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# THE $p$ -ADIC KUMMER–LEOPOLDT CONSTANT NORMALIZED $p$ -ADIC REGULATOR

GEORGES GRAS

ABSTRACT. The  $p$ -adic Kummer–Leopoldt constant  $\kappa_K$  of a number field  $K$  is (under the Leopoldt conjecture for  $p$  in  $K$ ) the least integer  $c$  such that for all  $n$  large enough, any global unit  $\varepsilon$  of  $K$ , which is locally (at the  $p$ -places) a  $p^{n+c}$ th power, is necessarily a  $p^n$ th power of a global unit of  $K$ . This constant has been computed by J. Assim and T. Nguyen Quang Do using Iwasawa’s technics. In this short Note, we give an elementary  $p$ -adic proof of their result and the class field theory interpretation of  $\kappa_K$  by means of Abelian  $p$ -ramification theory over  $K$ . Then, from this, we give a natural definition of the normalized  $p$ -adic regulator of  $K$  for any  $K$  and any  $p \geq 2$ .

## 1. NOTATIONS

Let  $K$  be a number field and let  $p \geq 2$  be a prime number. We assume in the sequel that  $K$  satisfies the Leopoldt conjecture at  $p$ . Denote by  $E_K$  the group of principal global units of  $K$  (i.e., units  $\varepsilon \equiv 1 \pmod{\prod_{\mathfrak{p}|p} \mathfrak{p}}$ ) and by

$$U_K := \left\{ u \in \prod_{\mathfrak{p}|p} K_{\mathfrak{p}}^{\times}, u = 1 + x, x \in \prod_{\mathfrak{p}|p} \mathfrak{p} \right\} \quad \& \quad W_K = \text{tor}_{\mathbb{Z}_p}(U_K),$$

the  $\mathbb{Z}_p$ -module of local units at  $p$  and its torsion subgroup, where  $K_{\mathfrak{p}}$  is the  $\mathfrak{p}$ -completion of  $K$ .

The  $p$ -adic logarithm  $\log$  is defined on  $1 + x$ ,  $x \in \prod_{\mathfrak{p}|p} \mathfrak{p}$ , by means of the series  $\log(1 + x) = \sum_{i \geq 1} (-1)^{i+1} \frac{x^i}{i} \in \prod_{\mathfrak{p}|p} K_{\mathfrak{p}}$ . The kernel of  $\log$  in  $U_K$  is  $W_K$ .

We consider the diagonal embedding  $E_K \rightarrow U_K$  and its natural extension  $E_K \otimes \mathbb{Z}_p \rightarrow U_K$  whose image is  $\overline{E}_K$ , the topological closure of  $E_K$  in  $U_K$ . In the sequel, these embeddings shall be understood.

## 2. THE KUMMER–LEOPOLDT CONSTANT

This notion comes from the Kummer lemma (see [W3, Theorem 5.36]), that is to say, if the prime number  $p$  is “regular”, the cyclotomic field  $K = \mathbb{Q}(\mu_p)$  satisfies the following property for the whole group  $E'_K$  of global units of  $K$ : *any  $\varepsilon \in E'_K$ , congruent to a rational integer modulo  $p$ , is a  $p$ th power in  $E'_K$ .*

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In fact,  $\varepsilon \equiv a \pmod{p}$ , with  $a \in \mathbb{Z}$ , implies  $\varepsilon^{p-1} \equiv a^{p-1} \equiv 1 \pmod{p}$ . So we shall write, equivalently, the Kummer property with principal units:

*any  $\varepsilon \in E_K$ , congruent to 1 modulo  $p$ , is a  $p$ th power in  $E_K$ .*

From [A], [L], [O], [S], [W1], [W2] (see the rather intricate history in [AN]) one can study this property and its generalizations with various technics. Give the following definition (from [AN]):

**Definition 2.1.** Let  $K$  be a number field satisfying the Leopoldt conjecture at the prime  $p \geq 2$ . Let  $E_K$  be the group of principal global units of  $K$ . We call Kummer–Leopoldt constant (denoted  $\kappa = \kappa_K$ ), the smallest integer  $c$  such that the following condition is fulfilled:

*for all  $n \gg 0$ , any unit  $\varepsilon \in E_K$ , such that  $\varepsilon \in U_K^{p^{n+c}}$ , is necessarily in  $E_K^{p^n}$ .*

**Remark 2.2.** The existence of  $\kappa$  comes from the classical characterization of Leopoldt’s conjecture given, for instance, in [Gr1, Theorem III.3.6.2 (iv)]. If the Leopoldt conjecture is not satisfied, we can find a sequence  $\varepsilon_n \in E_K \setminus E_K^p$  such that  $\log(\varepsilon_n) \rightarrow 0$  (i.e.,  $\varepsilon_n \in U_K^{p^m} \cdot W_K$ , with  $m \rightarrow \infty$  when  $n \rightarrow \infty$ ); since  $W_K$  is finite, taking a suitable power of  $\varepsilon_n$ , we see that  $\kappa$  does not exist in that case.

We have the following  $p$ -adic result giving  $p^\kappa$  under the Leopoldt conjecture:

**Theorem 2.3.** *Denote by  $E_K$  the group of principal global units of  $K$ , then by  $U_K$  the  $\mathbb{Z}_p$ -module of principal local units at  $p$  and by  $W_K$  its torsion subgroup. Let  $\kappa_K$  (Definition 2.1) be the Kummer–Leopoldt constant; then  $p^{\kappa_K}$  is the exponent of the finite group  $\text{tor}_{\mathbb{Z}_p}(\log(U_K)/\log(\overline{E}_K))$ , where  $\overline{E}_K$  is the closure of  $E_K$  in  $U_K$  (hence  $\log(\overline{E}_K) = \mathbb{Z}_p \log(E_K)$ ).*

*Proof.* (i) ( $\kappa$  is suitable). Let  $n \gg 0$  and let  $\varepsilon \in E_K$  be such that  $\varepsilon = u^{p^{n+\kappa}}$ ,  $u \in U_K$  (thus  $\varepsilon$  is arbitrary near from 1, depending on the choice of  $n \gg 0$ ). Then  $\log(\varepsilon) = p^n \cdot (p^\kappa \cdot \log(u)) = p^n \cdot \log(\overline{\eta})$  with  $\overline{\eta} \in \overline{E}_K$ . So, writting

$$\overline{\eta} = \eta(N) \cdot u_N, \quad \eta(N) \in E_K, \quad u_N \equiv 1 \pmod{p^N}, \quad N \rightarrow \infty,$$

we get  $\log(\varepsilon) = p^n \cdot \log(\eta(N)) + p^n \cdot \log(u_N)$  giving

$$\varepsilon = \eta(N)^{p^n} \cdot u_N^{p^n} \cdot \xi, \quad \xi \in W_K.$$

But  $\xi$  is near from 1, whence  $\xi = 1$ . Then

$$\varepsilon = \eta(N)^{p^n} \cdot u'_N, \quad u'_N \rightarrow 1 \text{ as } N \rightarrow \infty;$$

so  $u'_N = \varepsilon \cdot \eta(N)^{-p^n}$  is a global unit, arbitrary near from 1, hence, because of the Leopoldt conjecture (loc. cit. in Remark 2.2), of the form  $\varphi_N^{p^n}$  with  $\varphi_N \in E_K$ ,  $\varphi_N \rightarrow 1$  as  $N \rightarrow \infty$  (recall that  $n$  is arbitrary but fixed), giving  $\varepsilon = \eta(N)^{p^n} \cdot \varphi_N^{p^n}$ , whence:

$$\varepsilon \in E_K^{p^n}.$$

(ii) ( $\kappa$  is the least solution). Suppose there exists  $c < \kappa$  having the property given in Definition 2.1.

Let  $u_0 \in U_K$  be such that  $\log(u_0)$  is of order  $p^\kappa$  in  $\text{tor}_{\mathbb{Z}_p}(\log(U_K)/\log(\overline{E}_K))$ ; then  $\log(u_0^{p^\kappa}) = \log(\overline{\varepsilon}_0)$ ,  $\overline{\varepsilon}_0 \in \overline{E}_K$ . This is equivalent to

$$u_0^{p^\kappa} = \overline{\varepsilon}_0 \cdot \xi_0 = \varepsilon(N) \cdot u_N \cdot \xi_0, \quad \varepsilon(N) \in E_K, \quad u_N \in U_K, \quad \xi_0 \in W_K$$

(with  $u_N \rightarrow 1$  when  $N \rightarrow \infty$ ), hence, for any  $n \gg 0$ ,

$$u_0^{p^{n+\kappa}} = \varepsilon(N)^{p^n} \cdot u_N^{p^n}.$$

Taking  $N$  large enough, we can suppose that  $u_N = v_N^{p^{2\kappa}}$ ,  $v_N \in U_K$  arbitrary near from 1; because of the above relation,  $\log(v_N)$  is of finite order modulo  $\log(\overline{E}_K)$ , thus  $\log(v_N^{p^\kappa}) \in \log(\overline{E}_K)$ . This is sufficient, for  $u'_0 := u_0 \cdot v_N^{-p^\kappa}$ , to get  $\log(u'_0)$  of order  $p^\kappa$  modulo  $\log(\overline{E}_K)$ . So we can write:

$$\varepsilon(N)^{p^n} = u_0^{p^{n+\kappa}} \cdot u_N^{-p^n} = u_0^{p^{n+\kappa}} \cdot (v_N^{-p^\kappa})^{p^{n+\kappa}} = u_0^{p^{n+\kappa}} \in U_K^{p^{n+(\kappa-c)+c}},$$

but, by assumption on  $c$ , we obtain  $\varepsilon(N)^{p^n} = \eta_0^{p^{n+(\kappa-c)}}$ ,  $\eta_0 \in E_K$ ; thus, the above relation  $u_0^{p^{n+\kappa}} = u_0^{p^{n+(\kappa-c)+c}} = \eta_0^{p^{n+(\kappa-c)}}$ , yields to:

$$p^c \cdot \log(u'_0) = \log(\eta_0) \in \log(E_K) \text{ (absurd)}. \quad \square$$

### 3. INTERPRETATION OF $\kappa_K$ – FUNDAMENTAL EXACT SEQUENCE

Consider the following diagram, under the Leopoldt conjecture for  $p$  in  $K$  (diagram given in [Gr1], § III.2, (c), Fig. 2.2):

$$\begin{array}{ccccc}
 & & \mathcal{T}_K & & \\
 & & \curvearrowright & & \\
 & & \mathcal{T}_K^E & & \\
 \tilde{K} & \xrightarrow{\quad} & \tilde{K}H_K & \xrightarrow{\mathcal{R}_K} & H_K^E & \xrightarrow{\mathcal{W}_K} & H_K^{\text{pr}} \\
 | & & | & & | & & | \\
 \tilde{K} \cap H_K & \xrightarrow{\quad} & H_K & & & & \\
 | & & \mathcal{C}_K & & & & \\
 K & & & & & & 
 \end{array}$$

$\mathcal{U}_K/\overline{E}_K$  (arc from  $H_K$  to  $H_K^{\text{pr}}$ )

where  $\tilde{K}$  is the compositum of the  $\mathbb{Z}_p$ -extensions of  $K$ ,  $\mathcal{C}_K$  the  $p$ -class group in the ordinary sense,  $H_K$  the  $p$ -Hilbert class field,  $H_K^{\text{pr}}$  the maximal Abelian  $p$ -ramified (i.e., unramified outside  $p$ ) pro- $p$ -extension of  $K$ ; then we put  $\mathcal{T}_K^E := \text{tor}_{\mathbb{Z}_p}(\text{Gal}(H_K^{\text{pr}}/H_K)) \subseteq \mathcal{T}_K := \text{tor}_{\mathbb{Z}_p}(\text{Gal}(H_K^{\text{pr}}/K))$ , and  $\mathcal{R}_K \simeq \text{Gal}(H_K^E/\tilde{K}H_K)$ .

By class field theory,  $\text{Gal}(H_K^{\text{pr}}/H_K)$  is canonically isomorphic to  $U_K/\overline{E}_K$  in which the image of  $\mathcal{W}_K := W_K/\mu_K$  (where  $\mu_K$  is the group of roots of unity of  $K$  of  $p$ -power order) fixes  $H_K^E$ , which is (except possibly if  $p = 2$  in the “special case”) equal to the Bertrandias–Payan field  $H_K^{\text{bp}}$ ,  $\text{Gal}(H_K^{\text{bp}}/\tilde{K})$  being then the Bertrandias–Payan module (also evoked in [AN] about the calculation of  $\kappa$ ; see also [Gr2, Références] for some history about this module).

**Lemma 3.1.** *We have (without any assumption on  $K$  and  $p$ ) the following exact sequence:*

$$1 \longrightarrow W_K / \text{tor}_{\mathbb{Z}_p}(\overline{E}_K) \longrightarrow \text{tor}_{\mathbb{Z}_p}(U_K / \overline{E}_K) \xrightarrow{\log} \text{tor}_{\mathbb{Z}_p}(\log(U_K) / \log(\overline{E}_K)) \longrightarrow 0.$$

*Proof.* From [Gr1, Lemma 4.2.4]: the surjectivity comes from the fact that if  $u \in U_K$  is such that  $p^n \log(u) = \log(\overline{\varepsilon})$ ,  $\overline{\varepsilon} \in \overline{E}_K$ , then  $u^{p^n} = \overline{\varepsilon} \cdot \xi$  for  $\xi \in W_K$ , hence there exists  $m \geq n$  such that  $u^{p^m} \in \overline{E}_K$ , whence  $u \in \text{tor}_{\mathbb{Z}_p}(U_K / \overline{E}_K)$ ;  $u$  is a preimage. If  $u$  is such that  $\log(u) \in \log(\overline{E}_K)$ ,  $u = \overline{\varepsilon} \cdot \xi$  as above, giving the kernel equal to  $\overline{E}_K \cdot W_K / \overline{E}_K = W_K / \text{tor}_{\mathbb{Z}_p}(\overline{E}_K)$ .  $\square$

Assuming the Leopoldt conjecture, we have  $\text{tor}_{\mathbb{Z}_p}(\overline{E}_K) = \mu_K$  ([Gr1, Corollary III.3.6.3]). Thus,  $W_K / \text{tor}_{\mathbb{Z}_p}(\overline{E}_K) = W_K / \mu_K = \mathcal{W}_K$  and the exact sequence in the Lemma becomes, with the notations of the diagram:

$$1 \longrightarrow \mathcal{W}_K \longrightarrow \mathcal{T}_K^E \xrightarrow{\log} \mathcal{R}_K \longrightarrow 0,$$

**Corollary 3.2.** *Under the Leopoldt conjecture for  $p$  in  $K$ , the Kummer-Leopoldt constant  $\kappa$  of  $K$  is 0 if and only if  $\mathcal{R}_K = 1$ , which is equivalent to  $\#\mathcal{T}_K = [H_K : \tilde{K} \cap H_K] \cdot \#\mathcal{W}_K$ .*

**Corollary 3.3.** *If the prime number  $p$  is regular, then  $\kappa = 0$  for the field  $\mathbb{Q}(\mu_p)$  (Kummer's lemma).*

*Proof.* In this case we know that the cyclotomic field  $\mathbb{Q}(\mu_p)$  is regular in the meaning of [GJ, Théorème & Définition 2.1], so  $\mathcal{T}_K = 1$  giving the result.  $\square$

For instance, if  $\mathcal{T}_K = 1$  (in which case  $\kappa_K = 0$ ) then in any  $p$ -primitively ramified  $p$ -extension  $L$  of  $K$  (definition and examples in [Gr1, IV.3, (b); IV.3.5.1], after [Gr4, Theorem 1, II.2] in the direction of the notion of “ $p$ -rational field”), we get  $\mathcal{T}_L = 1$  whence  $\kappa_L = 0$ .

Many generalizations are available which are left to the reader.

#### 4. NORMALIZED $p$ -ADIC REGULATOR OF A NUMBER FIELD

**Definition 4.1.** Under the Leopoldt conjecture for  $p$  in  $K$ , we call

$$\mathcal{R}_K := \text{tor}_{\mathbb{Z}_p}(\log(U_K) / \log(\overline{E}_K)) \simeq \text{Gal}(H_K^E / \tilde{K}H_K),$$

(or its order) the *normalized  $p$ -adic regulator* of  $K$ , whatever the number field  $K$  and the prime number  $p \geq 2$ .

For instance, in the totally real case, from Coates's formula [C, Appendix], we get easily:

$$\#\mathcal{R}_K \sim \frac{1}{2} \cdot \frac{(\mathbb{Z}_p : \log(N_{K/\mathbb{Q}}(U_K)))}{\#\mathcal{W}_K \cdot \prod_{\mathfrak{p}|p} N_{\mathfrak{p}}} \cdot \frac{R_K}{\sqrt{D_K}},$$

where  $\sim$  means equality up to a  $p$ -adic unit factor, where  $R_K$  is the usual  $p$ -adic regulator and  $D_K$  the discriminant of  $K$ .

In the real Galois case, with  $p \neq 2$  unramified in  $K/\mathbb{Q}$ , we get, as defined in [Gr3, Definition 2.3],  $\#\mathcal{R}_K \sim \frac{R_K}{p^{[K:\mathbb{Q}]-1}}$ .

One computes that  $\#\mathcal{R}_K = 1$  for all imaginary quadratic fields and all  $p$ .

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