



# The p-adic Kummer-Leopoldt constant – Normalized p-adic regulator

Georges Gras

## ► To cite this version:

Georges Gras. The p-adic Kummer-Leopoldt constant – Normalized p-adic regulator. 2017. hal-01444560v2

**HAL Id: hal-01444560**

**<https://hal.science/hal-01444560v2>**

Preprint submitted on 31 Mar 2017

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

---

# THE $p$ -ADIC KUMMER–LEOPOLDT CONSTANT NORMALIZED $p$ -ADIC REGULATOR

by

Georges Gras

---

**Abstract.** — The  $p$ -adic Kummer–Leopoldt constant  $\kappa_K$  of a number field  $K$  is (assuming the Leopoldt conjecture) the least integer  $c$  such that for all  $n \gg 0$ , any global unit of  $K$ , which is locally a  $p^{n+c}$ th power at the  $p$ -places, is necessarily the  $p^n$ th power of a global unit of  $K$ . This constant has been computed by Assim & Nguyen Quang Do using Iwasawa’s techniques, after intricate studies and calculations by many authors. We give an elementary  $p$ -adic proof and an improvement of these results, then a class field theory interpretation of  $\kappa_K$ . We give some applications (including generalizations of Kummer’s lemma on regular  $p$ th cyclotomic fields) and a natural definition of the normalized  $p$ -adic regulator for any  $K$  and any  $p \geq 2$ . This is done without analytical computations, using only class field theory and especially the properties of the so-called  $p$ -torsion group  $\mathcal{T}_K$  of Abelian  $p$ -ramification theory over  $K$ .

**Résumé.** — La constante  $p$ -adique de Kummer–Leopoldt  $\kappa_K$  d’un corps de nombres  $K$  est (sous la conjecture de Leopoldt) le plus petit entier  $c$  tel que pour tout  $n \gg 0$ , toute unité globale de  $K$ , qui est localement une puissance  $p^{n+c}$ -ième en les  $p$ -places, est nécessairement puissance  $p^n$ -ième d’une unité globale de  $K$ . Cette constante a été calculée par Assim & Nguyen Quang Do en utilisant les techniques d’Iwasawa, après des études et calculs complexes par divers auteurs. Nous donnons une preuve  $p$ -adique élémentaire et une généralisation de ces résultats, puis une interprétation corps de classes de  $\kappa_K$ . Nous donnons certaines applications (dont des généralisations du lemme de Kummer sur les  $p$ -corps cyclotomiques réguliers) et une définition naturelle du régulateur  $p$ -adique normalisé pour tous  $K$  &  $p \geq 2$ . Ceci est fait sans calculs analytiques, en utilisant uniquement le corps de classes et tout spécialement les propriétés du fameux  $p$ -groupe de torsion  $\mathcal{T}_K$  de la théorie de la  $p$ -ramification Abélienne sur  $K$ .

## 1. Notations

Let  $K$  be a number field and let  $p \geq 2$  be a prime number; we denote by  $\mathfrak{p} \mid p$  the prime ideals of  $K$  dividing  $p$ . Consider the group  $E_K$  of  $p$ -principal global units of  $K$  (i.e., units  $\varepsilon \equiv 1 \pmod{\prod_{\mathfrak{p} \mid p} \mathfrak{p}}$ ), so that the index of  $E_K$  in the group of units is prime to  $p$ . For each  $\mathfrak{p} \mid p$ , let  $K_{\mathfrak{p}}$  be the  $\mathfrak{p}$ -completion of  $K$  and  $\bar{\mathfrak{p}}$  the corresponding prime ideal of the ring of integers of  $K_{\mathfrak{p}}$ ; then let

$$U_K := \left\{ u \in \bigoplus_{\mathfrak{p} \mid p} K_{\mathfrak{p}}^{\times}, \ u = 1 + x, \ x \in \bigoplus_{\mathfrak{p} \mid p} \bar{\mathfrak{p}} \right\} \quad \& \quad W_K := \text{tor}_{\mathbb{Z}_p}(U_K),$$

the  $\mathbb{Z}_p$ -module of principal local units at  $p$  and its torsion subgroup.

The  $p$ -adic logarithm  $\log$  is defined on  $1 + x$ ,  $x \in \bigoplus_{\mathfrak{p}|p} \bar{\mathfrak{p}}$ , by means of the series

$$\log(1 + x) = \sum_{i \geq 1} (-1)^{i+1} \frac{x^i}{i} \in \bigoplus_{\mathfrak{p}|p} K_{\mathfrak{p}}. \text{ Its kernel in } U_K \text{ is } W_K \text{ [15, Proposition 5.6].}$$

We consider the diagonal embedding  $E_K \longrightarrow U_K$  and its natural extension  $E_K \otimes \mathbb{Z}_p \longrightarrow U_K$  whose image is  $\bar{E}_K$ , the topological closure of  $E_K$  in  $U_K$ .

In the sequel, these embeddings shall be understood; moreover, we assume in this paper that  $K$  satisfies the Leopoldt conjecture at  $p$ , which is equivalent to the condition  $\text{rk}_{\mathbb{Z}_p}(\bar{E}_K) = \text{rk}_{\mathbb{Z}}(E_K)$  (see, e.g., [15, § 5.5, p. 75]).

## 2. The Kummer–Leopoldt constant

This notion comes from the Kummer lemma (see, e.g., [15, Theorem 5.36]), that is to say, if the odd prime number  $p$  is “regular”, the cyclotomic field  $K = \mathbb{Q}(\mu_p)$  of  $p$ th roots of unity satisfies the following property stated for the whole group  $E'_K$  of global units of  $K$ :

*any  $\varepsilon \in E'_K$ , congruent to a rational integer modulo  $p$ , is a  $p$ th power in  $E'_K$ .*

In fact,  $\varepsilon \equiv a \pmod{p}$  with  $a \in \mathbb{Z}$ , implies  $\varepsilon^{p-1} \equiv a^{p-1} \equiv 1 \pmod{p}$ . So we shall write the Kummer property with  $p$ -principal units in the more suitable equivalent statement:

*any  $\varepsilon \in E_K$ , congruent to 1 modulo  $p$ , is a  $p$ th power in  $E_K$ .*

From [1], [11], [13], [14], [16], [17] one can study this property and its generalizations with various techniques (see the rather intricate history in [2]). Give the following definition from [2]:

**Definition 2.1.** — *Let  $K$  be a number field satisfying the Leopoldt conjecture at the prime  $p \geq 2$ . Let  $E_K$  be the group of  $p$ -principal global units of  $K$  and let  $U_K$  be the group of principal local units at the  $p$ -places.*

*We call Kummer–Leopoldt constant (denoted  $\kappa_K =: \kappa$ ), the smallest integer  $c$  such that the following condition is fulfilled:*

*for all  $n \gg 0$ , any unit  $\varepsilon \in E_K$ , such that  $\varepsilon \in U_K^{p^{n+c}}$ , is necessarily in  $E_K^{p^n}$ .*

**Remark 2.1.** — *The existence of  $\kappa$  comes from various classical characterizations of Leopoldt’s conjecture proved for instance in [5, Theorem III.3.6.2], after [14], [11] and oldest Iwasawa papers. Indeed, if the Leopoldt conjecture is not satisfied, we can find a sequence  $\varepsilon_n \in E_K \setminus E_K^p$  such that  $\log(\varepsilon_n) \rightarrow 0$  (i.e.,  $\varepsilon_n \in U_K^{p^m} \cdot W_K$ , with  $m \rightarrow \infty$  as  $n \rightarrow \infty$ ); since  $W_K$  is finite, taking a suitable  $p$ -power of  $\varepsilon_n$ , we see that  $\kappa$  does not exist in that case.*

We shall prove (Theorem 3.1) that in the above definition, the condition “for all  $n \gg 0$ ” can be replaced by “for all  $n \geq 0$ ”, subject to introduce the group of global roots of unity of  $K$  and a suitable statement.

We have the following first  $p$ -adic result giving  $p^\kappa$  under the Leopoldt conjecture:

**Theorem 2.1.** — *Denote by  $E_K$  the group of  $p$ -principal global units of  $K$ , by  $U_K$  the  $\mathbb{Z}_p$ -module of principal local units at the  $p$ -places, and by  $W_K$  its torsion subgroup. Let  $\kappa_K$  be the Kummer–Leopoldt constant (Definition 2.1).*

*Then  $p^{\kappa_K}$  is the exponent of the finite group  $\text{tor}_{\mathbb{Z}_p}(\log(U_K)/\log(\bar{E}_K))$ , where  $\log$  is the  $p$ -adic logarithm and  $\bar{E}_K$  the topological closure of  $E_K$  in  $U_K$  (whence the relation  $\log(\bar{E}_K) = \mathbb{Z}_p \log(E_K)$ ).*

*Proof.* — Let  $p^\kappa$  be the exponent of  $\text{tor}_{\mathbb{Z}_p}(\log(U_K)/\log(\overline{E}_K))$ .

(i) ( $\kappa$  is suitable). Let  $n \gg 0$  and let  $\varepsilon \in E_K$  be such that

$$\varepsilon = u^{p^{n+\kappa}}, u \in U_K.$$

So  $\log(u)$  is of finite order modulo  $\log(\overline{E}_K)$  and  $\log(\varepsilon) = p^n \cdot (p^\kappa \cdot \log(u)) = p^n \cdot \log(\overline{\eta})$  with  $\overline{\eta} \in \overline{E}_K$ . By definition of  $\overline{E}_K$ , we can write in  $U_K$ , for all  $N \gg 0$ ,

$$\overline{\eta} = \eta(N) \cdot u_N, \quad \eta(N) \in E_K, \quad u_N \equiv 1 \pmod{p^N};$$

we get  $\log(\varepsilon) = p^n \cdot \log(\eta(N)) + p^n \cdot \log(u_N)$  giving in  $U_K$

$$\varepsilon = \eta(N)^{p^n} \cdot u_N^{p^n} \cdot \xi_N, \quad \xi_N \in W_K.$$

But  $\xi_N$  is near 1 (depending on the choice of  $n \gg 0$ ), whence  $\xi_N = 1$  for all  $N$ , and  $\varepsilon = \eta(N)^{p^n} \cdot u_N^{p^n}$ ,  $u_N^{p^n} \rightarrow 1$  as  $N \rightarrow \infty$ ; so  $u_N^{p^n} = \varepsilon \cdot \eta(N)^{-p^n}$  is a global unit, arbitrary close to 1, hence, because of Leopoldt's conjecture [5, Theorem III.3.6.2 (iii, iv)], of the form  $\varphi_N^{p^n}$  with  $\varphi_N \in E_K$  (recall that  $n$  is large enough, arbitrary, but fixed), giving

$$\varepsilon = \eta(N)^{p^n} \cdot \varphi_N^{p^n} \in E_K^{p^n}.$$

(ii) ( $\kappa$  is the least solution). Suppose there exists an integer  $c < \kappa$  having the property given in Definition 2.1. Let  $u_0 \in U_K$  be such that

$$\log(u_0) \text{ is of order } p^\kappa \text{ in } \text{tor}_{\mathbb{Z}_p}(\log(U_K)/\log(\overline{E}_K));$$

then  $\log(u_0^{p^\kappa}) = \log(\overline{\varepsilon}_0)$ ,  $\overline{\varepsilon}_0 \in \overline{E}_K$ . This is equivalent to

$$u_0^{p^\kappa} = \overline{\varepsilon}_0 \cdot \xi_0 = \varepsilon(N) \cdot u_N \cdot \xi_0, \quad \varepsilon(N) \in E_K, \quad u_N \equiv 1 \pmod{p^N}, \quad \xi_0 \in W_K,$$

hence, for any  $n \gg 0$ ,  $u_0^{p^{n+\kappa}} = \varepsilon(N)^{p^n} \cdot u_N^{p^n}$ . Taking  $N$  large enough, but fixed, we can suppose that  $u_N = v^{p^{2\kappa}}$ ,  $v \in U_K$  near 1; because of the above relations,  $\log(v)$  is of finite order modulo  $\log(\overline{E}_K)$ , thus  $\log(v^{p^\kappa}) \in \log(\overline{E}_K)$ . This is sufficient, for

$$u'_0 := u_0 \cdot v^{-p^\kappa},$$

to get  $\log(u'_0)$  of order  $p^\kappa$  modulo  $\log(\overline{E}_K)$ . So we can write:

$$\varepsilon(N)^{p^n} = u_0^{p^{n+\kappa}} \cdot u_N^{-p^n} = u_0^{p^{n+\kappa}} \cdot (v^{-p^\kappa})^{p^{n+\kappa}} = u_0'^{p^{n+\kappa}} \in U_K^{p^{n+(\kappa-c)+c}},$$

but, by assumption on  $c$  applied to the global unit  $\varepsilon(N)^{p^n}$ , we obtain

$$\varepsilon(N)^{p^n} = \eta_0^{p^{n+(\kappa-c)}}, \quad \eta_0 \in E_K;$$

thus, the above relation  $u_0'^{p^{n+\kappa}} = \varepsilon(N)^{p^n} = \eta_0^{p^{n+(\kappa-c)}}$  yields:

$$p^c \cdot \log(u'_0) = \log(\eta_0) \in \log(E_K),$$

which is absurd since  $\log(u'_0)$  is of order  $p^\kappa$  modulo  $\log(\overline{E}_K)$ .  $\square$

### 3. Interpretation of $\kappa_K$ – Fundamental exact sequence

The following  $p$ -adic result is valid without any assumption on  $K$  and  $p$ :

**Lemma 3.1.** — *We have the exact sequence (from [5, Lemma 4.2.4]):*

$$1 \rightarrow W_K / \text{tor}_{\mathbb{Z}_p}(\overline{E}_K) \longrightarrow \text{tor}_{\mathbb{Z}_p}(U_K / \overline{E}_K) \xrightarrow{\log} \text{tor}_{\mathbb{Z}_p}(\log(U_K) / \log(\overline{E}_K)) \rightarrow 0.$$

*Proof.* — The surjectivity comes from the fact that if  $u \in U_K$  is such that  $p^n \log(u) = \log(\bar{\varepsilon})$ ,  $\bar{\varepsilon} \in \bar{E}_K$ , then  $u^{p^n} = \bar{\varepsilon} \cdot \xi$  for  $\xi \in W_K$ , hence there exists  $m \geq n$  such that  $u^{p^m} \in \bar{E}_K$ , whence  $u$  gives a preimage in  $\text{tor}_{\mathbb{Z}_p}(U_K/\bar{E}_K)$ .

If  $u \in U_K$  is such that  $\log(u) \in \log(\bar{E}_K)$ , then  $u = \bar{\varepsilon} \cdot \xi$  as above, giving the kernel equal to  $\bar{E}_K \cdot W_K/\bar{E}_K = W_K/\text{tor}_{\mathbb{Z}_p}(\bar{E}_K)$ .  $\square$

**Corollary 3.1.** — *Let  $\mu_K$  be the group of global roots of unity of  $p$ -power order of  $K$ .*

*Then, under the Leopoldt conjecture for  $p$  in  $K$ , we have  $\text{tor}_{\mathbb{Z}_p}(\bar{E}_K) = \mu_K$ ; thus in that case  $W_K/\text{tor}_{\mathbb{Z}_p}(\bar{E}_K) = W_K/\mu_K$ .*

*Proof.* — From [5, Corollary III.3.6.3], [9, Définition 2.11, Proposition 2.12].  $\square$

Put

$$\mathcal{W}_K := W_K/\mu_K \quad \& \quad \mathcal{R}_K := \text{tor}_{\mathbb{Z}_p}(\log(U_K)/\log(\bar{E}_K)).$$

Then the exact sequence of Lemma 3.1 becomes:

$$1 \longrightarrow \mathcal{W}_K \longrightarrow \text{tor}_{\mathbb{Z}_p}(U_K/\bar{E}_K) \xrightarrow{\log} \mathcal{R}_K \longrightarrow 0.$$

Consider the following diagram (see [5], § III.2, (c), Fig. 2.2), under the Leopoldt conjecture for  $p$  in  $K$ :

$$\begin{array}{ccccc}
 & & \mathcal{T}_K & & \\
 & \curvearrowright & & \curvearrowright & \\
 \tilde{K} & \xrightarrow{\quad} & \tilde{K}H_K & \xrightarrow[\simeq \mathcal{R}_K]{\quad} & H_K^{\text{reg}} & \xrightarrow[\simeq \mathcal{W}_K]{\quad} & H_K^{\text{pr}} \\
 | & & | & & & & \\
 \tilde{K} \cap H_K & \xrightarrow{\quad} & H_K & \xrightarrow[\simeq \mathcal{U}_K/\bar{E}_K]{\quad} & & & \\
 | & & \curvearrowright & & & & \\
 K & & & & & & 
 \end{array}$$

$\simeq \mathcal{C}_K$

where  $\tilde{K}$  is the compositum of the  $\mathbb{Z}_p$ -extensions,  $\mathcal{C}_K$  the  $p$ -class group,  $H_K$  the  $p$ -Hilbert class field,  $H_K^{\text{pr}}$  the maximal Abelian  $p$ -ramified (i.e., unramified outside  $p$ ) pro- $p$ -extension, of  $K$ . These definitions are given in the ordinary sense when  $p = 2$  (so that the real infinite places of  $K$  are not complexified (= are unramified) in the class fields under consideration which are “real”).

By class field theory,  $\text{Gal}(H_K^{\text{pr}}/H_K) \simeq U_K/\bar{E}_K$  in which the image of  $\mathcal{W}_K$  fixes  $H_K^{\text{reg}}$ , the Bertrandias–Payan field,  $\text{Gal}(H_K^{\text{reg}}/\tilde{K})$  being then the Bertrandias–Payan module, except possibly if  $p = 2$  in the “special case” (cf. [2] about the calculation of  $\kappa$  and the Références in [6] for some history about this module).

But  $\mathcal{R}_K$  giving  $\kappa_K$  has, a priori, nothing to do with *the definition* of the Bertrandias–Payan module associated with  $p^r$ -cyclic extensions of  $K$ ,  $r \geq 1$ , which are embeddable in cyclic  $p$ -extensions of  $K$  of arbitrary large degree.

Then we put  $\mathcal{T}'_K := \text{tor}_{\mathbb{Z}_p}(\text{Gal}(H_K^{\text{pr}}/H_K)) \subseteq \mathcal{T}_K := \text{tor}_{\mathbb{Z}_p}(\text{Gal}(H_K^{\text{pr}}/K))$ . The group  $\mathcal{R}_K$  is then isomorphic to  $\text{Gal}(H_K^{\text{reg}}/\tilde{K}H_K)$ . Of course, for  $p \geq p_0$  (explicit),  $\mathcal{W}_K = [H_K : \tilde{K} \cap H_K] = 1$ , whence  $\mathcal{R}_K = \mathcal{T}_K$ . We shall see in the Section 5 that  $\mathcal{R}_K \simeq \mathcal{T}'_K/\mathcal{W}_K$  is closely related to the classical  $p$ -adic regulator of  $K$ .

**Corollary 3.2.** — Under the Leopoldt conjecture for  $p$  in  $K$ , the Kummer–Leopoldt constant  $\kappa_K$  of  $K$  is 0 if and only if  $\mathcal{R}_K = 1$  (i.e.,  $H_K^{\text{reg}} = \tilde{K}H_K$ ).

*Proof.* — From Theorem 2.1 using the new terminology of the “algebraic regulator”  $\mathcal{R}_K := \text{tor}_{\mathbb{Z}_p}(\log(U_K)/\log(\overline{E}_K))$  whose exponent is  $p^\kappa$ .  $\square$

**Corollary 3.3.** — If the prime number  $p$  is regular, then  $\kappa_K = 0$  for the field  $K = \mathbb{Q}(\mu_p)$  of  $p$ th roots of unity, and any unit  $\varepsilon \in E_K$  such that  $\varepsilon \equiv 1 \pmod{p}$  is in  $E_K^p$  (Kummer’s lemma).

*Proof.* — (i) We first prove that if the real unit  $\varepsilon$  is congruent to 1 modulo  $p$  then it is a  $p$ th power in  $U_K$ . Put  $\varepsilon = 1 + \alpha \cdot p$  for a  $p$ -integer  $\alpha \in K^\times$ . Let  $K_0$  be the maximal real subfield of  $K$  and let  $\pi_0$  be an uniformizing parameter of its  $p$ -completion. Put  $\alpha = a_0 + \beta \cdot \pi_0$  with  $a_0 \in [0, p-1]$  and a  $p$ -integer  $\beta$ . Since  $N_{K_0/\mathbb{Q}}(\varepsilon) = 1$ , this yields  $a_0 = 0$ , whence  $\varepsilon = 1 + \beta \cdot p \cdot \pi_0$ . The valuation of  $p \cdot \pi_0$ , calculated in  $K$ , is  $p+1$ , which is sufficient to get  $\varepsilon \in U_K^p$  (use [15, Proposition 5.7]).

(ii) Then we prove that  $\kappa = 0$ . The cyclotomic field  $K = \mathbb{Q}(\mu_p)$  is  $p$ -regular and  $p$ -rational in the meaning of [4, Théorème & Définition 2.1], so  $\mathcal{T}_K = 1$  giving  $\kappa = 0$ . In other words,  $\kappa = 0$  is given by a stronger condition ( $p$ -rationality of  $K$ ) than  $\mathcal{R}_K = 1$ .

One may preferably use the general well-known  $p$ -rank formula (the  $p$ -rank  $\text{rk}_p(A)$  of a finite Abelian group  $A$  is the  $\mathbb{F}_p$ -dimension of  $A/A^p$ ), valid for any field  $K$  under the Leopoldt conjecture, when the group  $\mu_K$  of  $p$ th roots of unity is nontrivial [5, Proposition III.4.2.2]:

$$\text{rk}_p(\mathcal{T}_K) = \text{rk}_p(\mathcal{C}_K^{S_K^{\text{res}}}) + \#S_K - 1,$$

where  $S_K$  is the set of prime ideals of  $K$  above  $p$  and  $\mathcal{C}_K^{S_K^{\text{res}}}$  the  $S_K$ -class group in the restricted sense (when  $p = 2$ ) equal to the quotient of the  $p$ -class group of  $K$  in the restricted sense by the subgroup generated by the classes of ideals of  $S_K$ ; so for  $K = \mathbb{Q}(\mu_p)$ , we immediately get  $\text{rk}_p(\mathcal{T}_K) = \text{rk}_p(\mathcal{C}_K)$ , which is by definition trivial for regular primes.  $\square$

**Theorem 3.1.** — Let  $\kappa_K$  be the Kummer–Leopoldt constant of  $K$  (Definition 2.1) and let  $p^\nu$  be the exponent of  $\mathcal{W}_K = W_K/\mu_K$ , where  $W_K = \text{tor}_{\mathbb{Z}_p}(U_K)$  and  $\mu_K$  is the group of global roots of unity of  $K$  of  $p$ -power order.<sup>(1)</sup>

The property defining  $\kappa_K$  can be improved as follows:

(i) If  $\nu \geq 1$ , for all  $n \geq 0$ , any  $\varepsilon \in E_K$  such that  $\varepsilon \in U_K^{p^{n+\kappa_K}}$  is necessarily of the form  $\varepsilon = \zeta \cdot \eta^{p^n}$ , with  $\zeta \in \mu_K \cap W_K^{p^n}$ ,  $\eta \in E_K$ .

(ii) If  $\nu = 0$ , for all  $n \geq 0$ , any  $\varepsilon \in E_K$  being in  $U_K^{p^{n+\kappa_K}}$  is necessarily in  $E_K^{p^n}$ .

*Proof.* — Let  $n \geq 0$ . Suppose that  $\varepsilon = u^{p^{n+\kappa}}$ ,  $u \in U_K$ . So  $\log(\varepsilon) = p^n \cdot p^\kappa \cdot \log(u) = p^n \cdot \log(\bar{\eta})$ ,  $\bar{\eta} \in \overline{E}_K$ ; thus  $\bar{\eta} = \eta(N) \cdot u_N$ , with  $\eta(N) \in E_K$ ,  $u_N \equiv 1 \pmod{p^N}$ , for all  $N \gg 0$ , and  $\log(\varepsilon) = p^n \cdot \log(\eta(N)) + p^n \cdot \log(u_N)$  giving in  $U_K$

$$\varepsilon = \eta(N)^{p^n} \cdot u_N^{p^n} \cdot \xi_N, \quad \xi_N \in W_K, \text{ for all } N \gg 0.$$

Taking  $N$  in a suitable infinite subset of  $\mathbb{N}$ , we can suppose  $\xi_N = \xi$  independent of  $N \rightarrow \infty$ . Then  $\xi = (\varepsilon \cdot \eta(N)^{-p^n}) \cdot u_N^{-p^n} \in \text{tor}_{\mathbb{Z}_p}(\overline{E}_K)$ , whence  $\xi = \zeta \in \mu_K$  because of Leopoldt’s conjecture (loc. cit. in proof of Corollary 3.1). Then

<sup>(1)</sup>In the case  $\nu = 0$ , if  $\mu_K = 1$ , then  $\mu_{K_p} = 1 \forall p \in S_K$ ; if  $\mu_K \neq 1$ , then  $S_K = \{p\}$  &  $\mu_{K_p} = \mu_K$ .

$$u^{p^{n+\kappa}} = \varepsilon = \eta(N)^{p^n} \cdot u_N^{p^n} \cdot \zeta = \eta(N)^{p^n} \cdot u'_N \cdot \zeta, \quad u'_N (\in E_K) \rightarrow 1 \text{ as } N \rightarrow \infty,$$

whence  $\varepsilon$  of the form  $\eta(N)^{p^n} \cdot \varphi_N^{p^n} \cdot \zeta$ ,  $\varphi_N \in E_K$ , for  $N \gg 0$ . So  $\varepsilon = \zeta \cdot \eta^{p^n}$ , with  $\eta \in E_K$  and  $\zeta = \varepsilon \cdot \eta^{-p^n} \in \mu_K \cap W_K^{p^n}$ , since  $\varepsilon$  is a local  $p^n$ th power.

If  $\nu = 0$ ,  $W_K = \mu_K$  and  $\zeta \in \mu_K \cap W_K^{p^n} = \mu_K^{p^n}$  is a  $p^n$ th power.  $\square$

#### 4. Remarks and applications

As above, we assume the Leopoldt conjecture for  $p$  in the fields under consideration.

(a) The condition  $\varepsilon \in U_K^{p^{n+\kappa}} = \bigoplus_{\mathfrak{p}|p} U_{\mathfrak{p}}^{p^{n+\kappa}}$ , where  $U_{\mathfrak{p}} := 1 + \bar{\mathfrak{p}}$ , may be translated,

in the framework of Kummer's lemma, into a less precise condition of the form  $\varepsilon \equiv 1 \pmod{\prod_{\mathfrak{p}|p} \mathfrak{p}^{m_{\mathfrak{p}}(n, \kappa)}}$  for suitable minimal exponents  $m_{\mathfrak{p}}(n, \kappa)$  giving local  $p^{n+\kappa}$ th powers. This was used by most of the cited references with  $p$ -adic analytical calculations using the fact that  $\#\mathcal{T}_K$  is, roughly speaking, a product “class number”  $\times$  “regulator” from  $p$ -adic  $L$ -functions, giving an upper bound for  $\kappa$  (it is the analytic way used in [16] and [13] to generalize Kummer's lemma when  $p$  is not regular).

(b) If  $\mathcal{T}_K = 1$  (in which case  $\kappa = 0$ ), the field  $K$  is said to be a  $p$ -rational field (see [5, § IV.3], [4], [10], [12]). Then in any  $p$ -primitively ramified  $p$ -extension  $L$  of  $K$  (definition and examples in [5, § IV.3, (b); § IV.3.5.1], after [8, Theorem 1, § II.2]), we get  $\mathcal{T}_L = 1$  whence  $\kappa_L = 0$ .

The following examples illustrate this principle:

(i) The  $p^m$ -cyclotomic fields. The above applies for the fields  $K_m := \mathbb{Q}(\mu_{p^m})$  of  $p^m$ -roots of unity when the prime  $p$  is regular, since we have seen that  $\mathcal{T}_{\mathbb{Q}(\mu_p)} = 1$ .

(ii) Some  $p$ -rational  $p$ -extensions of  $\mathbb{Q}$  ( $p = 2$  and  $p = 3$ ). The following fields have a Kummer-Leopoldt constant  $\kappa = 0$  ([5, Example IV.3.5.1], after [8, § III]):

– The real Abelian 2-extensions of  $\mathbb{Q}$ , subfields of the fields  $\mathbb{Q}(\mu_{2^\infty}) \cdot \mathbb{Q}(\sqrt{\ell})$ ,  $\ell \equiv 3 \pmod{8}$ , and  $\mathbb{Q}(\mu_{2^\infty}) \cdot \mathbb{Q}\left(\sqrt{\sqrt{\ell} \frac{a - \sqrt{\ell}}{2}}\right)$ ,  $\ell = a^2 + 4b^2 \equiv 5 \pmod{8}$ .

– The real Abelian 3-extensions of  $\mathbb{Q}$ , subfields of the fields  $\mathbb{Q}(\mu_{3^\infty}) \cdot k_\ell$ ,  $\ell \equiv 4, 7 \pmod{9}$ , where  $k_\ell$  is the cyclic cubic field of conductor  $\ell$ .

(c) When  $\mu_K = 1$ , the formula giving  $\text{rk}_p(\mathcal{T}_K)$ , used in the proof of Kummer's lemma (Corollary 3.3), must be replaced by a formula deduced from the “reflection theorem”: let  $K' := K(\zeta_p)$ , where  $\zeta_p$  is a primitive  $p$ th root of unity; then

$$\text{rk}_p(\mathcal{T}_K) = \text{rk}_\omega(\mathcal{C}_{K'}^{S_{K'}^{\text{res}}}) + \sum_{\mathfrak{p}|p} \delta_{\mathfrak{p}} - \delta,$$

which links the  $p$ -rank of  $\mathcal{T}_K$  to that of the  $\omega$ -component of the  $p$ -group of  $S_{K'}$ -ideal classes of the field  $K'$ , where  $\omega$  is the Teichmüller character of  $\text{Gal}(K'/K)$ ,  $\delta_{\mathfrak{p}} := 1$  or 0 according as the completion  $K_{\mathfrak{p}}$  contains  $\zeta_p$  or not,  $\delta := 1$  or 0 according as  $\zeta_p \in K$  or not (so that  $\omega = 1$  if and only if  $\zeta_p \in K$ ).

(d) Unfortunately,  $p^\kappa$  may be less than  $\#\mathcal{R}_K$  (hence a fortiori less than  $\#\mathcal{T}_K$ ) due to the unknown group structure of  $\mathcal{R}_K$ ; as usual, when  $K/\mathbb{Q}$  is Galois with Galois group  $G$ , the study of its  $G$ -structure may give more precise information:

Indeed, to simplify assume  $p > 2$  unramified in  $K$ , so that  $\log(U_K)$  is isomorphic to  $O_K$ , the direct sum of the rings of integers of the  $K_{\mathfrak{p}}$ ,  $\mathfrak{p} \mid p$ ; if  $\eta =: 1 + p \cdot \alpha \in E_K$  generates a sub- $G$ -module of  $E_K$ , of index prime to  $p$  (such a unit does exist since  $E_K \otimes \mathbb{Q}$  is a monogenic  $\mathbb{Q}[G]$ -module; cf. [5, Corollary I.3.7.2 & Remark I.3.7.3]),

the structure of  $\mathcal{R}_K$  can be easily deduced from the knowledge of  $P(\alpha) \equiv \frac{1}{p}\log(\eta)$  modulo a suitable power of  $p$ , where  $P(\alpha)$  is a rational polynomial expression of  $\alpha$  generating a sub- $G$ -module of  $O_K$ ; thus many numerical examples may be obtained.

(e) We have given in [7, §8.6] a conjecture saying that, in any fixed number field  $K$ , we have  $\mathcal{T}_K = 1$  for all  $p \gg 0$ , giving conjecturally  $\kappa = 0$  for all  $p \gg 0$ .

### 5. Normalized $p$ -adic regulator of a number field

The previous Section 3 shows that the good notion of  $p$ -adic regulator comes from the expression of the  $p$ -adic finite group  $\mathcal{R}_K$  associated with the class field theory interpretation of  $\text{Gal}(H_K^{\text{reg}}/\bar{K}H_K)$ .

For this, recall that  $\bar{E}_K$  is the topological closure, in the  $\mathbb{Z}_p$ -module  $U_K$  of principal local units at  $p$ , of the group of  $p$ -principal global units of  $K$ , and  $\log$  the  $p$ -adic logarithm:

**Definition 5.1.** — *Let  $K$  be any number field and let  $p \geq 2$  be any prime number. Under the Leopoldt conjecture for  $p$  in  $K$ , we call  $\mathcal{R}_K := \text{tor}_{\mathbb{Z}_p}(\log(U_K)/\log(\bar{E}_K))$  (or its order  $\#\mathcal{R}_K$ ) the normalized  $p$ -adic regulator of  $K$ .*

We have in the simplest case of *totally real* number fields (from Coates's formula [3, Appendix] and also [5, Remarks III.2.6.5] for  $p = 2$ ):

**Proposition 5.1.** — *For any totally real number field  $K \neq \mathbb{Q}$ , we have, under the Leopoldt conjecture for  $p$  in  $K$ ,*

$$\#\mathcal{R}_K \sim \frac{1}{2} \cdot \frac{(\mathbb{Z}_p : \log(N_{K/\mathbb{Q}}(U_K)))}{\#\mathcal{W}_K \cdot \prod_{\mathfrak{p}|p} N\mathfrak{p}} \cdot \frac{R_K}{\sqrt{D_K}},$$

where  $\sim$  means equality up to a  $p$ -adic unit factor, where  $R_K$  is the usual  $p$ -adic regulator [15, §5.5] and  $D_K$  the discriminant of  $K$ .

With this expression, we find again classical results obtained by means of analytic computations (e.g., [1, Theorem 6.5]). In the real Galois case, with  $p$  unramified in  $K/\mathbb{Q}$ , we get, as defined in [7, Définition 2.3],  $\#\mathcal{R}_K \sim \frac{R_K}{p^{[K:\mathbb{Q}]-1}}$  for  $p \neq 2$  and

$\#\mathcal{R}_K \sim \frac{1}{2^{d-1}} \frac{R_K}{2^{[K:\mathbb{Q}]-1}}$  for  $p = 2$ , where  $d$  is the number of prime ideals  $\mathfrak{p} \mid 2$  in  $K$ .

Of course,  $\#\mathcal{R}_K = \#\text{tor}_{\mathbb{Z}_p}(\log(U_K)) = 1$  for  $\mathbb{Q}$  and any imaginary quadratic field.

### References

- [1] B. Anglès, *Units and norm residue symbol*, Acta Arithmetica **98**(1) (2001), 33–51. <https://eudml.org/doc/278965>
- [2] J. Assim et T. Nguyen Quang Do, *Sur la constante de Kummer–Leopoldt d'un corps de nombres*, Manuscripta Math. **115**(1) (2004), 55–72. <http://link.springer.com/article/10.1007/s00229-004-0482-9>
- [3] J. Coates,  *$p$ -adic  $L$ -functions and Iwasawa's theory*, Algebraic number fields:  $L$ -functions and Galois properties (Proc. Sympos., Univ. Durham, Durham, 1975), Academic Press, London (1977), 269–353.
- [4] G. Gras et J-F. Jaulent, *Sur les corps de nombres réguliers*, Math. Z. **202** (1989), 343–365. <https://eudml.org/doc/174095>
- [5] G. Gras, *Class Field Theory: from theory to practice*, SMM, Springer-Verlag 2003; second corrected printing 2005. <https://www.researchgate.net/publication/268005797>



- [6] G. Gras, *Sur le module de Bertrandias–Payan dans une  $p$ -extension – Noyau de capitulation*, Publ. Mathématiques de Besançon, Algèbre et Théorie des Nombres (2016), 25–44. [http://pmb.cedram.org/item?id=PMB\\_2016\\_\\_\\_\\_25\\_0](http://pmb.cedram.org/item?id=PMB_2016____25_0)
- [7] G. Gras, *Les  $\theta$ -régulateurs locaux d'un nombre algébrique : Conjectures  $p$ -adiques*, Canadian Journal of Mathematics **68**(3) (2016), 571–624. <http://dx.doi.org/10.4153/CJM-2015-026-3>
- [8] G. Gras, *Remarks on  $K_2$  of number fields*, Jour. Number Theory **23** (1986), 322–335. <http://www.sciencedirect.com/science/article/pii/0022314X86900776>
- [9] J-F. Jaulent, *Théorie  $\ell$ -adique globale du corps de classes*, J. Théorie des Nombres de Bordeaux **10**(2) (1998), 355–397. [http://www.numdam.org/article/JTNB\\_1998\\_\\_10\\_2\\_355\\_0.pdf](http://www.numdam.org/article/JTNB_1998__10_2_355_0.pdf)
- [10] J-F. Jaulent et T. Nguyen Quang Do, *Corps  $p$ -rationnels, corps  $p$ -réguliers et ramification restreinte*, J. Théorie des Nombres de Bordeaux **5** (1993), 343–363. [http://www.numdam.org/article/JTNB\\_1993\\_\\_5\\_2\\_343\\_0.pdf](http://www.numdam.org/article/JTNB_1993__5_2_343_0.pdf)
- [11] F. Lorenz, *Some remarks on Leopoldt's conjecture*, Algebra i Analiz **10**(6) (1998), 144–155; translation in St. Petersburg Math. J. **10**(6) (1999), 1005–1013. <http://mi.mathnet.ru/eng/aa1036>
- [12] A. Movahhedi et T. Nguyen Quang Do, *Sur l'arithmétique des corps de nombres  $p$ -rationnels*, Séminaire de Théorie des Nombres, Paris 1987–88, Progress in Mathematics Volume 81, 1990, 155–200. [https://link.springer.com/chapter/10.1007%2F978-1-4612-3460-9\\_9](https://link.springer.com/chapter/10.1007%2F978-1-4612-3460-9_9)
- [13] M. Ozaki *Kummer's lemma for  $\mathbb{Z}_p$ -extensions over totally real number fields*, Acta Arithmetica **81**(1) (1997), 37–43. <https://eudml.org/doc/207053>
- [14] J. Sands, *Kummer's and Iwasawa's version of Leopoldt's conjecture*, Canad. Math. Bull. **31**(1) (1988), 338–346. <http://cms.math.ca/openaccess/cmb/v31/cmb1988v31.0338-0346.pdf>
- [15] L.C. Washington, *Introduction to cyclotomic fields*, Graduate Texts in Math. 83, Springer enlarged second edition 1997.
- [16] L.C. Washington, *Kummer's lemma for prime power cyclotomic fields*, Jour. Number Theory **40** (1992), 165–173. <http://www.sciencedirect.com/science/article/pii/0022314X9290037P>
- [17] L.C. Washington, *Units of irregular cyclotomic fields*, Ill. J. Math. **23** (1979), 635–647. [https://projecteuclid.org/download/pdf\\_1/euclid.ijm/1256047937](https://projecteuclid.org/download/pdf_1/euclid.ijm/1256047937)

---

GEORGES GRAS, Villa la Gardette, Chemin Château Gagnière, F-38520 Le Bourg d'Oisans, France  
 – [https://www.researchgate.net/profile/Georges\\_Gras](https://www.researchgate.net/profile/Georges_Gras) • E-mail : [g.mn.gras@wanadoo.fr](mailto:g.mn.gras@wanadoo.fr)