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THE p -ADIC KUMMER–LEOPOLDT CONSTANT NORMALIZED p -ADIC REGULATOR

GEORGES GRAS

ABSTRACT. The p -adic Kummer–Leopoldt constant κ_K of a number field K is (under the Leopoldt conjecture for p in K) the least integer c such that for all n large enough, any global unit ε of K , which is locally (at the p -places) a p^{n+c} th power, is necessarily a p^n th power of a global unit of K . This constant has been computed by J. Assim and T. Nguyen Quang Do using Iwasawa’s technics. In this short Note, we give an elementary p -adic proof of their result and the class field theory interpretation of κ_K by means of Abelian p -ramification theory over K . Then, from this, we give a natural definition of the normalized p -adic regulator of K for any K and any $p \geq 2$.

1. NOTATIONS

Let K be a number field and let $p \geq 2$ be a prime number. We assume in the sequel that K satisfies the Leopoldt conjecture at p . Denote by E_K the group of principal global units of K (i.e., units $\varepsilon \equiv 1 \pmod{\prod_{\mathfrak{p}|p} \mathfrak{p}}$) and by

$$U_K := \left\{ u \in \prod_{\mathfrak{p}|p} K_{\mathfrak{p}}^{\times}, u = 1 + x, x \in \prod_{\mathfrak{p}|p} \mathfrak{p} \right\} \quad \& \quad W_K = \text{tor}_{\mathbb{Z}_p}(U_K),$$

the \mathbb{Z}_p -module of local units at p and its torsion subgroup, where $K_{\mathfrak{p}}$ is the \mathfrak{p} -completion of K .

The p -adic logarithm \log is defined on $1 + x$, $x \in \prod_{\mathfrak{p}|p} \mathfrak{p}$, by means of the series $\log(1 + x) = \sum_{i \geq 1} (-1)^{i+1} \frac{x^i}{i} \in \prod_{\mathfrak{p}|p} K_{\mathfrak{p}}$. The kernel of \log in U_K is W_K .

We consider the diagonal embedding $E_K \rightarrow U_K$ and its natural extension $E_K \otimes \mathbb{Z}_p \rightarrow U_K$ whose image is \overline{E}_K , the topological closure of E_K in U_K . In the sequel, these embeddings shall be understood.

2. THE KUMMER–LEOPOLDT CONSTANT

This notion comes from the Kummer lemma (see [W3, Theorem 5.36]), that is to say, if the prime number p is “regular”, the cyclotomic field $K = \mathbb{Q}(\mu_p)$ satisfies the following property for the whole group E'_K of global units of K : *any $\varepsilon \in E'_K$, congruent to a rational integer modulo p , is a p th power in E'_K .*

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In fact, $\varepsilon \equiv a \pmod{p}$, with $a \in \mathbb{Z}$, implies $\varepsilon^{p-1} \equiv a^{p-1} \equiv 1 \pmod{p}$. So we shall write, equivalently, the Kummer property with principal units:

any $\varepsilon \in E_K$, congruent to 1 modulo p , is a p th power in E_K .

From [A], [L], [O], [S], [W1], [W2] (see the rather intricate history in [AN]) one can study this property and its generalizations with various technics. Give the following definition (from [AN]):

Definition 2.1. Let K be a number field satisfying the Leopoldt conjecture at the prime $p \geq 2$. Let E_K be the group of principal global units of K . We call Kummer–Leopoldt constant (denoted $\kappa = \kappa_K$), the smallest integer c such that the following condition is fulfilled:

for all $n \gg 0$, any unit $\varepsilon \in E_K$, such that $\varepsilon \in U_K^{p^{n+c}}$, is necessarily in $E_K^{p^n}$.

Remark 2.2. The existence of κ comes from the classical characterization of Leopoldt’s conjecture given, for instance, in [Gr1, Theorem III.3.6.2 (iv)]. If the Leopoldt conjecture is not satisfied, we can find a sequence $\varepsilon_n \in E_K \setminus E_K^p$ such that $\log(\varepsilon_n) \rightarrow 0$ (i.e., $\varepsilon_n \in U_K^{p^m} \cdot W_K$, with $m \rightarrow \infty$ when $n \rightarrow \infty$); since W_K is finite, taking a suitable power of ε_n , we see that κ does not exist in that case.

We have the following p -adic result giving p^κ under the Leopoldt conjecture:

Theorem 2.3. *Denote by E_K the group of principal global units of K , then by U_K the \mathbb{Z}_p -module of principal local units at p and by W_K its torsion subgroup. Let κ_K (Definition 2.1) be the Kummer–Leopoldt constant; then p^{κ_K} is the exponent of the finite group $\text{tor}_{\mathbb{Z}_p}(\log(U_K)/\log(\overline{E}_K))$, where \overline{E}_K is the closure of E_K in U_K (hence $\log(\overline{E}_K) = \mathbb{Z}_p \log(E_K)$).*

Proof. (i) (κ is suitable). Let $n \gg 0$ and let $\varepsilon \in E_K$ be such that $\varepsilon = u^{p^{n+\kappa}}$, $u \in U_K$ (thus ε is arbitrary near from 1, depending on the choice of $n \gg 0$). Then $\log(\varepsilon) = p^n \cdot (p^\kappa \cdot \log(u)) = p^n \cdot \log(\overline{\eta})$ with $\overline{\eta} \in \overline{E}_K$. So, witting

$$\overline{\eta} = \eta(N) \cdot u_N, \quad \eta(N) \in E_K, \quad u_N \equiv 1 \pmod{p^N}, \quad N \rightarrow \infty,$$

we get $\log(\varepsilon) = p^n \cdot \log(\eta(N)) + p^n \cdot \log(u_N)$ giving

$$\varepsilon = \eta(N)^{p^n} \cdot u_N^{p^n} \cdot \xi, \quad \xi \in W_K.$$

But ξ is near from 1, whence $\xi = 1$. Then

$$\varepsilon = \eta(N)^{p^n} \cdot u'_N, \quad u'_N \rightarrow 1 \text{ as } N \rightarrow \infty;$$

so $u'_N = \varepsilon \cdot \eta(N)^{-p^n}$ is a global unit, arbitrary near from 1, hence, because of the Leopoldt conjecture (loc. cit. in Remark 2.2), of the form $\varphi_N^{p^n}$ with $\varphi_N \in E_K$, $\varphi_N \rightarrow 1$ as $N \rightarrow \infty$ (recall that n is arbitrary but fixed), giving $\varepsilon = \eta(N)^{p^n} \cdot \varphi_N^{p^n}$, whence:

$$\varepsilon \in E_K^{p^n}.$$

(ii) (κ is the least solution). Suppose there exists $c < \kappa$ having the property given in Definition 2.1.

Let $u_0 \in U_K$ be such that $\log(u_0)$ is of order p^κ in $\text{tor}_{\mathbb{Z}_p}(\log(U_K)/\log(\overline{E}_K))$; then $\log(u_0^{p^\kappa}) = \log(\overline{\varepsilon}_0)$, $\overline{\varepsilon}_0 \in \overline{E}_K$. This is equivalent to

$$u_0^{p^\kappa} = \overline{\varepsilon}_0 \cdot \xi_0 = \varepsilon(N) \cdot u_N \cdot \xi_0, \quad \varepsilon(N) \in E_K, \quad u_N \in U_K, \quad \xi_0 \in W_K$$

(with $u_N \rightarrow 1$ when $N \rightarrow \infty$), hence, for any $n \gg 0$,

$$u_0^{p^{n+\kappa}} = \varepsilon(N)^{p^n} \cdot u_N^{p^n}.$$

Taking N large enough, we can suppose that $u_N = v_N^{p^{2\kappa}}$, $v_N \in U_K$ arbitrary near from 1; because of the above relation, $\log(v_N)$ is of finite order modulo $\log(\overline{E}_K)$, thus $\log(v_N^{p^\kappa}) \in \log(\overline{E}_K)$. This is sufficient, for $u'_0 := u_0 \cdot v_N^{-p^\kappa}$, to get $\log(u'_0)$ of order p^κ modulo $\log(\overline{E}_K)$. So we can write:

$$\varepsilon(N)^{p^n} = u_0^{p^{n+\kappa}} \cdot u_N^{-p^n} = u_0^{p^{n+\kappa}} \cdot (v_N^{-p^\kappa})^{p^{n+\kappa}} = u_0^{p^{n+\kappa}} \in U_K^{p^{n+(\kappa-c)+c}},$$

but, by assumption on c , we obtain $\varepsilon(N)^{p^n} = \eta_0^{p^{n+(\kappa-c)}}$, $\eta_0 \in E_K$; thus, the above relation $u_0^{p^{n+\kappa}} = u_0^{p^{n+(\kappa-c)+c}} = \eta_0^{p^{n+(\kappa-c)}}$, yields to:

$$p^c \cdot \log(u'_0) = \log(\eta_0) \in \log(E_K) \text{ (absurd)}. \quad \square$$

3. INTERPRETATION OF κ_K – FUNDAMENTAL EXACT SEQUENCE

Consider the following diagram, under the Leopoldt conjecture for p in K (diagram given in [Gr1], § III.2, (c), Fig. 2.2):

$$\begin{array}{ccccc}
 & & \mathcal{T}_K & & \\
 & & \curvearrowright & & \\
 & & \mathcal{T}_K^E & & \\
 \tilde{K} & \xrightarrow{\quad} & \tilde{K}H_K & \xrightarrow{\mathcal{R}_K} & H_K^E & \xrightarrow{\mathcal{W}_K} & H_K^{\text{pr}} \\
 | & & | & & & & \\
 \tilde{K} \cap H_K & \xrightarrow{\quad} & H_K & & & & \\
 | & & \mathcal{C}_K & & & & \\
 K & & & & & &
 \end{array}$$

$\mathcal{U}_K/\overline{E}_K$ (arc from H_K to H_K^{pr})

where \tilde{K} is the compositum of the \mathbb{Z}_p -extensions of K , \mathcal{C}_K the p -class group in the ordinary sense, H_K the p -Hilbert class field, H_K^{pr} the maximal Abelian p -ramified (i.e., unramified outside p) pro- p -extension of K ; then we put $\mathcal{T}_K^E := \text{tor}_{\mathbb{Z}_p}(\text{Gal}(H_K^{\text{pr}}/H_K)) \subseteq \mathcal{T}_K := \text{tor}_{\mathbb{Z}_p}(\text{Gal}(H_K^{\text{pr}}/K))$, and $\mathcal{R}_K \simeq \text{Gal}(H_K^E/\tilde{K}H_K)$.

By class field theory, $\text{Gal}(H_K^{\text{pr}}/H_K)$ is canonically isomorphic to U_K/\overline{E}_K in which the image of $\mathcal{W}_K := W_K/\mu_K$ (where μ_K is the group of roots of unity of K of p -power order) fixes H_K^E , which is (except possibly if $p = 2$ in the “special case”) equal to the Bertrandias–Payan field H_K^{bp} , $\text{Gal}(H_K^{\text{bp}}/\tilde{K})$ being then the Bertrandias–Payan module (also evoked in [AN] about the calculation of κ ; see also [Gr2, Références] for some history about this module).

Lemma 3.1. *We have (without any assumption on K and p) the following exact sequence:*

$$1 \longrightarrow W_K / \text{tor}_{\mathbb{Z}_p}(\overline{E}_K) \longrightarrow \text{tor}_{\mathbb{Z}_p}(U_K / \overline{E}_K) \\ \xrightarrow{\log} \text{tor}_{\mathbb{Z}_p}(\log(U_K) / \log(\overline{E}_K)) \longrightarrow 0.$$

Proof. From [Gr1, Lemma 4.2.4]: the surjectivity comes from the fact that if $u \in U_K$ is such that $p^n \log(u) = \log(\overline{\varepsilon})$, $\overline{\varepsilon} \in \overline{E}_K$, then $u^{p^n} = \overline{\varepsilon} \cdot \xi$ for $\xi \in W_K$, hence there exists $m \geq n$ such that $u^{p^m} \in \overline{E}_K$, whence $u \in \text{tor}_{\mathbb{Z}_p}(U_K / \overline{E}_K)$; u is a preimage. If u is such that $\log(u) \in \log(\overline{E}_K)$, $u = \overline{\varepsilon} \cdot \xi$ as above, giving the kernel equal to $\overline{E}_K \cdot W_K / \overline{E}_K = W_K / \text{tor}_{\mathbb{Z}_p}(\overline{E}_K)$. \square

Assuming the Leopoldt conjecture, we have $\text{tor}_{\mathbb{Z}_p}(\overline{E}_K) = \mu_K$ ([Gr1, Corollary III.3.6.3]). Thus, $W_K / \text{tor}_{\mathbb{Z}_p}(\overline{E}_K) = W_K / \mu_K = \mathcal{W}_K$ and the exact sequence in the Lemma becomes, with the notations of the diagram:

$$1 \longrightarrow \mathcal{W}_K \longrightarrow \mathcal{T}_K^E \xrightarrow{\log} \mathcal{R}_K \longrightarrow 0,$$

Corollary 3.2. *Under the Leopoldt conjecture for p in K , the Kummer-Leopoldt constant κ of K is 0 if and only if $\mathcal{R}_K = 1$, which is equivalent to $\#\mathcal{T}_K = [H_K : \tilde{K} \cap H_K] \cdot \#\mathcal{W}_K$.*

Corollary 3.3. *If the prime number p is regular, then $\kappa = 0$ for the field $\mathbb{Q}(\mu_p)$ (Kummer's lemma).*

Proof. In this case we know that the cyclotomic field $\mathbb{Q}(\mu_p)$ is regular in the meaning of [GJ, Théorème & Définition 2.1], so $\mathcal{T}_K = 1$ giving the result. \square

For instance, if $\mathcal{T}_K = 1$ (in which case $\kappa_K = 0$) then in any p -primitively ramified p -extension L of K (definition and examples in [Gr1, IV.3, (b); IV.3.5.1], after [Gr4, Theorem 1, II.2] in the direction of the notion of “ p -rational field”), we get $\mathcal{T}_L = 1$ whence $\kappa_L = 0$.

Many generalizations are available which are left to the reader.

4. NORMALIZED p -ADIC REGULATOR OF A NUMBER FIELD

Definition 4.1. Under the Leopoldt conjecture for p in K , we call

$$\mathcal{R}_K := \text{tor}_{\mathbb{Z}_p}(\log(U_K) / \log(\overline{E}_K)) \simeq \text{Gal}(H_K^E / \tilde{K}H_K),$$

(or its order) the *normalized p -adic regulator* of K , whatever the number field K and the prime number $p \geq 2$.

For instance, in the totally real case, from Coates's formula [C, Appendix], we get easily:

$$\#\mathcal{R}_K \sim \frac{1}{2} \cdot \frac{(\mathbb{Z}_p : \log(N_{K/\mathbb{Q}}(U_K)))}{\#\mathcal{W}_K \cdot \prod_{\mathfrak{p}|p} N_{\mathfrak{p}}} \cdot \frac{R_K}{\sqrt{D_K}},$$

where \sim means equality up to a p -adic unit factor, where R_K is the usual p -adic regulator and D_K the discriminant of K .

In the real Galois case, with $p \neq 2$ unramified in K/\mathbb{Q} , we get, as defined in [Gr3, Definition 2.3], $\#\mathcal{R}_K \sim \frac{R_K}{p^{[K:\mathbb{Q}]-1}}$.

One computes that $\#\mathcal{R}_K = 1$ for all imaginary quadratic fields and all p .

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