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Quantum mean-field asymptotics and multiscale analysis

Z. Ammari*, S. Breteaux† and F. Nier‡

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Abstract

We study, via multiscale analysis, some defect of compactness phenomena which occur in bosonic and fermionic quantum mean-field problems. The approach relies on a combination of mean-field asymptotics and second microlocalized semiclassical measures. The phase space geometric description is illustrated by various examples.

Keywords and phrases: Semiclassical and multiscale measures, reduced density matrices, second quantization, microlocal analysis.

2010 Mathematics subject classification: 81S30, 81Q20, 81V70, 81S05.

1 Introduction

Motivations: Over the past two decades, it becomes clear that microlocal analysis provides interesting mathematical tools for the study of quantum field theories and quantum many-body theory, see for instance [AmNi1, Rad]. In particular, in the analysis of general bosonic mean-field problems, as done in [AmNi1, AmNi2, AmNi3, AmNi4], the following defect of compactness problem arises. If $\gamma_\varepsilon^{(p)}$ denotes the p -particles reduced density matrix, one may have

$$\lim_{\varepsilon \rightarrow 0} \text{Tr}[\gamma_\varepsilon^{(p)} \tilde{b}] = \text{Tr}[\gamma_0^{(p)} \tilde{b}] \quad (1)$$

for any p -particle compact observable \tilde{b} , while it is not true for a general bounded \tilde{b} , e.g.

$$\lim_{\varepsilon \rightarrow 0} \text{Tr}[\gamma_\varepsilon^{(p)}] > \text{Tr}[\gamma_0^{(p)}].$$

In the fermionic case, it is even worse, because mean-field asymptotics cannot be described in terms of finitely many quantum states and the right-hand side of (1) is usually 0 while $\lim_{\varepsilon \rightarrow 0} \text{Tr}[\gamma_\varepsilon^{(p)}] > 0$. From the analysis of finite dimensional partial differential equations, it is known that such defect of compactness can be localized geometrically with accurate quantitative information by introducing

*zied.ammari@univ-rennes1.fr, IRMAR, Université de Rennes I, UMR-CNRS 6625, campus de Beaulieu, 35042 Rennes Cedex, France.

†sebastien.breteaux@ens-cachan.org, BCAM - Basque Center for Applied Mathematics, Alameda de Mazarredo 14, 48009 Bilbao, Spain.

‡francis.nier@math.univ-paris13.fr, LAGA, UMR-CNRS 9345, Université de Paris 13, av. J.B. Clément, 93430 Villetaneuse, France.

scales and small parameters within semiclassical techniques (e.g. [Ger, GMMP]). We are thus led to introduce two small parameters $\varepsilon > 0$ for the mean-field asymptotics and $h > 0$ for the semiclassical quantization of finite dimensional p -particles phase space. The small parameter ε stands for $1/n$, where $n \rightarrow \infty$ is the typical number of particles, while h is the rescaled Planck constant measuring the proximity of quantum mechanics to classical mechanics. The combined analysis of this article is concerned with the general situation when $\varepsilon = \varepsilon(h)$ with $\lim_{h \rightarrow 0} \varepsilon(h) = 0$. In order to keep track of the information at the quantum level especially in the bosonic case we also introduce finite dimensional multiscale observables in spirit of [Bon, FeGe, Fer, Nie].

Framework: The one particle space \mathcal{L} is a separable complex Hilbert space endowed with the scalar product $\langle \cdot, \cdot \rangle$ (anti-linear in the left-hand side). For a Hilbert space \mathfrak{h} the set of bounded operator is denoted by $\mathcal{L}(\mathfrak{h})$, while the Schatten class is denoted by $\mathcal{L}^p(\mathfrak{h})$, $1 \leq p \leq \infty$, the case $p = \infty$ corresponding to the space of compact operators. Let $\Gamma_{\pm}(\mathcal{L})$ be the bosonic (+ sign) or fermionic (− sign) Fock-space built on the separable Hilbert space \mathcal{L} :

$$\Gamma_{\pm}(\mathcal{L}) = \bigoplus_{n \in \mathbb{N}}^{\perp} \mathcal{S}_{\pm}^n \mathcal{L}^{\otimes n},$$

where tensor products and direct sum are Hilbert completed. The operator \mathcal{S}_{\pm}^n is the orthogonal projection given by

$$\mathcal{S}_{\pm}^n(f_1 \otimes \cdots \otimes f_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} s_{\pm}(\sigma) f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)},$$

where $s_+(\sigma)$ equals 1 while $s_-(\sigma)$ denotes the signature of the permutation σ and \mathfrak{S}_n is the n -symmetric group.

The dense set of many-body states with a finite number of particles is

$$\Gamma_{\pm}^{fin}(\mathcal{L}) = \bigoplus_{n \in \mathbb{N}}^{\perp, alg} \mathcal{S}_{\pm}^n \mathcal{L}^{\otimes n},$$

where the $^{\perp, alg}$ superscript stands for the algebraic orthogonal direct sum.

We shall use the notations $[A, B]_+ = [A, B] = \text{ad}_A B = AB - BA$ for the commutator of two operators and the notation $[A, B]_- = AB + BA$ for the anticommutator.

One way to investigate the mean-field asymptotics relies on parameter-dependent *CCR* (resp. *CAR*). The small parameter $\varepsilon > 0$ has to be thought of as the inverse of the typical number of particles and the Canonical Commutation (resp. Anticommutation) Relations are given by

$$[a_{\pm}(g), a_{\pm}(f)]_{\pm} = [a_{\pm}^*(g), a_{\pm}^*(f)]_{\pm} = 0 \quad , \quad [a_{\pm}(g), a_{\pm}^*(f)]_{\pm} = \varepsilon \langle g, f \rangle.$$

Let $(\varrho_{\varepsilon})_{\varepsilon > 0}$ be a family of normal states (i.e., non-negative and normalized trace-class operators) on the Fock space $\Gamma_{\pm}(\mathcal{L})$, depending on $\varepsilon > 0$, we want to investigate the asymptotic behaviour of reduced density matrices, defined below, as $\varepsilon \rightarrow 0$, by possibly introducing another scale $h > 0$ on the p -particles phase-space, with $\varepsilon = \varepsilon(h)$ and $\lim_{h \rightarrow 0} \varepsilon(h) = 0$.

Outline: In Section 2, we recall how Wick observables are used to define the reduced density matrices $\gamma_\varepsilon^{(p)}$. Note that it is much more convenient here, in the general grand canonical framework, to work with non normalized reduced density matrices. Some symmetrization formulas are also recalled in this section. In Section 3, we present the geometry of the classical p -particles phase space and introduce the formalism of double scale semiclassical measures, after [Fer, FeGe]. In Section 4, we combine the mean-field asymptotics with semiclassical analysis, the two parameters ε and h being related through $\varepsilon = \varepsilon(h)$ with $\lim_{h \rightarrow 0} \varepsilon(h) = 0$. Instead of studying the collection of non normalized reduced density matrices $(\gamma_{\varepsilon(h)}^{(p)})_{p \in \mathbb{N}}$, it is more convenient to associate generating functions $z \mapsto \text{Tr}[\varrho_{\varepsilon(h)} e^{z d\Gamma_\pm(a^{\mathcal{Q},h})}]$, and to use holomorphy arguments presented there. In Section 5, some classical examples with various asymptotics illustrate the general framework: coherent states in the bosonic setting; simple Gibbs states in the fermionic case; more involved Gibbs states in the bosonic case, which make explicit the separation of condensate and non condensate phases for rather general non interacting steady Bose gases. The Appendices collect or revisit known things about multiscale semiclassical measures, the (PI)-condition of bosonic mean-field problems, Wick composition formulas, and traces of non self-adjoint second quantized contractions.

2 Wick observables and reduced density matrices

2.1 Wick observables

Notation: For $n \in \mathbb{N}$, the operator \mathcal{S}_\pm^n is an orthogonal projection in $\mathcal{Z}^{\otimes n}$ so that $(\mathcal{S}_\pm^n)^* = \mathcal{S}_\pm^n$. However, we consider \mathcal{S}_\pm^n as a bounded operator from $\mathcal{Z}^{\otimes n}$ onto $\mathcal{S}_\pm^n \mathcal{Z}^{\otimes n}$ and its adjoint, denoted by $\mathcal{S}_\pm^{n,*} : \mathcal{S}_\pm^n \mathcal{Z}^{\otimes n} \rightarrow \mathcal{Z}^{\otimes n}$, is nothing but the natural embedding.

Let $\tilde{b} \in \mathcal{L}(\mathcal{S}_\pm^p \mathcal{Z}^{\otimes p}; \mathcal{S}_\pm^q \mathcal{Z}^{\otimes q})$, the Wick quantization of \tilde{b} is the operator on $\Gamma_\pm^{fin}(\mathcal{Z})$ defined by

$$\tilde{b}^{Wick} \Big|_{\mathcal{S}_\pm^{n+p} \mathcal{Z}^{\otimes(n+p)}} = \varepsilon^{\frac{p+q}{2}} \frac{\sqrt{(n+p)!(n+q)!}}{n!} \mathcal{S}_\pm^{n+q} (\tilde{b} \otimes \text{Id}_{\mathcal{Z}^{\otimes n}}) \mathcal{S}_\pm^{n+p,*}.$$

In the bosonic case, an element $\tilde{b} \in \mathcal{L}(\mathcal{S}_+^p \mathcal{Z}^{\otimes p}; \mathcal{S}_+^q \mathcal{Z}^{\otimes q})$ is determined by the symbol $\mathcal{Z} \ni z \mapsto b(z) = \langle z^{\otimes q}, \tilde{b} z^{\otimes p} \rangle$ owing to the relation $\tilde{b} = \frac{1}{q!p!} \partial_{\bar{z}}^q \partial_z^p b$. Observe that $b(z)$ admits higher Fréchet derivatives with the natural identification of $\partial_z^k b(z)$ as a continuous form on $\mathcal{S}_+^k \mathcal{Z}^{\otimes k}$ and $\partial_{\bar{z}}^k b(z)$ as a vector in $\mathcal{S}_+^k \mathcal{Z}^{\otimes k}$. We shall use also the notation $b^{Wick} = \tilde{b}^{Wick}$.

Examples:

- a) The annihilation operator $a_\pm(f)$, $f \in \mathcal{Z}$, is the Wick quantization of $\tilde{b} = \langle f | : \mathcal{Z}^{\otimes 1} = \mathcal{Z} \ni \varphi \mapsto \langle f, \varphi \rangle \in \mathcal{Z}^{\otimes 0} = \mathbb{C}$.
- b) The creation operator $a_\pm^*(f)$, $f \in \mathcal{Z}$, is the Wick quantization of $\tilde{b} = |f\rangle : \mathcal{Z}^{\otimes 0} = \mathbb{C} \ni \lambda \mapsto \lambda f \in \mathcal{Z}^{\otimes 1} = \mathcal{Z}$.
- c) For $\tilde{b} \in \mathcal{L}(\mathcal{Z})$ its Wick quantization \tilde{b}^{Wick} is nothing but

$$d\Gamma_\pm(\tilde{b}) \Big|_{\mathcal{S}_\pm^n \mathcal{Z}^{\otimes n}} = \varepsilon \left[\tilde{b} \otimes \text{Id}_{\mathcal{Z}} \otimes \cdots \otimes \text{Id}_{\mathcal{Z}} + \cdots + \text{Id}_{\mathcal{Z}} \otimes \cdots \otimes \text{Id}_{\mathcal{Z}} \otimes \tilde{b} \right].$$

When \tilde{b} is self-adjoint one has

$$d\Gamma_\pm(\tilde{b}) = i\partial_t e^{-itd\Gamma_\pm(\tilde{b})} \Big|_{t=0} = i\partial_t \Gamma_\pm(e^{-i\varepsilon t \tilde{b}}) \Big|_{t=0},$$

while for a contraction $C \in \mathcal{L}(\mathcal{Z}; \mathcal{Z})$,

$$\Gamma_{\pm}(C)|_{\mathcal{S}_{\pm}^n \mathcal{Z}^{\otimes n}} = C \otimes \cdots \otimes C.$$

A particular case is $\tilde{b} = \text{Id}_{\mathcal{Z}}$ associated with the scaled number operator ($\mathbf{N}_{\pm, \varepsilon=1}$ stands for the usual ε -independent number operator):

$$\tilde{b}^{Wick} = d\Gamma_{\pm}(\text{Id}_{\mathcal{Z}}) = \mathbf{N}_{\pm} = \varepsilon \mathbf{N}_{\pm, \varepsilon=1}.$$

From the definition of the Wick quantization one easily checks the following properties.

Proposition 2.1. For $\tilde{b} \in \mathcal{L}(\mathcal{S}_{\pm}^p \mathcal{Z}^{\otimes p}; \mathcal{S}_{\pm}^q \mathcal{Z}^{\otimes q})$:

- $[\tilde{b}^{Wick}]^* = [\tilde{b}^*]^{Wick}$.
- The operator $(1 + \mathbf{N}_{\pm})^{-m/2} \tilde{b}^{Wick} (1 + \mathbf{N}_{\pm})^{-m'/2}$ extends to a bounded operator on $\Gamma_{\pm}(\mathcal{Z})$ as soon as $m + m' \geq p + q$ with

$$\|(1 + \mathbf{N}_{\pm})^{-m/2} \tilde{b}^{Wick} (1 + \mathbf{N}_{\pm})^{-m'/2}\|_{\mathcal{L}(\Gamma_{\pm}(\mathcal{Z}))} \leq C_{m, m'} \|\tilde{b}\|_{\mathcal{L}(\mathcal{S}_{\pm}^p \mathcal{Z}; \mathcal{S}_{\pm}^q \mathcal{Z})}, \quad (2)$$

with $C_{m, m'}$ independent of \tilde{b} and of $\varepsilon \in (0, \varepsilon_0)$.

- $(\tilde{b} \geq 0) \Leftrightarrow (\tilde{b}^{Wick} \geq 0)$, while this makes sense only for $q = p$.

The Wick quantized operators generally are unbounded operators on $\Gamma_{\pm}(\mathcal{Z})$ (e.g. \mathbf{N}_{\pm}) but they are well defined on the dense set $\Gamma_{\pm}^{fin}(\mathcal{Z})$ which is preserved by their action. Hence $\tilde{b}_1^{Wick} \circ \tilde{b}_2^{Wick}$ makes sense at least on $\Gamma_{\pm}^{fin}(\mathcal{Z})$ and the following composition law holds true.

Proposition 2.2 (Composition of Wick operators). Let $\tilde{b}_j \in \mathcal{L}(\mathcal{S}_{\pm}^{p_j} \mathcal{Z}^{\otimes p_j}; \mathcal{S}_{\pm}^{q_j} \mathcal{Z}^{\otimes q_j})$, $j = 1, 2$, then

$$\tilde{b}_1^{Wick} \circ \tilde{b}_2^{Wick} = \sum_{k=0}^{\min\{p_1, q_2\}} (\pm 1)^{(p_1-k)(p_2+q_2)} \frac{\varepsilon^k}{k!} (\tilde{b}_1 \#^k \tilde{b}_2)^{Wick}, \quad (3)$$

where $\tilde{b}_1 \#^k \tilde{b}_2 := \frac{p_1!}{(p_1-k)!} \frac{q_2!}{(q_2-k)!} \mathcal{S}_{\pm}^{q_1+q_2-k} (\tilde{b}_1 \otimes \text{Id}^{\otimes q_2-k}) (\text{Id}^{\otimes p_1-k} \otimes \tilde{b}_2) \mathcal{S}_{\pm}^{p_1+p_2-k, *}$.

For reader's convenience, the proof of Prop. 2.2 is provided in Appendix C.

In the bosonic case the symbols $b(z) = \langle z^{\otimes q}, \tilde{b} z^{\otimes p} \rangle$ are convenient for writing the composition of Wick quantized operators. If $b_1 \#^{Wick} b_2$ denotes the symbol of $\tilde{b}_1^{Wick} \circ \tilde{b}_2^{Wick}$, the composition law is summarized below (see [AmNi1]).

Proposition 2.3 (Composition of Wick symbols in the bosonic case).

$$b_1 \#^{Wick} b_2(z) = e^{\varepsilon \partial_{z_1} \cdot \partial_{\bar{z}_2}} b_1(z_1) b_2(z_2) \Big|_{z_1=z_2=z} = \sum_{k=0}^{\min\{p_1, q_2\}} \frac{\varepsilon^k}{k!} \partial_z^k b_1(z) \cdot \partial_{\bar{z}}^k b_2(z).$$

The commutator of Wick operators in the bosonic case:

$$[b_1^{Wick}, b_2^{Wick}] = \left(\sum_{k=1}^{\max\{\min\{p_1, q_2\}, \min\{p_2, q_1\}\}} \frac{\varepsilon^k}{k!} \{b_1, b_2\}^{(k)} \right)^{Wick},$$

where the k -th order Poisson bracket is given by $\{b_1, b_2\}^{(k)} = \partial_z^k b_1(z) \cdot \partial_{\bar{z}}^k b_2(z) - \partial_{\bar{z}}^k b_2(z) \cdot \partial_z^k b_1(z)$.

Proposition 2.4. *Let $p, m, m' \in \mathbb{N}$, such that $m + m' \geq 2p - 2$. Then, there exist coefficients $C_{j_1, \dots, j_k} \geq 0$ such that, for any $\tilde{b} \in \mathcal{L}(\mathcal{Z}; \mathcal{Z})$,*

$$d\Gamma_{\pm}(\tilde{b})^p - (\tilde{b}^{\otimes p})^{Wick} = \sum_{k=1}^{p-1} \varepsilon^{p-k} \sum_{\substack{0 \leq j_1 \leq \dots \leq j_k \\ j_1 + \dots + j_k = p}} C_{j_1, \dots, j_k} \left(\mathcal{S}_{\pm}^k \tilde{b}^{j_1} \otimes \dots \otimes \tilde{b}^{j_k} \mathcal{S}_{\pm}^{k,*} \right)^{Wick} \quad (4)$$

and the estimate

$$\|(1 + \mathbf{N}_{\pm})^{-m/2} (d\Gamma_{\pm}(\tilde{b})^p - (\tilde{b}^{\otimes p})^{Wick}) (1 + \mathbf{N}_{\pm})^{-m'/2}\|_{\mathcal{L}(\Gamma_{\pm}(\mathcal{Z}))} \leq \varepsilon B_p \|\tilde{b}\|_{\mathcal{L}(\mathcal{Z})}^p$$

holds in both the bosonic case and the fermionic case, with B_p the p -th Bell number.

Remark 2.5. *The p -th Bell number B_p can be defined as the number of partitions of a set with p elements and satisfies $B_p < \left(\frac{0.792p}{\ln(p+1)}\right)^p$ (see [BeTa]), and hence it grows much slower than $p!$.*

Proof. We first prove Formula (4) by induction on $p \in \mathbb{N}^*$.

For $p = 1$, Formula (4) holds because $d\Gamma_{\pm}(\tilde{b}) = (\tilde{b})^{Wick}$.

We then set $r_p(\tilde{b}) := d\Gamma_{\pm}(\tilde{b})^p - (\tilde{b}^{\otimes p})^{Wick}$. Assuming the result holds for some $p \in \mathbb{N}^*$, one can compute

$$d\Gamma_{\pm}(\tilde{b})^{p+1} = (\tilde{b}^{\otimes p})^{Wick} (\tilde{b})^{Wick} + r_p(\tilde{b})^{Wick} (\tilde{b})^{Wick}$$

using the composition formula (3) for

$$(\tilde{b}^{\otimes p})^{Wick} (\tilde{b})^{Wick} = (\tilde{b}^{\otimes p+1})^{Wick} + p\varepsilon (\mathcal{S}_{\pm}^p \tilde{b}^{\otimes p-1} \otimes \tilde{b}^2 \mathcal{S}_{\pm}^{p,*})^{Wick}$$

and for

$$\begin{aligned} & \varepsilon^{p-k} (\mathcal{S}_{\pm}^k \tilde{b}^{j_1} \otimes \dots \otimes \tilde{b}^{j_k} \mathcal{S}_{\pm}^{k,*})^{Wick} (\tilde{b})^{Wick} \\ &= \varepsilon^{p+1-(k+1)} (\mathcal{S}_{\pm}^{k+1} \tilde{b} \otimes \tilde{b}^{j_1} \otimes \dots \otimes \tilde{b}^{j_k} \mathcal{S}_{\pm}^{k+1,*})^{Wick} \\ &+ k\varepsilon^{p+1-k} \left(\mathcal{S}_{\pm}^k (\tilde{b}^{j_1} \otimes \dots \otimes \tilde{b}^{j_k}) \mathcal{S}_{\pm}^{k,*} \mathcal{S}_{\pm}^k (\tilde{b} \otimes \text{Id}_{\mathcal{Z}^{\otimes j_1 + \dots + j_k - 1}}) \mathcal{S}_{\pm}^{k,*} \right)^{Wick}, \end{aligned}$$

which yields the expected form for $r_{p+1}(\tilde{b})$, and achieves the induction.

We then remark that the sum of coefficients of order k ,

$$S_2(p, k) = \sum_{\substack{0 \leq j_1 \leq \dots \leq j_k \\ j_1 + \dots + j_k = p}} C_{j_1, \dots, j_k},$$

satisfies the recurrence relation $S_2(p, k) = kS_2(p-1, k) + S_2(p-1, k-1)$, with $S_2(p, 1) = 1 = S_2(1, k)$ for all $p, k \in \mathbb{N}^*$, where the $S_2(p, k)$ are the Stirling numbers of the second kind. Observe that, for $M/2 \geq k$, and for any $\tilde{c} \in \mathcal{L}(\mathcal{S}_{\pm}^k \mathcal{Z}^{\otimes k})$,

$$\|\tilde{c}^{Wick} (1 + \mathbf{N}_{\pm})^{-M/2}\|_{\mathcal{L}(\Gamma_{\pm}(\mathcal{Z}))} \leq \|\tilde{c}\|_{\mathcal{L}(\mathcal{S}_{\pm}^k \mathcal{Z}^{\otimes k}; \mathcal{S}_{\pm}^k \mathcal{Z}^{\otimes k})}.$$

We thus get,

$$\|(1 + \mathbf{N}_{\pm})^{-m/2} (d\Gamma_{\pm}(\tilde{b})^p - (\tilde{b}^{\otimes p})^{Wick}) (1 + \mathbf{N}_{\pm})^{-m'/2}\|_{\mathcal{L}(\Gamma_{\pm}(\mathcal{Z}))} \leq \sum_{k=1}^{p-1} \varepsilon^{p-k} S_2(p, k) \|\tilde{b}\|_{\mathcal{L}(\mathcal{Z})}^p$$

and the estimate then follows from $\sum_{k=1}^{p-1} \varepsilon^{p-k} S_2(p, k) \leq \varepsilon \sum_{k=1}^p S_2(p, k) = \varepsilon B_p$ with B_p the p -th Bell number. \square

2.2 Reduced density matrices

Reduced density matrices emerge naturally in the study of correlation functions of quantum gases and in particular in the quantum mean-field theory they are the main quantities to be analysed. We shall work with non normalized reduced density matrices, which are easier to handle. Going back to the more natural reduced density matrices with trace equal to 1, requires attention when normalizing and taking the limits.

Definition 2.6. Let $\varrho_\varepsilon \in \mathcal{L}^1(\Gamma_\pm(\mathcal{Z}))$ ($\varepsilon > 0$ is fixed here) be such that $\varrho_\varepsilon \geq 0$, $\text{Tr}[\varrho_\varepsilon] = 1$ and $\text{Tr}(\varrho_\varepsilon e^{c\mathbf{N}_\pm}) < \infty$ for some $c > 0$. The non normalized reduced density matrix of order $p \in \mathbb{N}$, $\gamma_\varepsilon^{(p)} \in \mathcal{L}^1(\mathcal{S}_\pm^p \mathcal{Z}^{\otimes p})$, is defined by duality according to

$$\forall \tilde{b} \in \mathcal{L}(\mathcal{S}_\pm^p \mathcal{Z}^{\otimes p}; \mathcal{S}_\pm^p \mathcal{Z}^{\otimes p}), \quad \text{Tr}[\gamma_\varepsilon^{(p)} \tilde{b}] = \text{Tr}[\varrho_\varepsilon \tilde{b}^{Wick}].$$

The definition makes sense owing to the number estimate (2) and to $(1 + \mathbf{N}_\pm)^k e^{-c\mathbf{N}_\pm} \in \mathcal{L}(\Gamma_\pm(\mathcal{Z}))$.

When $\text{Tr}[\gamma_\varepsilon^{(p)}] \neq 0$, the normalized density matrix $\bar{\gamma}_\varepsilon^{(p)}$ is defined by $\bar{\gamma}_\varepsilon^{(p)} = \frac{\gamma_\varepsilon^{(p)}}{\text{Tr}[\gamma_\varepsilon^{(p)}}$, that is

$$\begin{aligned} \forall \tilde{b} \in \mathcal{L}(\mathcal{S}_\pm^p \mathcal{Z}^{\otimes p}), \quad \text{Tr}[\bar{\gamma}_\varepsilon^{(p)} \tilde{b}] &= \frac{\text{Tr}[\varrho_\varepsilon \tilde{b}^{Wick}]}{\text{Tr}[\varrho_\varepsilon (\text{Id}_{\mathcal{S}_\pm^p \mathcal{Z}^{\otimes p}})^{Wick}]} \\ &= \frac{\text{Tr}[\varrho_\varepsilon \tilde{b}^{Wick}]}{\text{Tr}[\varrho_\varepsilon N_\pm (N_\pm - \varepsilon) \dots (N_\pm - \varepsilon(p-1))]} . \end{aligned}$$

These normalized reduced density matrices $\bar{\gamma}_\varepsilon^{(p)}$ are commonly used, especially when $\varrho_\varepsilon \in \mathcal{L}^1(\mathcal{S}_\pm \mathcal{Z}^{\otimes n})$, with $n\varepsilon \sim 1$ (see [BGM, BEGMY, BPS, KnPi, LNR]), for the following reason: When $\varrho_\varepsilon \in \mathcal{L}^1(\mathcal{S}_\pm^n \mathcal{Z}^{\otimes n}) \subset \mathcal{L}^1(\mathcal{Z}^{\otimes n})$ lies in the n -particles sector (with $n\varepsilon \rightarrow 1$ for the mean-field regime) it is simply given by the partial trace of ϱ_ε when $n > p$. Actually

$$\tilde{b}^{Wick}|_{\mathcal{S}_\pm^n \mathcal{Z}^{\otimes n}} = \varepsilon^p \frac{n!}{(n-p)!} \mathcal{S}_\pm^n(\tilde{b} \otimes \text{Id}_{\mathcal{Z}^{\otimes(n-p)}}) \mathcal{S}_\pm^{n,*}$$

and, as $\varepsilon^p n(n-1) \dots (n-p+1) \rightarrow 1$ if $n\varepsilon \rightarrow 1$, it follows that

$$\lim_{\substack{n\varepsilon \sim 1 \\ \varepsilon \rightarrow 0}} \text{Tr}[\gamma_\varepsilon^{(p)} \tilde{b}] = \lim_{\substack{n\varepsilon \sim 1 \\ \varepsilon \rightarrow 0}} \text{Tr}[\varrho_\varepsilon(\tilde{b} \otimes \text{Id}_{\mathcal{Z}^{\otimes(n-p)}})] .$$

When $\mathcal{Z} = L^2(M; dv)$ one thus often considers:

$$\tilde{\gamma}_\varepsilon^{(p)}(x_1, \dots, x_p; x'_1, \dots, x'_p) = \int_{M^{n-p}} \varrho_\varepsilon(x_1, \dots, x_p, x; x'_1, \dots, x'_p, x) dv^{\otimes(n-p)}(x) .$$

But, if the states ϱ_ε are not localized on the n -particles sector, such an alternative definition does not coincide with $\gamma_\varepsilon^{(p)}$, even asymptotically in the mean-field regime.

As well there is no general relation between the non normalized density matrices $\gamma_\varepsilon^{(p+1)}$ and $\gamma_\varepsilon^{(p)}$. Actually we have

$$\begin{aligned} (\mathcal{S}_\pm^{p+1}(\tilde{b} \otimes \text{Id}_{\mathcal{Z}}) \mathcal{S}_\pm^{p+1,*})^{Wick}|_{\mathcal{S}_\pm^{n+p+1} \mathcal{Z}} &= \varepsilon^{p+1} \frac{(n+p+1)!}{n!} \mathcal{S}_\pm^{n+p+1}(\tilde{b} \otimes \text{Id}_{\mathcal{Z}^{\otimes n+1}}) \mathcal{S}_\pm^{n+p+1,*} \\ &= \varepsilon(n+1) \tilde{b}^{Wick}|_{\mathcal{S}_\pm^n \mathcal{Z}^{n+p+1}} , \end{aligned}$$

from which we deduce

$$\begin{aligned} \text{Tr} \left[\gamma_\varepsilon^{(p+1)}(\tilde{b} \otimes \text{Id}_{\mathcal{Z}}) \right] &= \text{Tr} \left[\varrho_\varepsilon \mathbf{N}_\pm \tilde{b}^{Wick} \right] + \mathcal{O}(\varepsilon), \\ \text{while} \quad \text{Tr} \left[\gamma_\varepsilon^{(p)} \tilde{b} \right] &= \text{Tr} \left[\varrho_\varepsilon \tilde{b}^{Wick} \right], \end{aligned}$$

where we have again identified $\gamma_\varepsilon^{(p+1)}$ as an element of $\mathcal{L}^1(\mathcal{Z}^{\otimes(p+1)})$. We thus conclude with the following important remark.

Remark 2.7. *Simple asymptotic relation between the $\gamma_\varepsilon^{(p)}$ and $\gamma_\varepsilon^{(p')}$ (or the normalized version $\bar{\gamma}_\varepsilon^{(p)}$ and $\bar{\gamma}_\varepsilon^{(p')}$) can be expected when $\varrho_\varepsilon = \varrho_\varepsilon \mathbf{1}_{[\nu-\delta(\varepsilon), \nu+\delta(\varepsilon)]}(\mathbf{N}_\pm)$ with $\nu > 0$ and $\lim_{\varepsilon \rightarrow 0} \delta(\varepsilon) = 0$ but not otherwise (of course the condition above is sufficient but not necessary).*

We shall use recurrently with variations the following lemma, with the following notations

$$\tilde{b}_1 \odot \cdots \odot \tilde{b}_p = \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} \tilde{b}_{\sigma(1)} \otimes \cdots \otimes \tilde{b}_{\sigma(p)},$$

for $\tilde{b}_1, \dots, \tilde{b}_p \in \mathcal{L}(\mathcal{Z})$.

We also write shortly $(\tilde{b}_1 \odot \dots \odot \tilde{b}_p)^{Wick}$ and $(\tilde{b}^{\otimes p})^{Wick}$ instead of $\left(\mathcal{S}_\pm^p(\tilde{b}_1 \odot \dots \odot \tilde{b}_p) \mathcal{S}_\pm^{p,*} \right)^{Wick}$ and $\left(\mathcal{S}_\pm^p(\tilde{b}^{\otimes p}) \mathcal{S}_\pm^{p,*} \right)^{Wick}$.

Lemma 2.8. Quantum symmetrization lemma: *In the bosonic and fermionic cases for any $p \in \mathbb{N}$, the equality*

$$\mathcal{S}_\pm^p(\tilde{b}_1 \otimes \cdots \otimes \tilde{b}_p) \mathcal{S}_\pm^{p,*} = \mathcal{S}_\pm^p(\tilde{b}_1 \odot \cdots \odot \tilde{b}_{\sigma(p)}) \mathcal{S}_\pm^{p,*}, \quad (5)$$

holds in $\mathcal{L}(\mathcal{S}_\pm^p \mathcal{Z}^{\otimes p}; \mathcal{S}_\pm^p \mathcal{Z}^{\otimes p})$ for all $\tilde{b}_1, \dots, \tilde{b}_p \in \mathcal{L}(\mathcal{Z}; \mathcal{Z})$.

As a consequence, under the assumptions of Definition 2.6, the non normalized (resp. normalized if possible) reduced density matrix $\gamma_\varepsilon^{(p)}$ (resp. $\bar{\gamma}_\varepsilon^{(p)}$), $p \in \mathbb{N}$, is completely determined by the set of quantities $\left\{ \text{Tr} \left[\varrho_\varepsilon (\tilde{b}^{\otimes p})^{Wick} \right], \tilde{b} \in \mathcal{B} \right\}$ when \mathcal{B} is any dense subset of $\mathcal{L}^\infty(\mathcal{Z}; \mathcal{Z})$.

Remark 2.9. *While computing $\text{Tr} \left[\gamma_\varepsilon^{(p)} \right]$ or studying $\bar{\gamma}_\varepsilon^{(p)}$ one can simply add to \mathcal{B} the element $\text{Id}_{\mathcal{Z}}$ owing to $\mathcal{S}_\pm^p \text{Id}_{\mathcal{Z}^{\otimes p}} \mathcal{S}_\pm^{p,*} = \text{Id}_{\mathcal{S}_\pm^p \mathcal{Z}^{\otimes p}}$. For $\varepsilon > 0$ fixed it is not necessary because compact observables are sufficient to determine the total trace owing to $\text{Tr} \left[\gamma_\varepsilon^{(p)} \right] = \sup_{B \in \mathcal{L}^\infty(\mathcal{S}_\pm^p \mathcal{Z}^{\otimes p}), 0 \leq B \leq \text{Id}} \text{Tr} \left[\gamma_\varepsilon^{(p)} B \right]$. However, while considering weak*-limits as $\varepsilon \rightarrow 0$, adding the identity operator $\text{Id}_{\mathcal{S}_\pm^p \mathcal{Z}^{\otimes p}}$ to the set of compact observables, or possibly replacing \mathcal{B} by the Calkin algebra $\mathbb{C}\text{Id}(\mathcal{Z}) \oplus \mathcal{L}^\infty(\mathcal{Z})$, is useful in order to control the asymptotic total mass.*

Proof. For $\tilde{b}_1, \dots, \tilde{b}_p \in \mathcal{L}(\mathcal{Z})$, we decompose $\mathcal{S}_\pm^p(\tilde{b}_1 \otimes \cdots \otimes \tilde{b}_p) \mathcal{S}_\pm^{p,*} \mathcal{S}_\pm^p(\psi_1 \otimes \cdots \otimes \psi_p)$ as

$$\begin{aligned} & \mathcal{S}_\pm^p \left[\frac{1}{p!} \sum_{\sigma' \in \mathfrak{S}_p} s_\pm(\sigma') (\tilde{b}_1 \psi_{\sigma'(1)}) \otimes \cdots \otimes (\tilde{b}_p \psi_{\sigma'(p)}) \right] \\ &= \frac{1}{p! p!} \left[\sum_{\sigma \in \mathfrak{S}_p} \sum_{\sigma' \in \mathfrak{S}_p} s_\pm(\sigma) s_\pm(\sigma') (\tilde{b}_{\sigma(1)} \psi_{\sigma \circ \sigma'(1)}) \otimes \cdots \otimes (\tilde{b}_{\sigma(p)} \psi_{\sigma \circ \sigma'(p)}) \right]. \end{aligned}$$

Setting $\sigma'' = \sigma \circ \sigma'$, with $s_{\pm}(\sigma'') = s_{\pm}(\sigma)s_{\pm}(\sigma')$ yields the Eq. (5), after noting that $\tilde{b}_1 \odot \cdots \odot \tilde{b}_p = \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} \tilde{b}_{\sigma(1)} \otimes \cdots \otimes \tilde{b}_{\sigma(p)}$ commutes with \mathcal{S}_{\pm}^p in both the bosonic case and the fermionic case. Now the non normalized reduced density matrix is determined by

$$\mathrm{Tr} \left[\gamma_{\varepsilon}^{(p)} \tilde{B} \right] = \mathrm{Tr} \left[\varrho_{\varepsilon} \tilde{B}^{Wick} \right]$$

for $\tilde{B} \in \mathcal{L}^{\infty}(\mathcal{S}_{\pm}^p \mathcal{Z}^{\otimes p})$ as $\mathcal{L}^1(\mathcal{S}_{\pm}^p \mathcal{Z}^{\otimes p})$ is the dual of $\mathcal{L}^{\infty}(\mathcal{S}_{\pm}^p \mathcal{Z}^{\otimes p})$. But $\tilde{B} \in \mathcal{L}^{\infty}(\mathcal{S}_{\pm}^p \mathcal{Z}^{\otimes p})$ means $\tilde{B} = \mathcal{S}_{\pm}^p \tilde{B}' \mathcal{S}_{\pm}^{p,*}$ with $\tilde{B}' \in \mathcal{L}^{\infty}(\mathcal{Z}^{\otimes p})$, while the algebraic tensor product $\mathcal{L}^{\infty}(\mathcal{Z})^{\otimes alg p}$ is dense in $\mathcal{L}^{\infty}(\mathcal{Z}^{\otimes p})$.

With the estimate

$$\begin{aligned} \left| \mathrm{Tr} \left[\varrho_{\varepsilon} \tilde{B}^{Wick} \right] \right| &= \left| \mathrm{Tr} \left[e^{c/2\mathbf{N}} \varrho_{\varepsilon} e^{c/2\mathbf{N}} e^{-c/2\mathbf{N}} \tilde{B}^{Wick} e^{-c/2\mathbf{N}} \right] \right| \\ &\leq C \mathrm{Tr} \left[\varrho_{\varepsilon} e^{c\mathbf{N}} \right] \|\tilde{B}'\|_{\mathcal{L}(\mathcal{S}_{\pm}^p \mathcal{Z}^{\otimes p}; \mathcal{S}_{\pm}^p \mathcal{Z}^{\otimes p})}, \end{aligned}$$

it suffices to consider $\tilde{B} = \mathcal{S}_{\pm}^p \tilde{B}' \mathcal{S}_{\pm}^{p,*}$ with $\tilde{B}' \in \mathcal{L}^{\infty}(\mathcal{Z})^{\otimes alg p}$. By linearity and density, $\gamma_{\varepsilon}^{(p)}$ is determined by the quantities $\mathrm{Tr}[\varrho_{\varepsilon} \tilde{B}^{Wick}]$ with $\tilde{B}' = \tilde{b}_1 \otimes \cdots \otimes \tilde{b}_p$, $\tilde{b}_i \in \mathcal{B}$. We conclude with

$$\mathcal{S}_{\pm}^p \left(\tilde{b}_1 \otimes \cdots \otimes \tilde{b}_p \right) \mathcal{S}_{\pm}^{p,*} = \mathcal{S}_{\pm}^p \left(\tilde{b}_1 \odot \cdots \odot \tilde{b}_p \right) \mathcal{S}_{\pm}^{p,*},$$

and the polarization identity

$$\tilde{b}_1 \odot \cdots \odot \tilde{b}_p = \frac{1}{2^p p!} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \cdots \varepsilon_p \left(\sum_{i=1}^p \varepsilon_i \tilde{b}_i \right)^{\otimes p}.$$

□

Remark 2.10. *In the bosonic case, the non normalized reduced density matrices $\gamma_{\varepsilon}^{(p)}$ are also characterized by the values of $\mathrm{Tr}[\gamma_{\varepsilon}^{(p)} B]$ for B in $\mathcal{B} = \{|\psi^{\otimes p}\rangle\langle\psi^{\otimes p}|, \psi \in \mathcal{Z}\} \cup \{\mathrm{Id}_{\mathcal{Z}}^{\otimes p}\}$. This does not hold in the fermionic case.*

The rest of the article is devoted to the asymptotic analysis of $\gamma_{\varepsilon}^{(p)}$ as $\varepsilon \rightarrow 0$. In particular we shall study their concentration at the quantum level while testing with fixed observable \tilde{b} (with \tilde{b} compact) and their semiclassical behaviour after taking semiclassically quantized observables, e.g. $a(x, hD_x)$ with some relation $\varepsilon = \varepsilon(h)$ between ε and h .

3 Classical phase-space and h -quantizations

When $\mathcal{Z} = L^2(M^1, dx)$, with $M^1 = M$ a manifold with volume measure dx , the classical one particle phase-space is $\mathcal{X}^1 = \mathcal{X} = T^*M^1$ and we will focus on the h -dependent quantizations which associates with a symbol $a(x, \xi) = a(X)$, $X \in \mathcal{X}^1$ an operator $a^{Q,h} = a(x, hD_x)$ with the standard semiclassical quantization or when $M^1 = \mathbb{R}^d$, $a^{Q,h} = a^{W,h} = a^W(h^t x, h^{1-t} D_x)$ by using the Weyl quantization, $t \in \mathbb{R}$ being fixed.

Note that in later sections the parameters ε and h will be linked through $\varepsilon = \varepsilon(h)$ with $\lim_{h \rightarrow 0} \varepsilon(h) = 0$. In relation with the symmetrization Lemma 2.8, we introduce the adapted p -particles phase-space which was also considered in [Der1], and the corresponding semiclassical observables.

3.1 Classical p -particles phase-space

A fundamental principle of quantum mechanics is that identical particles are indistinguishable. The classical description is thus concerned with indistinguishable classical particles. If one classical particle is characterized by its position-momentum $(x, \xi) \in \mathcal{X}^1 = T^*M^1$, $x \in M$ being the position coordinate and ξ the momentum coordinates, p indistinguishable particles will be described by their position-momentum coordinates $(X_1, \dots, X_p) = (x_1, \xi_1, \dots, x_p, \xi_p) \in \mathcal{X}^p/\mathfrak{S}_p = (T^*M)^p/\mathfrak{S}_p = T^*(M^p)/\mathfrak{S}_p$, where the quotient by \mathfrak{S}_p simply implements the identification

$$\forall \sigma \in \mathfrak{S}_p, \quad (X_{\sigma(1)}, \dots, X_{\sigma(p)}) \equiv (X_1, \dots, X_p).$$

The grand canonical description of a classical particles system then takes place in the disjoint union

$$\bigsqcup_{p \in \mathbb{N}} \mathcal{X}^p/\mathfrak{S}_p = \bigsqcup_{p \in \mathbb{N}} (T^*M)^p/\mathfrak{S}_p.$$

A p -particles classical observable will be a function on $\mathcal{X}^p/\mathfrak{S}_p$ and when the number of particles is not fixed a collection of functions $(a^{(p)})_{p \in \mathbb{N}}$ each $a^{(p)}$ being a function on $\mathcal{X}^p/\mathfrak{S}_p$. The situation is presented in this way in [Der1]. A p -particles observable is a function $a^{(p)}$ on $\mathcal{X}^p/\mathfrak{S}_p$ and a p -particles classical state is a probability measure (and when the normalization is forgotten a non negative measure) on $\mathcal{X}^p/\mathfrak{S}_p$.

However while quantizing a classical observable, it is better to work in \mathcal{X}^p which equals $T^*(M^p)$, a function $a^{(p)}$ on $\mathcal{X}^p/\mathfrak{S}_p$ being nothing but a function on \mathcal{X}^p which satisfies

$$\forall \sigma \in \mathfrak{S}_p, \quad \sigma^* a^{(p)} = a^{(p)} = \frac{1}{p!} \sum_{\tilde{\sigma} \in \mathfrak{S}_p} \tilde{\sigma}^* a^{(p)},$$

$$\text{with} \quad \forall X_1 \dots X_p \in \mathcal{X}^p, \quad \sigma^* a^{(p)}(X_1, \dots, X_p) = a^{(p)}(X_{\sigma(1)}, \dots, X_{\sigma(p)}).$$

In the same way, we define for a Borel measure ν on \mathcal{X}^p and $\sigma \in \mathfrak{S}_p$, the measure $\sigma_*\nu$ by $\int_X \sigma^* a^{(p)} d\nu = \int_{\mathcal{X}^p} a^{(p)} d(\sigma_*\nu)$ for all $a^{(p)} \in \mathcal{C}_c^0(\mathcal{X}^p)$, or alternatively $\sigma_*\nu(E) = \nu(\sigma^{-1}E)$ for all Borel subset E of \mathcal{X}^p . A non-negative measure on $\mathcal{X}^p/\mathfrak{S}_p$ is identified with a non-negative measure ν on \mathcal{X}^p such that

$$\forall \sigma \in \mathfrak{S}_p, \quad \sigma_*\nu = \nu = \frac{1}{p!} \sum_{\tilde{\sigma} \in \mathfrak{S}_p} \tilde{\sigma}_*\nu. \quad (6)$$

Lemma 3.1 (Classical symmetrization lemma). *Any Borel measure $\mu^{(p)}$ on $\mathcal{X}^p/\mathfrak{S}_p$ is characterized by the quantities $\{\int_{\mathcal{X}^p} a^{\otimes p} d\mu^{(p)}, \quad a \in \mathcal{C}\}$ where the tensor power $a^{\otimes p}$ means $a^{\otimes p}(X_1, \dots, X_p) = \prod_{i=1}^p a(X_i)$ and \mathcal{C} is any dense set in $\mathcal{C}_\infty^0(\mathcal{X}^1) = \{f \in \mathcal{C}^0(\mathcal{X}^1), \lim_{X \rightarrow \infty} f(X) = 0\}$.*

Proof. By Stone-Weierstrass Theorem the subalgebra generated by the algebraic tensor product $\mathcal{C}^{\otimes \text{alg } p}$ is dense in $\mathcal{C}_\infty^0(\mathcal{X}^p)$. Hence it suffices to consider

$$a_1 \odot \dots \odot a_p = \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} a_{\sigma(1)} \otimes \dots \otimes a_{\sigma(p)}, \quad a_i \in \mathcal{C}.$$

We conclude again with the polarization identity

$$a_1 \odot \dots \odot a_p = \frac{1}{2^p p!} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \dots \varepsilon_p \left(\sum_{i=1}^p \varepsilon_i a_i \right)^{\otimes p}.$$

□

We will work essentially with $M = \mathbb{R}^d$ and $\mathcal{X} = T^*\mathbb{R}^d$ and therefore on $\mathcal{X}^p = T^*\mathbb{R}^{dp} \sim \mathbb{R}^{2dp}$ and recall the invariance properties, if possible by a change of variable in order to extend it to the general case. Remember that on \mathbb{R}^{dp} , the standard and Weyl semiclassical quantization are asymptotically equivalent $a(x, hD_x) - a^W(x, hD_x) = O(h)$ when $a \in S(1, dX^2)$ ($\sup_{X \in T^*\mathbb{R}^{dp}} |\partial_X^\alpha a(X)| < \infty$ for all $\alpha \in \mathbb{N}^{2d}$). Moreover on \mathbb{R}^{dp} , $a^W(x, hD_x)$ is unitary equivalent to $a^W(h^t x, h^{1-t} D_x)$ for any fixed $t \in \mathbb{R}$ so that result can be adapted to different scalings.

3.2 Semiclassical and multiscale measures

We recall the notions of semiclassical (or Wigner) measures and multiscale measures in the finite dimensional case. We start with the results on $M = \mathbb{R}^D$ (think of $D = dp$) and review the invariance properties for applications to some more general manifolds M .

3.2.1 In the Euclidean Space

On \mathbb{R}^D the semiclassical Weyl quantization of a symbol $a \in \mathcal{S}'(\mathbb{R}^{2D})$ will be written $a^{W,h} = a^W(h^t x, h^{1-t} D_x)$ with $t > 0$ fixed and a kernel given by

$$[a^W(x, D_x)](x, y) = \int_{\mathbb{R}^d} e^{i\xi \cdot (x-y)} a\left(\frac{x+y}{2}, \xi\right) \frac{d\xi}{(2\pi)^d}.$$

Definition 3.2. Let $(\gamma_h)_{h \in \mathcal{E}}$ with $0 \in \bar{\mathcal{E}}$, $\mathcal{E} \subset (0, +\infty)$, be a family of trace-class non-negative operators on $L^2(\mathbb{R}^D)$ such that $\lim_{h \rightarrow 0} \text{Tr}[\gamma_h] < +\infty$. The semiclassical quantization $a \mapsto a^{W,h} = a^W(h^t x, h^{1-t} D_x)$ is said adapted to the family $(\gamma_h)_{h \in \mathcal{E}}$ if

$$\lim_{\delta \rightarrow 0^+} \limsup_{h \in \mathcal{E}, h \rightarrow 0} \text{Re} \text{Tr} \left[(1 - \chi(\delta \cdot)^{W,h}) \gamma_h \right] = 0$$

for some $\chi \in C_0^\infty(T^*\mathbb{R}^D)$ such that $\chi \equiv 1$ in a neighborhood of 0.

The set of Wigner measures $\mathcal{M}(\gamma_h, h \in \mathcal{E})$ is the set of non-negative measures ν on $T^*\mathbb{R}^D$ such that there exists $\mathcal{E}' \subset \mathcal{E}$, $0 \in \bar{\mathcal{E}'}$ such that

$$\forall a \in C_0^\infty(T^*\mathbb{R}^D), \quad \lim_{\substack{h \in \mathcal{E}' \\ h \rightarrow 0}} \text{Tr} \left[\gamma_h a^{W,h} \right] = \int_{T^*\mathbb{R}^D} a(X) d\nu(X).$$

The following well known statement (see [CdV, HMR, Ger, GMMP, LiPa, Sch]) results from the asymptotic positivity of the semiclassical quantization and it is actually the finite dimensional version of bosonic mean-field Wigner measures (with the change of parameter $\varepsilon = 2h$) (see [AmNi1]–Section 3.1).

Proposition 3.3. Let $(\gamma_h)_{h \in \mathcal{E}}$ with $0 \in \bar{\mathcal{E}}$, $\mathcal{E} \subset (0, +\infty)$, such that $\gamma_h \geq 0$ and $\lim_{h \rightarrow 0} \text{Tr}[\gamma_h] < +\infty$. The set of semiclassical measures $\mathcal{M}(\gamma_h, h \in \mathcal{E})$ is non-empty. The semiclassical quantization $a^{W,h}$ is adapted to the family $(\gamma_h)_{h \in \mathcal{E}}$, if, and only if, any $\nu \in \mathcal{M}(\gamma_h, h \in \mathcal{E})$ satisfies $\nu(\mathbb{R}^{2D}) = \lim_{h \rightarrow 0} \text{Tr}[\gamma_h]$.

Remark 3.4. The manifold version, with $a^{Q,h} = a(x, hD_x)$ instead of $a^{W,h}$, results from the semiclassical Egorov theorem.

By reducing \mathcal{E} to some subset \mathcal{E}' (think of subsequence extraction), one can always assume that there is a unique semiclassical measure. While considering a time evolution problem, or adding

another non countable parameter, $(\gamma_{t,h})_{h \in \mathcal{E}, t \in \mathbb{R}}$ finding simultaneously the subset \mathcal{E}' for all $t \in \mathbb{R}$ requires some compactness argument w.r.t the parameter $t \in \mathbb{R}$, usually obtained by equicontinuity properties.

We now review the multiscale measures introduced in [FeGe, Fer]. For reader's convenience, details are given in Appendix A, concerning the relationship between Prop. 3.5 below and the more general statement of [Fer].

The class of symbols $S^{(2)}$ is defined as the set of $a \in C^\infty(\mathbb{R}^{2D} \times \mathbb{R}^{2D})$, such that

- there exists $C > 0$ such that $\forall Y \in \mathbb{R}^{2D}$, $a(\cdot, Y) \in C_0^\infty(B(0, C))$;
- there exists a function $a_\infty \in C_0^\infty(\mathbb{R}^{2D} \times \mathbb{S}^{2D-1})$ such that $a(X, R\omega) \xrightarrow{R \rightarrow \infty} a_\infty(X, \omega)$ in $C^\infty(\mathbb{R}^{2D} \times \mathbb{S}^{2D-1})$.

Those symbols are quantized according to

$$a^{(2),h} = a_h^{W,h} \quad , \quad a_h(X) = a\left(X, \frac{X}{h^{1/2}}\right).$$

A geometrical interpretation of those double scale symbols can be given by matching the compactified quantum phase space with the blow-up at $r = 0$ of the macroscopic phase space, see Figure 1.

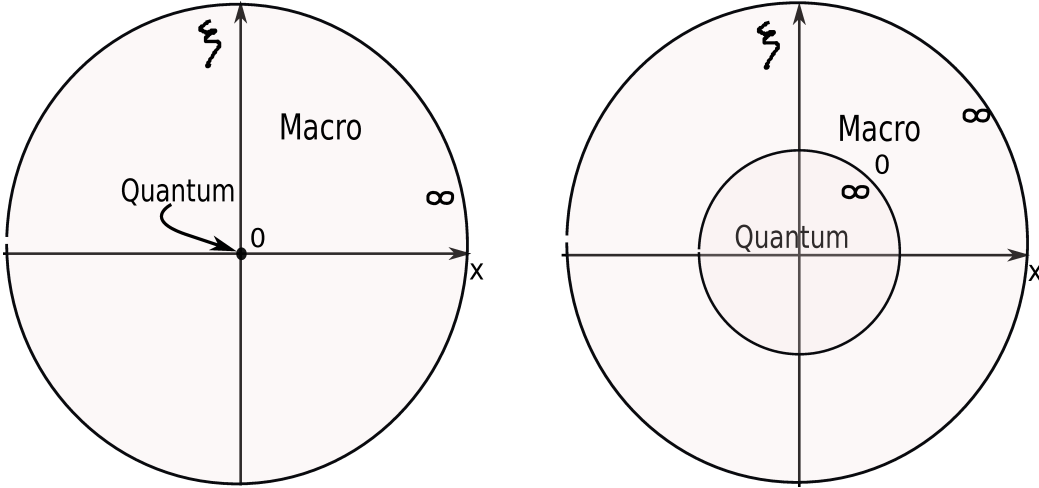


Figure 1: On the left-hand side, the macroscopic phase space with its sphere at infinity. On the right-hand side the matched quantum and macroscopic phase spaces for which the quantum sphere at infinity and the $r = 0$ macroscopic sphere coincide.

Proposition 3.5. *Let $(\gamma_h)_{h \in \mathcal{E}}$ be a bounded family of non-negative trace-class operators on $L^2(\mathbb{R}^D)$ with $\lim_{h \rightarrow 0} \text{Tr} [\gamma_h] < +\infty$. There exist $\mathcal{E}' \subset \mathcal{E}$, $0 \in \overline{\mathcal{E}'}$, non-negative measures ν and $\nu_{(I)}$ on \mathbb{R}^{2D} and \mathbb{S}^{2D-1} , and a $\gamma_0 \in \mathcal{L}^1(L^2(\mathbb{R}^D))$, such that $\mathcal{M}(\gamma_h, h \in \mathcal{E}') = \{\nu\}$ and, for all $a \in S^{(2)}$,*

$$\lim_{\substack{h \in \mathcal{E}' \\ h \rightarrow 0}} \text{Tr} [\gamma_h a^{(2),h}] = \int_{\mathbb{R}^{2D} \setminus \{0\}} a_\infty\left(X, \frac{X}{|X|}\right) d\nu(X) \\ + \int_{\mathbb{S}^{2D-1}} a_\infty(0, \omega) d\nu_{(I)}(\omega) + \text{Tr} [a(0, x, D_x) \gamma_0] .$$

Definition 3.6. $\mathcal{M}^{(2)}(\gamma_h, h \in \mathcal{E})$ denotes the set of all triples $(\nu, \nu_{(I)}, \gamma_0)$ which can be obtained in Prop. 3.5 for suitable choices of $\mathcal{E}' \subset \mathcal{E}$, $0 \in \overline{\mathcal{E}'}$.

Remark 3.7. Actually when $a^{W,h} = a^W(\sqrt{\hbar}x, \sqrt{\hbar}D_x)$, this trace class operator γ_0 is nothing but the weak*-limit of γ_h . Take simply $\tilde{a}(X, Y) = \chi(X)\alpha(Y)$ with $\chi, \alpha \in \mathcal{C}_0^\infty(\mathbb{R}^{2D})$, $\chi \equiv 1$ in a neighborhood of 0, for which $\lim_{h \rightarrow 0} \|\tilde{a}^{(2),h} - \alpha^W(x, D_x)\|_{\mathcal{L}(L^2)} = 0$. The above results says $\lim_{h \rightarrow 0} \text{Tr} [\gamma_h \alpha^W(x, D_x)] = \text{Tr} [\gamma_0 \alpha^W(x, D_x)]$ for all $\alpha \in \mathcal{C}_0^\infty(\mathbb{R}^{2D}) \subset L^2(\mathbb{R}^{2D}, dX)$, and by the density of the embeddings $\mathcal{C}_0^\infty(\mathbb{R}^{2D}) \subset L^2(\mathbb{R}^{2D}, dx) \sim \mathcal{L}^2(L^2(\mathbb{R}^D)) \subset \mathcal{L}^\infty(L^2(\mathbb{R}^D))$, the test observable $\alpha^W(x, D_x)$, can be replaced by any compact operator $K \in \mathcal{L}^\infty(L^2(\mathbb{R}^D, dx))$. Moreover the relationship between ν and the triple $(1_{(0,+\infty)}(|X|)\nu, \nu_{(I)}, \gamma_0)$ can be completed in this case by

$$\nu(\{0\}) = \int_{\mathbb{S}^{2D-1}} d\nu_{(I)}(\omega) + \text{Tr} [\gamma_0], \quad (7)$$

and $\nu_{(I)} \equiv 0$ is equivalent to $\nu(\{0\}) = \text{Tr} [\gamma_0]$.

Because products of spheres are not spheres, handling the par $\nu_{(I)}$ in the p -particles space, $D = dp$, is not straightforward within a tensorization procedure, see Figure 2. Actually we expect

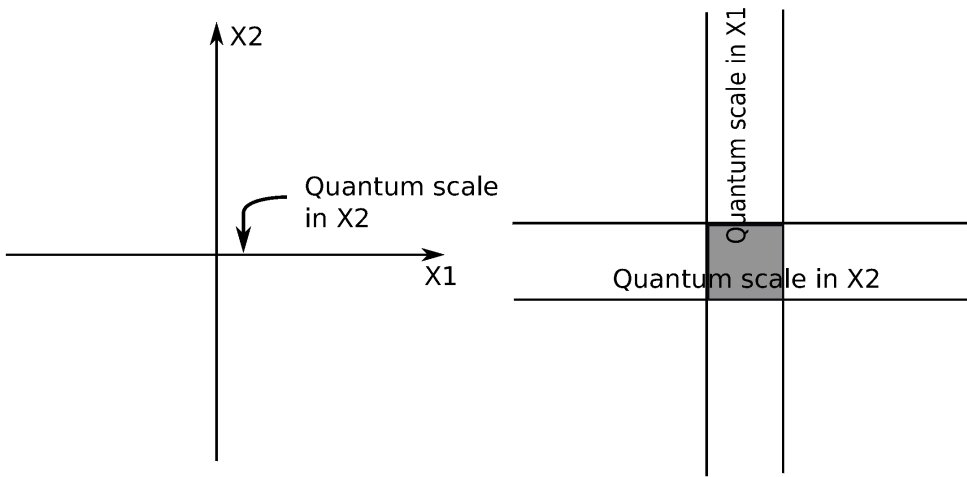


Figure 2: Tensor product of two blow-ups. The product of the two matching spheres is not a sphere: the corners of the grey square correspond to the case when the quantum variables $|X_1|$ and $|X_2|$ go to infinity without any proportionality rule.

in the applications that a well chosen quantization leads to $\nu_{(I)} = 0$. This leads to the following definition.

Definition 3.8. Assume that the quantization $a^{W,h} = a^W(\sqrt{\hbar}x, \sqrt{\hbar}D_x)$ is adapted to the family $(\gamma_h)_{h \in \mathcal{E}}$, $\gamma_h \geq 0$, $\text{Tr} [\gamma_h] = 1$. We say that the quantization $a^{W,h} = a^W(\sqrt{\hbar}x, \sqrt{\hbar}D_x)$ is separating for the family $(\gamma_h)_{h \in \mathcal{E}}$ if one of the three following (equivalent) conditions is satisfied:

1. For any triple $(\nu, \nu_{(I)}, \gamma_0) \in \mathcal{M}^{(2)}(\gamma_h, h \in \mathcal{E})$, $\nu_{(I)} = 0$.

$$2. \left. \begin{array}{l} \mathcal{M}(\gamma_h, h \in \mathcal{E}') = \{\nu\}, \\ \text{w}^* - \lim_{h \in \mathcal{E}', h \rightarrow 0} \gamma_h = \gamma_0 \text{ in } \mathcal{L}^1(L^2(\mathbb{R}^D)) \end{array} \right\} \Rightarrow \nu(\{0\}) = \text{Tr} [\gamma_0].$$

3. For any triple $(\nu, \nu_{(I)}, \gamma_0) \in \mathcal{M}^{(2)}(\gamma_h, h \in \mathcal{E})$, $\nu(\{0\}) = \text{Tr}[\gamma_0]$.

Remark 3.9. *This terminology expresses the fact that the mass localized at any intermediate scale vanishes asymptotically when $\nu_{(I)} \equiv 0$. Accordingly, the microscopic quantum scale and the macroscopic scale are well identified and separated.*

Hence we can get all the information by computing the weak*-limit of γ_h and the semiclassical measure ν and then by checking a posteriori the equality $\nu(\{0\}) = \text{Tr}[\gamma_0]$.

This will suffice when the quantum part corresponds within a macroscopic scale, to a point in the phase-space. When $M = \mathbb{R}^d$, we have enough flexibility by choosing the small parameter $h > 0$ and using some dilation in \mathbb{R}^D in order to reduce many problems to such a case. On a manifold M if we can first localize the analysis around a point $x_0 \in M$, the problem can be transferred to \mathbb{R}^D and then analyzed with the suitable scaling.

3.2.2 On a Compact Manifold

We now consider another interesting case of a compact manifold M with the semiclassical calculus $a^{Q,h} = a(x, hD_x)$. This case is not completely treated in [Fer] because the geometric invariance properties do not follow only from the microlocal equivariance of semiclassical calculus. We assume $\mathcal{L} = L^2(M, dx)$ to be defined globally on the compact manifold M (e.g. by introducing a metric, dx being the associated volume measure).

Remark 3.10. *When M is a general manifold, replace $a^{W,h}$ in Def. 3.2 by $a^{Q,h} = a(x, hD_x)$, and $\chi(\delta \cdot)$ with $\delta \rightarrow 0$ by some increasing sequence of compactly supported cut-off functions $(\chi_n)_{n \in \mathbb{N}}$, such that $\cup_{n \in \mathbb{N}} \chi_n^{-1}(\{1\}) = T^*M$.*

To adapt Prop. 3.5 to the case of a compact manifold, we consider another notion instead of the symbols $S^{(2)}$. For the observables we shall consider the pair (K, a) where $K \in \mathcal{L}^\infty(L^2(M, dx))$ and $a \in \mathcal{C}_0^\infty(S^*M \sqcup (T^*M \setminus M))$ with $S^*M \sqcup (T^*M \setminus M)$ being described in local coordinates through the identification

$$M \times [0, \infty) \times \mathbb{S}^{D-1} \ni (x, r, \omega) \mapsto \begin{cases} (x, \omega) \in S^*M & \text{if } r = 0, \\ (x, \xi = r\omega) \in T^*M \setminus M & \text{otherwise.} \end{cases}$$

We have identified the 0-section of the cotangent bundle T^*M with M . After introducing an additional parameter $\delta > 0$, $\delta \geq h$, and a \mathcal{C}^∞ partition of unity $(1 - \chi) + \chi \equiv 1$ on T^*M with $1 - \chi \in \mathcal{C}_0^\infty(T^*M)$, $1 - \chi \equiv 1$ in a neighborhood of M , we can quantize a as

$$a^{(2)Q,\delta,h} = [\chi(x, \xi)a(x, h\delta^{-1}\xi)]^{Q,\delta}.$$

Note that K and the quantization of a are geometrically defined modulo $\mathcal{O}(\delta)$ when $h \leq \delta$ in $\mathcal{L}(L^2(M, dx))$: Use local charts for the semiclassical calculus with parameter δ while $\mathcal{L}^\infty(L^2(M, dx))$ is globally defined like all natural spaces associated with $L^2(M, dx)$. Actually in local coordinates the seminorms of the symbol $\chi(x, \xi)a(x, h\delta^{-1}\xi)$ in $S(1, dx^2 + d\xi^2)$ are uniformly bounded w.r.t. $h \in (0, \delta]$ by seminorms of a in $\mathcal{C}_0^\infty((T^*M \setminus M) \sqcup S^*M)$. When $a \geq 0$ one also has

$$\|(\chi a(\cdot, h\delta^{-1}\cdot))^{Q,\delta} - \text{Re} \left[(\chi a(\cdot, h\delta^{-1}\cdot))^{Q,\delta} \right]\| \leq C_a \delta \quad (8)$$

$$\|a\|_{L^\infty} + C_a \delta \geq \text{Re} \left[(\chi a(\cdot, h\delta^{-1}\cdot))^{Q,\delta} \right] \geq -C_a \delta \quad (9)$$

uniformly w.r.t to $h \in (0, \delta]$.

Proposition 3.11. *Let $(\gamma_h)_{h \in \mathcal{E}}$ be a family of non-negative trace class operators on $L^2(M, dx)$, such that $\lim_{h \rightarrow 0} \text{Tr} [\gamma_h] < +\infty$. Then there exist $\mathcal{E}' \subset \mathcal{E}$, $0 \in \overline{\mathcal{E}'}$, with $\mathcal{M}(\gamma_h, h \in \mathcal{E}') = \{\nu\}$, a non-negative measure $\nu_{(I)}$ on S^*M and a non-negative $\gamma_0 \in \mathcal{L}^1(L^2(M, dx))$ such that, for any $K \in \mathcal{L}^\infty(L^2(M, dx))$,*

$$\lim_{\substack{h \in \mathcal{E}' \\ h \rightarrow 0}} \text{Tr} [\gamma_h K] = \text{Tr} [\gamma_0 K]$$

and, for any $a \in \mathcal{C}_0^\infty(S^*M \sqcup (T^*M \setminus M))$, and any partition of unity $(1 - \chi) + \chi \equiv 1$ with $1 - \chi \in \mathcal{C}_0^\infty(T^*M)$, $1 - \chi \equiv 1$ in a neighborhood of M ,

$$\lim_{\delta \rightarrow 0} \lim_{\substack{h \in \mathcal{E}' \\ h \rightarrow 0}} \text{Tr} \left[\gamma_h a^{(2)Q, \delta, h} \right] = \int_{T^*M \setminus M} a(X) d\nu(X) + \int_{S^*M} a(X) d\nu_{(I)}(X).$$

Additionally $(\nu_{(I)}, \gamma_0)$ is related to ν by

$$\nu(E) = \nu_{(I)}(\pi^{-1}(E)) + \nu_0(E),$$

for any Borel set $E \subset M$ identified with $E \times \{0\}$, when $\pi : S^*M \rightarrow M$ is the natural projection and ν_0 is defined by $\int_M \varphi(x) d\nu_0(x) = \text{Tr} [\gamma_0 \varphi]$, where $\varphi \in \mathcal{C}^\infty(M)$ is identified with the multiplication operator by the function φ .

Proof. When γ_h is bounded in $\mathcal{L}^1(L^2(M, dx))$, after extraction of a sequence $h_n \rightarrow 0$ from \mathcal{E} , $\mathcal{M}((\gamma_{h_n})_{n \in \mathbb{N}}) = \{\nu\}$, and the weak* limit γ_0 of (γ_{h_n}) , and the associated measure ν_0 are well-defined objects on the manifold M .

Let us construct a measure $\tilde{\nu}$ on

$$(T^*M \setminus M) \sqcup S^*M = \left\{ (x, r\omega), x \in M, \omega \in S^{d-1}, r \in [0, \infty) \right\}$$

and a subset $\mathcal{E}' \subset \mathcal{E}$, $0 \in \overline{\mathcal{E}'}$, such that

$$\lim_{\delta \rightarrow 0} \lim_{\substack{h \in \mathcal{E}' \\ h \rightarrow 0}} \text{Tr} \left[\gamma_h (\chi a(\cdot, h\delta^{-1}\cdot))^{Q, \delta} \right] = \int_{(T^*M \setminus M) \sqcup S^*M} a d\tilde{\nu} \quad (10)$$

holds for all $a \in \mathcal{C}_0^\infty((T^*M \setminus M) \sqcup S^*M)$.

Fix first the partition of unity $(1 - \chi) + \chi \equiv 1$, $1 - \chi \in \mathcal{C}_0^\infty(T^*M)$, $1 - \chi \equiv 1$ in a neighborhood of M , and $\delta = \delta_0 > 0$. For a given $a \in \mathcal{C}_0^\infty((T^*M \setminus M) \sqcup S^*M)$, the inequalities (8) and (9) imply that one can find a subsequence $(h_{k, \chi, \delta_0, a})_{k \in \mathbb{N}}$ of $(h_n)_{n \in \mathbb{N}}$, such that

$$\lim_{k \rightarrow \infty} \text{Tr} \left[\gamma_{h_{k, \chi, \delta_0, a}} (\chi a(\cdot, h_{k, \chi, \delta_0, a} \delta_0^{-1} \cdot))^{Q, \delta_0} \right] = \ell_{\chi, \delta_0, a} \in \mathbb{C}. \quad (11)$$

For a different partition of unity $(1 - \tilde{\chi}) + \tilde{\chi} \equiv 1$ the symbol $[\chi - \tilde{\chi}]a(x, h\delta_0^{-1}\xi)$ is supported in $C_{\chi, \tilde{\chi}, \delta_0}^{-1} \leq |\xi| \leq C_{\chi, \tilde{\chi}, \delta_0}$ and equals

$$[\chi - \tilde{\chi}]a(x, h\delta_0^{-1}\xi) = [\chi - \tilde{\chi}]a_0(x, \frac{\xi}{|\xi|}) + hr_{\chi, \tilde{\chi}, \delta_0, h}(x, \xi),$$

where $a_0 = a|_{S^*M}$ and with $r_{\chi, \tilde{\chi}, \delta_0, h}$ uniformly bounded in $S(1, dx^2 + d\xi^2)$. For $\delta_0 > 0$ fixed, the operator $[(\chi - \tilde{\chi})a_0]^{Q, \delta_0}$ is a compact operator and we obtain

$$\lim_{h \rightarrow 0} \text{Tr} \left[\gamma_h (\chi a(\cdot, h\delta_0^{-1}\cdot))^{Q, \delta_0} \right] - \text{Tr} \left[\gamma_h (\tilde{\chi} a(\cdot, h\delta_0^{-1}\cdot))^{Q, \delta_0} \right] = \text{Tr} \left[\gamma_0 ((\chi - \tilde{\chi})a_0)^{Q, \delta_0} \right].$$

Therefore the subsequence extraction, which ensures the convergence (11) can be done independently of the choice of $\tilde{\chi}$ and by taking $\tilde{\chi}(x, \xi) = \chi(x, \delta\delta_0^{-1}\xi)$ independently of $\delta > 0$. For $\mathcal{E}_a = (h_{k,a})_{k \in \mathbb{N}}$ such a sequence of parameters, the limits can be compared by

$$\begin{aligned} \ell_{\tilde{\chi}, \delta, a} - \ell_{\chi, \delta_0, a} &= \lim_{\substack{h \in \mathcal{E}_a \\ h \rightarrow 0}} \text{Tr} \left[(\gamma_h \tilde{\chi} a(\cdot, h\delta^{-1}\cdot))^{Q, \delta} \right] - \text{Tr} \left[(\gamma_h \chi a(\cdot, h\delta_0^{-1}\cdot))^{Q, \delta_0} \right] \\ &= \text{Tr} \left[((\tilde{\chi}(\delta\delta_0^{-1}) - \chi)a_0)^{Q, \delta_0} \gamma_0 \right]. \end{aligned} \quad (12)$$

By choosing $\tilde{\chi} = \chi$ above, the inequality $0 \leq (\chi - \chi(\delta\delta_0^{-1}))a_0 \leq \chi a_0$, for $a_0 \geq 0$ and $\delta \leq \delta_0$, and the δ_0 -Garding inequality implies

$$|\text{Tr} \left[((\tilde{\chi}(\delta\delta_0^{-1}) - \chi)a_0)^{Q, \delta_0} \gamma_0 \right]| \leq \text{Tr}[(\chi a_0)^{Q, \delta_0} \gamma_0] + \mathcal{O}(\delta_0)$$

uniformly with respect to $\delta \leq \delta_0$. Thus the quantity $\ell_{\chi, \delta, a}$ thus satisfies the Cauchy criterion as $\delta \rightarrow 0$ because $\lim_{\delta \rightarrow 0} (\chi a_0)^{Q, \delta_0} = 0$ and γ_0 is fixed in $\mathcal{L}^1(L^2(M, dx))$. Hence the limit

$$\ell_{\chi, a} = \lim_{\delta \rightarrow 0} \ell_{\chi, \delta, a} = \lim_{\delta \rightarrow 0} \lim_{\substack{h \in \mathcal{E}_a \\ h \rightarrow 0}} \text{Tr} \left[(\gamma_h \chi a(\cdot, h\delta^{-1}\cdot))^{Q, \delta} \right]$$

exists for any fixed $a \in \mathcal{C}_0^\infty((T^*M \setminus M) \sqcup S^*M)$. Using (12) with $\delta = \delta_0$ but a general pair $(\chi, \tilde{\chi})$ and taking the limit as $\delta \rightarrow 0$ shows $\ell_{\tilde{\chi}, a} = \ell_{\chi, a} = \ell_a$. The inequalities (8) and (9) give $0 \leq \ell_a \leq \|a\|_{L^\infty}$. By the usual diagonal extraction process according to a countable set $\mathcal{N} \subset \mathcal{C}_0^\infty((T^*M \setminus M) \sqcup S^*M)$ dense in the set of continuous functions with limit 0 at infinity, we have found a subset $\mathcal{E}' \subset \mathcal{E}$, $0 \in \overline{\mathcal{E}'}$, and a non-negative measure $\tilde{\nu}$ such that (10) holds. Note that we have also proved

$$\begin{aligned} \int_{(T^*M \setminus M) \sqcup S^*M} a \, d\tilde{\nu} &= \lim_{\delta \rightarrow 0} \lim_{\substack{h \in \mathcal{E}' \\ h \rightarrow 0}} \text{Tr} \left[\gamma_h (\chi a(\cdot, h\delta^{-1}\cdot))^{Q, \delta} \right] \\ &= \lim_{\delta \rightarrow 0} \lim_{\substack{h \in \mathcal{E}' \\ h \rightarrow 0}} \text{Tr} \left[(\gamma_h - \gamma_0)(\chi a(\cdot, h\delta^{-1}\cdot))^{Q, \delta} \right] \end{aligned}$$

where both limits do not depend on the partition of unity $(1 - \chi) + \chi \equiv 1$ with $1 - \chi \in \mathcal{C}_0^\infty(T^*M)$ equal to 1 in a neighborhood of M .

We still have to compare $\tilde{\nu}$ and ν . For this take $a \in \mathcal{C}_0^\infty(T^*M)$ and set $a_0(x, \omega) = \varphi(x) = a(x, 0)$. The symbol identity

$$a(x, h\delta^{-1}\xi) = a(x, h\delta^{-1}\xi)(1 - \chi) + a(x, h\delta^{-1}\xi)\chi = \varphi(x)(1 - \chi) + a(x, h\delta^{-1}\xi)\chi + hr_{a, \chi, \delta, h}$$

with $r_{a, \delta, \chi, h}$ uniformly bounded in $S(1, dx^2 + d\xi^2)$ w.r.t. h , leads after δ -quantization to

$$\begin{aligned} \int_{T^*M} a \, d\nu &= \lim_{\substack{h \in \mathcal{E}' \\ h \rightarrow 0}} \text{Tr} \left[\gamma_h a^{Q, h} \right] \\ &= \lim_{\substack{h \in \mathcal{E}' \\ h \rightarrow 0}} \text{Tr} \left[\gamma_h (\varphi(x)(1 - \chi))^{Q, \delta} \right] + \lim_{\substack{h \in \mathcal{E}' \\ h \rightarrow 0}} \text{Tr} \left[\gamma_h (\chi a(\cdot, h\delta^{-1}\cdot))^{Q, \delta} \right]. \end{aligned}$$

For $\delta > 0$ fixed $(\varphi(x)(1 - \chi))^{Q, \delta}$ is a fixed compact operator so that the first limit is

$$\lim_{\substack{h \in \mathcal{E}' \\ h \rightarrow 0}} \text{Tr} \left[\gamma_h (\varphi(x)(1 - \chi))^{Q, \delta} \right] = \text{Tr} \left[\gamma_0 (\varphi(x)(1 - \chi))^{Q, \delta} \right],$$

while the second one is exactly the quantity occurring in the definition of $\tilde{\nu}$. Taking the limit as $\delta \rightarrow 0$ with $s - \lim_{\delta \rightarrow 0} (\varphi(x)(1 - \chi))^{Q, \delta} = \varphi(x)$, yields $\nu|_{T^*M \setminus M} = \tilde{\nu}|_{T^*M \setminus M}$. Finally setting $\nu_{(I)} = \tilde{\nu}|_{S^*M}$ yields, for any $a \in \mathcal{C}_0^\infty(T^*M)$,

$$\int_{T^*M} a d\nu = \int_{T^*M \setminus M} a d\nu + \int_{S^*M} a_0 d\nu_{(I)} + \int_M \varphi d\nu_0$$

which imply the relation for the measures. \square

Definition 3.12. $\mathcal{M}^{(2)}(\gamma_h, h \in \mathcal{E})$ denotes the set of all triples $(\nu, \nu_{(I)}, \gamma_0)$ which can be obtained in Prop. 3.11 for suitable choices of $\mathcal{E}' \subset \mathcal{E}$, $0 \in \overline{\mathcal{E}'}$.

We note that the equality $\nu(M) = \text{Tr}[\gamma_0]$ implies $\nu_{(I)} \equiv 0$ and this leads like in the previous case to the following definition.

Definition 3.13. On a compact manifold M , assume that the quantization $a^{Q, h} = a(x, hD_x)$ is adapted to the family $(\gamma_h)_{h \in \mathcal{E}}$, with $\gamma_h \in \mathcal{L}^1(L^2(M))$, $\gamma_h \geq 0$ and $\lim_{h \rightarrow 0} \text{Tr}[\gamma_h] < \infty$. We say that the quantization is separating if for any $\mathcal{E}' \subset \mathcal{E}$, $0 \in \overline{\mathcal{E}'}$,

$$\left. \begin{array}{l} \mathcal{M}(\gamma_h, h \in \mathcal{E}') = \{\nu\}, \\ \text{w}^* - \lim_{h \in \mathcal{E}', h \rightarrow 0} \gamma_h = \gamma_0 \text{ in } \mathcal{L}^1(L^2(M)) \end{array} \right\} \Rightarrow \nu(\{\xi = 0\}) = \text{Tr}[\gamma_0].$$

While doing the double scale analysis of the non normalized reduced density matrices $\tilde{\gamma}_h^{(p)}$ especially with the help of tensorization arguments, we will simply study their weak* limit in \mathcal{L}^1 and their semiclassical measures. The equality of Definition 3.8 or Definition 3.13 will be checked a posteriori in order to ensure $\nu_{(I)} \equiv 0$.

4 Mean-field asymptotics with h -dependent observables

We now combine the mean-field asymptotics with semiclassically quantized observables. This means that the parameter ε appearing in CCR (resp. CAR) relations in Section 2 is bound to the semiclassical parameter h of Section 3 parametrizing observables $a^{W, h}$ (or $a^{Q, h}$):

$$\varepsilon = \varepsilon(h) > 0 \quad \text{with} \quad \lim_{h \rightarrow 0} \varepsilon(h) = 0.$$

Firstly, we give a sufficient condition in terms of semiclassical 1-particle observables and of the family $(\varrho_{\varepsilon(h)})_{h \in \mathcal{E}}$ so that a quantization $a^{W, h}$ defined on the p -particles phase-space \mathcal{X}^p is adapted to the non normalized reduced density matrix $\gamma_{\varepsilon(h)}^{(p)}$ for all $p \in \mathbb{N}$. If $\lim_{h \rightarrow 0} \text{Tr}[\gamma_{\varepsilon(h)}^{(p)}] = \lim_{h \rightarrow 0} \text{Tr}[\varrho_{\varepsilon(h)} \mathbf{N}_{\pm}^p] = T^{(p)}$ then the semiclassical measures $\nu^{(p)} \in \mathcal{M}(\gamma_{\varepsilon(h)}^{(p)}, h \in \mathcal{E})$ (or multiscale asymptotic triples $(\nu^{(p)}, \nu_{(I)}^{(p)}, \gamma_0^{(p)})$) have a total mass equal to $T^{(p)}$.

After this, the quantum and classical symmetrization Lemmas 2.8 and 3.1 then provide simple ways to identify the weak limits $\gamma_0^{(p)}$ or the semiclassical measures associated with the family $(\gamma_{\varepsilon(h)}^{(p)})_{h \in \mathcal{E}}$ for all $p \in \mathbb{N}$. According to the discussion in Section 2, about Definitions 3.8 and 3.13, a simple

mass argument allows to check that all the multiscale information has been classified. Remember that the non normalized reduced density matrices $\gamma_{\varepsilon(h)}^{(p)}$ are defined for $h > 0$ by

$$\forall \tilde{b} \in \mathcal{L}(\mathcal{S}_{\pm}^p \mathcal{Z}^{\otimes p}), \quad \text{Tr} \left[\gamma_{\varepsilon(h)}^{(p)} \tilde{b} \right] = \text{Tr} \left[\varrho_{\varepsilon(h)} \tilde{b}^{Wick} \right].$$

They are well defined and uniformly bounded trace-class operators w.r.t $h \in \mathcal{E}$, as soon as $\text{Tr} \left[\varrho_{\varepsilon(h)} \mathbf{N}_{\pm}^p \right]$ is bounded uniformly w.r.t $h \in \mathcal{E}$, for every $p \in \mathbb{N}$. Actually, it is more convenient in many cases, and not so restrictive, to work with exponential weights in terms of the number operator \mathbf{N}_{\pm} .

Hypothesis 4.1. *The family $(\varrho_{\varepsilon(h)})_{h \in \mathcal{E}}$ in $\mathcal{L}^1(\Gamma_{\pm}(\mathcal{Z}))$ satisfies*

- i) *For all $h \in \mathcal{E}$, $\varrho_{\varepsilon(h)} \geq 0$ and $\text{Tr} \left[\varrho_{\varepsilon(h)} \right] = 1$;*
- ii) *There exists $c, C > 0$ such that $\text{Tr} \left[\varrho_{\varepsilon(h)} e^{c\mathbf{N}_{\pm}} \right] \leq C$, for all $h \in \mathcal{E}$.*

When the one particle phase-space is $\mathcal{X}^1 = T^*\mathbb{R}^d$ we use the Weyl quantization on $\mathcal{X}^p = T^*\mathbb{R}^{dp}$, $a^{Q,h} = a^{W,h} = a^W(h^t x, h^{1-t} D_x)$, $x \in \mathbb{R}^{dp}$, and when M^1 is a compact manifold, $\mathcal{X}^p = T^*M^p$, we use $a^{Q,h} = a(x, hD_x)$, $x \in M^p$.

Proposition 4.2. *Assume Hypothesis 4.1. Let $\chi \in C_0^{\infty}(T^*M^1)$ satisfy $0 \leq \chi \leq 1$ and $\chi \equiv 1$ in a neighborhood of 0 (resp. in a neighborhood of the null section $\{(x, \xi) \in T^*M, \xi = 0\} = M$) when $M = \mathbb{R}^d$ (resp. M compact manifold) and let $\chi_{\delta}(X) = \chi(\delta X)$ (resp. $\chi_{\delta}(x, \xi) = \chi(x, \delta \xi)$). For $c' < c$, c given by Hypothesis 4.1-ii), If*

$$s_{c', \chi}(\delta) = \limsup_{h \rightarrow 0} \text{Re} \text{Tr} \left[\varrho_{\varepsilon(h)} (e^{c'\mathbf{N}_{\pm}} - e^{c'd\Gamma_{\pm}(\chi_{\delta}^{Q,h})}) \right] \rightarrow 0 \quad \text{as} \quad \delta \rightarrow 0, \quad (13)$$

then for all $p \in \mathbb{N}$, the quantization $a^{Q,h}$ is adapted to the family $\gamma_{\varepsilon(h)}^{(p)}$.

Lemma 4.3. *Let $A \in \mathcal{L}(\mathcal{Z})$ and $\alpha \geq \|A\|$. For z in the open disc $D(0, \frac{\alpha}{\|A\|}) \subset \mathbb{C}$, the operator $e^{zd\Gamma_{\pm}(A)} e^{-\alpha\mathbf{N}_{\pm}} = e^{d\Gamma_{\pm}(zA - \alpha\text{Id}_{\mathcal{Z}})}$ is a contraction in $\Gamma_{\pm}(\mathcal{Z})$.*

- 1. *The function $z \mapsto e^{d\Gamma_{\pm}(zA - \alpha\text{Id}_{\mathcal{Z}})}$ is holomorphic in $D(0, \frac{\alpha}{\|A\|})$ and*

$$\frac{1}{p!} d\Gamma_{\pm}(A)^p e^{-\alpha\mathbf{N}_{\pm}} = e^{-\alpha\mathbf{N}_{\pm}} \frac{1}{p!} d\Gamma_{\pm}(A)^p = \frac{1}{2i\pi} \int_{|z|=r} e^{d\Gamma_{\pm}(zA - \alpha\text{Id}_{\mathcal{Z}})} \frac{dz}{z^{p+1}}$$

holds in $\mathcal{L}(\Gamma_{\pm}(\mathcal{Z}))$ for all $p \in \mathbb{N}$ and all $r \in (0, \frac{\alpha}{\|A\|})$.

Assume moreover that $A, B \in \mathcal{L}(\mathcal{Z})$, and $\alpha > \alpha_0 = \max \{ \|A\|, \|B\| \}$, then:

- 2. *For all $z \in D(0, \frac{\alpha}{\alpha_0})$,*

$$\left\| (e^{zd\Gamma_{\pm}(B)} - e^{zd\Gamma_{\pm}(A)}) e^{-\alpha\mathbf{N}_{\pm}} \right\|_{\mathcal{L}(\Gamma_{\pm}(\mathcal{Z}))} \leq \frac{\alpha \|B - A\|_{\mathcal{L}(\mathcal{Z})}}{\alpha_0(\alpha - \alpha_0)e}.$$

- 3. *This contains, for all $p \in \mathbb{N}$ and $r \in (0, \frac{\alpha}{\alpha_0})$,*

$$\left\| (d\Gamma_{\pm}(B)^p - d\Gamma_{\pm}(A)^p) e^{-\alpha\mathbf{N}_{\pm}} \right\|_{\mathcal{L}(\Gamma_{\pm}(\mathcal{Z}))} \leq \frac{\alpha p! \|B - A\|_{\mathcal{L}(\mathcal{Z})}}{\alpha_0(\alpha - \alpha_0) e r^p}.$$

Proof of Lemma 4.3. After setting $A' = zA$ with $|z| < \frac{\alpha}{\|A\|}$ so that $\|A'\| < \alpha$, notice that $\alpha \text{Id}_{\mathcal{Z}} - A' = \alpha - A'$ is a bounded accretive operator so that $(e^{-t\varepsilon(\alpha - A')})_{t \geq 0}$ is a strongly continuous semigroup of contractions on \mathcal{Z} , the same holds for $\Gamma_{\pm}(e^{-t\varepsilon(\alpha - A')}) = e^{-\alpha \mathbf{N}_{\pm}} e^{t d\Gamma(A')} = e^{t d\Gamma(A')} e^{-t\alpha \mathbf{N}_{\pm}}$ in $\Gamma_{\pm}(\mathcal{Z})$. The holomorphy and the Cauchy formula are then standard.

For the second inequality, set $B' = zB$ and $A' = zA$, $|z| < \frac{\alpha}{\alpha_0}$, and use Duhamel's formula

$$\begin{aligned} e^{-d\Gamma_{\pm}(\alpha - B')} - e^{-d\Gamma_{\pm}(\alpha - A')} \\ = \int_0^1 e^{-(1-t)d\Gamma_{\pm}(\alpha_0 - A')} d\Gamma_{\pm}(B' - A') e^{-(\alpha - \alpha_0)\mathbf{N}_{\pm}} e^{-td\Gamma_{\pm}(\alpha_0 - B')} dt. \end{aligned}$$

Since $e^{-(1-t)d\Gamma_{\pm}(\alpha_0 - A')}$ and $e^{-td\Gamma_{\pm}(\alpha_0 - B')}$ are contractions, the inequality

$$\|d\Gamma_{\pm}(B' - A') e^{-(\alpha - \alpha_0)\mathbf{N}_{\pm}}\| \leq \frac{\alpha}{\alpha_0} \|B - A\| \sup_{n \in \mathbb{N}} \varepsilon n e^{-(\alpha - \alpha_0)\varepsilon n} \leq \frac{\alpha \|B - A\|}{\alpha_0(\alpha - \alpha_0)e},$$

yields Point 2.

Point (3) follows from Point (1) and Point (2). \square

Proof of Proposition 4.2. Fix $p \in \mathbb{N}$. We want to find $\tilde{\chi} \in \mathcal{C}_0^{\infty}(T^*M^p)$, $0 \leq \tilde{\chi} \leq 1$ and $\tilde{\chi} \equiv 1$ in a neighborhood of $\{X \in \mathbb{R}^{2dp}, X = 0\}$ (resp. $\{(x, \xi) \in T^*M^p, \xi = 0\} = M^p$) when $M^p = \mathbb{R}^{dp}$ (resp. when M is a compact manifold), such that

$$\begin{aligned} \lim_{\delta \rightarrow 0} \limsup_{h \rightarrow 0} \mathcal{T}(\delta, h) = 0 \\ \text{with } \mathcal{T}(\delta, h) := \text{Re Tr} \left[\gamma_{\varepsilon(h)}^{(p)} (\text{Id}_{\mathcal{S}_{\pm}^p \mathcal{Z}^{\otimes p}} - \tilde{\chi}_{\delta}^{Q, h}) \right] = \text{Re Tr} \left[\varrho_{\varepsilon(h)} (\text{Id}_{\mathcal{S}_{\pm}^p \mathcal{Z}^{\otimes p}} - \tilde{\chi}_{\delta}^{Q, h})^{Wick} \right]. \end{aligned}$$

We know that $\chi^{\otimes p} \in \mathcal{C}_0^{\infty}(T^*M^p)$, with $0 \leq \chi^{\otimes p} \leq 1$. Take $\tilde{\chi}$ such that $\chi^{\otimes p} \leq \tilde{\chi} \leq 1$. For a constant $\kappa_{\delta} > 0$ to be fixed, the inequalities of symbols

$$\begin{aligned} 0 \leq \chi_{\delta}^{\otimes p} \leq \tilde{\chi}_{\delta} \leq 1 \\ 0 \leq \chi_{\delta} + \kappa_{\delta} h \leq 1 + \kappa_{\delta} h \end{aligned}$$

and the semiclassical calculus imply

$$\begin{aligned} \|(1 - \tilde{\chi}_{\delta})^{Q, h} - \text{Re} \left[(1 - \tilde{\chi}_{\delta})^{Q, h} \right]\|_{\mathcal{L}(\mathcal{Z}^{\otimes p})} \leq C_{\delta} h, \quad \|\chi_{\delta}^{Q, h} - \text{Re} \left[\chi_{\delta}^{Q, h} \right]\| \leq C_{\delta} h, \\ 0 \leq \text{Re} \left[(1 - \chi_{\delta}^{\otimes p})^{Q, h} \right] + C'_{\delta} h = 1 - (\text{Re} \left[\chi_{\delta}^{Q, h} \right])^{\otimes p} + C'_{\delta} h \\ \leq (1 + 2\kappa_{\delta} h)^p - (\text{Re} \left[(\chi_{\delta} + \kappa_{\delta} h)^{Q, h} \right])^{\otimes p} + C''_{\delta} h \quad \text{in } \mathcal{L}(\mathcal{Z}^{\otimes p}), \end{aligned}$$

for some constants $C_{\delta}, C'_{\delta}, C''_{\delta} > 0$, chosen according to $p \in \mathbb{N}$, $\delta > 0$ and $\kappa_{\delta} > 0$. Moreover for $\delta > 0$ fixed, the constant κ_{δ} can be chosen so that

$$0 \leq \text{Re} \left[(\chi_{\delta} + \kappa_{\delta} h)^{Q, h} \right] \leq 1 + 2\kappa_{\delta} h.$$

With $\|(1 + \mathbf{N}_{\pm})^p e^{-c'/2\mathbf{N}_{\pm}}\|_{\mathcal{L}(\Gamma_{\pm}(\mathcal{Z}))} \leq C_{p, c'}$, the number estimate (2) and the positivity property ($\tilde{b} \geq 0 \Rightarrow (\tilde{b}^{Wick} \geq 0)$), writing

$$\varrho_{\varepsilon(h)} = e^{-c/2\mathbf{N}_{\pm}} e^{c/2\mathbf{N}_{\pm}} \varrho_{\varepsilon(h)} e^{c/2\mathbf{N}_{\pm}} e^{-c/2\mathbf{N}_{\pm}}$$

leads to

$$\begin{aligned}
\mathcal{T}(\delta, h) &:= \operatorname{Re} \operatorname{Tr} \left[\varrho_{\varepsilon(h)} (\operatorname{Id}_{\mathcal{S}_{\pm}^p \mathcal{F}^{\otimes p}} - \tilde{\chi}_{\delta}^{Q,h})^{Wick} \right] \\
&= \operatorname{Tr} \left[\varrho_{\varepsilon(h)} (\operatorname{Id}_{\mathcal{S}_{\pm}^p \mathcal{F}^{\otimes p}} - \operatorname{Re} \left[\tilde{\chi}_{\delta}^{Q,h} \right])^{Wick} \right] + \mathcal{O}_{\delta}(h) \\
&\leq \operatorname{Tr} \left[\varrho_{\varepsilon(h)} \left((1 + 2\kappa_{\delta}h)^p - (\operatorname{Re} \left[(\chi_{\delta} + \kappa_{\delta}h)^{Q,h} \right])^{\otimes p} \right)^{Wick} \right] + \mathcal{O}_{\delta}(h).
\end{aligned}$$

We now use Propostion 2.4 for

$$\mathcal{T}(\delta, h) \leq \operatorname{Tr} \left[\varrho_{\varepsilon(h)} \left(d\Gamma_{\pm}(1 + 2\kappa_{\delta}h)^p - d\Gamma_{\pm}(\operatorname{Re} \left[(\chi_{\delta} + \kappa_{\delta}h)^{Q,h} \right])^p \right) \right] + \mathcal{O}_{\delta}(h + \varepsilon(h)).$$

The two operators $\mathcal{A} = d\Gamma_{\pm}(1 + 2\kappa_{\delta}h)$ and $\mathcal{B} = d\Gamma_{\pm}(\operatorname{Re} \left[(\chi_{\delta} + \kappa_{\delta}h)^{Q,h} \right])$ are commuting self-adjoint operators such that $0 \leq \mathcal{B} \leq \mathcal{A}$, so that $0 \leq \mathcal{A}^p - \mathcal{B}^p \leq C_{p,c'}[e^{c'\mathcal{A}} - e^{c'\mathcal{B}}]$. We deduce

$$\begin{aligned}
\mathcal{T}(\delta, h) &\leq C_{p,c'} \operatorname{Tr} \left[\varrho_{\varepsilon(h)} e^{cN_{\pm}} e^{-cN_{\pm}} \left(e^{d\Gamma_{\pm}(c'(1+2\kappa_{\delta}h))} - e^{d\Gamma_{\pm}(c' \operatorname{Re} \left[(\chi_{\delta} + \kappa_{\delta}h)^{Q,h} \right])} \right) \right] + \mathcal{O}_{\delta}(h + \varepsilon(h)).
\end{aligned}$$

We apply Lemma 4.3 with $z = 1$, $A = c'(1 + 2\kappa_{\delta}h)$ and $B = c'$, or $A = c' \operatorname{Re} \left[(\chi_{\delta} + \kappa_{\delta}h)^{Q,h} \right]$ and $B = c' \chi_{\delta}^{Q,h}$, and finally

$$\alpha = c > \alpha_0 = \frac{c + c'}{2} \geq c' \max \left\{ 1 + 2\kappa_{\delta}h, \|(\chi_{\delta} + \kappa_{\delta}h)^{Q,h}\|, \|\chi_{\delta}^{Q,h}\| \right\} \quad \text{for } h \leq h_{\delta,c,c'}$$

and we get

$$\mathcal{T}(\delta, h) \leq \operatorname{Re} \operatorname{Tr} \left[\varrho_{\varepsilon(h)} \left(e^{c'N_{\pm}} - e^{c'd\Gamma_{\pm}(\chi_{\delta}^{Q,h})} \right) \right] + \mathcal{O}_{\delta}(h + \varepsilon(h)).$$

We thus obtain

$$\limsup_{h \rightarrow 0} \mathcal{T}(\delta, h) \leq s_{c',\chi}(\delta)$$

and our assumption $\lim_{\delta \rightarrow 0} s_{c',\chi}(\delta) = 0$ allows to conclude. \square

Notation For any open set $\Omega \subseteq \mathbb{C}$ the Hardy space $H^{\infty}(\Omega)$ is the space of bounded holomorphic functions on Ω .

Proposition 4.4. *Assume Hypothesis 4.1. The set \mathcal{E} can be reduced to \mathcal{E}' so that $\mathcal{M}(\gamma_{\varepsilon(h)}^{(p)}, h \in \mathcal{E}') = \{\nu^{(p)}\}$, where $\nu^{(p)}$ is a non-negative measure on T^*M^p/\mathfrak{S}_p , i.e. a measure on $(T^*M)^p$ with the invariance (6).*

*When (13) is satisfied, this implies $\lim_{h \rightarrow 0} \operatorname{Tr} \left[\gamma_{\varepsilon(h)}^{(p)} \right] = \int_{T^*M^p} d\nu^{(p)}(X)$ for all $p \in \mathbb{N}$.*

For any $a \in \mathcal{C}_0^{\infty}(\mathbb{R}^{2d})$ there exists $r_a > 0$ such that the function $\Phi_{a,h} : s \mapsto \operatorname{Tr} \left[\varrho_{\varepsilon(h)} e^{sd\Gamma_{\pm}(a^{W,h})} \right]$ is uniformly bounded in $H^{\infty}(D(0, r_a))$ and

$$\lim_{\substack{h \in \mathcal{E}' \\ h \rightarrow 0}} \Phi_{a,h}(s) = \Phi_{a,0}(s) := \sum_{p=0}^{\infty} \frac{s^p}{p!} \int_{T^*M^p} a^{\otimes p}(X) d\nu^{(p)}(X). \quad (14)$$

Reciprocally if $\Phi_{a,h}$ converges, pointwise on the interval $(-r_a, r_a)$ or in $\mathcal{D}'((-r_a, r_a))$, to some function $\Phi_{a,0}$, as $h \rightarrow 0, h \in \mathcal{E}$, then $\mathcal{M}(\gamma_{\varepsilon(h)}^{(p)}, h \in \mathcal{E}) = \{\nu^{(p)}\}$ for all $p \in \mathbb{N}$ and $\Phi_{a,0}$ is equal to (14).

Proof. The uniform bound

$$\mathrm{Tr} \left[\gamma_{\varepsilon(h)}^{(p)} \right] \leq \mathrm{Tr} \left[\varrho_{\varepsilon(h)} \langle \mathbf{N} \rangle^p \right] \leq C_{p,c} \mathrm{Tr} \left[\varrho_{\varepsilon(h)} e^{c\mathbf{N}} \right]$$

and Hypothesis 4.1 ensure for each $p \in \mathbb{N}$ the existence of $\mathcal{E}^{(p)} \subseteq \mathcal{E}^{(p-1)} \subseteq \mathcal{E}$, $0 \in \overline{\mathcal{E}^{(p)}}$, and $\mathcal{M}(\gamma_{\varepsilon(h)}^{(p)}, h \in \mathcal{E}^{(p)}) = \{\nu^{(p)}\}$. A diagonal extraction w.r.t to p determines $\mathcal{E}' \subset \mathcal{E}$, $0 \in \overline{\mathcal{E}'}$, such that $\mathcal{M}(\gamma_{\varepsilon(h)}^{(p)}, h \in \mathcal{E}') = \{\nu^{(p)}\}$ for all $p \in \mathbb{N}$.

The second statement is a straightforward application of Lemma 4.3.

Thanks to the classical symmetrization Lemma 3.1, the measures $\nu^{(p)}$ are determined after integrating with all the test functions $a^{\otimes p}$ with $a \in \mathcal{C}_0^\infty(T^*M)$.

Hypothesis 4.1 now combined with

$$\|e^{-c\mathbf{N}} e^{zd\Gamma(\alpha^{W,h})}\| = \|\Gamma(e^{\varepsilon(z\alpha^{W,h})-c})\| \leq e^{|z|\|a^{W,h}\|-c} \quad \text{and} \quad \|a^{W,h}\| \leq C_a,$$

provides the uniform boundedness w.r. t $h \in \mathcal{E}$ of $\Phi_{a,h}$ in $H^\infty(D(0, 2r_a))$. In any $\mathcal{E}_1 \subset \mathcal{E}$, $0 \in \tilde{E}_1$, we can find a subset \mathcal{E}_2 , $0 \in \overline{\mathcal{E}_2}$, such that $\Phi_{a,h}$, locally uniformly in $D(0, 2r_a)$ and therefore in $H^\infty(D(0, r_a))$, to some function $\Phi_{a,0}$. In particular when $\mathcal{E}_2 \subset \mathcal{E}_1 \subset \mathcal{E}'$, Corollary 2.4 implies

$$\begin{aligned} \int_{T^*M^p} a^{\otimes p} d\nu^{(p)} &= \lim_{\substack{h \in \mathcal{E}_2 \\ h \rightarrow 0}} \mathrm{Tr} \left[\gamma_{\varepsilon(h)}^{(p)} (a^{W,h})^{\otimes p} \right] = \lim_{\substack{h \in \mathcal{E}_2 \\ h \rightarrow 0}} \mathrm{Tr} \left[\varrho_{\varepsilon(h)} \left((a^{W,h})^{\otimes p} \right)^{Wick} \right] \\ &= \lim_{\substack{h \in \mathcal{E}_2 \\ h \rightarrow 0}} \mathrm{Tr} \left[\varrho_{\varepsilon(h)} d\Gamma(a^{W,h})^p \right] = \lim_{h \rightarrow 0} \frac{d^p \Phi_{a,h}}{ds^p}(0) = \frac{d^p \Phi_{a,0}}{ds^p}(0) \end{aligned}$$

Hence the limit $\Phi_{a,0} \in H^\infty(D(0, r_a))$, as $h \in \mathcal{E}_2$, $h \rightarrow 0$, equals the right-hand side of (14) and this uniqueness implies the convergence for the whole family $(\Phi_{a,h})_{h \in \mathcal{E}'}$.

Reciprocally assume the convergence of $\Phi_{a,h}$ to $\Phi_{a,0}$ in a weak topology on the interval $(-r_a, r_a)$ as $h \in \mathcal{E}$. With the a uniform bound on $\Phi_{a,h}$ in $H^\infty(D(0, 2r_a))$, $\Phi_{a,0}$ has a holomorphic extension in $D(0, r_a)$. Additionally we can extract a subset $\mathcal{E}' \subset \mathcal{E}$ such that $\mathcal{M}(\gamma_h^{(p)}, h \in \mathcal{E}') = \{\nu^{(p)}\}$ and (14) hold. Again the uniqueness of the limit $\Phi_{a,0}|_{(-r_a, r_a)}$ and of its holomorphic extension to $D(0, r_a)$ ends the proof. \square

Replacing the semiclassical symmetrization Lemma 3.1 by the quantum ones, Lemma 2.8 in the above proof leads to the following similar result for the quantum part.

Proposition 4.5. *Assume Hypothesis 4.1. For all $K \in \mathcal{L}^\infty(\mathcal{Z})$ there exists $r_K > 0$ such that the set $\{\Psi_{K,h}, h \in \mathcal{E}\}$ of functions $\Psi_{K,h}(s) := \mathrm{Tr} \left[\varrho_{\varepsilon(h)} e^{sd\Gamma^\pm(K)} \right]$ is bounded in $H^\infty(D(0, r_K))$.*

The pointwise or $\mathcal{D}'((-r_K, r_K))$ -convergence $\lim_{h \in \mathcal{E}} \lim_{h \rightarrow 0} \Psi_{K,h} = \Psi_{K,0}$ is equivalent to $w^ - \lim_{h \in \mathcal{E}} \lim_{h \rightarrow 0} \gamma_h^{(p)} = \gamma_0^{(p)}$ (remember $\mathcal{L}^1 = (\mathcal{L}^\infty)^*$) with*

$$\Psi_{K,0}(s) = \sum_{p=0}^{\infty} \mathrm{Tr} \left[\gamma_0^{(p)} K^{\otimes p} \right] \frac{s^p}{p!},$$

Let us consider the fermionic case:

Proposition 4.6. *Let $(\varrho_\varepsilon)_{\varepsilon \in \mathcal{E}}$ be a family of non-negative, trace 1 operators in $\mathcal{L}^1(\Gamma_-(\mathcal{Z}))$. Let $\gamma_\varepsilon^{(p)}$ denote the corresponding non normalized reduced density matrices of order p . If $\gamma_0^{(p)} \in \mathcal{L}^1(\mathcal{S}_-^p \mathcal{Z}^{\otimes p})$ is such that*

$$\forall K \in \mathcal{L}^\infty(\mathcal{S}_-^p \mathcal{Z}^{\otimes p}), \quad \lim_{\substack{\varepsilon \in \mathcal{E} \\ \varepsilon \rightarrow 0}} \text{Tr}[\gamma_\varepsilon^{(p)} K] = \text{Tr}[\gamma_0^{(p)} K],$$

then $\gamma_0^{(p)} = 0$.

As a consequence, the weak limits $\gamma_0^{(p)}$ always vanish in the fermionic case.

Proof. First consider K a non-negative finite rank operator. Then

$$\lim_{\substack{\varepsilon \in \mathcal{E} \\ \varepsilon \rightarrow 0}} \text{Tr}[\varrho_\varepsilon K^{Wick}] = \text{Tr}[\gamma_0^{(p)} K].$$

For fermions, $K^{Wick} \leq \varepsilon^p \text{Tr}[K]$, and hence $\text{Tr}[\varrho_\varepsilon K^{Wick}] \leq \varepsilon(h)^p \rightarrow 0$ as $\varepsilon \rightarrow 0$. Any finite rank operator being of the form $K = K_1 - K_2 + i(K_3 - K_4)$ for some non-negative finite rank operators K_j , $j \in \{1, 2, 3, 4\}$, the limit $\text{Tr}[\varrho_\varepsilon K^{Wick}] \rightarrow 0 = \text{Tr}[\gamma_0^{(p)} K]$ holds for any finite rank operator K . Hence, by density of the finite rank operators in the compact operators for the operator norm, $\text{Tr}[\gamma_0^{(p)} K] = 0$ for any $K \in \mathcal{L}^\infty(\mathcal{S}_-^p \mathcal{Z}^{\otimes p})$, i.e., $\gamma_0^{(p)} = 0$. \square

5 Examples

5.1 h -dependent coherent states in the bosonic case

We first recall our normalization for a coherent state. We need the notion of empty state: if we use the identification $\mathcal{S}_\pm^0 \mathcal{Z} \equiv \mathbb{C}$, then the empty state is defined as $\Omega = (1, 0, 0, \dots) \in \Gamma_\pm(\mathcal{Z})$. We then introduce the usual field operators $\Phi(f) = \frac{1}{\sqrt{2}}(a^*(f) + a(f))$, with $f \in \mathcal{Z}$ and the Weyl operators are the $W(f) = \exp(\frac{i}{\sqrt{2}}\Phi(f))$. A coherent state is a pure state $E_z = W(\frac{\sqrt{2}z}{i\varepsilon})\Omega$, with $z \in \mathcal{Z}$. One then also speak of coherent state for the corresponding density matrix $|E_z\rangle\langle E_z|$. One of the useful properties of coherent states is that

$$b(z) = \langle E(z), b^{Wick} E(z) \rangle. \quad (15)$$

(See e.g. [AmNi1, Prop. 2.10]) The case of coherent states is simple:

Proposition 5.1. *Let $(z_\varepsilon)_{\varepsilon \in (0,1]}$ a bounded family of \mathcal{Z} , choose the semiclassical quantization $a \mapsto a^{W,h} = a^W(\sqrt{h}x, \sqrt{h}D_x)$, and fix a function $\varepsilon = \varepsilon(h) \rightarrow 0$ as $h \rightarrow 0$. Up to an extraction, $z_{\varepsilon(h)} \rightarrow z_0 \in \mathcal{Z}$ weakly, and $\mathcal{M}(|z_{\varepsilon(h)}\rangle\langle z_{\varepsilon(h)}|, h \in \mathcal{E}) = \{\nu\}$. Assume that the semiclassical quantization $a^{W,h} = a^W(\sqrt{h}x, \sqrt{h}D_x)$ is adapted to $(|z_{\varepsilon(h)}\rangle\langle z_{\varepsilon(h)}|)_h$ and separating for $(|z_{\varepsilon(h)}\rangle\langle z_{\varepsilon(h)}|)_h$. Then the family $(\varrho_{\varepsilon(h)} = |E_{z_{\varepsilon(h)}}\rangle\langle E_{z_{\varepsilon(h)}}|)_{h \in \mathcal{E}}$ has $\gamma_{\varepsilon(h)}^{(p)} = |z_{\varepsilon(h)}^{\otimes p}\rangle\langle z_{\varepsilon(h)}^{\otimes p}|$ as (non normalized) reduced density matrices of order p , for which the quantization is adapted and separating, and*

$$\mathcal{M}^{(2)}(\gamma_{\varepsilon(h)}^{(p)}, h \in \mathcal{E}) = \{(\nu^{\otimes p}, 0, |z_0^{\otimes p}\rangle\langle z_0^{\otimes p}|)\}.$$

Proof. Formula (15) yields, for $B \in \mathcal{L}(\mathcal{S}_+^p \mathcal{Z}^{\otimes p})$,

$$\langle z_{\varepsilon(h)}^{\otimes p}, B z_{\varepsilon(h)}^{\otimes p} \rangle = \langle E_{z_{\varepsilon(h)}} | B^{Wick} | E_{z_{\varepsilon(h)}} \rangle = \text{Tr}[\varrho_{\varepsilon(h)} B^{Wick}] = \text{Tr}[\gamma_{\varepsilon(h)}^{(p)} B],$$

which implies the result. \square

The case of coherent states, although simple, can already exhibit interesting behaviors for some families $(z_\varepsilon)_{\varepsilon \in (0,1]}$. Indeed,

Remark 5.2. *Let $(z_{j,\varepsilon})_{\varepsilon \in (0,1]}$, $j \in \{1,2\}$, be families of \mathcal{Z} such that*

- $z_{1,\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} z_{1,0} \in \mathcal{Z}$, and
- $(z_{2,\varepsilon})_{\varepsilon \in (0,1]}$ is bounded, converges weakly to 0, $\lim_{R \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \|z_{2,\varepsilon} 1_{\mathcal{L}B(0,R)}\| = 0$ (no mass escaping at infinity), and $\mathcal{M}(|z_{2,\varepsilon(h)}\rangle\langle z_{2,\varepsilon(h)}|, h \in \mathcal{E}) = \{\nu_2\}$, with $\nu_2(\{0\}) = 0$.

Then $(|z_{1,\varepsilon} + z_{2,\varepsilon}\rangle\langle z_{1,\varepsilon} + z_{2,\varepsilon}|)_{\varepsilon \in (0,1]}$ satisfies the assumptions of Prop. 5.1, and $z_0 = z_{1,0}$, $\nu = \|z_{1,0}\|^2 \delta_0 + \nu_2$.

5.2 Gibbs states

For a given non-negative self-adjoint hamiltonian H defined in \mathcal{Z} with domain $D(H)$, the Gibbs state at positive temperature $\frac{1}{\beta}$ and with the chemical potential $\mu < 0$ is given by

$$\omega_\varepsilon(A) = \frac{\text{Tr} [\Gamma_\pm(e^{-\beta(H-\mu)})A]}{\text{Tr} [\Gamma_\pm(e^{-\beta(H-\mu)})]} = \text{Tr} [\varrho_\varepsilon A].$$

In general $\varrho_\varepsilon \in \mathcal{L}^1(\Gamma_\pm(\mathcal{Z}))$ as soon as $e^{-\beta(H-\mu)} \in \mathcal{L}^1(\mathcal{Z})$ (in the bosonic case $H \geq 0$ and $\mu < 0$ imply $\|e^{-\beta(H-\mu)}\|_{\mathcal{L}(\mathcal{Z})} < 1$, see Lemma D.1). Moreover the quasi-free state formula (see [BrRo2]) with ε -dependent quantization gives

$$\text{Tr} [\varrho_\varepsilon \mathbf{N}] = \varepsilon \text{Tr} \left[e^{-\beta(H-\mu)} (1 \mp e^{-\beta(H-\mu)})^{-1} \right]$$

and additionally, in the case of bosons,

$$\text{Tr} [\varrho_\varepsilon W(f)] = \exp \left[-\frac{\varepsilon}{4} \langle f, (1 + e^{-\beta(H-\mu)})(1 - e^{-\beta(H-\mu)})^{-1} f \rangle \right].$$

5.2.1 In the fermionic case

We begin by the fermionic case, which is simpler than the bosonic case for two reasons: first because the quantum part vanishes (see Prop. 4.6), and second because there is no singularity to handle. To fix the ideas we consider the simple case when H is the harmonic oscillator. Actually one can treat more general pseudo differential operators, and we do that below in the more interesting case of bosons and Bose-Einstein condensation.

Proposition 5.3. *Let $\beta > 0$, $H = \frac{1}{2}|X|^{2W,h}$, $\mu(\varepsilon)$ such that $\mu(\varepsilon) \geq C\varepsilon$ for some constant $C > 0$, and assume that $\varepsilon = \varepsilon(h) = h^d$. Let $\varrho_{\varepsilon(h)} = \frac{\Gamma_-(e^{-\beta(H-\mu(\varepsilon))})}{\text{Tr}[\Gamma_-(e^{-\beta(H-\mu(\varepsilon))})]}$ and $\gamma_{\varepsilon(h)}^{(p)}$ its non normalized reduced density matrix of order $p \geq 1$. Then $\mathcal{M}^{(2)}(\gamma_{\varepsilon(h)}^{(p)}, h \in (0,1]) = \{(\nu^{(p)}, 0, 0)\}$, where $d\nu^{(p)} = p! \left(\frac{e^{-\beta|X|^{2/2}}}{1+e^{-\beta|X|^{2/2}}} \frac{dX}{(2\pi)^d} \right)^{\otimes p}$.*

Proof. From Rem. 3.7 and Prop. 4.6, any $(\nu^{(p)}, \nu_I^{(p)}, \gamma_0^{(p)}) \in \mathcal{M}^{(2)}(\gamma_{\varepsilon(h)}^{(p)}, h \in (0,1])$ satisfies $\gamma_0^{(p)} = 0$. Since we are considering a Gibbs state, the Wick formula yields

$$\gamma_{\varepsilon(h)}^{(p)} = p! \varepsilon^p \mathcal{S}_\pm^p \gamma_{\varepsilon(h)}^{(1)\otimes p} \mathcal{S}_\pm^{p,*}.$$

Moreover, in the fermionic case, $\gamma_{\varepsilon(h)}^{(1)} = \frac{C}{1+C}$ for $\varrho_{\varepsilon(h)} = \frac{\Gamma_-(C)}{\text{Tr}[\Gamma_-(C)]}$, that is to say $\gamma_{\varepsilon(h)}^{(1)} = h^d \frac{e^{-\beta(H-\mu)}}{1+e^{-\beta(H-\mu)}}$ in our case. The semiclassical calculus combined with Helffer-Sjöstrand functional calculus formula yields

$$\frac{e^{-\beta(|X|^{2W,h}/2-\mu)}}{1+e^{-\beta(|X|^{2W,h}/2-\mu)}} = \left(\frac{e^{-\beta|X|^2/2}}{1+e^{-\beta|X|^2/2}} \right)^{W,h} + \mathcal{O}(h) \quad \text{in } \mathcal{L}(\mathcal{Z}).$$

For details we refer the reader e.g. to [DiSj, HeNi] or to the proof of Prop. 5.6. Again by the semiclassical calculus we know $h^d a^{W,h}$ is uniformly bounded in $\mathcal{L}^1(L^2(\mathbb{R}^d))$ for $a \in C_0^\infty(\mathbb{R}^{2d})$. This leads to

$$\text{Tr}[(a^{W,h})^{\otimes p} \gamma_{\varepsilon(h)}^{(p)}] = p! \text{Tr}[a^{W,h} \gamma_{\varepsilon(h)}^{(1)}]^p = p! \text{Tr} \left[\left(\frac{e^{-\beta|X|^2/2}}{1+e^{-\beta|X|^2/2}} \right)^{W,h} h^d a^{W,h} \right]^p + \mathcal{O}(h).$$

We finally use $h^d \text{Tr}[a^{W,h} b^{W,h}] = \int_{\mathbb{R}^{2d}} a(X) b(X) \frac{dX}{(2\pi)^d}$ which implies

$$\lim_{h \rightarrow 0} \text{Tr}[(a^{W,h})^{\otimes p} \gamma_{\varepsilon(h)}^{(p)}] = p! \left(\int_{\mathbb{R}^{2d}} \frac{e^{-\beta|X|^2/2}}{1+e^{-\beta|X|^2/2}} a(X) \frac{dX}{(2\pi)^d} \right)^p.$$

Hence $d\nu^{(p)}(X) = p! \left(\frac{e^{-\beta|X|^2/2}}{1+e^{-\beta|X|^2/2}} \frac{dX}{(2\pi)^d} \right)^{\otimes p}$. □

5.2.2 Parameter dependent Gibbs states and Bose-Einstein condensation in the bosonic case

The Bose-Einstein condensation phenomenon occurs when H has a ground state $\ker H = \mathbb{C}\psi_0$ and the chemical potential is scaled according to

$$-\beta\mu = \frac{\varepsilon}{\nu_C} \quad \text{for some fixed } \nu_C > 0.$$

An especially interesting case is when H is a semiclassically quantized symbol with semiclassical parameter h related to ε , or $\varepsilon = \varepsilon(h)$ according to our previous notations. The quantum and semiclassical parts arise simultaneously when $\varepsilon = h^d$. Two cases will be considered: the first one concerns $\mathcal{Z} = L^2(\mathbb{R}^d)$ with a non degenerate bottom well hamiltonian; the second one $\mathcal{Z} = L^2(M)$ with the semiclassical Laplace-Beltrami operator on the compact riemannian manifold M .

In the first case, let $S(\langle X \rangle^m, \frac{dX^2}{\langle X \rangle^2})$ denote the Hörmander class of symbols satisfying $|\partial_X^\beta a(X)| \leq C_\beta \langle X \rangle^{m-\beta}$, and let $\alpha \in S(\langle X \rangle^2, \frac{dX^2}{\langle X \rangle^2})$ be elliptic in this class with a unique non degenerate minimum at $X = 0$ (e.g. the symbol of the harmonic oscillator hamiltonian). We can even consider small perturbations of this situation after setting

$$H = \alpha^{W,h} + B^h - \lambda_0(\alpha^{W,h} + B^h) \quad , \quad \alpha^{W,h} = \alpha(\sqrt{h}x, \sqrt{h}D_x) \quad , \quad \varepsilon = h^d,$$

where $B^h = B^{h*} \in \mathcal{L}(L^2(\mathbb{R}^d))$, $\|B^h\| = o(h)$ and $\lambda_0(\alpha^{W,h} + B^h) = \inf \sigma(\alpha^{W,h} + B^h)$. It is convenient in this case to introduce the linear symplectic transformation $T \in Sp_{2d}(\mathbb{R})$ such that ${}^t X^t T^{-1} \text{Hess } \alpha(0) T^{-1} X = \sum_{j=1}^d \beta_j X_j^2$ and to introduce some unitary quantization U_T of T , i.e. a unitary operator on $L^2(\mathbb{R}^d)$ such that $U_T^* b^W U_T = b(T^{-1})^W$.

Proposition 5.4. *Under the above assumptions with dimension $d \geq 2$, for any $p \in \mathbb{N}$, $\mathcal{M}^{(2)}(\gamma_{\varepsilon(h)}^{(p)}, h \in \mathcal{E}) = \{(\nu^{(p)}, 0, \gamma_0^{(p)})\}$ (see Def. 3.12), where*

$$\begin{aligned} \gamma_0^{(p)} &= p! |\psi_0^{\otimes p}\rangle \langle \psi_0^{\otimes p}| \quad \text{with } \psi_0(x) = U_T \frac{e^{-x^2/2}}{\pi^{d/4}} \\ \nu^{(p)} &= \sum_{\sigma \in \mathfrak{S}_p} \sigma_* \left[\sum_{k=0}^p \frac{1}{(p-k)!} \nu_C^k \delta_0^{\otimes k} \otimes \nu(\beta, \cdot)^{\otimes p-k} \right], \\ \text{with } \quad d\nu(\beta, X) &= \frac{e^{-\beta\alpha(X)}}{1 - e^{-\beta\alpha(X)}} \frac{dX}{(2\pi)^d}. \end{aligned}$$

The proof is, given in Section 5.2.4, needs some preliminaries given in Prop. 5.6 and Lemma 5.7.

Another even simpler case, related to the example $M = \mathbb{T}^d$ presented in [AmNi1], is $\mathcal{L} = L^2(M, dv_g(x))$ when (M, g) is a compact Riemannian manifold with volume $dv_g(x)$ and

$$H = -h^2 \Delta_g + B_h - \lambda_0(-h^2 \Delta_g + B_h),$$

where Δ_g is the Laplace Beltrami operator on (M, g) and $B_h = B_h^* \in \mathcal{L}(L^2(M))$, $\|B_h\| = \underline{o(h^2)}$.

Proposition 5.5. *Under the above assumptions with $d \geq 3$, for any $p \in \mathbb{N}$, $\mathcal{M}^{(2)}(\gamma_{\varepsilon(h)}^{(p)}, h \in \mathcal{E}) = \{(\nu^{(p)}, 0, \gamma_0^{(p)})\}$ where*

$$\begin{aligned} \gamma_0^{(p)} &= p! |\psi_0^{\otimes p}\rangle \langle \psi_0^{\otimes p}| \quad , \quad \psi_0 = \frac{1}{v_g(M)^{1/2}}, \\ \nu^{(p)} &= \sum_{\sigma \in \mathfrak{S}_p} \sigma_* \left[\sum_{k=0}^p \frac{1}{(p-k)!} \nu_C^k \left(\frac{1}{v_g(M)} dv_g(x) \otimes \delta_0(\xi) \right)^{\otimes k} \otimes \nu(\beta)^{\otimes (p-k)} \right], \\ \text{with } \quad d\nu(\beta, X) &= \frac{e^{-\beta|\xi|_{g(x)}^2}}{1 - e^{-\beta|\xi|_{g(x)}^2}} \frac{dx d\xi}{(2\pi)^d}, \\ \text{and } \quad |\xi|_{g(x)}^2 &= \sum_{i,j \leq d} g^{ij}(x) \xi_i \xi_j \quad \text{when } \quad g = \sum_{i,j \leq d} g_{ij}(x) dx^i dx^j \quad , \quad (g_{ij})^{-1} = (g^{ij}). \end{aligned}$$

We shall focus on the first case which requires a more careful analysis, while $\sigma(-h^2 \Delta_g) = h^2 \sigma(-\Delta_g)$ reduces even more easily the problem to the integrability of $\frac{e^{-\beta|\xi|_{g(x)}^2}}{1 - e^{-\beta|\xi|_{g(x)}^2}}$ valid when $d \geq 3$. The proof of Proposition 5.5 is left as an exercise, which requires the adaptation of the following arguments in the case of Proposition 3.11 with the associated Definitions 3.13 and 3.12.

5.2.3 Semiclassical asymptotics with a singularity at $X = 0$

We give here a general semiclassical result in $T^*\mathbb{R}^d$, which involves traces and symbols with a singularity at $X = 0$.

Proposition 5.6. *Consider the hamiltonian $H = \alpha^{W,h} + B_h - \lambda_0(\alpha^{W,h} + B_h)$, with $\alpha^{W,h} = \alpha(\sqrt{h}x, \sqrt{h}D_x)$, $\alpha \in S(\langle X \rangle^2, \frac{dX^2}{\langle X \rangle^2})$ elliptic and real such that $\alpha(0) = 0$ is the unique non degenerate minimum, $B_h = B_h^* \in \mathcal{L}(L^2(\mathbb{R}^d))$, $\|B_h\| = o(h)$, and $\lambda_0(\alpha^{W,h} + B_h) = \inf \sigma(\alpha^{W,h} + B_h)$.*

Assume that $f \in C^\infty((0, +\infty); \mathbb{R})$ is decreasing and satisfies

$$0 \leq f(u) \leq Cu^{-\kappa_\infty}, \quad \lim_{u \rightarrow 0^+} u^{\kappa_0} f(u) = f_0 \in \mathbb{R}, \quad 0 < \kappa_0 < d < \kappa_\infty.$$

For $c > 0$, the operator $f(H + ch^{d/\kappa_0})$ is trace class with

$$\limsup_{h \rightarrow 0^+} h^d \|f(H + ch^{d/\kappa_0})\|_{\mathcal{L}^1(L^2(\mathbb{R}^d))} < +\infty.$$

Moreover the convergence

$$\lim_{h \rightarrow 0} h^d \text{Tr} \left[f(H + ch^{d/\kappa_0}) a^{W,h} \right] = \frac{f_0}{c^{\kappa_0}} a(0) + \int_{\mathbb{R}^{2d}} f(\alpha(X)) a(X) \frac{dX}{(2\pi)^d},$$

holds for all $a \in S(1, dX^2)$. Finally, all the above estimates and convergences hold uniformly with respect to $c \in (\frac{1}{A}, A)$ for any fixed $A > 1$.

The following Lemma gives in a simple way useful inequalities for our purpose which are deduced with elementary arguments, and in a robust way w.r.t the perturbation B_h , from more accurate and sophisticated results on the spectrum of $\alpha^{W,h}$ (see [ChVN][DiSj] and references therein).

Lemma 5.7. *Let $\alpha \in S(\langle X \rangle^2, \frac{dX^2}{\langle X \rangle^2})$ be real-valued, elliptic, which means $1 + \alpha(X) \geq C^{-1} \langle X \rangle^2$, with a unique non degenerate minimum at $X = 0$ and set $\alpha_0(X) = \frac{|X|^2}{2}$. Let $B_h = B_h^* \in \mathcal{L}(L^2(\mathbb{R}^d))$ be such that $\|B_h\| = o(h)$. The ordered eigenvalues are denoted by $\lambda_j(\alpha^{W,h} + B_h)$ and $\lambda_j(\alpha_0^{W,h})$ for $j \in \mathbb{N}$.*

- For $j = 0$, $\lambda_0(\alpha^{W,h} + B_h) = \text{Tr} [\text{Hess } \alpha(0)] h + o(h)$ and the associated spectral projection satisfies

$$\lim_{h \rightarrow 0} 1_{\{\lambda_0(\alpha^{W,h} + B_h)\}}(\alpha^{W,h} + B_h) = (\pi^{-d} e^{-|TX|^2})^W(x, D_x), \quad \text{in } \mathcal{L}^1(L^2(\mathbb{R}^d)),$$

where $T \in Sp_{2d}(\mathbb{R})$ is such that ${}^t X^t T^{-1} \text{Hess } \alpha(0) T^{-1} X = \sum_{j=1}^d \beta_j X_j^2$.

- There exist $h_0 > 0$ and $C' \geq 1$ such that, for all $j > 0$ and $h \in (0, h_0)$,

$$C'^{-1} h d / 2 \leq C'^{-1} \lambda_j(\alpha_0^{W,h}) \leq \lambda_j(\alpha^{W,h} + B_h) - \lambda_0(\alpha^{W,h} + B_h) \leq C' \lambda_j(\alpha_0^{W,h}). \quad (16)$$

Remark 5.8. *Of course $\sigma(\alpha_0^{W,h}) = \{h(d/2 + |n|), n \in \mathbb{N}^d\}$ and the bounds (16) are actually written in order to use this later. But for an easy use of the min-max principle it is better to write the eigenvalues $\lambda_j(\alpha_0^{W,h})$ in the increasing order, with repetition according to their multiplicity.*

Proof of Lemma 5.7. We start by noting that $1 + \alpha \in S(\langle X \rangle^2, \frac{dX^2}{\langle X \rangle^2})$ is fully elliptic in the sense that $(1 + \alpha)^{-1} \in S(\langle X \rangle^{-2}, \frac{dX^2}{\langle X \rangle^2})$. Therefore

$$(1 + \alpha) \sharp^{W,h} \frac{1}{1 + \alpha} = 1 + h^2 R_+(h) \quad , \quad \frac{1}{1 + \alpha} \sharp^{W,h} (1 + \alpha) = 1 + h^2 R_-(h)$$

with $R_\pm(h)$ uniformly bounded in $S(\langle X \rangle^{-2}, \frac{dX^2}{\langle X \rangle^2})$. The semiclassical calculus with the metric $\frac{dX^2}{\langle X \rangle^2}$, then says

$$(1 + \alpha^{W,h})^{-1} = [(1 + \alpha)^{-1}]^{W,h} + \mathcal{O}(h^2) \quad \text{in } S\left(\langle X \rangle^{-2}, \frac{dX^2}{\langle X \rangle^2}\right). \quad (17)$$

The same of course also holds for $\tau\alpha_0(X) = \tau\frac{|X|^2}{2}$ with $\tau \in (0, +\infty)$ fixed. Therefore $\alpha^{W,h} + B_h$ and $\alpha_0^{W,h}$ are self-adjoint with the same domain $D(\alpha^{W,h}) = D(\alpha_0^{W,h}) = D(\alpha_0^{W,1})$, and they have a compact resolvent. We shall collect all the necessary information by comparing the eigenvalues of $\alpha^{W,h} + B_h$ and $\alpha_0^{W,h}$ in the intervals $(-\infty, 2|\beta|h]$, $[0, 2]$ and $[1, +\infty[$, with $|\beta| = \sum_{j=1}^d \beta_j$. For the first part, we refer to the ready-made simple statement of [ChVN]-Theorem 4.5 and complete the other parts with simple pseudodifferential calculus and the min-max principle.

Interval $(-\infty, 2|\beta|h]$: By Theorem 4.5 of [ChVN], there exist a family of real numbers $(\omega_n^h)_{h>0, n \in \mathbb{N}^d}$ and, for any $t > 0$, a constant $C_t > 0$ such that

$$\sigma(\alpha^{W,h}) \cap (-\infty, th] = \left\{ \omega_n^h, n \in \mathbb{N}^d \right\} \cap [|\beta|h/2, th]$$

and

$$\left| \omega_n^h - \sum_{j=1}^d h\beta_j \left(\frac{1}{2} + n_j \right) \right| \leq C_t h^{3/2}.$$

As $\|B_h\| = o(h)$, the min-max principle with $\alpha^{W,h}$ and $\alpha^{W,h} + B_h$ then gives,

$$\sigma(\alpha^{W,h} + B_h) \cap (-\infty, th] = \left\{ \omega_n^h + o(h), n \in \mathbb{N} \right\} \cap [0, th].$$

By choosing $t = 2|\beta|$, the operator $\alpha^{W,h} + B_h$ is non-negative with $\lambda_0(\alpha^{W,h} + B_h) = |\beta|h/2 + o(h)$ and the spectral gap is bounded from below by

$$\begin{aligned} \forall j \in \mathbb{N} \setminus \{0\}, \quad \lambda_j(\alpha^{W,h} + B_h) - \lambda_0(\alpha^{W,h} + B_h) &\geq \lambda_1(\alpha^{W,h} + B_h) - \lambda_0(\alpha^{W,h} + B_h) \\ &\geq \beta_m h + o(h) \geq \beta_m h/2, \end{aligned} \quad (18)$$

with $\beta_m = \min\{\beta_1, \dots, \beta_d\}$.

Let $T \in Sp_{2d}(\mathbb{R}^d)$ be such that ${}^t X^t T^{-1} \text{Hess } \alpha(0) T^{-1} X = \sum_{j=1}^d \beta_j X_j^2$, let U_T be a unitary operator such that $U_T^* b^W U_T = b(T^{-1} \cdot)^W$ and set $\varphi_T(x) = (\pi)^{-d/4} U_T e^{-\frac{x^2}{2}}$. We compute

$$\begin{aligned} \langle \varphi_T, (\alpha^{W,h} + B_h) \varphi_T \rangle &= \text{Tr} \left[U_T^* \alpha^{W,h} U_T |\varphi_{\text{Id}}\rangle \langle \varphi_{\text{Id}}| \right] + o(h) \\ &= \int_{\mathbb{R}^{2d}} \alpha(\sqrt{h} T^{-1} X) e^{-|X|^2} \frac{dX}{\pi^d} + o(h). \end{aligned}$$

But since $\alpha(T^{-1}X) = \sum_{j=1}^d \beta_j |X_j|^2/2 + P_3(X) + \mathcal{O}(|X|^4)$, with P_3 a homogeneous polynomial of degree 3, we obtain

$$\langle \varphi_T, (\alpha^{W,h} + B_h) \varphi_T \rangle = h|\beta|/2 + o(h) = \lambda_0(\alpha^{W,h} + B_h) + o(h).$$

With the spectral gap (18) this implies that the ground state ψ_0^h of $\alpha^{W,h} + B_h$ satisfies $\lim_{h \rightarrow 0} \|\psi_0^h - \varphi_T\|_{L^2} = 0$ and

$$\lim_{h \rightarrow 0} \|1_{\{\lambda_0(\alpha^{W,h} + B_h)\}} (\alpha^{W,h} + B_h) - \pi^{-d} (e^{-|TX|^2})^{W,1}\|_{\mathcal{L}^1} = 0.$$

Interval $[0, 2]$: Our assumptions on α provide a constant $C_2 \geq 1$ such that $C_2^{-1}\alpha_0 \leq \alpha \leq C_2\alpha_0$ and therefore $\frac{C_2^{-1}\alpha_0}{1+C_2^{-1}\alpha_0} \leq \frac{\alpha}{1+\alpha} \leq \frac{C_2\alpha_0}{1+C_2\alpha_0}$, as $x \mapsto \frac{x}{1+x}$ is increasing on \mathbb{R}^* . Since all those symbols

belong to $S(1, dX^2)$, the semiclassical Fefferman-Phong inequality for the constant metric dX^2 (see [Hor3]-Lemma 18.6.10) says

$$\frac{C_2^{-1}\alpha_0^{W,h}}{1 + C_2^{-1}\alpha_0^{W,h}} - \mathcal{O}(h^2) \leq \frac{\alpha^{W,h}}{1 + \alpha^{W,h}} \leq \frac{C_2\alpha_0^{W,h}}{1 + C_2\alpha_0^{W,h}} + \mathcal{O}(h^2),$$

after using $\left(\frac{\alpha}{1+\alpha}\right)^{W,h} = \frac{\alpha^{W,h}}{1+\alpha^{W,h}} + \mathcal{O}(h^2)$. With $\|(1+\alpha^{W,h})^{-1} - (1+\alpha^{W,h} + B_h)^{-1}\| = \mathcal{O}(\|B_h\|) = o(h)$ and $\frac{x}{1+x} = 1 - \frac{1}{1+x}$, we deduce

$$\frac{C_2^{-1}\alpha_0^{W,h}}{1 + C_2^{-1}\alpha_0^{W,h}} - o(h) \leq \frac{\alpha^{W,h} + B_h}{1 + \alpha^{W,h} + B_h} \leq \frac{C_2\alpha_0^{W,h}}{1 + C_2\alpha_0^{W,h}} + o(h).$$

For $r = 2(1 + C_2)$ and $h_0 > 0$ small enough the above operators have a discrete spectrum in $[0, \frac{r}{1+r}]$ with eigenvalues in this interval, while the function $x \mapsto \frac{x}{1+x}$ increases on $[0, +\infty)$. Hence the min-max principle implies that there exists $C_2' \geq 1$ such that

$$\begin{aligned} & \left(\lambda_j(\alpha^{W,h} + B_h) \leq 2 \right) \\ & \Rightarrow \left(C_2'^{-1}\lambda_j(\alpha_0^{W,h}) - o(h) \leq \lambda_j(\alpha^{W,h} + B_h) \leq C_2'\lambda_j(\alpha_0^{W,h}) + o(h) \right) \quad (19) \end{aligned}$$

holds for all $j \in \mathbb{N}$. With the spectral gap (18) and $\lambda_0(\alpha^{W,h} + B_h) = |\beta|h/2 + o(h)$ we conclude that (16) holds when $\lambda_j(\alpha^{W,h} + B_h) \leq 2$.

Interval $[1, +\infty)$: Our assumptions on α provide a constant $C_1 \geq 1$ such that $C_1^{-2} \leq \left(\frac{1+\alpha_0}{1+\alpha}\right)^2 \leq C_1^2$. With (17), the semiclassical Garding inequality then gives for h_0 small enough:

$$\max \left\{ \|(1 + \alpha_0^{W,h})(1 + \alpha^{W,h})^{-1}\|, \|(1 + \alpha^{W,h})(1 + \alpha_0^{W,h})^{-1}\| \right\} \leq 2C_1$$

Owing to $\|B_h\| = o(h)$ this is also true when $\alpha^{W,h}$ is replaced by $\alpha^{W,h} + B_h$. We obtain

$$\begin{aligned} & \forall \psi \in D(\alpha_0^{W,1}), \\ & (2C_1)^{-2} \langle \psi, (1 + \alpha_0^{W,h})^2 \psi \rangle \leq \langle \psi, (1 + \alpha^{W,h} + B_h)^2 \psi \rangle \leq (2C_1)^2 \langle \psi, (1 + \alpha_0^{W,h})^2 \psi \rangle. \end{aligned}$$

and the min-max principle gives

$$\forall j \in \mathbb{N}, (2C_1)^{-2} \lambda_j((1 + \alpha_0^{W,h})^2) \leq \lambda_j((1 + \alpha^{W,h} + B_h)^2) \leq (2C_1)^2 \lambda_j((1 + \alpha_0^{W,h})^2).$$

By taking the square roots

$$\forall j \in \mathbb{N}, (2C_1)^{-1} (1 + \lambda_j(\alpha_0^{W,h})) \leq 1 + \lambda_j(\alpha^{W,h} + B_h) \leq 2C_1 (1 + \lambda_j(\alpha_0^{W,h})).$$

yields (16) for $\lambda_j(\alpha^{W,h} + B_h) \geq 1$. □

Proposition 5.6. With $H = \alpha^{W,h} + B_h - \lambda_0(\alpha^{W,h} + B_h)$, Lemma 5.7 provides a constant $C' > 0$ such that

$$\forall j \in \mathbb{N} \setminus \{0\}, \quad C'^{-1} \lambda_j(\alpha_0^{W,h}) \leq \lambda_j(H) \leq C' \lambda_j(\alpha_0^{W,h})$$

while $\lambda_0(H) = 0$ and the ground state of H is the same as the one of $\alpha^{W,h} + B_h$. When the function f is non-negative and decaying, we deduce

$$\begin{aligned} \text{Tr} \left[f(H + ch^{d/\kappa_0}) \right] &= f(ch^{d/\kappa_0}) + \sum_{j=1}^{\infty} f(\lambda_j(H) + ch^{d/\kappa_0}) \\ &\leq f(ch^{d/\kappa_0}) + \sum_{j=1}^{\infty} f(\lambda_j(H)) \leq f(ch^{d/\kappa_0}) + \sum_{\substack{n \in \mathbb{N}^d \\ n \neq 0}} f(C_4^{-1}h|n|), \end{aligned} \quad (20)$$

with $C_4 = C_3 \left(1 + \frac{4|\beta|}{\beta_m}\right)$; and for $R > 0$,

$$\text{Tr} \left[f(H + ch^{d/\kappa_0}) 1_{[R, +\infty)}(H) \right] = \sum_{\lambda_j(H) \geq R} f(\lambda_j(H) + ch^{d/\kappa_0}) \leq \sum_{\substack{n \in \mathbb{N}^d \\ h|n| \geq R/(2C_3)}} f(C_4^{-1}h|n|).$$

Apply (20) first, with $f = s^{-\kappa_0} \langle s \rangle^{-\kappa_\infty + \kappa_0}$:

$$h^d \text{Tr} \left[f(H + ch^{d/\kappa_0}) \right] \leq c^{-\kappa_0} + Ch^d \sum_{n \in \mathbb{N}^d, n \neq 0} (h|n|)^{-\kappa_0} \langle h|n| \rangle^{-\kappa_\infty + \kappa_0}$$

after splitting the sum into $\sum_{h|n| \leq 1}$ and $\sum_{h|n| \geq 1}$ and with $\#\{n \in \mathbb{N}^d, |n| = m\} = C_{m+d-1}^{d-1} = \mathcal{O}(m^{d-1})$, it becomes

$$\begin{aligned} h^d \text{Tr} \left[f(H + ch^{d/\kappa_0}) \right] &\leq c^{-\kappa_0} + C'h^d \sum_{m=1}^{\lceil h^{-1} \rceil} h^{-\kappa_0} m^{d-1-\kappa_0} + C'h^d \sum_{m=\lfloor h^{-1} \rfloor}^{\infty} h^{-\kappa_\infty} m^{d-1-\kappa_\infty} \\ &\leq c^{-\kappa_0} + C''h^{d-\kappa_0} \lceil h^{-1} \rceil^{d-\kappa_0} + C'''h^{d-\kappa_\infty} \lfloor h^{-1} \rfloor^{d-\kappa_\infty} \leq c^{-\kappa_0} + C'''. \end{aligned}$$

owing to $\kappa_\infty > d$ and $\kappa_0 \in (0, d)$. With a function $f(s) = s^{-\kappa_0} \chi(s/\delta)$ with $0 \leq \chi \leq 1$ compactly supported and decaying on $[0, +\infty)$ we get similarly

$$\lim_{\delta \rightarrow 0^+} \limsup_{h \rightarrow 0} h^d \text{Tr} \left[f(H + ch^{d/\kappa_0}) \right] - c^{-\kappa_0} = 0,$$

while with $f(s) = \langle s \rangle^{-\kappa_\infty}$, the truncated trace $\text{Tr} \left[f(H + ch^{d/\kappa_0}) 1_{[\delta^{-1}, +\infty)}(H) \right]$ satisfies

$$\lim_{\delta \rightarrow 0^+} \limsup_{h \rightarrow 0} h^d \text{Tr} \left[f(H + ch^{d/\kappa_0}) 1_{[\delta^{-1}, +\infty)}(H) \right] = 0.$$

The comparison of $\lambda_j(H)$ with $\lambda_j(\alpha_0^{W,h})$, $j \in \mathbb{N}$, stated in Lemma 5.7 does not depend on the parameter c . Neither do the constants C_3, C_4, C, C', C'' and C''' (f is non-negative and decaying) depend on c . Therefore the previous asymptotic trace estimates are uniform with respect to $c \in (\frac{1}{A}, A)$ for any fixed $A > 1$.

Thus if $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ is a cut-off function such that $0 \leq \chi \leq 1$, $\chi \equiv 1$ in $(-1, 1)$ and if a general $f \in \mathcal{C}^\infty((0, +\infty))$ fulfills all the assumptions of Prop. 5.6, then

$$\lim_{\delta \rightarrow 0^+} \limsup_{h \rightarrow 0^+} h^d \left\| [f(H + ch^{d/\kappa_0}) 1_{(0, +\infty)}(H) [\chi(\delta^{-1}H) + (1 - \chi(\delta H))]] \right\|_{\mathcal{L}^1} = 0. \quad (21)$$

For $g \in \mathcal{C}_0^\infty(\mathbb{R})$, with an almost analytic extension $\tilde{g} \in \mathcal{C}_0^\infty(\mathbb{C})$, Helffer-Sjöstrand formula

$$g(\alpha^{W,h}) = \frac{1}{2i\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{g}(z) (z - \alpha^{W,h}) dz \wedge d\bar{z},$$

combined with the semiclassical Beals criterion [DiSj, HeNi, NaNi] with the constant metric dX^2 implies that

$$g(\alpha^{W,h}) - g(\alpha)^{W,h} = h r(h)^{W,h},$$

with $r(h)$ uniformly bounded (with respect to h) in $S(1, dX^2)$. Since $(1 + \alpha) \in S(\langle X \rangle^2, \frac{dX^2}{\langle X \rangle^2})$ is an invertible elliptic symbol, $(1 + \alpha^{W,h})^{-N} - [(1 + \alpha)^{-N}]^{W,h} = h^2 r'(h)^{W,h}$ with $r'(h)$ uniformly bounded in $S(\langle X \rangle^{-2N-2}, \frac{dX^2}{\langle X \rangle^2}) \subset S(\langle X \rangle^{-2N}, dX^2)$. For a function $f_\delta \in \mathcal{C}_0^\infty((0, +\infty))$, we take $g(s) = (1 + s)^N f_\delta(s)$ and write

$$f_\delta(\alpha^{W,h}) = g(\alpha^{W,h})(1 + \alpha^{W,h})^{-N},$$

so that

$$\begin{aligned} f_\delta(\alpha^{W,h}) - f_\delta(\alpha)^{W,h} &= \left[g(\alpha^{W,h}) - g(\alpha)^{W,h} \right] (1 + \alpha^{W,h})^{-N} \\ &\quad + g(\alpha)^{W,h} (1 + \alpha^{W,h})^{-N} - f_\delta(\alpha)^{W,h} = h r''(h)^{W,h}, \end{aligned}$$

with $r''(h)$ uniformly bounded in $S(\langle X \rangle^{-2N}, dX^2)$. In particular, $h^d r''(h)^{W,h}$ is uniformly bounded in $\mathcal{L}^1(L^2(\mathbb{R}^d))$ if we choose $N > d$.

Similarly, Helffer-Sjöstrand formula can be used to prove $g(H + ch^{d/\kappa_0}) - g(\alpha^{W,h}) = o(h)$ in $\mathcal{L}(L^2(\mathbb{R}^d))$. With $h^d [(1 + H + ch^{d/\kappa})^{-N} - (1 + \alpha^{W,h})^{-N}] = o(h)$ in $\mathcal{L}^1(L^2(\mathbb{R}^d))$ due to

$$(1 + H + ch^{d/\kappa_0})^{-1} = \left[1 + (1 + \alpha^{W,h})^{-1} (B_h + ch^{d/\kappa_0}) \right]^{-1} (1 + \alpha^{W,h})^{-1}$$

the same trick as above transforms the $\mathcal{L}(L^2(\mathbb{R}^d))$ estimate into

$$h^d \left[f_\delta(H + ch^{d/\kappa_0}) - f_\delta(\alpha^{W,h}) \right] = o(h) \quad \text{in } \mathcal{L}^1(L^2(\mathbb{R}^d)) \quad (22)$$

Note again that this holds uniformly with respect to $c \in (\frac{1}{A}, A)$ for any fixed $A > 1$.

Now take $f_\delta(s) = (1 - \chi(\delta^{-2}s))\chi(\delta^2s)f(s)$ for which we note that the inequality

$$\forall s \geq 0, \quad 1 - (1 - \chi(\delta^{-2}s))\chi(\delta^2s) \leq \chi(\delta^{-1}s) + (1 - \chi(\delta s))$$

as soon as $\delta < \delta_{chi}$ implies

$$\forall s \geq 0, \quad 0 \leq f(s) - f_\delta(s) \leq f(s)[\chi(\delta^{-1}s) + (1 - \chi(\delta s))]. \quad (23)$$

In the expression $h^d \text{Tr}[f(H + ch^{d/\kappa_0}) a^{W,h}]$, decompose $f(H + ch^{d/\kappa_0})$ into

$$\underbrace{f_\delta(H + ch^{d/\kappa_0})}_{(I)} + \underbrace{(f(H + ch^{d/\kappa_0}) - f_\delta(H + ch^{d/\kappa_0}))1_{(0,+\infty)}(H)}_{(II)} + \underbrace{1_{\{0\}}(H)f(ch^{d/\kappa_0})}_{(III)}.$$

We now conclude with the following steps:

- The estimate (22) yields

$$\begin{aligned} \lim_{h \rightarrow 0} h^d \text{Tr} \left[f_\delta(H + ch^{d/\kappa_0}) a^{W,h} \right] &= \lim_{h \rightarrow 0} h^d \text{Tr} \left[f_\delta(\alpha)^{W,h} a^{W,h} \right] \\ &= \int_{\mathbb{R}^{2d}} f_\delta(\alpha(X)) a(X) \frac{dX}{(2\pi)^d}, \end{aligned}$$

which provides the contribution of (I).

- The upperbound (23) combined with (21) leads to

$$\lim_{\delta \rightarrow 0^+} \limsup_{h \rightarrow 0} \left| h^d \text{Tr} \left[[f(H + ch^{d/\kappa_0}) - f_\delta(H + ch^{d/\kappa_0})] 1_{(0,+\infty)}(H) a^{W,h} \right] \right| = 0,$$

which says that (II) has a null contribution in the limit $\delta \rightarrow 0$.

- The contribution of (III) is simply computed as

$$h^d \text{Tr} \left[f(H + ch^{d/\kappa_0}) 1_{\{0\}}(H) a^{W,h} \right] = \frac{f_0}{c^{\kappa_0}} \langle \psi_0^h, a^{W,h} \psi_0^h \rangle$$

where ψ_0^h is the ground state of $H + ch^{d/\kappa_0}$ with $\|\psi^h - \pi^{-d/4} e^{-x^2/2}\| \xrightarrow{h \rightarrow 0} 0$. This implies $\lim_{h \rightarrow 0} \langle \psi^h, a^{W,h} \psi^h \rangle = a(0)$.

- Finally, the assumptions on f ensure $f(\alpha) \in L^1(\mathbb{R}^{2d})$ and

$$\lim_{\delta \rightarrow 0} \int_{\mathbb{R}^{2d}} f_\delta(\alpha(X)) a(X) dX = \int_{\mathbb{R}^{2d}} f(\alpha(X)) a(X) dX.$$

□

5.2.4 Semiclassical analysis of the reduced density matrices in the bosonic case

Proof of Proposition 5.4. This will be made in two parts: We first compute the semiclassical measures $\nu^{(p)}$ and then identify the weak-* limit $\gamma_0^{(p)}$.

For the first part Proposition 4.4 says that it suffices to find the limit $\Phi_{a,0}(s)$ of $\Phi_{a,h}(s)$ for $a \in \mathcal{C}_0^\infty(T^*\mathbb{R}^d)$, real-valued, and $s \in (-r_a, r_a)$. Actually Proposition 5.6 allows to consider more generally $a \in S(1, dX^2)$. For $a \in S(1, dX^2)$, real-valued, take $s \in \mathbb{R}$, $|s| < r_a = \frac{1}{\nu_C C_a}$, $4\|a^{W,h}\| \leq C_a$ and set

$$DT_{a,h}(s) = \log \text{Tr} [\varrho_\varepsilon \Gamma(e^{\varepsilon s a})] = -\text{Tr} [\log(1 - CB_s)] + \text{Tr} [\log(1 - C)]$$

$$\text{and } \Phi_{a,h}(s) = \text{Tr} \left[\varrho_\varepsilon \Gamma(e^{\varepsilon s a^{W,h}}) \right] = \exp DT_{a,h}(s), \quad \varepsilon = h^d,$$

with $C = e^{-\beta(H + \frac{\varepsilon}{\beta \nu_C})}$ and $B_s = e^{\varepsilon s a^{W,h}}$.

Assume $s \in (-r_a, r_a)$ and compute

$$\begin{aligned} DT_{a,h}(s) &= \int_0^1 \text{Tr} \left[\frac{C_{ts} \tilde{B}_{ts}}{1 - C_{ts} \tilde{B}_{ts}} \varepsilon s a^{W,h} \right] dt \\ &= \int_0^1 \text{Tr} \left[\varepsilon s f \left(H + \frac{\varepsilon}{\beta} (\nu_C^{-1} - t s a(0)) \right) a^{W,h} \right] dt \\ &\quad + \int_0^1 \text{Tr} \left[\varepsilon s [-(1 - C_{ts})^{-1} + (1 - C_{ts} \tilde{B}_{ts})^{-1}] a^{W,h} \right] dt \end{aligned}$$

with $C_{ts} = e^{-\beta(H + \frac{\varepsilon}{\beta}(\nu_C^{-1} - tsa(0)))}$, $\tilde{B}_{ts} = e^{\varepsilon ts(a-a(0))^{W,h}}$ and $f(u) = \frac{e^{-\beta u}}{1 - e^{-\beta u}}$. Note that for $t \in [0, 1]$ the parameter $\frac{1}{\beta}(\nu_C^{-1} - tsa(0))$ remains in a compact subset of $(0, +\infty)$. Prop. 5.6 implies for all $t \in [0, 1]$:

$$\begin{aligned} \lim_{h \rightarrow 0} \text{Tr} \left[\varepsilon s f \left(H + \frac{\varepsilon}{\beta} (\nu_C^{-1} - tsa(0)) \right) a^{W,h} \right] \\ = \frac{\nu_C sa(0)}{1 - t\nu_C sa(0)} + s \int_{\mathbb{R}^{2d}} \frac{e^{-\beta\alpha(X)}}{1 - e^{-\beta\alpha(X)}} a(X) \frac{dX}{(2\pi)^d}. \end{aligned}$$

With the uniform control with respect to $\frac{1}{\beta}(\nu_C^{-1} - tsa(0)) = c \in [\frac{1}{A}, A]$ in Proposition 5.6, we obtain for the first term

$$\begin{aligned} \lim_{h \rightarrow 0} \int_0^1 \text{Tr} \left[\varepsilon s f \left(H + \frac{\varepsilon}{\beta} (\nu_C^{-1} - tsa(0)) \right) a^{W,h} \right] dt \\ = -\log(1 - s\nu_C a(0)) + s \int_{\mathbb{R}^{2d}} \frac{e^{-\beta\alpha(X)}}{1 - e^{-\beta\alpha(X)}} a(X) \frac{dX}{(2\pi)^d}. \end{aligned}$$

For the remainder term, introduce $\Pi_0^h = |\psi_0^h\rangle\langle\psi_0^h|$, where $\psi_0^h = U_T(\pi^{-d/4}e^{-\frac{x^2}{2}}) + o(h^0)$ is the ground state of H , and write

$$\begin{aligned} (1 - C_{ts}\tilde{B}_{ts}) \\ = 1 - C_{ts} - C_{ts}(\tilde{B}_{ts} - 1) &= (1 - C_{ts}) \left[1 + \frac{C_{ts}}{1 - C_{ts}}(1 - \tilde{B}_{ts}) \right] \\ = (1 - C_{ts}) \left[1 + \frac{C_{ts}}{1 - C_{ts}}\Pi_0^h(1 - \tilde{B}_{ts}) + \frac{C_{ts}}{1 - C_{ts}}(1 - \Pi_0^h)(1 - \tilde{B}_{ts}) \right] \\ = (1 - C_{ts}) \left[\underbrace{1 + f\left(\frac{\varepsilon}{\beta}(\nu_C^{-1} - tsa(0))\right)\Pi_0^h(1 - \tilde{B}_{ts})}_I + \underbrace{\frac{C_{ts}}{1 - C_{ts}}(1 - \Pi_0^h)(1 - \tilde{B}_{ts})}_{II} \right]. \end{aligned}$$

We know

$$\varepsilon \times f\left(\frac{\varepsilon}{\beta}(\nu_C^{-1} - tsa(0))\right) = \frac{1}{\nu_C^{-1} - tsa(0)} + O(\varepsilon) = \frac{1}{\nu_C^{-1} - tsa(0)} + o(h).$$

We write

$$\varepsilon^{-1}(1 - \tilde{B}_{ts})\psi_0^h = - \int_0^1 e^{\varepsilon uts(a-a(0))^{W,h}} ts(a-a(0))^{W,h}\psi_0^h du,$$

where $\psi_0^h = \pi^{-d/4}U_T e^{-\frac{x^2}{2}} + o(h^0)$ and $a(X) - a(0) \leq C \min\{1, |X|\}$ for some $C > 0$ imply $\lim_{h \rightarrow 0} \|(a - a(0))^{W,h}\psi_0^h\|_{L^2(\mathbb{R}^d)} = 0$. Therefore the second term in the above bracket satisfies

$$I = f\left(\frac{\varepsilon}{\beta}(\nu_C^{-1} - tsa(0))\right)\Pi_0^h(1 - \tilde{B}_{ts}) = o(h^0) \quad \text{in } \mathcal{L}^1(L^2(\mathbb{R}^d)).$$

Note that we have also proved

$$(1 - \tilde{B}_{ts})\Pi_0^h - \Pi_0^h(1 - \tilde{B}_{ts}) = o(\varepsilon) \quad \text{in } \mathcal{L}(L^2(\mathbb{R}^d)).$$

By using

$$\|1 - \tilde{B}_{ts}\| = \mathcal{O}(\varepsilon) \quad , \quad \left\| \frac{C_{ts}}{1 - C_{ts}}(1 - \Pi_0^h) \right\| = \mathcal{O}\left(\frac{1}{h}\right),$$

and

$$\begin{aligned} \lim_{h \rightarrow 0} \left\| \varepsilon \frac{C_{ts}}{1 - C_{ts}} \right\|_{\mathcal{L}^1} &= \lim_{h \rightarrow 0} \text{Tr} \left[\varepsilon f \left(H + \frac{\varepsilon}{\beta} (\nu_C^{-1} - tsa(0)) \right) \right] \\ &= \frac{\nu_C}{1 - t\nu_C sa(0)} + s \int_{\mathbb{R}^{2d}} \frac{e^{-\beta\alpha(X)}}{1 - e^{-\beta\alpha(X)}} \frac{dX}{(2\pi)^d}, \end{aligned}$$

the third term in the above bracket satisfies

$$\begin{aligned} \|II\|_{\mathcal{L}^1} &= \mathcal{O}(1) \quad , \quad \|II\| = \mathcal{O}(\varepsilon h^{-1}) = o(h^0), \\ \|II - \frac{C_{ts}}{1 - C_{ts}}(1 - \Pi_0^h)(1 - \tilde{B}_{ts})(1 - \Pi_0^h)\|_{\mathcal{L}^1} &= o(h^0). \end{aligned}$$

Again all these estimates are uniform with respect to $t \in [0, 1]$ owing to the uniformity of the estimates in Proposition 5.6 with respect to $c = \frac{1}{\beta}(\nu_C^{-1} - tsa(0))$. By expanding the Neumann series $(1 + I + II)^{-1} = \sum_{k=0}^{\infty} (-1)^k (I + II)^k$ we deduce

$$\left[1 + \frac{C_{ts}}{1 - C_{ts}}(1 - \tilde{B}_{ts}) \right]^{-1} = 1 - \frac{C_{ts}}{1 - C_{ts}}(1 - \Pi_0^h)(1 - \tilde{B}_{ts})(1 - \Pi_0^h) + R_h$$

with $\|R_h\|_{\mathcal{L}^1} = o(h^0)$. With $\|\varepsilon(1 - C_{ts})^{-1}\| = \mathcal{O}(1)$ we finally obtain

$$\begin{aligned} \varepsilon s [(1 - C_{ts})^{-1} - (1 - C_{ts}\tilde{B}_{ts})^{-1}] \\ = \frac{s\varepsilon C_{ts}}{1 - C_{ts}}(1 - \Pi_0^h)(1 - \tilde{B}_{ts})(1 - \Pi_0^h)(1 - C_{ts})^{-1} + R'_h, \quad \|R'_h\|_{\mathcal{L}^1} = o(h^0), \end{aligned}$$

while $\|\varepsilon C_{ts}(1 - C_{ts})^{-1}\|_{\mathcal{L}^1} = \mathcal{O}(1)$, $\|1 - \tilde{B}\| = \mathcal{O}(\varepsilon)$ and $\|(1 - \Pi_0^h)(1 - C_{ts})^{-1}\| = \mathcal{O}(h^{-1})$. With $4\|a^{W,h}\| \leq C_a$, the remainder term tends to 0 as $h \rightarrow 0$ and we have proved

$$\begin{aligned} \forall s \in (-r_a, r_a), \quad \lim_{h \rightarrow 0} \Phi_{a,h}(s) &= \Phi_{a,0}(s) \\ &= \frac{1}{1 - s\nu_C a(0)} \exp \left[s \int_{\mathbb{R}^{2d}} \frac{e^{-\beta\alpha(X)}}{1 - e^{-\beta\alpha(X)}} a(X) \frac{dX}{(2\pi)^d} \right]. \end{aligned}$$

By expanding the generating function according to Proposition 4.4, we obtain

$$\lim_{h \rightarrow 0} \text{Tr} \left[\varrho_\varepsilon((a^{W,h})^{\otimes p})^{Wick} \right] = \sum_{k=0}^p \frac{1}{(p-k)!} \nu_C^k a(0)^k \int_{\mathbb{R}^{2d(p-k)}} a^{\otimes(p-k)} d\nu(\beta)^{\otimes p-k}.$$

with $d\nu(\beta) = \frac{e^{-\beta\alpha(X)}}{1 - e^{-\beta\alpha(X)}} \frac{dX}{(2\pi)^d}$. The possibility to take $a \in S(1, dX^2)$ contains the fact that our quantization is adapted to all the $\gamma_h^{(p)}$.

Now in order to identify the weak* limits of the $\gamma_h^{(p)}$ we compute the Wigner measure associated with $\varrho_{\varepsilon(h)}$. Remember (see (24) and (25))

$$\text{Tr} \left[\varrho_\varepsilon W(\sqrt{2\pi}f) \right] = \exp \left[-\frac{\varepsilon\pi^2}{2} \left\langle f, \frac{1 + e^{-\beta(H + \frac{\varepsilon}{\beta\nu_C})}}{1 - e^{-\beta(H + \frac{\varepsilon}{\beta\nu_C})}} f \right\rangle \right].$$

By using the orthonormal basis of eigenvectors $(\psi_j^h)_{j \in \mathbb{N}}$ of H with associated eigenvalues λ_j^h , $\lambda_0^h = 0$, $\lambda_j^h \geq ch$ for $j > 0$, we obtain

$$\log \left(\text{Tr} \left[\varrho_{\varepsilon(h)} W(\sqrt{2\pi}f) \right] \right) = -\pi^2 \nu_C |\langle f, \psi_0^h \rangle|^2 + \mathcal{O}(\varepsilon h^{-1}).$$

With $\|\psi_0^h - \psi_0\|_{L^2} = o(h)$, $\psi_0(x) = \pi^{-d/4} U_T e^{-\frac{x^2}{2}}$, we obtain after decomposing $f = f_0 \psi_0 \oplus^\perp f'$

$$\int_{L^2} e^{2i\pi \text{Re} \langle f, z \rangle} d\mu(z) = \lim_{h \rightarrow 0} \text{Tr} \left[\varrho_{\varepsilon(h)} W(\sqrt{2\pi}f) \right] = e^{-\pi^2 \nu_C |f_0|^2}.$$

We deduce, like in [AmNi1, Section 7.5] or [AmNi3, Section 4.4],

$$\mathcal{M}(\varrho_{\varepsilon(h)}, h \in \mathcal{E}) = \left\{ \left(\frac{e^{-\frac{|z_0|^2}{\nu_C}}}{\pi \nu_C} L(dz_0) \right) \otimes \delta_0(z') \right\} \quad (z = z_0 \psi_0 \oplus^\perp z'),$$

$$\text{and} \quad \gamma_0^{(p)} = p! \nu_C^p |\psi_0^{\otimes p}\rangle \langle \psi_0^{\otimes p}|, \forall p \in \mathbb{N}.$$

The fact that $\nu_{(I)}^{(p)} \equiv 0$ for all $p \in \mathbb{N}$, now comes from

$$\text{Tr} \left[\gamma_0^{(p)} \right] = p! \nu_C^p = \nu^{(p)}(\{0\}).$$

□

A Multiscale Measures

We now recall facts about multiscale measures, introduced in [FeGe, Fer]. For this we need a new class of symbols. Let $D', D'', D''' \in \mathbb{N}$ be such that $D' + D'' + D''' = D$ and set $F = \{X = (x', x'', x''', \xi', \xi'', \xi''') \in \mathbb{R}^{2D}, x' = 0, x'' = \xi'' = 0\}$. The class of symbols $S_F^{(2)}$ is defined as the set of $(X, Y) \rightarrow a(X, Y) \in \mathcal{C}^\infty(\mathbb{R}^{2D} \times \mathbb{R}^{D'+2D''})$, (note that $\mathbb{R}^{D'+2D''} \cong F^\perp$, hence the notation $S_F^{(2)}$) such that

- there exists $C > 0$ such that: $\forall Y \in \mathbb{R}^{D'+2D''}$, $a(\cdot, Y) \in \mathcal{C}_0^\infty(B(0, C))$;
- there exists a function $a_\infty \in \mathcal{C}_0^\infty(\mathbb{R}^{2D} \times \mathbb{S}^{D'+2D''-1})$ such that $a(X, R\omega) \xrightarrow{R \rightarrow \infty} a_\infty(X, \omega)$ in $\mathcal{C}^\infty(\mathbb{R}^{2D} \times \mathbb{S}^{D'+2D''-1})$.

Those symbols are quantized according to

$$a^{(2),h} = a_h^{W,h} \quad , \quad a_h(X) = a\left(X, \frac{x'}{h^{1/2}}, \frac{X''}{h^{1/2}}\right) \quad X = (x', x'', x''', \xi', \xi'', \xi''').$$

Theorem 0.1 in [Fer], which also considers the case when $(\frac{x'}{h^{1/2}}, \frac{X''}{h^{1/2}})$ is replaced by $(\frac{x'}{h^s}, \frac{X''}{h^s})$, $s < \frac{1}{2}$, says the following.

Proposition A.1. *Let $(\gamma_h)_{h \in \mathcal{E}}$ be a bounded family of non-negative trace-class operators on $L^2(\mathbb{R}^{2D})$ with $\lim_{h \rightarrow 0} \text{Tr}[\gamma_h] < +\infty$. There exist $\mathcal{E}' \subset \mathcal{E}$, $0 \in \overline{\mathcal{E}'}$, with $\mathcal{M}(\gamma_h, h \in \mathcal{E}') = \{\nu\}$, a non-negative measure $\nu_{(I)}$ on $F \times \mathbb{S}^{D'+2D''-1}$ and a $\mathcal{L}^1(L^2(\mathbb{R}^{2D'}))$ -measure m on $F \times \mathbb{R}^{D'}$, such that the convergence*

$$\begin{aligned} \lim_{h \in \mathcal{E}', h \rightarrow 0} \text{Tr} \left[\gamma_h a^{(2),h} \right] &= \int_{\mathbb{R}^{2D} \setminus F} a_\infty \left(X, \frac{(x', X'')}{|(x', X'')|} \right) d\nu(X) \\ &+ \int_{F \times \mathbb{S}^{D'+2D''-1}} a_\infty(X, \omega) d\nu_{(I)}(X, \omega) + \text{Tr} \left[\int_{F \times \mathbb{R}^{D'}} a(X, x', z, D_z) dm(X, x') \right] \end{aligned}$$

holds for all $a \in S_F^{(2)}$.

Remark A.2. *With this scaling and when $a^{W,h} = a^W(x, hD_x) = a(x, hD_x) + O(h)$, $t = 0$, Fermionic checked in [Fer] the equivariance by the semiclassical Egorov theorem. Hence, this construction is naturally extended to the case when $T^*\mathbb{R}^D$ is replaced by T^*M and F is replaced by a submanifold of $T^*\mathbb{R}^D$ on which the symplectic form has constant rank.*

In Prop. 3.5 we use the simple case of the above result when $D' = D''' = 0$ and $D'' = D$. Note that in this case $F \times \mathbb{R}^{D'} = \{0\}$ and the trace-class-valued measure is nothing but a trace-class operator γ_0 .

B Mean-field Wigner measures in the bosonic case and condition (PI)

The bosonic mean-field analysis is like a semiclassical analysis in infinite dimension. Let \mathcal{Z} be a separable complex Hilbert space and $\Gamma_+(\mathcal{Z})$ be the associated bosonic Fock space. With the scaled CCR relations

$$[a_+(g), a_+^*(f)] = \varepsilon \langle g, f \rangle \quad , \quad [a_+(g), a_+(f)] = [a_+^*(g), a_+^*(f)] = 0$$

and after setting

$$\Phi(f) = \frac{a_+(f) + a_+^*(f)}{\sqrt{2}} \quad , \quad W(f) = e^{i\Phi(h)} \quad , \quad (24)$$

mean-field Wigner measures were introduced in [AmNil]. Actually the parameter ε^{-1} represents the typical number of particles. Let $(\varrho_\varepsilon)_{\varepsilon \in \mathcal{E}}$, $0 \in \overline{\mathcal{E}}$, be a family of normal states (normalized non-negative trace-class operators) in $\Gamma_+(\mathcal{Z})$. Under the sole uniform estimate $\text{Tr}[\varrho_\varepsilon(1 + \mathbf{N})^\delta] \leq C_\delta$ for some $\delta > 0$, Wigner measures are defined as Borel probability measures on \mathcal{Z} and characterized by their characteristic function as follows: $\mu \in \mathcal{M}(\varrho_\varepsilon, \varepsilon \in \mathcal{E})$, iff there exists $\mathcal{E}' \subset \mathcal{E}$, $0 \in \overline{\mathcal{E}'}$, such that

$$\forall f \in \mathcal{Z} \quad , \quad \lim_{\substack{\varepsilon \in \mathcal{E}' \\ \varepsilon \rightarrow 0}} \text{Tr} \left[\varrho_\varepsilon W(\sqrt{2\pi}f) \right] = \int_{\mathcal{Z}} e^{2i\pi \text{Re} \langle f, z \rangle} d\mu(z) \quad . \quad (25)$$

Assuming $\text{Tr}[\varrho_\varepsilon \mathbf{N}_+^k] \leq C^k$ for all $k \in \mathbb{N}$ (or as we do in Hypothesis 4.1, $\text{Tr}[\varrho_\varepsilon e^{c\mathbf{N}_+}] \leq C$), $\mathcal{M}(\varrho_\varepsilon, \varepsilon \in \mathcal{E}) = \{\mu\}$ implies that

$$\lim_{\varepsilon \rightarrow 0} \text{Tr} \left[\varrho_\varepsilon \tilde{b}^{Wick} \right] = \int_{\mathcal{Z}} \langle z^{\otimes p}, \tilde{b}z^{\otimes p} \rangle d\mu(z) \quad (26)$$

holds for all compact $\tilde{b} \in \mathcal{L}^\infty(\mathcal{S}_+^p \mathcal{Z}^{\otimes p})$. In particular with the definition of non normalized reduced density matrices we obtain

$$\forall p \in \mathbb{N}, \quad w^* - \lim_{\varepsilon \rightarrow 0} \gamma_\varepsilon^{(p)} = \gamma_0^{(p)} = \int_{\mathcal{Z}} |z^{\otimes p}\rangle \langle z^{\otimes p}| d\mu(z).$$

This w^* -limit can be transformed to a $\|\cdot\|_{\mathcal{L}^1}$ iff the restriction to compact \tilde{b} in (26) can be removed. It actually suffices to check that (26) holds for $\tilde{b} \in \mathcal{L}^\infty(\mathcal{S}_+^p \mathcal{Z}^{\otimes p})$ and $\tilde{b} = \text{Id}_{\mathcal{S}_+^p \mathcal{Z}^{\otimes p}}$, as shows the following result.

Proposition B.1. *For a family $(\varrho_\varepsilon)_{\varepsilon \in \mathcal{E}}$ in $\mathcal{L}^1(\mathcal{H})$, $0 \in \bar{\mathcal{E}}$, such that $\varrho_\varepsilon \geq 0$, $\text{Tr}[\varrho_\varepsilon] = 1$, $\mathcal{M}(\varrho_\varepsilon, \varepsilon \in \mathcal{E}) = \{\mu\}$, the conditions (PI) and (P) are equivalent:*

$$\begin{aligned} \left((PI) : \quad \forall \alpha \in \mathbb{N}, \quad \lim_{\varepsilon \rightarrow 0} \text{Tr} [\varrho_\varepsilon \mathbf{N}^\alpha] = \int_{\mathcal{Z}} |z|^{2\alpha} d\mu(z) < \infty \right) \\ \Leftrightarrow \left((P) : \quad \forall b \in \mathcal{P}_{alg}(\mathcal{Z}), \quad \lim_{\varepsilon \rightarrow 0} \text{Tr} [\varrho_\varepsilon b^{Wick}] = \int_{\mathcal{Z}} b d\mu \right), \end{aligned}$$

where $\mathcal{P}_{p,q}(\mathcal{Z}) = \left\{ b : \mathcal{Z} \ni z \mapsto b(z) = \langle z^{\otimes q}, \tilde{b} z^{\otimes p} \rangle \in \mathbb{C}, \tilde{b} \in \mathcal{L}(\mathcal{S}_+^p \mathcal{Z}^{\otimes p}; \mathcal{S}_+^q \mathcal{Z}^{\otimes q}) \right\}$, and $\mathcal{P}_{alg}(\mathcal{Z}) = \bigoplus_{p,q \in \mathbb{N}}^{alg} \mathcal{P}_{p,q}(\mathcal{Z})$.

We give below the proof, which rectifies a minor mistake in [AmNi3].

Proof. For $\alpha \in \mathbb{N}^*$, $(|z|^{2\alpha})^{Wick} = \mathbf{N}(\mathbf{N}-\varepsilon) \dots (\mathbf{N}-(\alpha-1)\varepsilon)$. Hence the condition (PI) is equivalent to

$$(PI)' : \quad \forall \alpha \in \mathbb{N}, \quad \lim_{\varepsilon \rightarrow 0} \text{Tr} \left[\varrho_\varepsilon (|z|^{2\alpha})^{Wick} \right] = \int_{\mathcal{Z}} |z|^{2\alpha} d\mu(z) < \infty.$$

Hence the condition (PI) is a particular case of (P) and it is sufficient to prove $(PI)' \Rightarrow (P)$. From now, assume $(PI)'$.

We want to prove (P) for a general $b \in \mathcal{P}_{alg}(\mathcal{Z}) = \bigoplus_{p,q \in \mathbb{N}}^{alg} \mathcal{P}_{p,q}(\mathcal{Z})$. Let us first consider the “diagonal” case $b \in \mathcal{P}_{p,p}(\mathcal{Z})$, $p \in \mathbb{N}^*$. Using the decomposition $\tilde{b} = \tilde{b}_{R,+} - \tilde{b}_{R,-} + i\tilde{b}_{I,+} - i\tilde{b}_{I,-}$ with all the $\tilde{b}_\bullet \geq 0$ we can assume $\tilde{b} \geq 0$. For such a \tilde{b} , there exists a non-decreasing sequence $(\tilde{b}_n)_{n \geq 0}$ of non-negative compact operators in $\mathcal{L}^\infty(\mathcal{S}_+^p \mathcal{Z}^{\otimes p})$ such that $\lim_{n \rightarrow \infty} \tilde{b}_n = \tilde{b}$ in the weak operator topology. Recall from [AmNi3, Prop. 2.9] that the convergence in the (P) condition always holds when the kernel \tilde{b} is compact, thus

$$\forall n \in \mathbb{N}, \quad \int_{\mathcal{Z}} b_n d\mu = \lim_{\varepsilon \rightarrow 0} \text{Tr}[\varrho_\varepsilon b_n^{Wick}] \leq \liminf_{\varepsilon \rightarrow 0} \text{Tr}[\varrho_\varepsilon b^{Wick}].$$

Using $b_n(z) = \langle z^{\otimes p}, \tilde{b}_n z^{\otimes p} \rangle \rightarrow \langle z^{\otimes p}, \tilde{b} z^{\otimes p} \rangle = b(z)$ as $n \rightarrow \infty$ and Fatou’s lemma yield

$$\int_{\mathcal{Z}} b d\mu \leq \liminf_{\varepsilon \rightarrow 0} \text{Tr}[\varrho_\varepsilon b^{Wick}]. \quad (27)$$

The same arguments with \tilde{b} replaced by $|b|_{\mathcal{P}_{p,p}} \text{Id}_{\mathcal{S}_+^p \mathcal{Z}^{\otimes p}} - \tilde{b} \geq 0$ provide

$$\liminf_{\varepsilon \rightarrow 0} \text{Tr}[\varrho_\varepsilon (|\tilde{b}|_{\mathcal{S}_+^p \mathcal{Z}^{\otimes p}} |z|^{2p} - b(z))^{Wick}] \geq \int (|\tilde{b}|_{\mathcal{S}_+^p \mathcal{Z}^{\otimes p}} |z|^{2p} - b(z)) d\mu(z).$$

With $(PI)'$ condition, the $|z|^{2p}$ terms can be removed on both sides and thus

$$\limsup_{\varepsilon \rightarrow 0} \text{Tr}[\varrho_\varepsilon b^{Wick}] \leq \int_{\mathcal{Z}} b d\mu. \quad (28)$$

The inequalities (27) and (28) show that the convergence in the (P) condition holds for all $b \in \mathcal{P}_{p,p}(\mathcal{Z})$ such that $\tilde{b} \geq 0$, and hence for all $b \in \mathcal{P}_{p,p}(\mathcal{Z})$.

We now consider the general case $b \in \mathcal{P}_{p,q}(\mathcal{Z})$. There exists a sequence of compact operators $\tilde{b}_n \in \mathcal{L}^\infty(\mathcal{S}^p \mathcal{Z}^{\otimes p}, \mathcal{S}_+^q \mathcal{Z}^{\otimes q})$ such that:

$$\begin{aligned} \forall n \in \mathbb{N}, \quad |b_n|_{\mathcal{P}_{p,q}} = |\tilde{b}_n|_{\mathcal{L}(\mathcal{S}_+^p \mathcal{Z}^{\otimes p}, \mathcal{S}_+^q \mathcal{Z}^{\otimes q})} &\leq |\tilde{b}|_{\mathcal{L}(\mathcal{S}_+^p \mathcal{Z}^{\otimes p}, \mathcal{S}_+^q \mathcal{Z}^{\otimes q})} = |b|_{\mathcal{P}_{p,q}} \\ \text{and} \quad \forall z \in \mathcal{Z}, \quad \lim_{n \rightarrow \infty} b_n(z) = \lim_{n \rightarrow \infty} \langle z^{\otimes q}, \tilde{b}_n z^{\otimes p} \rangle &= \langle z^{\otimes q}, \tilde{b} z^{\otimes p} \rangle = b(z). \end{aligned}$$

For any fixed $n \in \mathbb{N}$,

$$\begin{aligned} &\limsup_{\varepsilon \rightarrow 0} \left| \text{Tr} \left[\varrho_\varepsilon b^{Wick} \right] - \int_{\mathcal{Z}} b(z) d\mu(z) \right| \\ &\leq \limsup_{\varepsilon \rightarrow 0} \left| \text{Tr} \left[\varrho_\varepsilon (b^{Wick} - b_n^{Wick}) \right] \right| + \limsup_{\varepsilon \rightarrow 0} \left| \text{Tr} \left[\varrho_\varepsilon b_n^{Wick} \right] - \int_{\mathcal{Z}} b_n d\mu \right| + \int_{\mathcal{Z}} |b_n - b| d\mu, \quad (29) \end{aligned}$$

where the second term of the right-hand side vanishes because \tilde{b}_n is a fixed compact operator. Using the Cauchy-Schwarz inequality with $\text{Tr}[\varrho_\varepsilon] = 1$ gives

$$\left| \text{Tr} \left[\varrho_\varepsilon (b^{Wick} - b_n^{Wick}) \right] \right| \leq \text{Tr} \left[\varrho_\varepsilon (b^{Wick} - b_n^{Wick})(b^{Wick,*} - b_n^{Wick,*}) \right]^{1/2}.$$

From the proved result when $p = q$, we deduce:

$$\limsup_{\varepsilon \rightarrow 0} \left| \text{Tr} \left[\varrho_\varepsilon (b^{Wick} - b_n^{Wick}) \right] \right| \leq \left[\int_{\mathcal{Z}} |b - b_n|^2 d\mu(z) \right]^{1/2}. \quad (30)$$

With $\int_{\mathcal{Z}} |z|^{r(p+q)} d\mu(z) < \infty$ and

$$\forall n \in \mathbb{N}, \forall z \in \mathcal{Z}, \quad |b(z) - b_n(z)|^r \leq (2|b|_{\mathcal{P}_{p,q}})^r |z|^{r(p+q)},$$

Lebesgue's convergence theorem yields

$$\lim_{n \rightarrow \infty} \int_{\mathcal{Z}} |b - b_n|^r d\mu = 0 \quad (31)$$

for $r \in \{1, 2\}$. Combining (29), (30) and (31) proves (P) for any $b \in \mathcal{P}_{p,q}(\mathcal{Z})$. \square

C The Composition Formula of Wick Quantized Operators

We give an algebraic proof for the composition formula (3) of two Wick quantized operators on a finite or infinite dimensional separable complex Hilbert space \mathcal{Z} . This proof holds in both the bosonic and fermionic cases. It uses only the definition of the Wick quantization, and it involves neither creation and annihilation operators, nor the canonical commutation or anticommutation relations.

We note $\llbracket m, n \rrbracket := \{m, \dots, n\}$ for $m \leq n \in \mathbb{N}$. The action of the symmetric group $\mathfrak{S}_{\llbracket 1, n \rrbracket}$ on product vectors in $\mathcal{Z}^{\otimes n}$, $\sigma \cdot (z_1 \otimes \dots \otimes z_n) = z_{\sigma_1} \otimes \dots \otimes z_{\sigma_n}$, $z_j \in \mathcal{Z}$, is extended to $\mathcal{Z}^{\otimes n}$ by linearity and density. With this notation $\mathcal{S}_{\pm}^n = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_{\llbracket 1, n \rrbracket}} s_{\pm}(\sigma) \sigma \cdot$.

We begin with a preliminary lemma on a special set of permutations.

Lemma C.1. *Let $k, p, q, K \in \mathbb{N}$ such that $k \in \llbracket \max\{0, p + q - K\}, \min\{p, q\} \rrbracket$, and*

$$\mathfrak{S}(k) := \left\{ \sigma \in \mathfrak{S}_{\llbracket 1, K \rrbracket} \mid \text{card}(\sigma(\llbracket p - k + 1, p - k + q \rrbracket) \cap \llbracket 1, p \rrbracket) = k \right\}.$$

1. *The cardinal of $\mathfrak{S}(k)$ is $\text{card } \mathfrak{S}(k) = \binom{q}{k} \binom{p}{k} k! \frac{(K-q)!(K-p)!}{(K-(q+p-k))!}$.*
2. *Any permutation $\sigma \in \mathfrak{S}(k)$ can be factorized as $\sigma = \sigma^{(1)} \sigma^{(2)} \sigma^{(3)} \sigma^{(4)}$, where $\sigma^{(1)} \in \mathfrak{S}_{\llbracket 1, p \rrbracket}$, $\sigma^{(2)} \in \mathfrak{S}_{\llbracket p+1, K \rrbracket}$, $\sigma^{(3)} \in \mathfrak{S}_{\llbracket p-k+1, p-k+q \rrbracket}$, $\sigma^{(4)} \in \mathfrak{S}_{\llbracket 1, K \rrbracket \setminus \llbracket p-k+1, p-k+q \rrbracket}$.*

Note that:

- There is no uniqueness of such a decomposition.
- For $A \subset B$ an element of \mathfrak{S}_A is identified with the corresponding element of \mathfrak{S}_B which is the identity on $B \setminus A$.
- The permutations $\sigma^{(1)}$ and $\sigma^{(2)}$ commute, and so do $\sigma^{(3)}$ and $\sigma^{(4)}$.

Proof. For Point 1: We count the number of permutation in $\mathfrak{S}(k)$. We first choose k integers out of $\llbracket p - k + 1, p - k + q \rrbracket$ and k integers out of $\llbracket 1, p \rrbracket$. There is $\binom{q}{k} \binom{p}{k}$ such possible choices and $k!$ possible permutations for each of these choices. Then the remaining $q - k$ integers of $\llbracket p - k + 1, p - k + q \rrbracket$ have to be sent in $\llbracket p + 1, K \rrbracket$. There is $(q - k)! \binom{K-p}{q-k}$ possibilities for that. In the same way we have $(p - k)! \binom{K-q}{p-k}$ possibilities for the remaining integers of $\llbracket 1, p \rrbracket$ that come from $\llbracket 1, K \rrbracket \setminus \llbracket p - k + 1, p - k + q \rrbracket$. Finally the $K - k - (q - k) - (p - k)$ remaining integers on both sides can be permuted in $(K - q - p + k)!$ different ways. So that

$$\text{card } \mathfrak{S}(k) = \binom{q}{k} \binom{p}{k} k! (q - k)! \binom{K-p}{q-k} (p - k)! \binom{K-q}{p-k} (K - q - p + k)!$$

and this gives the result.

For Point 2: Let $A = \sigma^{-1}(\llbracket 1, p \rrbracket) \cap \llbracket p - k + 1, p - k + q \rrbracket$. There exists $\sigma^{(3)} \in \mathfrak{S}_{\llbracket p-k+1, p-k+q \rrbracket}$ such that $\sigma^{(3)}(A) = \llbracket p - k + 1, p \rrbracket$. Then

$$\sigma \sigma^{(3)-1}(\llbracket p - k + 1, p \rrbracket) = \sigma(A) \subseteq \llbracket 1, p \rrbracket.$$

Hence there exists $\sigma^{(1)} \in \mathfrak{S}_{\llbracket 1, p \rrbracket}$ such that $\sigma^{(1)}(j) = \sigma \sigma^{(3)-1}(j)$ on $\llbracket p - k + 1, p \rrbracket$. And, similarly, there exists $\sigma^{(2)} \in \mathfrak{S}_{\llbracket p+1, K \rrbracket}$ such that $\sigma^{(2)}(j) = \sigma \sigma^{(3)-1}(j)$ on $\llbracket p + 1, p - k + q \rrbracket$. Note that $\sigma^{(1)}$ and $\sigma^{(2)}$ commute. Finally, we set $\sigma^{(4)} = \sigma^{(2)-1} \sigma^{(1)-1} \sigma \sigma^{(3)-1}$. By construction, $\sigma^{(4)}(j) = j$ for $j \in \llbracket p - k + 1, p - k + q \rrbracket$, hence $\sigma^{(4)} \in \mathfrak{S}_{\llbracket 1, K \rrbracket \setminus \llbracket p-k+1, p-k+q \rrbracket}$ and $\sigma = \sigma^{(1)} \sigma^{(2)} \sigma^{(3)} \sigma^{(4)}$ (as $\sigma^{(4)}$ and $\sigma^{(3)}$ commute). \square

Notation: On $\mathcal{L}(\mathcal{Z}^{\otimes p}; \mathcal{Z}^{\otimes q})$, the equivalence relation \cong is defined by

$$A \cong B \quad \Leftrightarrow \quad \mathcal{S}_{\pm}^q A \mathcal{S}_{\pm}^{p,*} = \mathcal{S}_{\pm}^q B \mathcal{S}_{\pm}^{p,*}.$$

Lemma C.2. Let $\tilde{b}_j \in \mathcal{L}(\mathcal{S}_\pm^{p_j} \mathcal{Z}^{\otimes p_j}; \mathcal{S}_\pm^{q_j} \mathcal{Z}^{\otimes q_j})$ and n_j such that $n_1 + p_1 = n_2 + q_2 =: K$. Then

$$(\tilde{b}_1 \otimes \text{Id}^{\otimes n_1}) \mathcal{S}_\pm^{K,*} \mathcal{S}_\pm^K (\tilde{b}_2 \otimes \text{Id}^{\otimes n_2}) \cong \sum_k (\pm 1)^{(p_2+q_2)(k-p_1)} \frac{n_2! n_1!}{K'! K! k!} (\tilde{b}_1 \#^k \tilde{b}_2) \otimes \text{Id}^{\otimes K'},$$

where $k \in \llbracket \max\{0, p_1 + q_2 - K\}, \min\{p_1, q_2\} \rrbracket$, and $K' = K - q_2 - p_1 + k$.

Proof. Using the partition $\mathfrak{S}_{\llbracket 1, K \rrbracket} = \bigsqcup_k \tilde{\mathfrak{S}}(k)$ in subsets

$$\tilde{\mathfrak{S}}(k) := \left\{ \sigma \in \mathfrak{S}_{\llbracket 1, K \rrbracket} \mid \text{card}(\sigma(\llbracket 1, q_2 \rrbracket) \cap \llbracket 1, p_1 \rrbracket) = k \right\}$$

for $k \in \llbracket \max\{0, p_1 + q_2 - K\}, \min\{p_1, q_2\} \rrbracket$, yields

$$(\tilde{b}_1 \otimes \text{Id}^{\otimes n_1}) \mathcal{S}_\pm^{K,*} \mathcal{S}_\pm^K (\tilde{b}_2 \otimes \text{Id}^{\otimes n_2}) = \frac{1}{K!} \sum_k \sum_{\tilde{\sigma} \in \tilde{\mathfrak{S}}(k)} (\tilde{b}_1 \otimes \text{Id}^{\otimes n_1}) s_\pm(\tilde{\sigma}) \tilde{\sigma} \cdot (\tilde{b}_2 \otimes \text{Id}^{\otimes n_2}).$$

We fix k and $\tilde{\sigma} \in \tilde{\mathfrak{S}}(k)$. A cyclic permutation $\tau_r := (1\ 2\ 3 \cdots r)$ acting on $\mathcal{Z}^{\otimes r}$ defines the shift operator $\tau_r \cdot = (1\ 2\ 3 \cdots r) \cdot$ and then $\sigma := \tilde{\sigma} \tau_K^{k-p_1}$ is in $\mathfrak{S}(k)$ (with $p = p_1$ and $q = q_2$) and

$$\begin{aligned} & (\tilde{b}_1 \otimes \text{Id}^{\otimes n_1}) s_\pm(\tilde{\sigma}) \tilde{\sigma} \tau_K^{k-p_1} \tau_K^{p_1-k} \cdot (\tilde{b}_2 \otimes \text{Id}^{\otimes n_2}) \tau_{p_2+n_2}^{k-p_1} \tau_{p_2+n_2}^{p_1-k} \cdot \\ & \cong (\tilde{b}_1 \otimes \text{Id}^{\otimes n_1}) s_\pm(\sigma) \sigma \cdot (\pm 1)^{K(k-p_1)} (\text{Id}^{\otimes p_1-k} \otimes \tilde{b}_2 \otimes \text{Id}^{\otimes K'}) (\pm 1)^{(p_2+n_2)(k-p_1)} \\ & \cong (\pm 1)^{(K+p_2+n_2)(k-p_1)} (\tilde{b}_1 \otimes \text{Id}^{\otimes n_1}) s_\pm(\sigma) \sigma \cdot (\text{Id}^{\otimes p_1-k} \otimes \tilde{b}_2 \otimes \text{Id}^{\otimes K'}) \end{aligned}$$

holds for operators in $\mathcal{L}(\mathcal{Z}^{\otimes q_1+n_1}; \mathcal{Z}^{\otimes p_2+n_2})$. We used

$$\begin{aligned} s_\pm(\sigma) &= s_\pm(\tilde{\sigma}) s_\pm(\tau_K^{k-p_1}) = s_\pm(\tilde{\sigma}) (\pm 1)^{K(k-p_1)} \\ \text{and } (\tau_{p_2+n_2}^{p_1-k} \cdot) \circ \mathcal{S}_\pm^{p_2+n_2} &= (\pm 1)^{(p_2+n_2)(p_1-k)} \mathcal{S}_\pm^{p_2+n_2}. \end{aligned}$$

Owing to the factorization $\sigma = \sigma^{(1)} \sigma^{(2)} \sigma^{(3)} \sigma^{(4)}$ of Lemma C.1 with $\sigma^{(i)} \sigma^{(i+1)} = \sigma^{(i+1)} \sigma^{(i)}$ for $i \in \{1, 3\}$, we get

$$\begin{aligned} & (\tilde{b}_1 \otimes \text{Id}^{\otimes n_1}) s_\pm(\sigma) \sigma \cdot (\text{Id}^{\otimes p_1-k} \otimes \tilde{b}_2 \otimes \text{Id}^{\otimes K'}) \\ & \cong (\tilde{b}_1 \otimes \text{Id}^{\otimes n_1}) s_\pm(\sigma) (\sigma^{(1)} \sigma^{(2)} \sigma^{(3)} \sigma^{(4)}) \cdot (\text{Id}^{\otimes p_1-k} \otimes \tilde{b}_2 \otimes \text{Id}^{\otimes K'}) \\ & \cong s_\pm(\sigma) ((b_1 \sigma^{(1)} \cdot) \otimes \text{Id}^{\otimes n_1} \sigma^{(2)} \cdot) \sigma^{(4)} \cdot (\text{Id}^{\otimes p_1-k} \otimes (\sigma^{(3)} \cdot \tilde{b}_2) \otimes \text{Id}^{\otimes K'}) \\ & \cong s_\pm(\sigma) (\tilde{b}_1 s_\pm(\sigma^{(1)}) \otimes s_\pm(\sigma^{(2)}) \text{Id}^{\otimes n_1}) s_\pm(\sigma^{(4)}) (\text{Id}^{\otimes p_1-k} \otimes s_\pm(\sigma^{(3)}) \tilde{b}_2 \otimes \text{Id}^{\otimes K'}) \\ & \cong (\tilde{b}_1 \otimes \text{Id}^{\otimes n_1}) (\text{Id}^{\otimes p_1-k} \otimes \tilde{b}_2 \otimes \text{Id}^{\otimes K'}) \\ & \cong [(\tilde{b}_1 \otimes \text{Id}^{\otimes q_2-k}) (\text{Id}^{\otimes p_1-k} \otimes \tilde{b}_2)] \otimes \text{Id}^{\otimes K'} \\ & \cong \left(\frac{p_1!}{(p_1-k)!} \frac{q_2!}{(q_2-k)!} \right)^{-1} (\tilde{b}_1 \#^k \tilde{b}_2) \otimes \text{Id}^{\otimes K'}. \end{aligned}$$

We conclude with the first statement of Lemma C.1 which counts the terms in $\sum_{\tilde{\sigma} \in \tilde{\mathfrak{S}}(k)}$ because $\text{card}(\tilde{\mathfrak{S}}(k)) = \text{card}(\mathfrak{S}(k))$. \square

Proof of Proposition 2.2. For n_1, n_2 such that $n_1 + p_1 = n_2 + q_2 =: K$, using Lemma C.2,

$$\begin{aligned}
& \varepsilon^{-\frac{p_1+q_1+p_2+q_2}{2}} \times \tilde{b}_1^{Wick} \tilde{b}_2^{Wick} \Big|_{\mathcal{S}_{\pm}^{n_2+p_2} \mathcal{Z}^{\otimes n_2+p_2}} \\
&= \frac{\sqrt{K!(n_1+q_1)!}}{n_1!} \frac{\sqrt{(n_2+p_2)!K!}}{n_2!} \mathcal{S}_{\pm}^{q_1+n_1} (\tilde{b}_1 \otimes \text{Id}^{\otimes n_1}) \mathcal{S}_{\pm}^{p_1+n_1,*} \mathcal{S}_{\pm}^{p_2+q_2} (\tilde{b}_2 \otimes \text{Id}^{\otimes n_2}) \mathcal{S}_{\pm}^{p_2+n_2,*} \\
&= \sum_k (\pm 1)^{(p_2+q_2)(k-p_1)} \frac{\sqrt{(n_1+q_1)(n_2+p_2)!}}{n_1!n_2!} K! \frac{n_2!n_1!}{K'!K!k!} \mathcal{S}_{\pm}^{q_1+n_1} ((\tilde{b}_1 \#^k \tilde{b}_2) \otimes \text{Id}^{\otimes K'}) \mathcal{S}_{\pm}^{p_2+n_2,*} \\
&= \sum_k (\pm 1)^{(p_2+q_2)(k-p_1)} \frac{\sqrt{(q_2+q_1-k+K')!(p_2+p_1-k+K')!}}{K'!k!} \mathcal{S}_{\pm}^{q_1+n_1} ((\tilde{b}_1 \#^k \tilde{b}_2) \otimes \text{Id}^{\otimes K'}) \mathcal{S}_{\pm}^{p_2+n_2,*}
\end{aligned}$$

where $K' := K - q_2 - p_1 + k$.

With $p_2 + n_2 = p_2 + p_1 - k + K'$ and $q_1 + n_1 = q_2 + q_1 - k + K'$, we thus obtain the equality of operators

$$\tilde{b}_1^{Wick} \tilde{b}_2^{Wick} = \sum_k (\pm 1)^{(p_2+q_2)(k-p_1)} \frac{\varepsilon^k}{k!} (\tilde{b}_1 \#^k \tilde{b}_2)^{Wick}$$

restricted to $\mathcal{S}_{\pm}^{n_2+p_2} \mathcal{Z}^{\otimes n_2+p_2}$. \square

D A general formula for $\text{Tr} [\Gamma_{\pm}(C)]$

The following result about traces of the second quantized operator $\Gamma_{\pm}(C)$ is often presented for self-adjoint trace-class operators, although it is valid without self-adjointness. We recall here the general version for the sake of completeness. It relies on a simple holomorphy argument and can be compared with Lidskii's Theorem which says that for any trace-class operator T , $\text{Tr}[T] = \sum_{\lambda \in \sigma(T)} \lambda$.

Lemma D.1. *For any trace-class operator $C \in \mathcal{L}^1(\mathcal{Z})$ (which is assumed to be a strict contraction in the bosonic case, $\pm = +$), its second quantized version $\Gamma_{\pm}(C)$ is trace-class in $\Gamma_{\pm}(\mathcal{Z})$ and*

$$\text{Tr} [\Gamma_{\pm}(C)] = \exp(\mp \text{Tr} [\log(1 \mp C)]) .$$

Proof. When $C = C^* \in \mathcal{L}^1(\mathcal{Z})$ using an orthonormal basis of eigenvectors $(e_n)_{n \in \mathbb{N}}$ in \mathcal{Z} with the corresponding eigenvalues $(\lambda_n)_{n \in \mathbb{N}}$, and $\Gamma_{\pm}(\mathcal{Z}) = \otimes_{n \in \mathbb{N}} \Gamma_{\pm}(\mathbb{C}e_n)$, we obtain

- in the bosonic case with $\|C\| < 1$,

$$\begin{aligned}
\text{Tr} [\Gamma_+(C)] &= \prod_{n \in \mathbb{N}} \text{Tr} [\Gamma_+(\lambda_n \text{Id}_{\mathbb{C}})] = \prod_{n \in \mathbb{N}} \frac{1}{1 - \lambda_n} = \exp \left(- \sum_{n \in \mathbb{N}} \log(1 - \lambda_n) \right) \\
&= \exp(-\text{Tr} [\log(1 - C)]) ,
\end{aligned}$$

- in the fermionic case,

$$\begin{aligned}
\text{Tr} [\Gamma_-(C)] &= \prod_{n \in \mathbb{N}} \text{Tr} [\Gamma_-(\lambda_n \text{Id}_{\mathbb{C}})] = \prod_{n \in \mathbb{N}} (1 + \lambda_n) = \exp \left(+ \sum_{n \in \mathbb{N}} \log(1 + \lambda_n) \right) \\
&= \exp(\text{Tr} [\log(1 + C)]) .
\end{aligned}$$

The functoriality of Γ_{\pm} for the polar decomposition $C = U|C|$, reads $\Gamma_{\pm}(C) = \Gamma_{\pm}(U)\Gamma_{\pm}(|C|)$, while $\|C\| < 1 \Leftrightarrow \||C|\| < 1$ in the bosonic case. Hence $\Gamma_{\pm}(C)$ is trace-class when $C \in \mathcal{L}^1(\mathcal{Z})$ (and $\|C\| < 1$ in the bosonic case).

Set $\mathcal{C} = \mathcal{L}^1(\mathcal{Z})$ in the fermionic case and $\mathcal{C} = \mathcal{L}^1(\mathcal{Z}) \cap \{C \in \mathcal{L}(\mathcal{Z}), \|C\| < 1\}$ in the bosonic case. In both cases \mathcal{C} is an open convex set, on which the two sides of the equality are holomorphic functions. Actually the holomorphy of the left-hand side comes from series expansion

$$\mathrm{Tr} [\Gamma_{\pm}(C)] = \sum_{n=0}^{\infty} \mathrm{Tr} [\mathcal{S}_{\pm}^n C^{\otimes n} \mathcal{S}_{\pm}^{n,*}] ,$$

which converges uniformly in $B(C_0, \delta_{C_0}) = \{C \in \mathcal{L}^1(\mathcal{Z}), \|C - C_0\|_{\mathcal{L}^1(\mathcal{Z})} < \delta_{C_0}\}$ for $\delta_{C_0} > 0$ small enough, for any $C_0 \in \mathcal{L}^1(\mathcal{Z})$ (satisfying additionally $\|C_0\| < 1$ in the bosonic case). Actually the estimate $\|C\|_{\mathcal{L}^1(\mathcal{Z})} \leq A$ (and $\|C\| \leq \varrho$ with $\varrho < 1$ in the bosonic case) imply $\||C|\|_{\mathcal{L}^1(\mathcal{Z})} \leq A$ (and $\||C|\| \leq \varrho$ in the bosonic case). Now the inequality

$$|\mathrm{Tr} [\mathcal{S}_{\pm}^n C^{\otimes n} \mathcal{S}_{\pm}^{n,*}]| \leq \mathrm{Tr} [\mathcal{S}_{\pm}^n |C|^{\otimes n} \mathcal{S}_{\pm}^{n,*}] ,$$

and the formula in the self-adjoint case with

$$\begin{aligned} & \sum_{n=0}^{\infty} \mathrm{Tr} [\mathcal{S}_{-}^n |C|^{\otimes n} \mathcal{S}_{-}^{n,*}] \leq \exp(A) \quad (\text{fermions}) \\ \text{resp.} \quad & \sum_{n=0}^{\infty} \mathrm{Tr} [\mathcal{S}_{+}^n |C|^{\otimes n} \mathcal{S}_{+}^{n,*}] \leq \exp\left(\frac{A}{1-\varrho}\right) \quad (\text{bosons}) , \end{aligned}$$

ensure the uniform convergence of the series.

For any $C \in \mathcal{C}$, C and $\mathrm{Re} C = \frac{C+C^*}{2}$ belong to \mathcal{C} so that $C(s) = \mathrm{Re} C + is\mathrm{Im} C$ belong to \mathcal{C} when $s \in \omega_0 = (-\delta, \delta) + i(-\delta, \delta)$ and when $s \in \omega_1 = (1 - \delta, 1 + \delta) + i(-\delta, \delta)$ for $\delta > 0$ small enough. By convexity of \mathcal{C} , $C(s) \in \mathcal{C}$ for all $s \in \omega = (-\delta, 1 + \delta) + i(-\delta, \delta)$. When $s \in i(-\delta, \delta)$, $C(s)$ is self-adjoint and the equality holds. The holomorphy of both sides w.r.t $s \in \omega$ implies that the equality holds true for all $s \in \omega$ in particular when $s = 1$. \square

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References

- [Am] Z. Ammari. Scattering theory for a class of fermionic Pauli-Fierz models. *J. Funct. Anal.* 208 (2004), no. 2, 302–359.
- [AmNi1] Z. Ammari, F. Nier. Mean field limit for bosons and infinite dimensional phase-space analysis. *Ann. Henri Poincaré* 9 (2008), 1503–1574.
- [AmNi2] Z. Ammari, F. Nier. Mean field limit for bosons and propagation of Wigner measures. *J. Math. Phys.* 50 (2009).

- [AmNi3] Z. Ammari, F. Nier. Mean field propagation of Wigner measures and BBGKY hierarchies for general bosonic states. *J. Math. Pures Appl.* (9) 95 (2011), no. 6, 585–626.
- [AmNi4] Z. Ammari, F. Nier. Mean field propagation of infinite dimensional Wigner measures with a singular two-body interaction potential. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) 14 (2015), no. 1, 155–220.
- [BGM] C. Bardos, F. Golse, N. Mauser. Weak coupling limit of the n -particle Schrödinger equation. *Methods Appl. Anal.* 7 (2000), 275–293.
- [BEGMY] C. Bardos, L. Erdős, F. Golse, N. Mauser, H-T. Yau. Derivation of the Schrödinger-Poisson equation from the quantum N -body problem. *C.R. Math. Acad. Sci. Paris* 334 (2002), 515–520.
- [BPS] N. Benedikter, M. Porta, B. Schlein. Mean-field evolution of fermionic systems. *Comm. Math. Phys.*, 331, (2014), 1087–1131.
- [BeTa] D. Berend, T. Tassa. Improved bounds on Bell numbers and on moments of sums of random variables. *Probab. Math. Stat.* 30 (2010), no. 2, 185–205.
- [Bon] J.M. Bony. Second microlocalization and propagation of singularities for semi-linear hyperbolic equations, *Hyperbolic equations and related topics*, Mizohata ed. Kinokuya, (1986), 11–49.
- [BrRo2] O. Brattelli, D. Robinson. *Operator Algebras and Quantum Statistical Mechanics Vol. 2*. Springer, Berlin, (1981).
- [ChVN] L. Charles, S. Vu Ngoc. Spectral asymptotics via the semiclassical Birkhoff normal form. *Duke Math. J.* 143 (2008), no. 3, 463–511.
- [CdV] Y. Colin de Verdière. Ergodicité et fonctions propres du Laplacien. *Comm. Math. Phys.* 102 (1985), 497–502.
- [DiSj] M. Dimassi, J. Sjöstrand. *Spectral asymptotics in the semi-classical limit* London Math. Soc. Lect. Notes Series, 268, Cambridge University Press (1999).
- [Der1] J. Dereziński. Asymptotic completeness in quantum field theory. A class of Galilei covariant models. *Rev. Math. Phys.* 10, (1998), 191–233.
- [Fer] C. Fermanian. Analyse deux échelles d’une suite bornée de L^2 sur une sous-variété du cotangent. *C.R. Math. Acad. Sci. Paris* 340 (2005), no. 4, 269–274.
- [FeGe] P. Gérard, C. Fermanian. Mesures semi-classiques et croisement de modes. *Bull. SMF* 130 (2002), no. 1, 123–168.
- [Ger] P. Gérard. Mesures semiclassiques et ondes de Bloch. “Séminaire sur les Equations aux Dérivées Partielles”, Exp. no. XVI. Ecole Polytechnique (1981).
- [GMMP] P. Gérard, P. Markowich, N. Mauser, F. Poupaud. Homogenization limits and Wigner transforms. *Comm. Pure Appl. Math.* 50 (1997), 323–379.

- [HMR] B. Helffer, A. Martinez, D. Robert. Ergodicité et limite semi-classique. *Comm. Math. Phys.* 109 (1987), no. 2, 313–326.
- [HeNi] B. Helffer, F. Nier. *Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten Laplacians*. Lecture Notes in Mathematics 1862, Springer Verlag (2005).
- [Hor3] L. Hörmander. *The analysis of linear partial differential operators, vol 3*. Springer Verlag (1985).
- [KnPi] A. Knowles, P. Pickl. Mean-field dynamics: singular potential and rate of convergence, *Comm. Math. Phys.* 298 (2010), 101–138.
- [LNR] M. Lewin, P. T. Nam, N. Rougerie, Remarks on the quantum de Finetti theorem for bosonic systems, *Appl. Math. Res. Express. AMRX* 1 (2015), 48–63.
- [LiPa] P.L. Lions, T. Paul. Sur les mesures de Wigner. *Rev. Mat. Iberoamericana* 9 (1993), no. 3, 553–618.
- [NaNi] F. Nataf, F. Nier, Convergence of domain decomposition methods via semi-classical calculus, *Comm. Partial Differential Equations* 23 (1998), no. 5-6, 1007–1059.
- [Nie] F. Nier, A semiclassical picture of quantum scattering, *Ann. Sci. École Norm. Sup. (4)* 29, (1996), no. 2, 149–183.
- [Rad] M. J. Radzikowski, Microlocal approach to the Hadamard condition in quantum field theory on curved space-time, *Comm. Math. Phys.*, 179, (1996), 529–553.
- [Sch] A. Schnirelman. Ergodic properties of eigenfunctions. *Uspehi Mat. Nauk.* 29 (1974), 181–182.
- [Tar] L. Tartar, H-measures, a new approach for studying homogenization, oscillations and concentration effects in partial differential equations. *Proc. Roy. Soc. Edinburgh Sect. A* 115 (1990), no. 3-4, 193–230.