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# THE $p$ -ADIC KUMMER–LEOPOLDT CONSTANT NORMALIZED $p$ -ADIC REGULATOR

by

Georges Gras

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**Abstract.** — The  $p$ -adic Kummer–Leopoldt constant  $\kappa_K$  of a number field  $K$  is (assuming the Leopoldt conjecture) the least integer  $c$  such that for all  $n \gg 0$ , any global unit of  $K$ , which is locally a  $p^{n+c}$ th power at the  $p$ -places, is necessarily the  $p^n$ th power of a global unit of  $K$ . This constant has been computed by Assim & Nguyen Quang Do using Iwasawa’s techniques, after intricate studies and calculations by many authors. We give an elementary  $p$ -adic proof and an improvement of these results, then a class field theory interpretation of  $\kappa_K$ . We give some applications (including generalizations of Kummer’s lemma on regular  $p$ th cyclotomic fields) and a natural definition of the normalized  $p$ -adic regulator for any  $K$  and any  $p \geq 2$ . This is done without analytical computations, using only class field theory and especially the properties of the so-called  $p$ -torsion group  $\mathcal{T}_K$  of Abelian  $p$ -ramification theory over  $K$ .

**Résumé.** — La constante  $p$ -adique de Kummer–Leopoldt  $\kappa_K$  d’un corps de nombres  $K$  est (sous la conjecture de Leopoldt) le plus petit entier  $c$  tel que pour tout  $n \gg 0$ , toute unité globale de  $K$ , qui est localement une puissance  $p^{n+c}$ -ième en les  $p$ -places, est nécessairement puissance  $p^n$ -ième d’une unité globale de  $K$ . Cette constante a été calculée par Assim & Nguyen Quang Do en utilisant les techniques d’Iwasawa, après des études et calculs complexes par divers auteurs. Nous donnons une preuve  $p$ -adique élémentaire et une généralisation de ces résultats, puis une interprétation corps de classes de  $\kappa_K$ . Nous donnons certaines applications (dont des généralisations du lemme de Kummer sur les  $p$ -corps cyclotomiques réguliers) et une définition naturelle du régulateur  $p$ -adique normalisé pour tous  $K$  &  $p \geq 2$ . Ceci est fait sans calculs analytiques, en utilisant uniquement le corps de classes et tout spécialement les propriétés du fameux  $p$ -groupe de torsion  $\mathcal{T}_K$  de la théorie de la  $p$ -ramification Abélienne sur  $K$ .

## 1. Notations

Let  $K$  be a number field and let  $p \geq 2$  be a prime number; we denote by  $\mathfrak{p} \mid p$  the prime ideals of  $K$  dividing  $p$ . Consider the group  $E_K$  of  $p$ -principal global units of  $K$  (i.e., units  $\varepsilon \equiv 1 \pmod{\prod_{\mathfrak{p} \mid p} \mathfrak{p}}$ ), so that the index of  $E_K$  in the group of units is prime to  $p$ . For each  $\mathfrak{p} \mid p$ , let  $K_{\mathfrak{p}}$  be the  $\mathfrak{p}$ -completion of  $K$  and  $\bar{\mathfrak{p}}$  the corresponding prime ideal of the ring of integers of  $K_{\mathfrak{p}}$ ; then let

$$U_K := \left\{ u \in \bigoplus_{\mathfrak{p} \mid p} K_{\mathfrak{p}}^{\times}, u = 1 + x, x \in \bigoplus_{\mathfrak{p} \mid p} \bar{\mathfrak{p}} \right\} \quad \& \quad W_K := \text{tor}_{\mathbb{Z}_p}(U_K),$$

the  $\mathbb{Z}_p$ -module of principal local units at  $p$  and its torsion subgroup.

The  $p$ -adic logarithm  $\log$  is defined on  $1 + x$ ,  $x \in \bigoplus_{\mathfrak{p}|p} \bar{\mathfrak{p}}$ , by means of the series

$$\log(1 + x) = \sum_{i \geq 1} (-1)^{i+1} \frac{x^i}{i} \in \bigoplus_{\mathfrak{p}|p} K_{\mathfrak{p}}. \text{ Its kernel in } U_K \text{ is } W_K \text{ [15, Proposition 5.6].}$$

We consider the diagonal embedding  $E_K \longrightarrow U_K$  and its natural extension  $E_K \otimes \mathbb{Z}_p \longrightarrow U_K$  whose image is  $\overline{E}_K$ , the topological closure of  $E_K$  in  $U_K$ .

In the sequel, these embeddings shall be understood; moreover, we assume in this paper that  $K$  satisfies the Leopoldt conjecture at  $p$ , which is equivalent to the condition  $\text{rk}_{\mathbb{Z}_p}(\overline{E}_K) = \text{rk}_{\mathbb{Z}}(E_K)$  (see, e.g., [15, § 5.5, p. 75]).

## 2. The Kummer–Leopoldt constant

This notion comes from the Kummer lemma (see, e.g., [15, Theorem 5.36]), that is to say, if the odd prime number  $p$  is “regular”, the cyclotomic field  $K = \mathbb{Q}(\mu_p)$  of  $p$ th roots of unity satisfies the following property stated for the whole group  $E'_K$  of global units of  $K$ :

*any  $\varepsilon \in E'_K$ , congruent to a rational integer modulo  $p$ , is a  $p$ th power in  $E'_K$ .*

In fact,  $\varepsilon \equiv a \pmod{p}$  with  $a \in \mathbb{Z}$ , implies  $\varepsilon^{p-1} \equiv a^{p-1} \equiv 1 \pmod{p}$ . So we shall write the Kummer property with  $p$ -principal units in the more suitable equivalent statement:

*any  $\varepsilon \in E_K$ , congruent to 1 modulo  $p$ , is a  $p$ th power in  $E_K$ .*

From [1], [11], [13], [14], [16], [17] one can study this property and its generalizations with various techniques (see the rather intricate history in [2]). Give the following definition from [2]:

**Definition 2.1.** — *Let  $K$  be a number field satisfying the Leopoldt conjecture at the prime  $p \geq 2$ . Let  $E_K$  be the group of  $p$ -principal global units of  $K$  and let  $U_K$  be the group of principal local units at the  $p$ -places.*

*We call Kummer–Leopoldt constant (denoted  $\kappa_K =: \kappa$ ), the smallest integer  $c$  such that the following condition is fulfilled:*

*for all  $n \gg 0$ , any unit  $\varepsilon \in E_K$ , such that  $\varepsilon \in U_K^{p^{n+c}}$ , is necessarily in  $E_K^{p^n}$ .*

**Remark 2.1.** — *The existence of  $\kappa$  comes from various classical characterizations of Leopoldt’s conjecture proved for instance in [5, Theorem III.3.6.2], after [14], [11] and oldest Iwasawa papers. Indeed, if the Leopoldt conjecture is not satisfied, we can find a sequence  $\varepsilon_n \in E_K \setminus E_K^p$  such that  $\log(\varepsilon_n) \rightarrow 0$  (i.e.,  $\varepsilon_n \in U_K^{p^m} \cdot W_K$ , with  $m \rightarrow \infty$  as  $n \rightarrow \infty$ ); since  $W_K$  is finite, taking a suitable  $p$ -power of  $\varepsilon_n$ , we see that  $\kappa$  does not exist in that case.*

We shall prove (Theorem 3.1) that in the above definition, the condition “for all  $n \gg 0$ ” can be replaced by “for all  $n \geq 0$ ”, subject to introduce the group of global roots of unity of  $K$  and a suitable statement.

We have the following first  $p$ -adic result giving  $p^\kappa$  under the Leopoldt conjecture:

**Theorem 2.1.** — *Denote by  $E_K$  the group of  $p$ -principal global units of  $K$ , by  $U_K$  the  $\mathbb{Z}_p$ -module of principal local units at the  $p$ -places, and by  $W_K$  its torsion subgroup. Let  $\kappa_K$  be the Kummer–Leopoldt constant (Definition 2.1).*

*Then  $p^{\kappa_K}$  is the exponent of the finite group  $\text{tor}_{\mathbb{Z}_p}(\log(U_K)/\log(\overline{E}_K))$ , where  $\log$  is the  $p$ -adic logarithm and  $\overline{E}_K$  the topological closure of  $E_K$  in  $U_K$  (whence the relation  $\log(\overline{E}_K) = \mathbb{Z}_p \log(E_K)$ ).*

*Proof.* — Let  $p^\kappa$  be the exponent of  $\text{tor}_{\mathbb{Z}_p}(\log(U_K)/\log(\overline{E}_K))$ .

(i) ( $\kappa$  is suitable). Let  $n \gg 0$  and let  $\varepsilon \in E_K$  be such that

$$\varepsilon = u^{p^{n+\kappa}}, u \in U_K.$$

So  $\log(u)$  is of finite order modulo  $\log(\overline{E}_K)$  and  $\log(\varepsilon) = p^n \cdot (p^\kappa \cdot \log(u)) = p^n \cdot \log(\overline{\eta})$  with  $\overline{\eta} \in \overline{E}_K$ . By definition of  $\overline{E}_K$ , we can write in  $U_K$ , for all  $N \gg 0$ ,

$$\overline{\eta} = \eta(N) \cdot u_N, \quad \eta(N) \in E_K, \quad u_N \equiv 1 \pmod{p^N};$$

we get  $\log(\varepsilon) = p^n \cdot \log(\eta(N)) + p^n \cdot \log(u_N)$  giving in  $U_K$

$$\varepsilon = \eta(N)^{p^n} \cdot u_N^{p^n} \cdot \xi_N, \quad \xi_N \in W_K.$$

But  $\xi_N$  is near 1 (depending on the choice of  $n \gg 0$ ), whence  $\xi_N = 1$  for all  $N$ , and  $\varepsilon = \eta(N)^{p^n} \cdot u_N^{p^n}$ ,  $u_N^{p^n} \rightarrow 1$  as  $N \rightarrow \infty$ ; so  $u_N^{p^n} = \varepsilon \cdot \eta(N)^{-p^n}$  is a global unit, arbitrary close to 1, hence, because of Leopoldt's conjecture [5, Theorem III.3.6.2 (iii, iv)], of the form  $\varphi_N^{p^n}$  with  $\varphi_N \in E_K$  (recall that  $n$  is large enough, arbitrary, but fixed), giving

$$\varepsilon = \eta(N)^{p^n} \cdot \varphi_N^{p^n} \in E_K^{p^n}.$$

(ii) ( $\kappa$  is the least solution). Suppose there exists an integer  $c < \kappa$  having the property given in Definition 2.1. Let  $u_0 \in U_K$  be such that

$$\log(u_0) \text{ is of order } p^\kappa \text{ in } \text{tor}_{\mathbb{Z}_p}(\log(U_K)/\log(\overline{E}_K));$$

then  $\log(u_0^{p^\kappa}) = \log(\overline{\varepsilon}_0)$ ,  $\overline{\varepsilon}_0 \in \overline{E}_K$ . This is equivalent to

$$u_0^{p^\kappa} = \overline{\varepsilon}_0 \cdot \xi_0 = \varepsilon(N) \cdot u_N \cdot \xi_0, \quad \varepsilon(N) \in E_K, \quad u_N \equiv 1 \pmod{p^N}, \quad \xi_0 \in W_K,$$

hence, for any  $n \gg 0$ ,  $u_0^{p^{n+\kappa}} = \varepsilon(N)^{p^n} \cdot u_N^{p^n}$ . Taking  $N$  large enough, but fixed, we can suppose that  $u_N = v^{p^{2\kappa}}$ ,  $v \in U_K$  near 1; because of the above relations,  $\log(v)$  is of finite order modulo  $\log(\overline{E}_K)$ , thus  $\log(v^{p^\kappa}) \in \log(\overline{E}_K)$ . This is sufficient, for

$$u'_0 := u_0 \cdot v^{-p^\kappa},$$

to get  $\log(u'_0)$  of order  $p^\kappa$  modulo  $\log(\overline{E}_K)$ . So we can write:

$$\varepsilon(N)^{p^n} = u_0^{p^{n+\kappa}} \cdot u_N^{-p^n} = u_0^{p^{n+\kappa}} \cdot (v^{-p^\kappa})^{p^{n+\kappa}} = u_0^{p^{n+\kappa}} \in U_K^{p^{n+(\kappa-c)+c}},$$

but, by assumption on  $c$  applied to the global unit  $\varepsilon(N)^{p^n}$ , we obtain

$$\varepsilon(N)^{p^n} = \eta_0^{p^{n+(\kappa-c)}}, \quad \eta_0 \in E_K;$$

thus, the above relation  $u_0^{p^{n+\kappa}} = \varepsilon(N)^{p^n} = \eta_0^{p^{n+(\kappa-c)}}$  yields:

$$p^c \cdot \log(u'_0) = \log(\eta_0) \in \log(E_K),$$

which is absurd since  $\log(u'_0)$  is of order  $p^\kappa$  modulo  $\log(\overline{E}_K)$ . □

### 3. Interpretation of $\kappa_K$ – Fundamental exact sequence

The following  $p$ -adic result is valid without any assumption on  $K$  and  $p$ :

**Lemma 3.1.** — *We have the exact sequence (from [5, Lemma 4.2.4]):*

$$1 \rightarrow W_K / \text{tor}_{\mathbb{Z}_p}(\overline{E}_K) \longrightarrow \text{tor}_{\mathbb{Z}_p}(U_K / \overline{E}_K) \xrightarrow{\log} \text{tor}_{\mathbb{Z}_p}(\log(U_K) / \log(\overline{E}_K)) \rightarrow 0.$$

*Proof.* — The surjectivity comes from the fact that if  $u \in U_K$  is such that  $p^n \log(u) = \log(\bar{\varepsilon})$ ,  $\bar{\varepsilon} \in \bar{E}_K$ , then  $u^{p^n} = \bar{\varepsilon} \cdot \xi$  for  $\xi \in W_K$ , hence there exists  $m \geq n$  such that  $u^{p^m} \in \bar{E}_K$ , whence  $u$  gives a preimage in  $\text{tor}_{\mathbb{Z}_p}(U_K/\bar{E}_K)$ .

If  $u \in U_K$  is such that  $\log(u) \in \log(\bar{E}_K)$ , then  $u = \bar{\varepsilon} \cdot \xi$  as above, giving the kernel equal to  $\bar{E}_K \cdot W_K/\bar{E}_K = W_K/\text{tor}_{\mathbb{Z}_p}(\bar{E}_K)$ .  $\square$

**Corollary 3.1.** — *Let  $\mu_K$  be the group of global roots of unity of  $p$ -power order of  $K$ .*

*Then, under the Leopoldt conjecture for  $p$  in  $K$ , we have  $\text{tor}_{\mathbb{Z}_p}(\bar{E}_K) = \mu_K$ ; thus in that case  $W_K/\text{tor}_{\mathbb{Z}_p}(\bar{E}_K) = W_K/\mu_K$ .*

*Proof.* — From [5, Corollary III.3.6.3], [9, Définition 2.11, Proposition 2.12].  $\square$

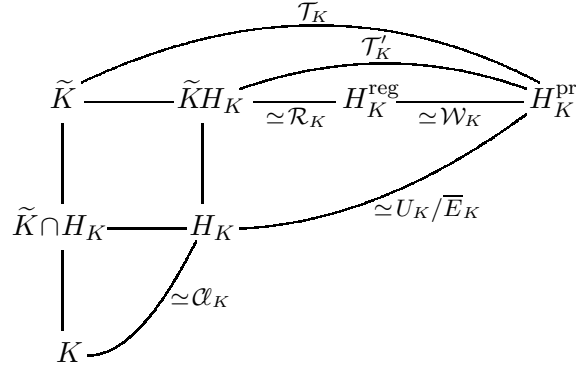
Put

$$\mathcal{W}_K := W_K/\mu_K \quad \& \quad \mathcal{R}_K := \text{tor}_{\mathbb{Z}_p}(\log(U_K)/\log(\bar{E}_K)).$$

Then the exact sequence of Lemma 3.1 becomes:

$$1 \longrightarrow \mathcal{W}_K \longrightarrow \text{tor}_{\mathbb{Z}_p}(U_K/\bar{E}_K) \xrightarrow{\log} \mathcal{R}_K \longrightarrow 0.$$

Consider the following diagram (see [5], §III.2, (c), Fig. 2.2), under the Leopoldt conjecture for  $p$  in  $K$ :



where  $\tilde{K}$  is the compositum of the  $\mathbb{Z}_p$ -extensions,  $\mathcal{C}_K$  the  $p$ -class group,  $H_K$  the  $p$ -Hilbert class field,  $H_K^{\text{pr}}$  the maximal Abelian  $p$ -ramified (i.e., unramified outside  $p$ ) pro- $p$ -extension, of  $K$ . These definitions are given in the ordinary sense when  $p = 2$  (so that the real infinite places of  $K$  are not complexified (= are unramified) in the class fields under consideration which are “real”).

By class field theory,  $\text{Gal}(H_K^{\text{pr}}/H_K) \simeq U_K/\bar{E}_K$  in which the image of  $\mathcal{W}_K$  fixes  $H_K^{\text{reg}}$ , the Bertrandias–Payan field,  $\text{Gal}(H_K^{\text{reg}}/\tilde{K})$  being then the Bertrandias–Payan module, except possibly if  $p = 2$  in the “special case” (cf. [2] about the calculation of  $\kappa$  and the Références in [6] for some history about this module).

But  $\mathcal{R}_K$  giving  $\kappa_K$  has, a priori, nothing to do with *the definition* of the Bertrandias–Payan module associated with  $p^r$ -cyclic extensions of  $K$ ,  $r \geq 1$ , which are embeddable in cyclic  $p$ -extensions of  $K$  of arbitrary large degree.

Then we put  $\mathcal{T}'_K := \text{tor}_{\mathbb{Z}_p}(\text{Gal}(H_K^{\text{pr}}/H_K)) \subseteq \mathcal{T}_K := \text{tor}_{\mathbb{Z}_p}(\text{Gal}(H_K^{\text{pr}}/K))$ . The group  $\mathcal{R}_K$  is then isomorphic to  $\text{Gal}(H_K^{\text{reg}}/\tilde{K}H_K)$ . Of course, for  $p \geq p_0$  (explicit),  $\mathcal{W}_K = [H_K : \tilde{K} \cap H_K] = 1$ , whence  $\mathcal{R}_K = \mathcal{T}_K$ . We shall see in the Section 5 that  $\mathcal{R}_K \simeq \mathcal{T}'_K/\mathcal{W}_K$  is closely related to the classical  $p$ -adic regulator of  $K$ .

**Corollary 3.2.** — Under the Leopoldt conjecture for  $p$  in  $K$ , the Kummer–Leopoldt constant  $\kappa_K$  of  $K$  is 0 if and only if  $\mathcal{R}_K = 1$  (i.e.,  $H_K^{\text{reg}} = \widetilde{K}H_K$ ).

*Proof.* — From Theorem 2.1 using the new terminology of the “algebraic regulator”  $\mathcal{R}_K := \text{tor}_{\mathbb{Z}_p}(\log(U_K)/\log(\overline{E}_K))$  whose exponent is  $p^\kappa$ .  $\square$

**Corollary 3.3.** — If the prime number  $p$  is regular, then  $\kappa_K = 0$  for the field  $K = \mathbb{Q}(\mu_p)$  of  $p$ th roots of unity, and any unit  $\varepsilon \in E_K$  such that  $\varepsilon \equiv 1 \pmod{p}$  is in  $E_K^p$  (Kummer’s lemma).

*Proof.* — (i) We first prove that if the real unit  $\varepsilon$  is congruent to 1 modulo  $p$  then it is a  $p$ th power in  $U_K$ . Put  $\varepsilon = 1 + \alpha \cdot p$  for a  $p$ -integer  $\alpha \in K^\times$ . Let  $K_0$  be the maximal real subfield of  $K$  and let  $\pi_0$  be an uniformizing parameter of its  $p$ -completion. Put  $\alpha = a_0 + \beta \cdot \pi_0$  with  $a_0 \in [0, p-1]$  and a  $p$ -integer  $\beta$ . Since  $N_{K_0/\mathbb{Q}}(\varepsilon) = 1$ , this yields  $a_0 = 0$ , whence  $\varepsilon = 1 + \beta \cdot p \cdot \pi_0$ . The valuation of  $p \cdot \pi_0$ , calculated in  $K$ , is  $p+1$ , which is sufficient to get  $\varepsilon \in U_K^p$  (use [15, Proposition 5.7]).

(ii) Then we prove that  $\kappa = 0$ . The cyclotomic field  $K = \mathbb{Q}(\mu_p)$  is  $p$ -regular and  $p$ -rational in the meaning of [4, Théorème & Définition 2.1], so  $\mathcal{T}_K = 1$  giving  $\kappa = 0$ . In other words,  $\kappa = 0$  is given by a stronger condition ( $p$ -rationality of  $K$ ) than  $\mathcal{R}_K = 1$ .

One may preferably use the general well-known  $p$ -rank formula (the  $p$ -rank  $\text{rk}_p(A)$  of a finite Abelian group  $A$  is the  $\mathbb{F}_p$ -dimension of  $A/A^p$ ), valid for any field  $K$  under the Leopoldt conjecture, when the group  $\mu_K$  of  $p$ th roots of unity is nontrivial [5, Proposition III.4.2.2]:

$$\text{rk}_p(\mathcal{T}_K) = \text{rk}_p(\mathcal{C}_K^{S_K^{\text{res}}}) + \#S_K - 1,$$

where  $S_K$  is the set of prime ideals of  $K$  above  $p$  and  $\mathcal{C}_K^{S_K^{\text{res}}}$  the  $S_K$ -class group in the restricted sense (when  $p = 2$ ) equal to the quotient of the  $p$ -class group of  $K$  in the restricted sense by the subgroup generated by the classes of ideals of  $S_K$ ; so for  $K = \mathbb{Q}(\mu_p)$ , we immediately get  $\text{rk}_p(\mathcal{T}_K) = \text{rk}_p(\mathcal{C}_K)$ , which is by definition trivial for regular primes.  $\square$

**Theorem 3.1.** — Let  $\kappa_K$  be the Kummer–Leopoldt constant of  $K$  (Definition 2.1) and let  $p^\nu$  be the exponent of  $\mathcal{W}_K = W_K/\mu_K$ , where  $W_K = \text{tor}_{\mathbb{Z}_p}(U_K)$  and  $\mu_K$  is the group of global roots of unity of  $K$  of  $p$ -power order. <sup>(1)</sup>

The property defining  $\kappa_K$  can be improved as follows:

(i) If  $\nu \geq 1$ , for all  $n \geq 0$ , any  $\varepsilon \in E_K$  such that  $\varepsilon \in U_K^{p^{n+\kappa_K}}$  is necessarily of the form  $\varepsilon = \zeta \cdot \eta^{p^n}$ , with  $\zeta \in \mu_K \cap W_K^{p^n}$ ,  $\eta \in E_K$ .

(ii) If  $\nu = 0$ , for all  $n \geq 0$ , any  $\varepsilon \in E_K$  being in  $U_K^{p^{n+\kappa_K}}$  is necessarily in  $E_K^{p^n}$ .

*Proof.* — Let  $n \geq 0$ . Suppose that  $\varepsilon = u^{p^{n+\kappa}}$ ,  $u \in U_K$ . So  $\log(\varepsilon) = p^n \cdot p^\kappa \cdot \log(u) = p^n \cdot \log(\bar{\eta})$ ,  $\bar{\eta} \in \overline{E}_K$ ; thus  $\bar{\eta} = \eta(N) \cdot u_N$ , with  $\eta(N) \in E_K$ ,  $u_N \equiv 1 \pmod{p^N}$ , for all  $N \gg 0$ , and  $\log(\varepsilon) = p^n \cdot \log(\eta(N)) + p^n \cdot \log(u_N)$  giving in  $U_K$

$$\varepsilon = \eta(N)^{p^n} \cdot u_N^{p^n} \cdot \xi_N, \quad \xi_N \in W_K, \text{ for all } N \gg 0.$$

Taking  $N$  in a suitable infinite subset of  $\mathbb{N}$ , we can suppose  $\xi_N = \xi$  independent of  $N \rightarrow \infty$ . Then  $\xi = (\varepsilon \cdot \eta(N)^{-p^n}) \cdot u_N^{-p^n} \in \text{tor}_{\mathbb{Z}_p}(\overline{E}_K)$ , whence  $\xi = \zeta \in \mu_K$  because of Leopoldt’s conjecture (loc. cit. in proof of Corollary 3.1). Then

<sup>(1)</sup>In the case  $\nu = 0$ , if  $\mu_K = 1$ , then  $\mu_{K_p} = 1 \forall p \in S_K$ ; if  $\mu_K \neq 1$ , then  $S_K = \{p\}$  &  $\mu_{K_p} = \mu_K$ .

$$u^{p^{n+\kappa}} = \varepsilon = \eta(N)^{p^n} \cdot u_N^{p^n} \cdot \zeta = \eta(N)^{p^n} \cdot u'_N \cdot \zeta, \quad u'_N (\in E_K) \rightarrow 1 \text{ as } N \rightarrow \infty,$$

whence  $\varepsilon$  of the form  $\eta(N)^{p^n} \cdot \varphi_N^{p^n} \cdot \zeta$ ,  $\varphi_N \in E_K$ , for  $N \gg 0$ . So  $\varepsilon = \zeta \cdot \eta^{p^n}$ , with  $\eta \in E_K$  and  $\zeta = \varepsilon \cdot \eta^{-p^n} \in \mu_K \cap W_K^{p^n}$ , since  $\varepsilon$  is a local  $p^n$ th power.

If  $\nu = 0$ ,  $W_K = \mu_K$  and  $\zeta \in \mu_K \cap W_K^{p^n} = \mu_K^{p^n}$  is a  $p^n$ th power. □

#### 4. Remarks and applications

As above, we assume the Leopoldt conjecture for  $p$  in the fields under consideration.

(a) The condition  $\varepsilon \in U_K^{p^{n+\kappa}} = \bigoplus_{\mathfrak{p}|p} U_{\mathfrak{p}}^{p^{n+\kappa}}$ , where  $U_{\mathfrak{p}} := 1 + \bar{\mathfrak{p}}$ , may be translated,

in the framework of Kummer’s lemma, into a less precise condition of the form  $\varepsilon \equiv 1 \pmod{\prod_{\mathfrak{p}|p} \mathfrak{p}^{m_{\mathfrak{p}}(n, \kappa)}}$  for suitable minimal exponents  $m_{\mathfrak{p}}(n, \kappa)$  giving local  $p^{n+\kappa}$ th powers. This was used by most of the cited references with  $p$ -adic analytical calculations using the fact that  $\#\mathcal{T}_K$  is, roughly speaking, a product “class number”  $\times$  “regulator” from  $p$ -adic  $L$ -functions, giving an upper bound for  $\kappa$  (it is the analytic way used in [16] and [13] to generalize Kummer’s lemma when  $p$  is not regular).

(b) If  $\mathcal{T}_K = 1$  (in which case  $\kappa = 0$ ), the field  $K$  is said to be a  $p$ -rational field (see [5, § IV.3], [4], [10], [12]). Then in any  $p$ -primitively ramified  $p$ -extension  $L$  of  $K$  (definition and examples in [5, § IV.3, (b); § IV.3.5.1], after [8, Theorem 1, § II.2]), we get  $\mathcal{T}_L = 1$  whence  $\kappa_L = 0$ .

The following examples illustrate this principle:

(i) The  $p^m$ -cyclotomic fields. The above applies for the fields  $K_m := \mathbb{Q}(\mu_{p^m})$  of  $p^m$ -roots of unity when the prime  $p$  is regular, since we have seen that  $\mathcal{T}_{\mathbb{Q}(\mu_p)} = 1$ .

(ii) Some  $p$ -rational  $p$ -extensions of  $\mathbb{Q}$  ( $p = 2$  and  $p = 3$ ). The following fields have a Kummer-Leopoldt constant  $\kappa = 0$  ([5, Example IV.3.5.1], after [8, § III]):

– The real Abelian 2-extensions of  $\mathbb{Q}$ , subfields of the fields  $\mathbb{Q}(\mu_{2^\infty}) \cdot \mathbb{Q}(\sqrt{\ell})$ ,  $\ell \equiv 3 \pmod{8}$ , and  $\mathbb{Q}(\mu_{2^\infty}) \cdot \mathbb{Q}\left(\sqrt{\sqrt{\ell} \frac{a - \sqrt{\ell}}{2}}\right)$ ,  $\ell = a^2 + 4b^2 \equiv 5 \pmod{8}$ .

– The real Abelian 3-extensions of  $\mathbb{Q}$ , subfields of the fields  $\mathbb{Q}(\mu_{3^\infty}) \cdot k_\ell$ ,  $\ell \equiv 4, 7 \pmod{9}$ , where  $k_\ell$  is the cyclic cubic field of conductor  $\ell$ .

(c) When  $\mu_K = 1$ , the formula giving  $\text{rk}_p(\mathcal{T}_K)$ , used in the proof of Kummer’s lemma (Corollary 3.3), must be replaced by a formula deduced from the “reflection theorem”: let  $K' := K(\zeta_p)$ , where  $\zeta_p$  is a primitive  $p$ th root of unity; then

$$\text{rk}_p(\mathcal{T}_K) = \text{rk}_\omega(\mathcal{C}_{K'}^{S_{K'}^{\text{res}}}) + \sum_{\mathfrak{p}|p} \delta_{\mathfrak{p}} - \delta,$$

which links the  $p$ -rank of  $\mathcal{T}_K$  to that of the  $\omega$ -component of the  $p$ -group of  $S_{K'}$ -ideal classes of the field  $K'$ , where  $\omega$  is the Teichmüller character of  $\text{Gal}(K'/K)$ ,  $\delta_{\mathfrak{p}} := 1$  or 0 according as the completion  $K_{\mathfrak{p}}$  contains  $\zeta_p$  or not,  $\delta := 1$  or 0 according as  $\zeta_p \in K$  or not (so that  $\omega = 1$  if and only if  $\zeta_p \in K$ ).

(d) Unfortunately,  $p^\kappa$  may be less than  $\#\mathcal{R}_K$  (hence a fortiori less than  $\#\mathcal{T}_K$ ) due to the unknown *group structure* of  $\mathcal{R}_K$ ; as usual, when  $K/\mathbb{Q}$  is Galois with Galois group  $G$ , the study of its  $G$ -structure may give more precise information:

Indeed, to simplify assume  $p > 2$  unramified in  $K$ , so that  $\log(U_K)$  is isomorphic to  $O_K$ , the direct sum of the rings of integers of the  $K_{\mathfrak{p}}$ ,  $\mathfrak{p} | p$ ; if  $\eta =: 1 + p \cdot \alpha \in E_K$  generates a sub- $G$ -module of  $E_K$ , of index prime to  $p$  (such a unit does exist since  $E_K \otimes \mathbb{Q}$  is a monogenic  $\mathbb{Q}[G]$ -module; cf. [5, Corollary I.3.7.2 & Remark I.3.7.3]),

the structure of  $\mathcal{R}_K$  can be easily deduced from the knowledge of  $P(\alpha) \equiv \frac{1}{p}\log(\eta)$  modulo a suitable power of  $p$ , where  $P(\alpha)$  is a rational polynomial expression of  $\alpha$  generating a sub- $G$ -module of  $O_K$ ; thus many numerical examples may be obtained.

(e) We have given in [7, §8.6] a conjecture saying that, in any fixed number field  $K$ , we have  $\mathcal{T}_K = 1$  for all  $p \gg 0$ , giving conjecturally  $\kappa = 0$  for all  $p \gg 0$ .

### 5. Normalized $p$ -adic regulator of a number field

The previous Section 3 shows that the good notion of  $p$ -adic regulator comes from the expression of the  $p$ -adic finite group  $\mathcal{R}_K$  associated with the class field theory interpretation of  $\text{Gal}(H_K^{\text{reg}}/\widehat{K}H_K)$ .

For this, recall that  $\overline{E}_K$  is the topological closure, in the  $\mathbb{Z}_p$ -module  $U_K$  of principal local units at  $p$ , of the group of  $p$ -principal global units of  $K$ , and  $\log$  the  $p$ -adic logarithm:

**Definition 5.1.** — *Let  $K$  be any number field and let  $p \geq 2$  be any prime number. Under the Leopoldt conjecture for  $p$  in  $K$ , we call  $\mathcal{R}_K := \text{tor}_{\mathbb{Z}_p}(\log(U_K)/\log(\overline{E}_K))$  (or its order  $\#\mathcal{R}_K$ ) the normalized  $p$ -adic regulator of  $K$ .*

We have in the simplest case of *totally real* number fields (from Coates's formula [3, Appendix] and also [5, Remarks III.2.6.5] for  $p = 2$ ):

**Proposition 5.1.** — *For any totally real number field  $K \neq \mathbb{Q}$ , we have, under the Leopoldt conjecture for  $p$  in  $K$ ,*

$$\#\mathcal{R}_K \sim \frac{1}{2} \cdot \frac{(\mathbb{Z}_p : \log(N_{K/\mathbb{Q}}(U_K)))}{\#\mathcal{W}_K \cdot \prod_{\mathfrak{p}|p} N\mathfrak{p}} \cdot \frac{R_K}{\sqrt{D_K}},$$

where  $\sim$  means equality up to a  $p$ -adic unit factor, where  $R_K$  is the usual  $p$ -adic regulator [15, §5.5] and  $D_K$  the discriminant of  $K$ .

With this expression, we find again classical results obtained by means of analytic computations (e.g., [1, Theorem 6.5]). In the real Galois case, with  $p$  unramified in  $K/\mathbb{Q}$ , we get, as defined in [7, Définition 2.3],  $\#\mathcal{R}_K \sim \frac{R_K}{p^{[K:\mathbb{Q}]-1}}$  for  $p \neq 2$  and

$\#\mathcal{R}_K \sim \frac{1}{2^{d-1}} \frac{R_K}{2^{[K:\mathbb{Q}]-1}}$  for  $p = 2$ , where  $d$  is the number of prime ideals  $\mathfrak{p} | 2$  in  $K$ .

Of course,  $\#\mathcal{R}_K = \#\text{tor}_{\mathbb{Z}_p}(\log(U_K)) = 1$  for  $\mathbb{Q}$  and any imaginary quadratic field.

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