



HAL
open science

Piecewise linear bounding of energy conversion functions and resulting MILP-based solution methods

Sandra Ulrich Ngueveu

► **To cite this version:**

Sandra Ulrich Ngueveu. Piecewise linear bounding of energy conversion functions and resulting MILP-based solution methods. [Research Report] 16358, INP Toulouse; LAAS-CNRS. 2016. hal-01444317v1

HAL Id: hal-01444317

<https://hal.science/hal-01444317v1>

Submitted on 23 Jan 2017 (v1), last revised 30 May 2018 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Piecewise linear bounding of energy conversion functions and resulting MILP-based solution methods

Sandra Ulrich Ngueveu

LAAS-CNRS, Université de Toulouse, CNRS, INP, F-31400 Toulouse, France
ngueveu@laas.fr

08/11/2016

Technical report LAAS-CNRS n° 16358

Abstract

Different energy sources can have very different characteristics in terms of power range and energy demand/cost function also known as efficiency function or energy conversion function. Introducing these energy sources characteristics in combinatorial optimization problems such as energy resource allocation problems or energy-consuming activity scheduling problems results into mixed-integer non-linear problems neither convex or concave. Approximations via piecewise linear functions have been proposed in the literature. Non-convex optimization models and heuristics exist to compute optimal breakpoint systems subject to the condition that the piecewise linear continuous approximator (under- and overestimator) never deviates more than a given delta-tolerance from the original continuous separable function over a given finite interval, or to minimize the area between the approximator and the function. We present an alternative solution method based on the upper and lower bounding of energy conversion expressions using discontinuous piecewise linear functions with a relative epsilon-tolerance. We prove that such approach yields a pair of mixed integer linear programs with a performance guarantee. Models and heuristics to compute the discontinuous piecewise linear functions with a relative epsilon-tolerance will also be presented. Computational results have shown the efficiency of the method in comparison to state-of-the-art methods on instances derived from the literature and on real-world instances from various energy optimization problems such as energy optimization in hybrid electric vehicles.

1 Introduction

Various optimization problems resulting from the introduction of energy conversion functions into combinatorial optimization problems can be modeled as non linear problems or mixed integer non linear problems (MINLP). Let us consider for example energy optimization in hybrid electric vehicles. In such vehicles the electrical powertrain system has multiple energy sources that it can gather power from to satisfy the propulsion power requested by the vehicle

at each instant. The problem usually consists in finding at each instant the optimal power split between the multiple energy sources to satisfy the power demand of a driver on a predefined road section. The objective is to minimize the total fuel consumption of the vehicle overall the mission, taking into account the characteristics and the limitations of each energy source, such as the energy losses happening during any energy transfer. Problem (P) models such problem for a HEV operating with two energy sources: (1) a Fuel Cell stack (FC) able to produce power up to P_{\max}^1 at each instant, (2) a storage element (SE) able to retrieve power up to P_{\min}^1 and to produce power up to P_{\max}^2 at each instant. In addition, the SE can only work if the quantity of energy it stores is between E_{\min} and E_{\max} , typically 25% and 100% of its energy capacity. The amount of energy stored in the SE is also called state of charge (SOC). Problem (P) can be modeled with equations (1)-(5). Problem (P) is a (MI)NLP because of non-linear energy conversion functions f^1, f^2 and f^3 continuous on intervals $]0, P_{\max}^1]$, $]0, P_{\max}^1]$ and $]0, P_{\max}^1]$ respectively and verifying $f^1(0) = f^2(0) = f^3(0) = 0$.

$$(P) \quad \min \sum_{i \in \mathbb{I}} f^1(x_i^1) \quad (1)$$

subject to (s.t.)

$$x_i^1 + x_i^2 - x_i^3 \geq P_i, \quad \forall i \in \mathbb{I} \quad (2)$$

$$\sum_{i \in \mathbb{I}} (f^2(x_i^2) - f^3(x_i^3)) \leq 0 \quad (3)$$

$$E_0 - E_{\max} \leq \sum_{k=1}^i (f^2(x_k^2) - f^3(x_k^3)) \leq E_0 - E_{\min}, \quad \forall i \in \mathbb{I} \quad (4)$$

$$x_i^1 \in [0, P_{\max}^1], x_i^2 \in [0, P_{\max}^2], x_i^3 \in [P_{\min}^2, 0], \quad \forall i \in \mathbb{I} \quad (5)$$

It is possible for the resulting (MI)NLP to be neither convex or concave even though all energy functions were convex or concave. Therefore, only small instances may be tractable using standard MINLP solvers. This is the case for various energy optimization problems. Several real-world applications have been addressed using piecewise linear approximations of the non linear functions of the MINLP to obtain a MILP easier to solve (for example Borghetti et al. (2008), Camponogara et al. (2007)). Such approach presents the main advantage of producing solutions faster than purely MINLP-based approaches if not too many additional binary variables or combinatorial conditions have been introduced in the process, meaning that the number of pieces of the piecewise linear (pwl) functions used should be kept to a minimum. Geißler et al. (2012) explains that the approach suffers from a few drawbacks due to the fact that the nature of the nonlinearities inherent to the problem may be lost and the solutions obtained from the MILP may have no meaning for the MINLP. If the solution obtained is not satisfactory, a new pwl approximation may be performed using a higher number of pieces, to obtain a new MILP to solve. This leads to an iterative solution procedure with a number of iterations unknown a priori which may translate into high computing times, either because several iterations needed to be performed, or because an unnecessary large number of pieces were chosen resulting into an unnecessarily large MILP requiring a high solution time.

As an alternative we propose a straightforward two-phase solution method. The first phase consists in bounding each non linear term from above and below using a pair of piecewise

linear functions satisfying conditions that will be specified in the core of the paper. Contrary to most publications on piecewise linear approximation which focus on the minimization of the approximation error for a given number of pieces or breakpoints that may or may not be equidistant, we aim at minimizing the number of pieces for a given error. The second phase of the solution method proposed consists in solving two MILP obtained from the replacement of the non linear functions with either one of the two piecewise linear functions. This paper focusses on the first phase of the iterative procedure, assuming that the resulting MILPs can be solved efficiently with a MILP solver. If it is not the case, then a specific solution method for the resulting MILP may need to be designed and an example of such case is presented in Ngueveu et al. (2016).

2 State of the art

Several publications exists on the application of piecewise linear (pwl) approximation to solve MINLP problems, but only a few focused on the specific problem of minimizing the number of breakpoints with a given precision.

Rosen and Pardalos (1986) first proposed the computation of breakpoints for a given error tolerance for concave quadratic functions. The pwl interpolators were built using equidistant breakpoints and by concavity their interpolators were underestimators. Rosen and Pardalos identified conditions on the number of breakpoints they required to achieve a given error tolerance. Geißler et al. (2012) showed that certain cases of general MINLPs can be solved by just applying techniques purely from the mixed integer linear programming by approximating the nonlinearities with pwl functions. They proposed to compute a priori errors for pwl approximations or a priori errors for over- and under- pwl estimators. However, they did not focus on the computation of optimal (minimal) breakpoint systems.

Rebennack and Kallrath (2015) propose two exact approaches and two heuristics for the computation of optimal continuous pwl approximators for univariate continuous functions over a compactum $\mathbb{D} = \{X_-, X_+\}$. Their methodology is also applicable if the function has finitely many discontinuities. Contrary to work of Rosen and Pardalos (1986), the algorithms proposed handle more general functions. In addition, the breakpoints are distributed freely and shifts from the function are allowed at breakpoints, which were shown to be important degrees of freedom contributing to a significant reduction of the number of breakpoints. The work of Rebennack and Kallrath (2015) differs from Geißler et al. (2012) in the following aspects. Geißler et al. did not target on computing minimal breakpoint systems whereas Rebennack and Kallrath do so and solve non-convex NLP problems to global optimality leading to the tightests approximators, tightest in the sense of minimizing the largest deviation. In addition, they consider shift variables at breakpoints, which adds an additional degree of freedom. Since their work is close to ours in some aspects, let us focus on their contributions before highlighting the main differences and contributions of this paper.

Rebennack and Kallrath (2015) show that ensuring that the approximator and the original function do not deviate more than a predefined tolerance δ from each other leads to sets of constraints which have to hold over a continuum, resulting in a semi-infinite programming (SIP) problem denoted OBSC for “Optimal Breakpoint System using a Continuum approach for x ”. The authors show that it is NP-hard to compute a δ -approximator for an arbitrary

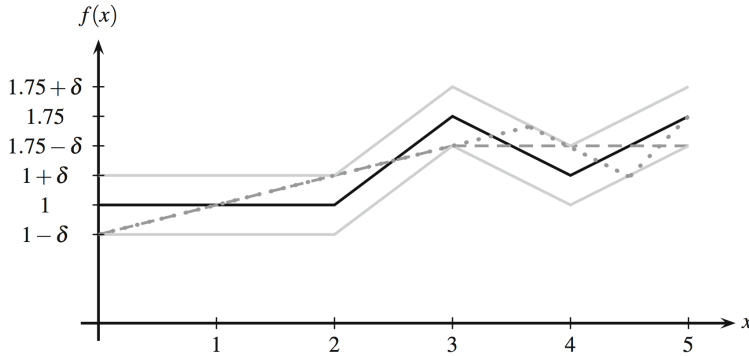


Figure 1: Maximizing the length of the intervals successively is not optimal, in general - $f(x)$ - δ -tube around $f(x)$ - (unique) optimal δ -approximator . . . δ -approximator maximizing interval length successively (source: Rebennack and Kallrath (2015), page 628)

continuous function and propose an iterative solution procedure based on the evaluation of the continuum conditions only at discrete points, resulting into a MINLP model which is a relaxation of the SIP. The feasibility of the resulting solution with regards to OBSC is then evaluated by solving NLPs to compute the true maximal deviation between the approximator and the original function on each piece of the pwl approximator. The algorithm stops if the true deviation is less or equal to δ . In this case the solution obtained is optimal for OBSC. Otherwise the grid is refined to obtain a new MINLP to be solved. Two different discretization strategies were proposed, resulting in MINLP models denoted OBSD and OBSI respectively. OBSC, OBSD and OBSI are in general too large and difficult to solve to optimality even for small numbers of breakpoints and discretization points, therefore Rebennack and Kallrath (2015) proposed two heuristic methods. The heuristics methods were based on the successive computation of the breakpoints, maximizing at each iteration the length of the successive intervals corresponding to the projections of the pwl approximations on the x -axis. This meant solving at each iteration a problem denoted BSB that can be expressed as follows: given the breakpoint x_i ending the i^{th} piece (which corresponds by continuity to the beginning of the $i + 1^{\text{th}}$ piece), compute the next breakpoint x_{i+1} (end of the $i + 1^{\text{th}}$ piece) so as to maximize x_{i+1} while ensuring a deviation of at most δ between the original function and the linear approximation on interval $[x_i, x_{i+1}]$. Rebennack and Kallrath provided a counter-example showing that maximizing the length of the intervals do not necessarily lead to an optimal breakpoint system i.e., a δ -approximator with the least number of breakpoints. Indeed, it might be beneficial, in certain cases, to consider intervals between two breakpoints which are not of maximal length. This counter-example is illustrated on Figure 1. It shows the function $f(x)$ defined on $[X-, X+] = [0, 5]$ that the authors proposed, together with a (unique) optimal $\delta = 0.25$ -approximator using three breakpoints and a $\delta = 0.25$ -approximator using four breakpoints obtained by a method maximizing the interval length successively from $X-$ to $X+$. Using the heuristics, it was possible to obtain breakpoint systems satisfying the required δ -tolerance, and more so, an upper bound on the minimal number of breakpoints.

In the view of the state-of-the-art, the contributions of the paper are the following: (1) instead of using continuous pwl δ -approximators, we propose to approximate general univari-

ate continuous functions with non-necessarily continuous piecewise linear δ -approximators, adding an additional degree of freedom to obtain a breakpoint system of equal or less breakpoints, (2) we prove that when discontinuity is allowed, maximising the interval length leads to an optimal pwl δ -approximator, leading to an exact solution procedure based on the iterative solution of adapted BSB problems, (3) we introduce relative ϵ -tolerance and show the benefit of using it instead of the absolute δ -tolerance, (4) we show that when using relative ϵ -tolerance, it is not possible to shift up (down) an optimal approximator to derive an optimal over- (under-) estimator, contrary to what can be done for when using absolute δ -tolerance, (5) for solving MINLP involving non linear univariate energy conversion functions, we propose a solution method based on the upper and lower bounding of energy conversion expressions using discontinuous piecewise linear functions with a relative ϵ -tolerance, (6) we prove that such approach yields a pair of mixed integer linear programs with a performance guarantee and dominates solution procedures based on approximations with absolute δ -tolerance, (7) models and heuristics to compute the discontinuous pwl under- and over-estimator with a relative ϵ -tolerance are presented and finally (8) Computational results show the efficiency of the method in comparison to state-of-the-art methods on instances derived from the literature and on real-world instances from various energy optimization problems such as energy optimization in hybrid electric vehicles.

3 From continuous to non-necessarily continuous pwl δ -approximation of continuous non-linear functions

Let $f : \mathbb{D} = [X_-, X_+] \rightarrow \mathbb{R}$ be a function on the compact interval $\mathbb{D} \subset \mathbb{R}$. A function $g : \mathbb{D} = [X_-, X_+] \rightarrow \mathbb{R}$ is a pwl function with $n_g \in \mathbb{N}$ line-segments if $\exists a \in \mathbb{R}^{n_g}, b \in \mathbb{R}^{n_g}, x^{\min} \in [X_-, X_+]^{n_g}$ and if $\forall i \in [1..n_g], \exists x_i^{\max} \in [x_i^{\min}, X_+]$, such that equations (6)-(9) are verified. Such pwl function is said to be defined by $G = \bigcup_{i=1}^{n_g} ([a_i, b_i], [x_i^{\min}, x_i^{\max}])$ and the two end-points x_i^{\min} and x_i^{\max} of each line-segment i are called breakpoints.

$$g(x) = a_i x + b_i, \quad \forall i \in [1..n_g], \forall x \in [x_i^{\min}, x_i^{\max}] \quad (6)$$

$$x_i^{\max} = x_{i+1}^{\min}, \quad \forall i \in [1..n_g - 1] \quad (7)$$

$$x_1^{\min} = X_- \quad (8)$$

$$x_{n_g}^{\max} = X_+ \quad (9)$$

A pwl function g is:

- continuous iff it verifies all equations (10)
- or discontinuous if it does not verify all equations (10), i.e $\exists j \in [1, n_g - 1]$ such that $a_j x_j^{\max} + b_j \neq a_{j+1} x_{j+1}^{\min} + b_{j+1}$
- or non-necessarily continuous if the satisfaction of equations (10) is neither required or forbidden

$$a_i x_i^{\max} + b_i = a_{i+1} x_{i+1}^{\min} + b_{i+1}, \forall i \in [1, n_g - 1] \quad (10)$$

Definition 3.1 (δ -approximator): A pwl function $g : \mathbb{D} = [X_-, X_+] \rightarrow \mathbb{R}$ is called δ -approximator of a function $f : \mathbb{D} = [X_-, X_+] \rightarrow \mathbb{R}$ with $\delta \in \mathbb{R}^+$, iff inequation (11) is verified.

$$\max_{x \in \mathbb{D}} |l(x) - f(x)| \leq \delta \quad (11)$$

Proposition 3.2 Any optimal pwl continuous δ -approximator with n^* line-segments can be converted into a pwl non-necessarily continuous δ -approximator with $n \leq n^*$ line-segments where the projection of the first line-segment on interval \mathbb{D} is of maximal length.

Proof Let $l = \bigcup_{i=1}^{n_l} ([a_i, b_i], [x_i^{\min}, x_i^{\max}])$ be an optimal continuous δ -pwl approximator ($n_l = n^*$) of a continuous function $f : \mathbb{D} = [X_-, X_+] \rightarrow \mathbb{R}$. Let $y \in \mathbb{D}$ be the solution value of problem

$$\begin{aligned} & \max && y \\ & \text{subject to} && |a_1 x + b_1 - f(x)| \leq \delta, \quad \forall x \in [x_1^{\min}, y] \end{aligned}$$

Let $q \in [1 \dots n_l]$ be the piece number that verifies $x_q^{\min} \leq y \leq x_q^{\max}$. Breakpoint x_1^{\max} verifies $|a_1 x_1^{\max} + b_1 - f(x)| \leq \delta$, therefore $y \geq x_1^{\max}$. If $y = x_1^{\max}$, then $\tilde{l} = l$ is a pwl continuous δ -approximator of f with $n = n^*$ pieces where the projection of the first line-segment on interval \mathbb{D} is of maximal length. Otherwise $y > x_1^{\max}$ in which case a discontinuous pwl δ -approximator \tilde{l} with $n_{\tilde{l}} \leq n^*$ line-segments where the projection of the first line-segment on interval \mathbb{D} is of maximal length can be defined by $\tilde{L} = \bigcup_{i=1}^{n_{\tilde{l}}} ([\tilde{a}_i, \tilde{b}_i], [\tilde{x}_i^{\min}, \tilde{x}_i^{\max}]) = ([a_1, b_1], [x_1^{\min}, y]) \cup ([a_q, b_q], [y, x_q^{\max}]) \cup \left(\bigcup_{i=q+1}^{n_{\tilde{l}}} ([a_i, b_i], [x_i^{\min}, x_i^{\max}]) \right)$. ■

Figure 3 illustrates Proposition 3.2 and it can be proven that $n \in [\frac{n^*+1}{2}, n^*]$.

Theorem 3.3 For any continuous fonction $f : \mathbb{D} = [X_-, X_+] \rightarrow \mathbb{R}$ and any scalar $\delta \in \mathbb{R}^+$, there exists an optimal non-necessarily continuous pwl δ -approximator g defined by $G = \bigcup_{i=1}^{n_g} ([a_i, b_i], [x_i^{\min}, x_i^{\max}])$ such that each line-segment i has a maximal length projection on the interval $[x_i^{\min}, X_+]$.

Proof There exists a continuous δ -approximator function for any continuous fonction f on a compactum \mathbb{D} and any scalar $\delta > 0$ (Duistermaat and Kol, 2004). Proposition 3.2 can be applied iteratively on an optimal continuous δ -approximator to obtain an optimal non-necessarily continuous pwl δ -approximator with intervals of maximal length on \mathbb{D} . i.e Applying proposition 3.2 ensures that the first line-segment has a maximal length projection on interval \mathbb{D} . Let $y = x_{\max}^1$ be the resulting breakpoint, then proposition 3.2 can be applied for function $\tilde{f} = f : [x_1^{\max}, X_+]$ to find the next line-segment of maximal length projection. The procedure can be repeated until the end of the compactum \mathbb{D} is reached, i.e $y = X_+$. ■

Let us investigate the implications of Theorem 3.3 on the example from Figure 1. Let g be the pwl 0.25-approximator of f obtained by maximizing the length of the successive intervals corresponding to the projections of the pwl approximations on the x -axis. Maximizing the length of the first interval given $x_1^{\min} = X_- = 0$ results into $x_1^{\max} = \frac{11}{3}$ and a

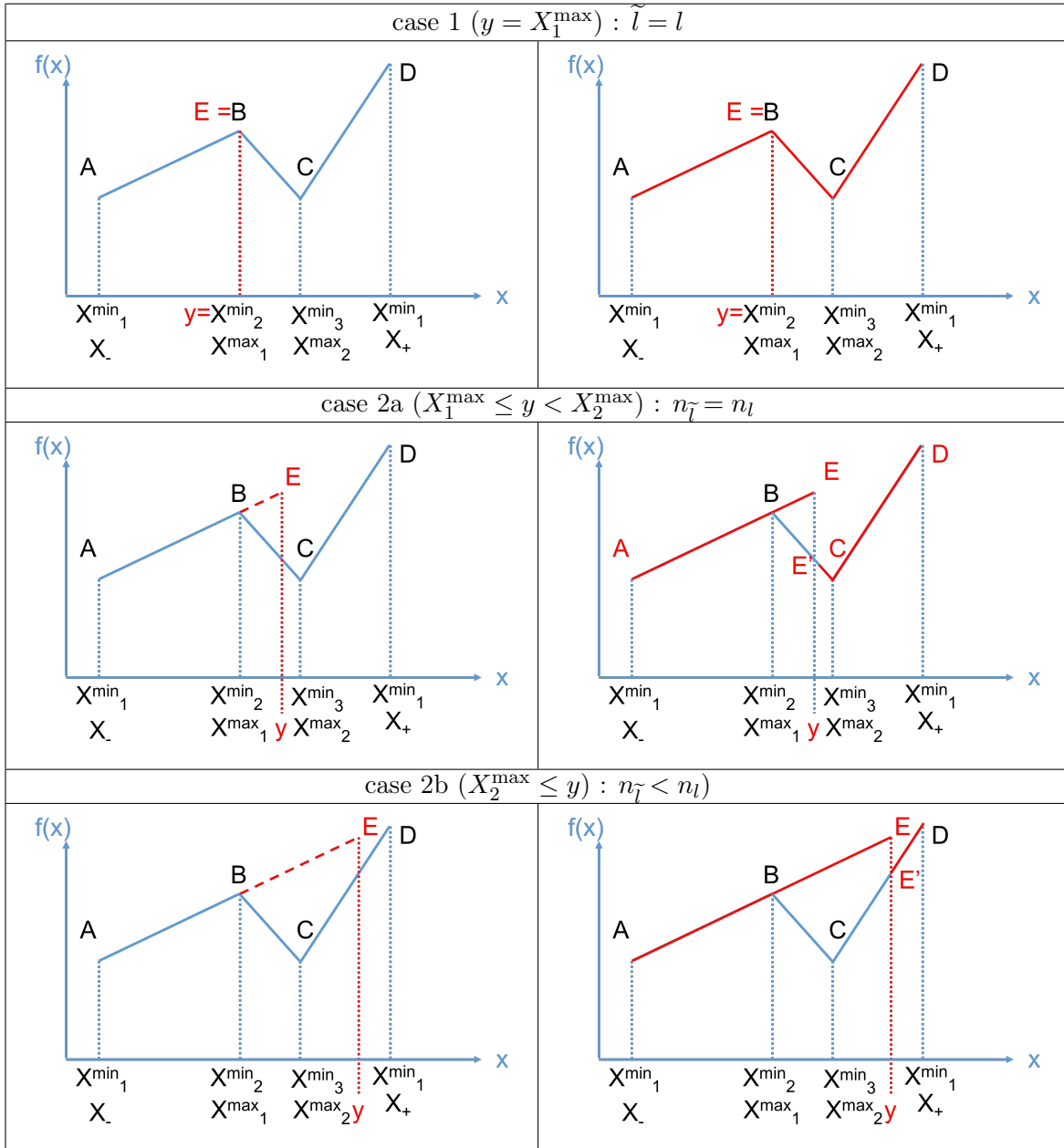


Figure 2: Maximization of the first interval (projection on \mathbb{D} of the first line-segment)

line-segment defined by affine function $y = 0.25x + 0.75$. Then, given $x_2^{\min} = x_1^{\max}$ maximizing the length of the second interval results into $x_2^{\max} = X^+ = 5$. Therefore discontinuous pwl 0.25-approximator g is composed of $n_g = 2$ line-segments. The existence of a 0.25-approximator g' of f composed of $n_{g'} = 1$ line-segment would contradict the assumption that $[0, \frac{11}{3}]$ is the maximal length first interval. Therefore 2 is a lower bound of the optimal number of line-segments of a 0.25-approximator of f , thereby g is an optimal non-necessarily continuous 0.25-approximator of f .

It is possible for all mathematical models and algorithms proposed by Rebennack and Kallrath (2015) for computing continuous pwl δ -approximators to be modified to compute non-necessarily continuous pwl δ -approximators by considering two shifts per breakpoint instead of one, namely s_b^- used for the line-segment ending a breakpoint b and s_b^+ used for the line-segment starting at breakpoint b . In particular, thanks to Theorem 3.3, Algorithm 1 resulting from the adaptation of Rebennack and Kallrath's Forward Heuristic with Moving Breakpoints yields an exact solution method for computing optimal non-necessarily continuous δ -approximators.

Algorithm 1 Compute an optimal non-necessarily continuous pwl approximation

Input: Function f ; domain $\mathbb{D} = [X_-; X_+]$; scalar $\delta > 0$

Output: pwl function g defined by G

- 1: $n_g := 0; y = X_-; G := \emptyset; x^{\text{end}} := X_+$
 - 2: **while** $y < X_+$ **do**
 - 3: $x^{\text{begin}} := y$
 - 4: solve SIP (12)-(14) to obtain a, b and y {for example using algorithm 2}
 - 5: $G := G \cup [(a, b), (x^{\text{begin}}, y)]$
 - 6: $n_g := n_g + 1$
 - 7: **end while**
-

$$\max y \tag{12}$$

s.t.

$$|ax + b - f(x)| \leq \delta, \quad \forall x \in [x^{\text{begin}}, y] \tag{13}$$

$$a \in \mathbb{R}, b \in \mathbb{R}, y \in [x^{\text{begin}}, x^{\text{end}}] \tag{14}$$

Algorithm 2 solve SIP (12)-(14) using discrete grid points x_k

Input: Function f ; $x^{\text{begin}}; x^{\text{end}}; \delta$; initial grid size $|\mathcal{K}_0|$

Output: $a; b; y$

- 1: $|\mathcal{K}| = |\mathcal{K}_0|; z = +\infty$
 - 2: **while** $z > \delta$ **do**
 - 3: increase grid size $|\mathcal{K}|$
 - 4: solve NLP (15)-(19) to obtain a, b and y
 - 5: solve NLP $z = \max_{x \in [x^{\text{begin}}, y]} |\tilde{a}x + \tilde{b} - f(x)|$ using $\tilde{a} = a$ and $\tilde{b} = b$
 - 6: **end while**
-

$$\max y \tag{15}$$

s.t.

$$|ax_k + b - f(x_k)| \leq \delta, \quad \forall k \in \mathcal{K} \tag{16}$$

$$x_k = x^{\text{begin}} + \frac{k}{|\mathcal{K}| + 1}(y - x^{\text{begin}}), \quad \forall k \in \mathcal{K} \tag{17}$$

$$x_k \in [x^{\text{begin}}, y] \quad \forall k \in \mathcal{K} \tag{18}$$

$$a \in \mathbb{R}, b \in \mathbb{R}, y \in [x^{\text{begin}}, x^{\text{end}}] \tag{19}$$

4 From pwl δ -approximation to pwl δ -bounding

A benefit of computing optimal δ -approximators in terms of number of breakpoints or line-segments is to minimize the number of additional binary variables added to obtain the MILP resulting from the replacement of non linear terms with their pwl approximators. A downside of applying δ -approximation is that the optimal solution of the resulting MILP can be infeasible with respect to the original MINLP. In this case the solution may not be of interest for the practitioners who formulated the original problem. An alternative is to use over- and under-estimators defined as follows.

Definition 4.1 (δ -underestimator): A pwl function $\underline{l} : \mathbb{D} \rightarrow \mathbb{R}$ is a δ -underestimator of a function $f : \mathbb{D} \rightarrow \mathbb{R}$ with $\delta \in \mathbb{R}^+$ iff it verifies inequation (20).

$$\underline{l}(x) \leq f(x) \leq \underline{l}(x) + \delta, \forall x \in \mathbb{D} \tag{20}$$

Definition 4.2 (δ -overestimator): A pwl function $\bar{l} : \mathbb{D} \rightarrow \mathbb{R}$ is a δ -overestimator of a function $f : \mathbb{D} \rightarrow \mathbb{R}$ with $\delta \in \mathbb{R}^+$ iff it verifies inequation (21).

$$\bar{l}(x) \geq f(x) \geq \bar{l}(x) - \delta, \forall x \in \mathbb{D} \tag{21}$$

In the context of energy optimization for hybrid electric vehicles for example, replacing each non-linear energy loss function with its pwl δ -overestimator yields a $\overline{\text{MILP}}$ solution where a sufficient amount of energy is produced at each time-step. Since excess energy can be dissipated as heat (in a resistance inserted in the braking system or in mechanical brakes present for security reasons) therefore the $\overline{\text{MILP}}$ solution is applicable on a test bench or on the real life hybrid electric vehicle considered. Having ensured the feasibility of the solution obtained, it is of interest to provide an estimate of the quality of the solution with respect to the optimum. This can be obtained using a pwl δ -underestimator of the energy loss function and solving the resulting $\underline{\text{MILP}}$ problem to obtain a lower bound of the MINLP. Each non-linear function is therefore bounded from above and below with two pwl functions, yielding two MILPs whose optimal solutions cost verify: $z_{\underline{\text{MILP}}} \leq z_{\text{MINLP}} \leq z_{\overline{\text{MILP}}}$. Table 1 summarizes which estimator should be used for each non-linear term of a MINLP to obtain a $\underline{\text{MILP}}$ or a $\overline{\text{MILP}}$ in the general case.

Given an optimal continuous pwl δ -approximator g of a function f , an optimal continuous pwl 2δ -over- (resp. under-) estimator can be obtained by shifting g by δ (resp. $-\delta$) (Rebennack and Kallrath, 2015), i.e. $g(x) + \delta$ (resp. $g(x) - \delta$) is a 2δ -over (resp. under-) estimator of f .

Origin	non-linear term	$\overline{\text{MILP}}$	$\underline{\text{MILP}}$
Objective-function to minimize	$+f(x)$	$+\bar{l}(x)$	$+\underline{l}(x)$
	$-f(x)$	$-\underline{l}(x)$	$-\bar{l}(x)$
Constraint of type \geq	$+f(x)$	$+\bar{l}(x)$	$+\bar{l}(x)$
	$-f(x)$	$-\bar{l}(x)$	$-\underline{l}(x)$
Constraint of type \leq	$+f(x)$	$+\bar{l}(x)$	$+\underline{l}(x)$
	$-f(x)$	$-\underline{l}(x)$	$-\bar{l}(x)$

Table 1: Estimator in replacement of each non-linear term to generate $\overline{\text{MILP}}$ or $\underline{\text{MILP}}$

This result can be extended to non-necessarily continuous pwl approximators and estimators. In addition, in the case of energy optimization for example, the MINLP can be neither convex or concave even if the individual non-linear energy loss functions or energy demand/cost conversion functions are convex or concave. Yet, if a non-linear continuous function $f : \mathbb{D} \rightarrow \mathbb{R}$ is convex (resp. concave) over \mathbb{D} with a derivative efficiently computable at any point of \mathbb{D} , then taking advantage of equation (22) it is possible to compute an optimal δ -underestimator (resp. δ -overestimator) more efficiently. Then shifting can be applied to obtain an optimal δ -overestimator (resp. δ -underestimator).

$$f \text{ is convex over } \mathbb{D} \Leftrightarrow f(y) \geq f(x) + f'(x)(y - x), \forall x, y \in \mathbb{D} \quad (22)$$

Let us focus on the computation of the optimal δ -underestimator of a non-linear convex function f for example. The reasoning hereafter can later be adapted to concave functions since their opposite is convex. Thanks to Theorem 3.3 the objective at each iteration i is to maximize x_i^{\max} for a given x_i^{\min} . Each line-segment i is tangent to f (otherwise it can be replaced with a line-segment of identical slope but tangent to f). Therefore the objective at iteration i is to identify a tangent point q_i that defines the slope $a_i = f'(q_i)$ and the y-intercept $b_i = f(q_i) - f'(q_i)q_i$ of line-segment i so that x_i^{\max} is maximized. In this context proposition 4.3 can be enunciated.

Proposition 4.3 *Finding the line-segment that maximizes x_i^{\max} for a given x_i^{\min} is equivalent to solving sequentially two separate problems: (1) maximizing q_i for the given x_i^{\min} , and then (2) maximizing x_i^{\max} for the previously computed q_i .*

Proof Let $x_i^{\max}(q_i^1)$ be the maximal value possible of x_i^{\max} for a given q_i^2 . The proof of property 4.3 consists in proving that $x_i^{\max}(q_i^2) \geq x_i^{\max}(q_i^1) \Rightarrow q_i^2 \geq q_i^1$. This is done using equation (22) which leads to $f(y) - (a_i^2 y + b_i^2) \leq f(y) - (a_i^1 y + b_i^1), \forall y \geq q_i^2$. ■

Also, because the gap $f - g$ between the estimator g and the function f increases the more the distance to the tangent point increases, therefore, the maximum gap is always at the extremities, so the test of equation (20) can be restricted to the extremities (no need to solve an SIP) as illustrated on Algorithm 3. If interval \mathbb{D} is discretized into a grid $|\mathcal{X}|$ and if step 4 is replaced by Algorithm 4 then the resulting heuristic can either fails or produce a feasible pwl g with $n_g^* \leq n_g \leq 2n_g^* - 1$ where $n_g^* - 1$ is the optimal number of line-segments of a δ -underestimator of f . It is also possible to replace step 5 of Algorithm 3 with Algorithm 5.

Algorithm 3 Compute an optimal non-necessarily continuous pwl δ -underestimator

Input: Convex function f ; domain $\mathbb{D} = [X_-; X_+]$; scalar $\delta > 0$

Output: pwl function g defined by G

1: $n_g := 0; y = X_-; G := \emptyset$

2: **while** $y < X_+$ **do**

3: $x^{\text{begin}} := y$

4: solve problem (PQ) with $q \in \mathbb{D}$ and deduce $a = f'(q), b = f(q) - f'(q)q$

$$\begin{aligned} \text{(PQ)} \quad & \max \quad q \\ & \text{subject to} \quad f(x^{\text{begin}}) - (f(q) + f'(q)(x^{\text{begin}} - q)) \leq \delta \end{aligned}$$

5: solve problem (PY) with $y \in \mathbb{D}$

$$\begin{aligned} \text{(PY)} \quad & \max \quad y \\ & \text{subject to} \quad f(y) - (ay + b) \leq \delta \end{aligned}$$

6: $G := G \cup ((a, b), (x^{\text{begin}}, y))$

7: $n_g := n_g + 1$

8: **end while**

Algorithm 4 Compute q, a, b using grid \mathcal{X}

Input: Convex function f ; $x^{\text{begin}}; \delta > 0$; grid \mathcal{X}

Output: slope a , y-intercept b , tangent point q

1: $n = |\mathcal{X}| + 1$; gap = $+\infty$

2: **while** $n \geq 2$ **and** gap $\geq \delta$ **do**

3: $n = n - 1; q := \mathcal{X}_n; a = f'(q); b = f(q) - f'(q)q$

4: gap := $f(x^{\text{begin}}) - ax^{\text{begin}} + b$

5: **end while**

6: **if** gap $\leq \delta$ **then**

7: **return** a,b,q

8: **else**

9: **print** Failed.

10: **end if**

Algorithm 5 Compute y using grid \mathcal{X}

Input: Convex function f ; $X_+; x^{\text{begin}}; \delta > 0$; slope a , y-intercept b ; grid \mathcal{X}

1: $n = |\mathcal{X}| + 1$; gap = $+\infty$

2: **while** $n \geq 2$ **and** gap $\geq \delta$ **do**

3: $n = n - 1; y := \mathcal{X}_n$

4: gap := $f(y) - ay + b$

5: **end while**

5 Drawbacks and limitations of pwl δ -bounding

One main challenge when applying a “ δ -bounding + MILP solution” procedure is to choose relevant δ values, which should be orders of magnitude smaller than the resulting solution value to provide an acceptable precision level. Once the MILP has been solved, it is possible to assess whether the chosen tolerance value δ was sufficiently small in the view of the resulting solution values. If it was not the case then the value of δ is decreased before a new round of pwl bounding then MILP solution. However, such iterative procedure is not satisfactory because the number of iterations is unknown a priori and the pwl bounding and/or the MILP solution may require significant computational efforts during each single iteration. An alternative is to identify a target precision Δ of the solution obtained from the MILP with regards to the optimal MINLP solution value and precompute the corresponding δ values in function of the chosen Δ . But even if the target precision Δ is known, the “ δ -bounding + MILP solution” procedure presents major drawbacks and limitations described in the remainder of this section.

5.1 Data dependence leading to multiple δ values

Let us consider the problem (1)-(5) for a specific HEV, which, for simplicity, is assumed to have an ideal supercapacitor ($\tilde{f}^2(x) = x, \tilde{f}^3(x) = x$). Therefore the non-linearity in the problem comes from the objective-function which is composed of n univariate non-linear terms $f^1(x)$, where n is the number of time-period of the time horizon considered. Ensuring a final precision of at least Δ requires to bound f^1 with a tolerance $\delta = \Delta/n$. Let us assume that the chosen HEV has to perform $|\mathcal{M}|$ different tests drive cycles of durations ($n^{(m)}, \forall m \in \mathcal{M}$ time periods), which translate into m different power profiles ($P_i = P_i^k, \forall i \in [1..n^{(m)}], \forall m \in \mathcal{M}$).

Despite the use of the same HEV which translates into the same non-linear function, and even if the same target precision Δ is chosen: (1) different power profiles translate into different δ values; (2) Even if a profile is an portion of another (i.e. $n^{(k)} < n^{(k')}$ and $P_i^k = P_i^{k'}, \forall i \in [1..n^{(k)}]$) the full “ δ -bounding + MILP solution” procedure has to be restarted to solve each resulting instance. In addition, even if two power profiles have the same duration, different δ values may be required because different power profiles may translate into solution costs which can differ significantly in order of magnitude, requiring different Δ values, and therefore different δ values.

To summarize, for any new power profile provided, the pwl bounding of the same function f^1 may need to be redone with a value of δ suitable to the new data set, even though the non-linear terms themselves have not changed. In addition, since the δ errors on the non linear terms are additive, a longer time horizon means a higher number of univariate terms ($f^1(x_i)$) in equation (1), which means that a smaller tolerance δ may be required. This translates into an increase of the number pieces for the pwl functions, and therefore more binary variables which adds to the complexity of a MILP that was already penalized by the fact that long time horizon meant a large set \mathbb{I} thus more decision variables x_i^1, x_i^2, x_i^3 . These all contribute to a substantial reduction of the size of instances that can be solved efficiently.

5.2 Solution dependence leading to unknown δ values

There exists several cases of MINLP where knowing Δ is not sufficient to infer the corresponding δ values. Let us consider for example problem (CF) modeling a scheduling problem

with a single energy source. There are a set of activities \mathcal{A} , each activity a having a release date r_a , a due date d_a , a duration p_a and an instantaneous energy demand b_a . The efficiency function of the energy source used to satisfy the total demand of activities scheduled at each instant is denoted ρ , i.e a cost or energy consumption of $\rho(x)$ produces an amount of usable energy x . Therefore ρ is defined on $[0, \sum_{i \in \mathcal{A}} b_i]$ and verifies $\rho(0) = 0$. The goal is to schedule the tasks so as to minimize the total energy cost. A resulting mathematical model requires binary decision variable x_{at} that is equal to 1 iff activity a is active at instant $t \in \mathcal{T}$. Continuous variables w_t represent the total energy demand at instant $t \in \mathcal{T}$. The objective function (23) aims at minimizing the total energy cost. Constraints (24) set each task a in process during at least p_a instants. Constraints (25) link variables w_t and x_{at} . Constraints (26)-(27) give the domains of the variables. In the resulting problem non-linearity comes from the objective-function which is comprised of $|\mathcal{T}|$ univariate non-linear terms.

$$(CF) \min \sum_{t \in \mathcal{T}} \rho(w_t) \quad (23)$$

s.t.

$$\sum_{t \in \mathcal{T}} \alpha_{at} x_{at} \geq p_a, \quad \forall a \in \mathcal{A} \quad (24)$$

$$w_t - \sum_{a \in \mathcal{A}} b_a x_{at} = 0, \quad \forall t \in \mathcal{T} \quad (25)$$

$$x_{at} \in \{0, 1\}, \quad \forall a \in \mathcal{A}, t \in \mathcal{T} \quad (26)$$

$$w_t \in \mathbb{R}^+, \quad \forall t \in \mathcal{T} \quad (27)$$

Let $(CF)_\Delta$ refer to the MILP derived from problem (CF) with a target tolerance Δ . Identifying a relevant δ value for obtaining $(CF)_\Delta$ is not straightforward. Indeed, in the general case, the duration of the schedule for the optimal solution is not known a priori. The only information available is the time horizon $|\mathcal{T}|$, but this value may be far from the real ending time of the optimal schedule. The “ δ -bounding + MILP solution” methodology requires the computation of a δ -estimator for each single term of the objective-function so as to satisfy the global tolerance value of Δ . Doing so requires each term to be bounded with a precision of $\Delta/|\mathcal{T}|$. Since $|\mathcal{T}|$ may be far from the optimal schedule duration, this can result into unnecessarily small δ values which lead to pwl functions with a higher number of pieces, which translates into more binary variables and MILPs more difficult to solve than necessary given the target tolerance. As an illustration, consider an instance with a time horizon $|\mathcal{T}| = 100$ having an optimal solution with an ending time of 10 (i.e an optimal solution x^* that verifies $\sum_{a \in \mathcal{A}} x_{a,10}^* \geq 1$ and $\sum_{a \in \mathcal{A}} \sum_{t > 10} x_{at}^* = 0$). In this case only 10 terms are active in the objective-function, therefore bounding each term with a tolerance $\delta = \Delta/10$ should have been sufficient to achieve the requested global tolerance Δ . Instead, because the duration is unknown a priori, each term of the objective function has to be bounded with precision $\delta = \Delta/100$. In summary, a pwl bounding tolerance 10 times higher than necessary would be requested, leading to pwl bounding functions with much more pieces than necessary, leading to a significant increase of the number of binary variables, leading to MILPs more time consuming than necessary to solve given the tolerance target.

5.3 Dependence on the instance

The two previous subsections focussed on problems where the non-linearity appeared in an objective-function which could be decomposed into a sum of non-linear univariate terms. In this case the given target tolerance Δ can be split among the terms of the univariate terms objective-function to deduce the δ -bounding tolerance. In the case where non-linearity exists in the constraints, for example if the HEV vehicle uses a super-capacitor with non linear loss functions f^2 and f^3 , then the link between the target solution tolerance Δ and the individual δ -bounding tolerance to apply to functions f^2 and f^3 is not obvious.

The various drawbacks described in the current Section participate to the difficulty of choosing a priori a relevant tolerance value δ when applying the “ δ -bounding + MILP solution” procedure to solve a MINLP resulting from the introduction of non-linear energy conversion functions into a combinatorial optimization problem.

6 Using ϵ -relative tolerance

To counter the drawbacks identified in Section 5, we propose to perform the upper and lower bounding of energy conversion expressions using non-necessarily continuous pwl functions with a *relative* ϵ -tolerance.

Definition 6.1 *Pwl bounding a function f on a compactum \mathbb{D} with a tolerance value $\epsilon \in [0, 100\%]$ consists in identifying two pwl functions $(\bar{f}^\epsilon, \underline{f}^\epsilon)$ that verify equations (28)-(30).*

$$\underline{f}^\epsilon(x) \leq f(x) \leq \bar{f}^\epsilon(x), \quad \forall x \in \mathbb{D} \quad (28)$$

$$|f(x) - \underline{f}^\epsilon(x)| \leq \epsilon |f(x)|, \quad \forall x \in \mathbb{D} \quad (29)$$

$$|\bar{f}^\epsilon(x) - f(x)| \leq \epsilon |f(x)|, \quad \forall x \in \mathbb{D} \quad (30)$$

Definition 6.1 generalizes the one of Ngueveu et al. (2016) by taking into account $\mathbb{D} \subset \mathbb{R}$ instead of $\mathbb{D} \subset \mathbb{R}^+$.

The resulting “ ϵ relative bounding + MILP solution” methodology presents several advantages. Firstly, pwl bounding can be applied directly on the non-linear energy conversion function before its insertion into the mathematical model, which is often a univariate function that may be convex or concave and therefore easier to approximate or bound. Secondly, two MILP problems denoted $(\overline{\text{MILP}}^\epsilon)$ and $(\underline{\text{MILP}}^\epsilon)$ can be obtained which provide upper and lower bounds to the original MINLP. Finally, thanks to the use of relative tolerance, guarantees can be obtained on the quality of resulting bounds (see for example proposition 6.2). Models and algorithms from Sections 3 and 4 can be adapted to ensure satisfaction of the relative ϵ tolerance constraint instead of the absolute δ tolerance constraint.

Proposition 6.2 *Let (P) be a (MI)NLP linearly constrained and with an objective-function decomposable into a sum of univariate positive linear or non linear functions, i.e verifying equation (31) where $g_{ki} : \mathbb{D} \rightarrow \mathbb{R}^+, \forall k, \forall i$ are linear or non linear functions. And let \bar{P}^ϵ and \underline{P}^ϵ be the MILP resulting from the replacement of g_{ki} with over-estimator \bar{g}_{ki}^ϵ or under-estimator $\underline{g}_{ki}^\epsilon$ respectively. Then the solution values of the optimal solutions $z_{(P)}$, $z_{(\bar{P}^\epsilon)}$ and*

$z_{(\overline{P}^\epsilon)}$ of the corresponding problems verify equations (32).

$$(P) \min \text{ or } \max \quad z_{(P)} = g(x) = \sum_k \sum_i g_{ki}(x_i) \quad s.t. \quad Ax \leq B \quad (31)$$

$$\max \left\{ z_{(\underline{P}^\epsilon)}, \frac{z_{(\overline{P}^\epsilon)}}{1 + \epsilon} \right\} \leq z_{(P)} \leq \min \left\{ z_{(\overline{P}^\epsilon)}, \frac{z_{(\underline{P}^\epsilon)}}{1 - \epsilon} \right\} \quad (32)$$

Proof It results from the combination of equations (28) with $z_{(P)} \leq z_{(\overline{P}^\epsilon)} \leq (1 + \epsilon)z_{(P)}$ and $(1 - \epsilon)z_{(P)} \leq z_{(\underline{P}^\epsilon)} \leq z_{(P)}$ ■

Given an optimal pwl ϵ -under- (resp. over-) estimator g of a function f , it is not straightforward to deduce a valid ϵ -over- (resp. under-) estimator of f because it is no longer possible to apply a shift g by ϵ . Indeed $\overline{f}^\epsilon(x) - \epsilon$ (resp. $\underline{f}(x)^\epsilon + \epsilon$) is not an under- (resp. over-) estimator of $f(x)$. Heuristics to obtain pwl upper and lower bounding functions for convex or concave functions defined on \mathbb{R}^+ are available in Ngueveu et al. (2014).

7 Preliminary computational results

Results on a scheduling problem under a non-reversible energy source are available in (Ngueveu et al., 2016). Algorithms proposed in (Ngueveu et al., 2014) were applied on convex or concave sections of a realistic energy conversion function provided by researchers in Electrical Engineering. The function expressed the energy demand in function of the energy cost and could not be easily inverted. The resulting pwl functions could be easily inverted without any loss of precision before their introduction into the mathematical model of the preemptive scheduling problem. The computational evaluation on a large set of instances show final gaps between the best upper and lower bounds found, in the best cases even smaller than half of $\epsilon\%$ and in the worst cases equal to $\epsilon\%$.

Hereafter are preliminary results on the problem of energy optimization for hybrid electric vehicles introduced in Section 1. Recall that the hybrid electric system of the vehicle considered has two energy sources: a Fuel Cell stack (FC) and a Storage Element (SE). The energy efficiency of each component is measured by the ratio between the amount of useful energy produced by the component and the total energy spent by the component. The amount of useful and usable energy is equal to the energy spent minus the energy losses. Figure 3 illustrates an instance used in computational evaluations in the literature. It is composed of an efficiency curve of the FC, a loss curve of the SE and a power demand profile of a vehicle on a highway. The goal is to minimize the total FC fuel cost for the vehicle which follows a given profile of power demands, by optimizing, the distribution of the power at each instant on the two sources taking into account their state of charge limitations and efficiency functions. With regards to the mathematical model (1)-(5), $f^1 = f_{FC}$ is a given polynomial whereas $f^2(x) = a^{\text{in}}x$ and $f^2 = a^{\text{out}}x$ where a^{in} and a^{out} are given constants.

To the best of our knowledge the best known solution method for the problem was proposed by (Gaoua et al., 2013) and was based on the reformulation of the problem as a MILP based on the use of efficiency points from the original data obtained experimentally for the FC.

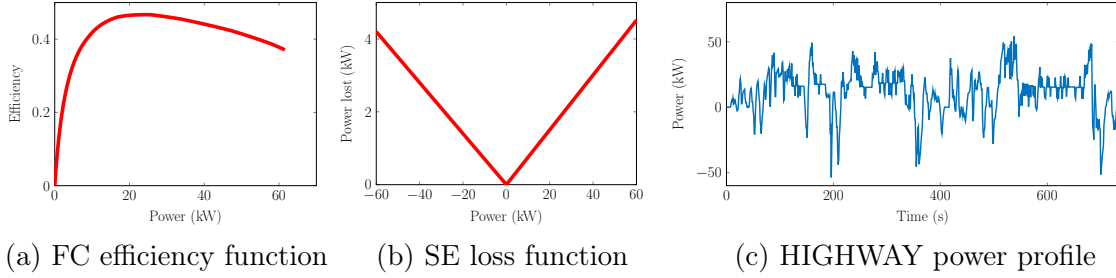


Figure 3: A hybrid electric vehicle instance

This approach was replicated by (Chauvin et al., 2015) in an iterative procedure based on the discretization of the non linear continuous efficiency function in equidistant efficiency points and solution of the resulting sub-problems with branch-and-bound. Tables 2 and 3 compare the results obtained from that efficiency points based method and our piecewise-linear based method on the instance illustrated on Figure 3. The total duration of the profile is 750s with a time step of 1s which provides $|\mathbb{I}| = 750$ instants.

Applying the efficiency points solution method on problem (P) consists in defining a set \mathcal{K}_{FC} of efficiency points k denoted $(\tilde{x}^k, f^1(\tilde{x}^k))$, defining binary variables y_i^k equal to 1 iff efficiency point k is used at time i , replacing x_i with $\sum_{k \in \mathcal{K}_{FC}} y_i^k$ in the constraints and replacing each non linear term $f^1(x_i)$ of the objective-function with the linear term $\sum_{k \in \mathcal{K}_{FC}} f^1(\tilde{x}^k) y_i^k$. The resulting MILP is denoted $(P_{\mathcal{K}_{FC}})$. Applying the pwl bounding solution method on problem (P) consists in identifying two pwl functions \bar{f}_{FC}^ϵ and $\underline{f}_{FC}^\epsilon$ verifying equations (28)-(30) to replace f^1 . The two resulting MILPs are denoted (\bar{P}^ϵ) and (\underline{P}^ϵ) . Different ϵ values lead to different number of pieces \bar{n}^ϵ and \underline{n}^ϵ of pwl functions \bar{f}_{FC}^ϵ et $\underline{f}_{FC}^\epsilon$. To ensure a fair comparison between the solution two methods being compared, the same number of binary variables have been used by enforcing either $|\mathcal{K}_{FC}| = \bar{n}^\epsilon$ or $|\mathcal{K}_{FC}| = \underline{n}^\epsilon$.

Table 2 show that the pwl bounding based method produced better feasible solutions than the efficiency points based method. Table 3 show that the best upper and lower bounds computed have a gap smaller than ϵ .

ϵ	\bar{n}^ϵ	$z_{(\bar{P}^\epsilon)}$	$\text{cpu}_{(\bar{P}^\epsilon)}$	$ \mathcal{K}_{FC} $	$z_{(P_{\mathcal{K}_{FC}})}$	$\text{cpu}_{(P_{\mathcal{K}_{FC}})}$
0.5 %	9	18691.6 kW	63.9 s	9	18694.8 kW	197.5 s
0.3 %	12	18653.8 kW	89.5 s	12	18662.1 kW	122.7 s
0.1 %	20	18629.4 kW	54.4 s	20	18638.8 kW	183.4 s

Table 2: Comparison of feasible solutions from efficiency points and pwl solution methods

8 Conclusion

This paper presents a two-phase solution method for combinatorial optimization problems involving non linear energy conversion functions. The first phase consists in bounding the non linear univariate functions from above and below with two piece-wise linear non-necessarily

ϵ	\underline{n}^ϵ	$z(\underline{P}^\epsilon)$	$\text{cpu}(\underline{P}^\epsilon)$	UB	LB	gap
0.5 %	9	18597.0 kW _s	24.5 s	18598.6 kW _s	18690.4 kW _s	0.49%
0.3 %	12	18597.3 kW _s	46.6 s	18598.0 kW _s	18653.3 kW _s	0.30%
0.1 %	20	18611.1 kW _s	53.1 s	18611.1 kW _s	18629.4 kW _s	0.10%

$$\text{LB} = \max \left\{ z(\underline{P}^\epsilon), \frac{z(\overline{P}^\epsilon)}{(1+\epsilon)} \right\}, \text{UB} = \min \left\{ z(\overline{P}^\epsilon), \frac{z(\underline{P}^\epsilon)}{(1-\epsilon)} \right\}, \text{gap} = (\text{UB} - \text{LB}) / \text{LB}$$

Table 3: Lower bounds and best bounds from the pwl bounding-based solution method

continuous functions with a relative ϵ tolerance. The second phase consists in solving the two mixed integer linear programs obtained when replacing the non linear terms with their overestimators or underestimators. Models and algorithms to perform the pwl bounding are presented. Preliminary computational results show the efficiency of the solution method on energy optimization problems.

References

- Borghetti, A., D’Ambrosio, C., Lodi, A., and Martello, S. (2008). An milp approach for short-term hydro scheduling and unit commitment with head-dependent reservoir. *Power Systems, IEEE Transactions on*, 23(3):1115–1124.
- Camponogara, E., de Castro, M., and Plucenio, A. (2007). Compressor scheduling in oil fields: A piecewise-linear formulation. In *Automation Science and Engineering, 2007. CASE 2007. IEEE International Conference on*, pages 436–441.
- Chauvin, A., Hijazi, A., Bideaux, E., and Sari, A. (2015). Combinatorial approach for sizing and optimal energy management of hev including durability constraints. In *24th International Symposium on Industrial Electronics (ISIE)*, pages 1236–1241. IEEE.
- Gaoua, Y., Caux, S., and Lopez, P. (2013). Energy management for an electric vehicle based on combinatorial modeling. In *Industrial Engineering and Systems Management (IESM), Proceedings of 2013 International Conference on*, pages 1–6. IEEE.
- Geißler, B., Martin, A., Morsi, A., and Schewe, L. (2012). *Using Piecewise Linear Functions for Solving MINLPs*, pages 287–314. Springer New York, New York, NY.
- Ngueveu, S. U., Artigues, C., and Lopez, P. (2016). Scheduling under a non-reversible energy source: An application of piecewise linear bounding of non-linear demand/cost functions. *Discrete Applied Mathematics*, 208:98–113.
- Ngueveu, S. U., Sareni, B., and Roboam, X. (2014). Piecewise bounding and integer linear programming for the optimal management of a water pumping and desalination process. Technical Report 14144, LAAS-CNRS.
- Rebennack, S. and Kallrath, J. (2015). Continuous piecewise linear delta-approximations for univariate functions: Computing minimal breakpoint systems. *J. Optim. Theory Appl.*, 167(2):617–643.

Rosen, J. B. and Pardalos, P. M. (1986). Global minimization of large-scale constrained concave quadratic problems by separable programming. *Mathematical Programming*, 34(2):163–174.