| > \varepsilon$ if

and only if $y = F(x) > 1 + \frac{\varepsilon}{2}$. From the previous properties of the jacobian matrix DF it follows that if $v \in W_0^{1,p}(B_2 \setminus B_\varepsilon)$ then $\tilde{v} = v \circ F^{-1} \in W_0^{1,p}(B_2 \setminus B_{1+\frac{\varepsilon}{2}})$. By the hypothesis on u and a variable changes we have

$$\int_{B_2 \setminus B_{1+\frac{\varepsilon}{2}}} a_b^F(y, \nabla u) \nabla \tilde{v} dy = \int_{B_2 \setminus B_\varepsilon} a(x, \nabla_x(u \circ F)) \nabla_x v dx = 0$$

for every $\varepsilon > 0$, Thus the locally bounded function $u \circ F$ on B_2 is a -harmonic on $B_2 \setminus \{0\}$, since the point $\{0\}$ is a -polar (i.e polar for quasilinear elliptic structure given by a , (cf. [10] theorem 10.1), then by the removable singularity theorem (cf. [10] theorem 7.36) the function $u \circ F$ is extendable to an a -harmonic Function. Since $u \circ F = u = \varphi$ at the boundary of B_2 , we hence obtain $u \circ F = P\varphi$ and therefore $u = P\varphi \circ F^{-1}$ on $B_2 \setminus \bar{B}_1$

Let $v \in C_c^\infty(B_2)$ and $v_1(y) = v(0)$ for $|y| \leq 1$ and $v_1(y) = v \circ F^{-1}(y)$ for $|y| > 1$. It is easy to verify that $v_1 \in W_0^{1,p}(B_2)$ and by the hypothesis we have

$$\int_{\bar{B}_1} b(y, \nabla_y u) \nabla_y v_1 dy + \int_{B_2 \setminus \bar{B}_1} a_F(y, \nabla_y u) \nabla_y v_1 dy = 0 \quad \text{for every } v \in C_c^\infty(B_2).$$

By the change of variables $y = F(x)$ we get

$$\int_{B_2 \setminus \bar{B}_1} a_F(y, \nabla_y u) \nabla_y v_1 dy = \int_{B_2 \setminus \{o\}} a(x, \nabla_x(u \circ F)) \nabla_x v dx.$$

By the previous calculus we have $u \circ F = P\varphi$ and then $\int_{B_2 \setminus \{o\}} a(x, \nabla_x(u \circ F)) \nabla_x v dx = 0$,

hence $\int_{\bar{B}_1} b(y, \nabla_y u) \nabla_y v dy = 0$ for every $v \in C_c^\infty(B_2)$, it follows that $b(y, \nabla_y u) = 0$ almost every where in \bar{B}_1 . Since the Function b satisfies a structural assumptions of section 3, by the assumption 2 we then obtain $\nabla_y u = 0$ a.e and u is constant a.e in B_1 .

Corollary 5.18 *Let $\varphi \in W^{1-\frac{1}{p},p}(\partial B_2)$ and v the function from B_2 to $\bar{\mathbb{R}}$ such that*

$$\begin{cases} v(y) = P\varphi(F^{-1}(y)) & \text{if } 1 < |y| \leq 2 \\ v(y) = P\varphi(0) & \text{for } y \in \bar{B}_1 \end{cases} \quad (19)$$

Then

1. v is the unique continuous function satisfying the conditions of the previous theorem
2. v is the potentiel outside the cloaked region with Dirichlet data $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$ obtained in (cf. [11] pages 17 and 19) in the linear case modelling the electric impedance tomographie.

By theorem 2, the existence of the limit of Λ_b^ρ , as ρ tends to zero, leads to the following definition of the Dirichlet-to-Neumann map:

Definition 5.19 *The mapping from $W^{1-\frac{1}{p},p}(\partial B_2)$ to its dual $W^{-(1-\frac{1}{p}),p'}(\partial B_2)$ defined, for every $\varphi, \psi \in W^{1-\frac{1}{p},p}(\partial B_2)$, by*

$$\langle \Lambda_b^F \varphi, \psi \rangle := \lim_{\rho \rightarrow 0} \langle \Lambda_b^\rho \varphi, \psi \rangle$$

will be called the Dirichlet-to-Neumann map associated with the singular cloak a_b^F (Definition 5.17).

Theorem 5

For every function b from $B_2 \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfying a structural assumptions of section 3, we have

1. $\Lambda_b^F = \Lambda$; where Λ is the Dirichlet-to-Neumann-map associated with a .
2. For every $\varphi, \psi \in W^{1-\frac{1}{p}, p}(\partial B_2)$ we have

$$\langle \Lambda_b^F \varphi, \psi \rangle = \int_{B_2 \setminus \bar{B}_1} a_F(y, \nabla_y P\varphi \circ F^{-1}) \nabla_y P\psi \circ F^{-1} dy$$

Proof

Let φ, ψ in $W^{1-\frac{1}{p}, p}(\partial B_2)$, by the Proposition 3.9 for the Dirichlet-to-Neumann-map associated with the function a and by the change of variables F from $B_2 \setminus \{o\}$ to $B_2 \setminus B_1$, we have

$$\begin{aligned} \langle \Lambda \varphi, \psi \rangle &= \int_{B_2 \setminus \{o\}} a(x, \nabla_x P\varphi) \nabla_x P\psi dx \\ &= \int_{B_2 \setminus \bar{B}_1} a_F(y, \nabla_y P\varphi \circ F^{-1}) \nabla_y P\psi \circ F^{-1} dy \\ &= \int_{B_2 \setminus \bar{B}_1} \frac{1}{\det DF(F^{-1})} a(x, DF(F^{-1}) \nabla_y P\varphi \circ F^{-1}) DF(F^{-1}) \nabla_y P\psi \circ F^{-1} dy \\ &\geq \alpha \int_{B_2 \setminus \bar{B}_1} \frac{1}{\det DF(F^{-1}(y))} |DF(F^{-1}(y)) \nabla_y P\varphi \circ F^{-1}|^p dy. \end{aligned} \tag{20}$$

This yields by the Hoelder Inequality that the function

$$\frac{1}{\det DF(x)} a(F^{-1}(y), DF(F^{-1}(y)) \nabla_y P\varphi \circ F^{-1}(y)) DF(F^{-1}(y)) \nabla_y P\psi \circ F^{-1}(y)$$

is integrable on $B_2 \setminus \bar{B}_1$

and hence by a change of variables $y = F(x)$ we get

$$\int_{B_2 \setminus \bar{B}_1} a_F(y, \nabla_y P\varphi \circ F^{-1}) \nabla_y P\psi \circ F^{-1} dy = \int_{B_2} a(x, \nabla_x P\varphi) \nabla_x P\psi dx = \langle \Lambda \varphi, \psi \rangle .$$

By the theorem 2, we have $\langle \Lambda \varphi, \psi \rangle = \lim_{\rho \rightarrow 0} \langle \Lambda_b^\rho \varphi, \psi \rangle$ and we thus obtain the sought results.

we end this section by the following very interesting result.

Corollary 5.20 *For every function a satisfying a structural assumptions of section 3, every singular cloak a_b^F (Definition 5.16), associated with the singular change of variables F and a , Cloaks (makes perfectly invisible) the unit ball B_1 in the sense of definition 5.3*

6 Applications

6.1 Electric Impedance Tomography

As given in Remark 3.1 item 5, differential operators in divergence form known in physic as the PDE in electrostatics is a mathematical modelisation for the Electric Impedance Tomography as follows:

$$\nabla \cdot (\sigma \nabla u) = \sum_{i,j} \frac{\partial}{\partial x_i} (\sigma_{ij}(x)) \frac{\partial u}{\partial x_j} \text{ in } \Omega$$

Where σ is a continuous and strongly elliptic symmetric matrix on Ω in the sense that for some constants $0 < m < M < \infty$:

$m|\xi|^2 \leq \langle \sigma(x)\xi, \xi \rangle \leq M|\xi|^2$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^d$. It is then easy to see that we have the case $p = 2$ and that the differential operator $\nabla \cdot (\sigma \nabla)$ is the second order quasi-linear elliptic operator corresponding to the following function satisfying a structural assumptions with $\alpha = m$ and $\beta = M$

$a(x, \xi) = \sigma(x)\xi$ for every $x \in \Omega$ and $\xi \in \mathbb{R}^d$.

We have then easily obtained for the singular cloak (see definition 5.16) in this case

$$a_F(y, \xi) = \frac{1}{\det DF(F^{-1}(y))} DF(F^{-1}(y)) \sigma(F^{-1}(y)) DF(F^{-1}(y))$$

Our theorem 5 is then a generalisation, in the framework of the class of Leray-Lions operators, of theorem 3 of R.V.Kohn, H Shen, M.S.Vogelius, M.I.Weinstein (cf. [11]) in the linear setting of operators in divergence form.

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