HAL
open science

A PROOF OF THE MUIR-SUFFRIDGE CONJECTURE FOR CONVEX MAPS OF THE UNIT BALL IN C n

Filippo Bracci, Hervé Gaussier

To cite this version:

Filippo Bracci, Hervé Gaussier. A PROOF OF THE MUIR-SUFFRIDGE CONJECTURE FOR CONVEX MAPS OF THE UNIT BALL IN C n. 2017. hal-01442635

HAL Id: hal-01442635

https://hal.science/hal-01442635v1

Preprint submitted on 20 Jan 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A PROOF OF THE MUIR-SUFFRIDGE CONJECTURE FOR CONVEX MAPS OF THE UNIT BALL IN $\mathbb{C}^n$

FILIPPO BRACCI† AND HERVÉ GAUSSIER

Abstract. We prove (and improve) the Muir-Suffridge conjecture for holomorphic convex maps. Namely, let $F : \mathbb{B}^n \to \mathbb{C}^n$ be a univalent map from the unit ball whose image D is convex. Let $\mathcal{S} \subset \partial\mathbb{B}^n$ be the set of points ξ such that $\lim_{z\to\xi} \|F(z)\| = \infty$. Then we prove that $\mathcal{S}$ is either empty, or contains one or two points and F extends as a homeomorphism $\tilde{F} : \overline{\mathbb{B}^n} \setminus \mathcal{S} \to \overline{D}$. Moreover, $\mathcal{S} = \emptyset$ if D is bounded, $\mathcal{S}$ has one point if D has one connected component at ∞ and $\mathcal{S}$ has two points if D has two connected components at ∞ and, up to composition with an automorphism of the ball and renormalization, F is an extension of the strip map in the plane to higher dimension.

1. Introduction

Let $\mathbb{B}^n := \{z \in \mathbb{C}^n : \|z\| < 1\}$ be the unit ball of $\mathbb{C}^n$, $n \geq 1$. A domain $D \subset \mathbb{C}^n$ is *convex* if for every $z, w \in D$ the real segment joining z and w is contained in D.

In the paper [12] J. Muir and T. Suffridge made the following conjecture:

Conjecture 1.1. *Let $F : \mathbb{B}^n \to \mathbb{C}^n$ be a univalent map such that $F(0) = 0$ and $dF_0 = \mathsf{id}$. Suppose that $D := F(\mathbb{B}^n)$ is convex. Then*

(a) *D is bounded and F extends continuously to $\partial\mathbb{B}^n$, or*
(b) *F extends continuously to $\partial\mathbb{B}^n$ except for one point that is an infinite discontinuity, or*
(b) *up to pre-composing with an automorphism of $\mathbb{B}^n$ and post-composing with an affine transformation, there exists a holomorphic function $H : \mathbb{B}^{n-1} \to \mathbb{C}^n$, continuous up to $\partial\mathbb{B}^{n-1}$, such that*

$$F(z_1, z') = \left(\frac{1}{2}\log\frac{1+z_1}{1-z_1}\right)e_1 + H\left(\frac{z'}{\sqrt{1-z_1^2}}\right). \tag{1.1}$$

Here, $z = (z_1, z') \in \mathbb{C}\times\mathbb{C}^{n-1}$, $e_1 = (1, 0, \ldots, 0)$ and a point $\xi \in \partial\mathbb{B}^n$ is a point of infinite discontinuity for F if $\lim_{z\to\xi} \|F(z)\| = \infty$.

Key words and phrases. holomorphic convex maps of the unit balls; boundary extension; Gromov hyperbolicity; semigroups of holomorphic maps; commuting maps.

† Partially supported by GNSAGA of INDAM.

This paper was written as part of the 2016-17 CAS project *Several Complex Variables and Complex Dynamics*.

We point out explicitly that no regularity assumptions on the boundary of D are made.

Since the boundary of a convex domain in $\mathbb{C}$ is a Jordan curve, the conjecture is true for $n = 1$ because of Carathéodory's extension theorem. In case $n > 1$, in [12, Theorem 2.13], Muir and Suffridge proved that if there exists $v \in \mathbb{C}^n$, $\|v\| = 1$, such that $tv \subset D$ for all $t \in \mathbb{R}$, then $\xi_1 := \lim_{t\to-\infty} F^{-1}(tv)$ and $\xi_2 := \lim_{t\to\infty} F^{-1}(tv)$ exist and, if $\xi_1 \neq \xi_2$, then F has the form (1.1) (although they could not prove that H extends continuously up to $\partial\mathbb{B}^{n-1}$). In [13], the same authors proved that, in case $\xi_1 = \xi_2$ and there is only one infinite singularity whose span contains the direction v, then F has a simple form for which the Conjecture holds.

In the recent paper [5] we proved, as a consequence of a prime ends-type theory (called "Horosphere theory") we developed there, that the conjecture is true if D is bounded and, moreover, in that case, F extends as a homeomorphism up to $\overline{\mathbb{B}^n}$.

In this paper we prove Conjecture 1.1. In fact, we prove a stronger result:

Theorem 1.2. *Let $F : \mathbb{B}^n \to D \subset \mathbb{C}^n$ be a biholomorphism. Suppose that D is convex. Then*

(1) *D is bounded and F extends as a homeomorphism from $\overline{\mathbb{B}^n}$ onto $\overline{D}$, or*
(2) *D is unbounded and has one connected component at ∞, there exists a unique point $\xi \in \partial\mathbb{B}^n$ such that $\lim_{z\to\xi} \|F(z)\| = \infty$ and F extends as a homeomorphism from $\overline{\mathbb{B}^n} \setminus \{\xi\}$ onto $\overline{D}$, or*
(3) *D is unbounded and has two connected components at ∞, there exist $\xi_1, \xi_2 \in \partial\mathbb{B}^n$, $\xi_1 \neq \xi_2$, such that $\lim_{z\to\xi_1} \|F(z)\| = \lim_{z\to\xi_2} \|F(z)\| = \infty$ and F extends as a homeomorphism from $\overline{\mathbb{B}^n} \setminus \{\xi_1, \xi_2\}$ onto $\overline{D}$.*

Here, we say that an unbounded convex domain $D \subset \mathbb{C}^n$ has one connected component at ∞ if for every compact ball B, the set $D \setminus B$ has one unbounded connected component, otherwise we say that it has two connected components at ∞ (see Section 2 for precise definitions and statements).

The Muir-Suffridge Conjecture 1.1 follows then from Theorem 1.2. Indeed, we show that a convex domain has two connected components at ∞ if and only if there exists $v \in \mathbb{C}^n$, $\|v\| = 1$, such that $z + tv \subset D$ for all $z \in D$ and $t \in \mathbb{R}$, v is the unique "direction at ∞" for D and, in such a case, $\lim_{t\to-\infty} F^{-1}(tv) \neq \lim_{t\to\infty} F^{-1}(tv)$ (see Lemma 2.2 and Proposition 5.2). Therefore, up to composing with an automorphism of $\mathbb{B}^n$ in order to have the infinite discontinuities in e_1 and $-e_1$ and post-composing with an affine transformation in such a way that $F(0) = 0$ and $dF_0 = \mathsf{id}$, F has precisely the form (1.1) by [12, Theorem 2.13].

The proof of Theorem 1.2 uses many ingredients: the properties of real geodesics for the Kobayashi distance of $\mathbb{B}^n$, Gromov's theory of quasi-geodesics in D, the theory of continuous one-parameter semigroups of $\mathbb{B}^n$ and that of commuting holomorphic self-maps of $\mathbb{B}^n$, and properties of horospheres and horospheres sequences of D.

We remark that, since all the ingredients we use are available for strongly convex domains with C^3 boundary, using Lempert's theory [10], one can prove that Theorem 1.2 holds replacing $\mathbb{B}^n$ with any strongly convex domain with C^3 boundary. We leave the details of this generalization to the interested readers.

The outline of the paper is the following. In Section 2 we prove some simple geometrical properties of unbounded convex domains. In Section 3, in order to make the paper as self-contained as possible, we collect the results and known facts we need for our proof.

Then, in Proposition 4.4 we show that F extends continuously to $\partial\mathbb{B}^n$ with image the closure of D in the one-point compactification of $\mathbb{C}^n$. The proof of such a result uses Gromov's theory of quasi-geodesics and properties of the Kobayashi distance in D.

Next, in Proposition 5.2, we deal with points in $\partial\mathbb{B}^n$ which are mapped to ∞ by F and we prove that those points correspond to the connected components at ∞ of D (which might be one or two). The proof uses results from the theory of continuous one-parameter semigroups in $\mathbb{B}^n$: indeed, if $\{p_n\} \subset D$ is a sequence converging to ∞ and $\frac{p_n}{\|p_n\|} \to v$, then $z + tv \in D$ for all $z \in D$ and all $t \geq 0$ (and we call such a v a "direction at ∞" for D). Thus one can define a (continuous one-parameter) semigroup in $\mathbb{B}^n$ with no fixed points by setting $\phi_t(\zeta) := F^{-1}(F(\zeta) + tv)$, $\zeta \in \mathbb{B}^n$ and $t \geq 0$. Using horospheres and properties of horospheres defined by sequences in D, we show essentially that the infinite discontinuities of F are the Denjoy-Wolff points of such semigroups. We then conclude our proof by using the fact that every two "directions at ∞" for D give rise to two commuting semigroups and commuting holomorphic self-maps of $\mathbb{B}^n$ do share the same Denjoy-Wolff point, unless they have very particular forms. The case of two connected components at ∞ for D is handled by using simple properties of real geodesics for the Kobayashi distance in $\mathbb{B}^n$.

Finally, Theorem 1.2 is proved in Section 6 (see Theorem 6.1).

This paper was written while both authors were visiting the Center for Advanced Studies in Oslo for the 2016-17 CAS project *Several Complex Variables and Complex Dynamics*. They both thank CAS for the support and for the wonderful atmosphere experienced there.

2. Geometry of unbounded convex domains

In this section we collect some simple results about geometry of unbounded convex domains.

If $x, y \in \mathbb{C}^n$, we denote by $[x, y]$ the real segment joining x, y, *i.e.*,

$$[x, y] := \{z \in \mathbb{C}^n : z = tx + (1-t)y, t \in [0, 1]\}.$$

For $z \in \mathbb{C}^n$ and $R > 0$ we let

$$B(z, R) := \{w \in \mathbb{C}^n : \|w - z\| < R\}$$

be the Euclidean ball of center z and radius R.

Definition 2.1. Let $D \subset \mathbb{C}^n$ be an unbounded convex domain. A vector $v \in \mathbb{C}^n$, $\|v\| = 1$, is called a *direction at ∞ for D* if there exists $x \in D$ such that $x + tv \in D$ for all $t \geq 0$.

By convexity of D, if v is a direction at ∞ for D, then for every $z \in D$ and all $t \geq 0$ it holds $z + tv \in D$.

Lemma 2.2. *Let $D \subset \mathbb{C}^n$ be an unbounded convex domain. Then there exists at least one direction v at ∞ for D. Moreover*

(1) *either $D \setminus \overline{B(0,R)}$ has only one unbounded connected component for all $R > 0$,*
(2) *or there exists $R_0 > 0$ such that $D \setminus \overline{B(0,R)}$ has two unbounded connected components for all $R \geq R_0$. This is the case if and only if the only directions at ∞ for D are v and $-v$.*

Proof. Since D is unbounded, there exists a sequence $\{p_k\} \subset D$ such that $\lim_{k\to\infty} \|p_k\| = \infty$. Up to subsequences, we can assume that $\lim_{k\to\infty} \frac{p_k}{\|p_k\|} = v$. Let $z \in D$. Since D is convex, the real segment $[z, p_k] \subset D$ for all k. Hence, by convexity of D, it follows that $z + tv \in D$ for all $t \geq 0$.

Next, assume that there exists $R > 0$ such that $D \setminus \overline{B(0,R)}$ is not connected. We claim that $D \setminus \overline{B(0,R)}$ has at most two unbounded components. Indeed, if U is an unbounded connected component of $D \setminus \overline{B(0,R)}$ and $\{p_k\} \subset U$ converges to ∞ and $\lim_{k\to\infty} \frac{p_k}{\|p_k\|} = v$ then for every $z \in U$ such that $\|z\| > R$ and for every $t \geq 0$ it holds $z + tv \in U$. Hence, for every unbounded connected component of $D \setminus \overline{B(0,R)}$ there exists $v \in \mathbb{C}^n$, $\|v\| = 1$, such that $z + tv$ belongs to such a component for every $t \geq 0$ and some $z \in D$. If the unbounded components were more than two there would exist two components U and U' and two directions v and w at ∞ for D which are $\mathbb{R}$-linearly independent and such that $z_0 + tv \in U$ for all $t \geq 0$ and some $z_0 \in D$, and $z_1 + tw \in U'$ for all $t \geq 0$ and some $z_1 \in D$. But then, since v and w are $\mathbb{R}$-linearly independent, for a, b sufficiently large the intersection $[z_0 + av, z_1 + bw] \cap \overline{B(0,R)}$ is empty and connects U with U', a contradiction.

Therefore, if $D \setminus \overline{B(0,R)}$ is not connected, then it has at most two unbounded connected components. If $D \setminus \overline{B(0,R)}$ contains two unbounded connected components, then it is easy to see that for every $R' > R$, also $D \setminus \overline{B(0,R')}$ contains two unbounded connected components.

Moreover, the previous argument shows that if there are two $\mathbb{R}$-linearly independent directions at ∞, then for every $R > 0$, $D \setminus \overline{B(0,R)}$ has only one unbounded connected component.

Therefore, if $D \setminus \overline{B(0,R)}$ has two unbounded connected components, then there are only two directions at ∞ for D, namely, v and $-v$ for some $v \in \mathbb{C}^n$, $\|v\| = 1$.

Conversely, assume $v, -v$ are the only directions at ∞ for D. Suppose by contradiction that for every $R > 0$ the open set $D \setminus \overline{B(0,R)}$ had only one unbounded connected component. Let $z_0 \in D$, and $R > \|z_0\|$. Then there exists $t_R \in (0, \infty)$ such that $z_0 + tv, z_0 - tv \in D \setminus \overline{B(0,R)}$ for all $t \geq t_R$. Since $D \setminus \overline{B(0,R)}$ has only one unbounded

connected component, the points $z_0 + t_R v$ and $z_0 - t_R v$ can be joined by a continuous path γ_R in $D \setminus \overline{B(0,R)}$. Let H be the real affine hyperplane through z_0 orthogonal to v. Then, by construction, $H \cap \gamma_R \neq \emptyset$ for all $R \geq \|z_0\|$. Therefore, there exists a sequence $\{p_k\} \subset H \cap D$ converging to ∞ such that $w := \lim_{k\to\infty} \frac{p_k}{\|p_k\|}$ is a direction at ∞ for D which is $\mathbb{R}$-linearly independent of v, a contradiction. □

The previous lemma allows us to give the following definition:

Definition 2.3. Let $D \subset \mathbb{C}^n$ be an unbounded convex domain. We say that D *has one connected component at* ∞ if for every $R > 0$ the open set $D \setminus \overline{B(0,R)}$ has only one unbounded connected component. Otherwise we say that D *has two connected components at* ∞.

3. Hyperbolic geometry of the unit ball

In this section we briefly recall what we need from hyperbolic geometry of the unit ball $\mathbb{B}^n$. We refer the reader to [1, 9] for details.

3.1. **Kobayashi distance and real Kobayashi geodesics.** Given a domain $D \subset \mathbb{C}^n$, for every $z, w \in D$, we denote by $K_D(z,w)$ the *Kobayashi distance* in D between z and w. If $D = \mathbb{D} \subset \mathbb{C}$ the unit disc, then $K_{\mathbb{D}}$ is the Poincaré distance of $\mathbb{D}$. If $D_1, D_2 \subset \mathbb{C}^n$ are two domains and $f : D_1 \to D_2$ is a biholomorphism, then $K_{D_1}(z,w) = K_{D_2}(f(z), f(w))$ for all $z, w \in D_1$.

Let $D \subset \mathbb{C}^n$. A *real (Kobayashi) geodesic* for D is a piecewise C^1 curve $\gamma : (a,b) \to D$ such that for every $t, t' \in (a,b)$ it holds $K_D(\gamma(t), \gamma(t')) = |t - t'|$, where $-\infty \leq a < b \leq +\infty$.

In the unit ball $\mathbb{B}^n$, given $z, w \in \mathbb{B}^n$, there exists a real geodesic $\gamma : [a,b] \to \mathbb{B}^n$ such that $\gamma(a) = z$ and $\gamma(b) = w$. Such a real geodesic is *unique up to reparametrization*, in the sense that, if $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \to \mathbb{B}^n$ is another real geodesic such that $\tilde{\gamma}(\tilde{a}) = z$ and $\tilde{\gamma}(\tilde{b}) = w$, then there exists $c \in \mathbb{R}$ such that $\tilde{\gamma}(t) = \gamma(\pm t + c)$.

The image of the real geodesic joining $z, w \in \mathbb{B}^n$ can be described as follows. Let $\Delta := \mathbb{B}^n \cap (\mathbb{C}(z-w) + w)$. Then Δ is an affine disc contained in the affine complex line $\mathbb{C}(z-w) + w$. Therefore, there exists an affine biholomorphic map $\varphi : \mathbb{D} \to \Delta$. Up to precomposing φ with an automorphism of the unit disc, we can assume that $\varphi(0) = z$ and $\varphi(t) = w$ for some $t > 0$. Then, $\gamma([a,b)] = \varphi([0,t])$. In particular, it means that $\gamma([a,b])$ is either a real segment contained in a real line passing through the center of Δ, or an arc of a circle in $\mathbb{C}(z-w) + w$ orthogonal to $\partial\Delta$. In other words, real geodesics in $\mathbb{B}^n$ are exactly the real geodesics for the Poincaré distance in Δ.

In particular, if $D \subset \mathbb{C}^n$ is a domain biholomorphic to $\mathbb{B}^n$, then for every $z, w \in D$ there exists a unique (up to reparametrization) real (Kobayashi) geodesic joining z and w. Moreover, if $F : \mathbb{B}^n \to D$ is a biholomorphism and $\gamma : [a,b] \to \mathbb{B}^n$ is a real geodesic in $\mathbb{B}^n$ joining $F^{-1}(a)$ and $F^{-1}(b)$, then $F \circ \gamma : [a,b] \to D$ is a real geodesic in D joining z and w.

By the previous consideration, it follows easily that, given $\xi \in \partial\mathbb{B}^n$ and $x_0 \in \mathbb{B}^n$, there exists a unique real geodesic $\gamma_\xi : [0, +\infty) \to \mathbb{B}^n$ such that $\gamma_\xi(0) = x_0$ and $\lim_{t\to\infty} \gamma_\xi(t) = \xi$. Moreover, let $x_0 \in \mathbb{B}^n$ and let $\{\zeta_k\} \subset \overline{\mathbb{B}^n}$ be a sequence converging to some point $\zeta \in \partial\mathbb{B}^n$. For each $k \in \mathbb{N}$, denote by $\gamma_k : [0, a_k) \to \mathbb{B}^n$ the unique real geodesic such that $\gamma_k(0) = x_0$ and either $\gamma_k(a_k) = \zeta_k$ in case $\zeta_k \in \mathbb{B}^n$ (and in such a case, necessarily $0 < a_k < \infty$), or $\lim_{t\to\infty} \gamma_k(t) = \zeta_k$ in case $\zeta_k \in \partial\mathbb{B}^n$ (and in such a case, necessarily $a_k = \infty$). Finally, let $\gamma : [0, \infty) \to \mathbb{B}^n$ be the only real geodesic such that $\gamma(0) = x_0$ and $\lim_{t\to\infty} \gamma(t) = \zeta$. Then $\{\gamma_k\}$ converges uniformly on compacta of $[0, \infty]$ to γ. In other words, for every $\epsilon > 0$ and $R > 0$ there exists $k_0 \in \mathbb{N}$ such that $a_k \geq R$ for every $k \geq k_0$ and, for every $s \in [0, R]$ and every $k \geq k_0$, it holds $K_{\mathbb{B}^n}(\gamma(s), \gamma_k(s)) < \epsilon$.

Finally, if $\xi_1, \xi_2 \in \partial\mathbb{B}^n$, $\xi_1 \neq \xi_2$, there exists a unique (up to reparametrization) real geodesic $\gamma : (-\infty, +\infty) \to \mathbb{B}^n$ such that $\lim_{t\to-\infty} \gamma(t) = \xi_1$ and $\lim_{t\to\infty} \gamma(t) = \xi_2$. By the previous considerations, it is easy to see that, if $\{\xi_k\} \subset \partial\mathbb{B}^n$ is a sequence converging to $\xi \in \partial\mathbb{B}^n$ and $\gamma_k : (-\infty, +\infty)$ is the unique (up to reparametrization) real geodesic whose closure contains ξ_k and ξ, then for every $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that $\gamma_k(-\infty, +\infty) \subset B(\xi, \epsilon)$ for all $k \geq k_0$.

3.2. **Horospheres.** For $\xi \in \partial\mathbb{B}^n$ and $R > 0$, let

$$E^{\mathbb{B}^n}(\xi, R) := \{z \in \mathbb{B}^n : \frac{|1 - \langle z, \xi\rangle|^2}{1 - \|z\|^2} < R\}.$$

The open set $E^{\mathbb{B}^n}(\xi, R)$ is called a *horosphere of center ξ and radius $R > 0$*, and it is a complex ellipsoid affinely biholomorphic to $\mathbb{B}^n$. For the aim of this paper, we need to recall the following properties of horospheres (see, *e.g.*, [1, Section 2] or [5]):

(1) $\overline{E^{\mathbb{B}^n}(\xi, R)} \cap \partial\mathbb{B}^n = \{\xi\}$ for every $R > 0$,
(2) $\bigcup_{R>0} E^{\mathbb{B}^n}(\xi, R) = \mathbb{B}^n$,
(3) $\bigcap_{R>0} \overline{E^{\mathbb{B}^n}(\xi, R)} = \{\xi\}$,
(4) if $0 < R < R'$ then $E^{\mathbb{B}^n}(\xi, R) \subset E^{\mathbb{B}^n}(\xi, R')$,
(5) if $\xi_1, \xi_2 \in \partial\mathbb{B}^n$ and $\xi_1 \neq \xi_2$, then there exists $R > 0$ such that $E^{\mathbb{B}^n}(\xi_1, R) \cap E^{\mathbb{B}^n}(\xi_2, R) = \emptyset$.

If $\xi \in \partial\mathbb{B}^n$ and $\{\zeta_k\} \subset \mathbb{B}^n$ is any sequence converging to ξ, then for every $R > 0$ it holds (see [1, Prop. 2.2.20]):

$$E^{\mathbb{B}^n}(\xi, R) = \{z \in \mathbb{B}^n : \limsup_{k\to\infty} [K_{\mathbb{B}^n}(z, \zeta_k) - K_{\mathbb{B}^n}(0, \zeta_k)] < \frac{1}{2} \log R\}.$$

If $F : \mathbb{B}^n \to D$ is a biholomorphism, and $\{z_k\} \subset D$ is any sequence such that $\{F^{-1}(z_k)\}$ converges to some $\xi \in \partial\mathbb{B}^n$, recalling that F is an isometry with respect to the Kobayashi distance, the previous equation allows us to define the *horosphere in D of radius $R > 0$*,

base point $F(0)$ relative to the sequence $\{z_k\}$ by setting

$$E^D(\{z_k\}, R) := F(E^{\mathbb{B}^n}(\xi, R))$$
$$= \{z \in D : \limsup_{k\to\infty}[K_D(z, z_k) - K_D(F(0), z_k)] < \frac{1}{2}\log R\}.$$

Horospheres defined using sequences can be used in general for defining a new topology (the *horosphere topology*) and a new boundary for complete hyperbolic manifolds (see [5]). We content here to state the following result which is needed later on (see [5, Proposition 6.1]):

Lemma 3.1. *Let $F : \mathbb{B}^n \to D$ be a biholomorphism. Suppose $D \subset \mathbb{C}^n$ is a convex domain. Let $\{z_k\} \subset D$ be a sequence such that $\{F^{-1}(z_k)\}$ converges to some $\xi \in \partial\mathbb{B}^n$. Then for every $R > 0$ the horosphere $E^D(\{z_k\}, R)$ is convex.*

3.3. **Iteration in the unit ball.** Let $f : \mathbb{B}^n \to \mathbb{B}^n$ be a holomorphic map. If f has no fixed points in $\mathbb{B}^n$ (namely, $f(z) \neq z$ for every $z \in \mathbb{B}^n$), then by the Denjoy-Wolff theorem for $\mathbb{B}^n$ (due to M. Hervé [8], see also [11] and [1, Theorem 2.2.31]), there exists a unique point $\tau \in \partial\mathbb{B}^n$, which is called the *Denjoy-Wolff point of f*, such that for every $z \in \mathbb{B}^n$ it holds $\lim_{k\to\infty} f^{\circ k}(z) = \tau$, where $f^{\circ k} := f^{\circ(k-1)} \circ f$, $f^1 = f$. In our argument we need the following result about Denjoy-Wolff points of commuting mappings (see [3, Theorem 3.3]):

Theorem 3.2. *Let $f, g : \mathbb{B}^n \to \mathbb{B}^n$ be holomorphic and assume $f \circ g = g \circ f$. Let $\tau \in \partial\mathbb{B}^n$ be the Denjoy-Wolff point of f and let $\sigma \in \partial\mathbb{B}^n$ be the Denjoy-Wolff point of g. Suppose $\tau \neq \sigma$. Let $\Delta := \mathbb{B}^n \cap (\mathbb{C}(\sigma - \tau) + \tau)$. Then $f(\Delta) = g(\Delta) = \Delta$ and $f|_\Delta, g|_\Delta$ are hyperbolic automorphisms of Δ with fixed points σ and τ.*

Note that Δ in the previous theorem is a disc contained in the affine complex line $\mathbb{C}(\sigma - \tau) + \tau$, hence there exists an affine biholomorphism $\varphi : \mathbb{D} \to \Delta$, and, saying that $f|_\Delta$ is a hyperbolic automorphism of Δ, we mean that $\varphi^{-1} \circ f \circ \varphi : \mathbb{D} \to \mathbb{D}$ is a hyperbolic automorphism of $\mathbb{D}$. Recall also that a hyperbolic automorphism of $\mathbb{D}$ is an automorphism of $\mathbb{D}$ (hence a Möbius transform) having exactly two fixed points on $\partial\mathbb{D}$ and no fixed points in $\mathbb{D}$.

Finally, recall that a continuous one-parameter group (h_t) of hyperbolic automorphisms of $\mathbb{D}$ is a continuous groups-homomorphism between the additive group of real numbers $\mathbb{R}$ endowed with the Euclidean topology and the group of automorphisms of $\mathbb{D}$ endowed with the topology of uniform convergence on compacta, such that, for every $t \neq 0$, the automorphism h_t is hyperbolic (see, *e.g.*, [1] or [14]). Then, there exist $\tau, \sigma \in \partial\mathbb{D}$, $\tau \neq \sigma$ such that $h_t(\tau) = \tau$ and $h_t(\sigma) = \sigma$ for all $t \in \mathbb{R}$. Also, τ is the Denjoy-Wolff point of h_t for all $t > 0$, while σ is the Denjoy-Wolff point of h_t for all $t < 0$ (or vice versa).

Remark 3.3. Let $\gamma : (-\infty, +\infty) \to \mathbb{D}$ be a real (Poincaré) geodesic such that $\lim_{s\to-\infty} \gamma(s) = \tau$ and $\lim_{s\to\infty} \gamma(s) = \sigma$, and let $\Gamma := \gamma(-\infty, +\infty)$. Since h_t is an isometry for $K_\mathbb{D}$ and τ, σ are fixed points for h_t for all $t \in \mathbb{R}$, then $h_t(\Gamma) = \Gamma$. Moreover, for $t > 0$,

given $s \in \mathbb{R}$ there exists $s' \in (-\infty, s)$ such that $h_t(\gamma(s)) = \gamma(s')$ (that is, $h_t(\gamma(s))$ is closer to τ than $\gamma(s)$ along γ), while, for $t < 0$, given $s \in \mathbb{R}$ there exists $s' \in (s, \infty)$ such that $h_t(\gamma(s)) = \gamma(s')$. Also, for every $\zeta_0, \zeta_1 \in \Gamma$ there exists $t \in \mathbb{R}$ such that $h_t(\zeta_0) = \zeta_1$.

4. Extension of convex maps

In this section we prove that every convex map of the unit ball extends continuously up to the boundary from the closed unit ball to the one-point compactification of $\mathbb{C}^n$. To this aim, we need some preliminary lemmas.

The following lemma was proved in [5, Lemma 6.16]:

Lemma 4.1. *Let $F : \mathbb{B}^n \to D$ be a biholomorphism. Suppose D is convex. Let $p, q \in \partial D$, $p \neq q$. Then for every sequences $\{p_n\}, \{q_n\} \subset D$ such that $\lim_{n\to\infty} p_n = p$ and $\lim_{n\to\infty} q_n = q$ it holds*

$$\lim_{n\to\infty} K_D(p_n, q_n) = \infty.$$

If $V \subset D$ and $\epsilon > 0$, we let

$$\mathcal{N}_\epsilon(V) := \{z \in D : \exists w \in V, K_D(z, w) < \epsilon\}.$$

Lemma 4.2. *Let $F : \mathbb{B}^n \to D$ be a biholomorphism. Suppose D is convex. Let $x \in D$ and let $p \in \partial D$. Then there exist an open set U containing p and $M > 0$ such that for every sequence $\{p_k\} \subset D \cap U$ converging to p, the real (Kobayashi) geodesic $\gamma_k : [0, R_k] \to D$ such that $\gamma_n(0) = x$ and $\gamma_k(R_k) = p_k$ satisfies*

$$\gamma_k(s) \in \mathcal{N}_M([x, p_k]) \quad \forall s \in [0, R_k].$$

Proof. By [5, Lemma 6.17], there exist $A > 0$ and $B > 0$ such that for every $k \in \mathbb{N}$, the real segment $[p_k, x]$ is a (A, B)-quasi-geodesic in the sense of Gromov (see, *e.g.*, [5, Section 6.2], [15], [6], [7]). Therefore the statement of the lemma follows immediately from Gromov's shadowing lemma (see [6, Théorème 11 p. 86]), since (D, K_D) is Gromov hyperbolic because so is $(\mathbb{B}^n, K_{\mathbb{B}^n})$ and F is an isometry for the Kobayashi distance. □

Definition 4.3. For a domain $D \subset \mathbb{C}^n$ we denote by $\overline{D}^*$ its closure in the one point compactification of $\mathbb{C}^n$.

Clearly, if D is relatively compact in $\mathbb{C}^n$ then $\overline{D}^* = \overline{D}$, while, if D is unbounded, then $\overline{D}^* = \overline{D} \cup \{\infty\}$.

Proposition 4.4. *Let $F : \mathbb{B}^n \to D$ be a biholomorphism. Suppose D is convex. Then there exists a continuous map $\tilde{F} : \overline{\mathbb{B}^n} \to \overline{D}^*$ such that $\tilde{F}|_{\mathbb{B}^n} = F$.*

Proof. We show, and it is enough, that for every $\xi \in \partial\mathbb{B}^n$ either the limit $\lim_{z\to\xi} F(z)$ exists and belongs to ∂D, or $\lim_{z\to\xi} \|F(z)\| = \infty$.

Assume by contradiction that this is not true. Then there exists $\xi \in \partial\mathbb{B}^n$ and two sequences $\{z_k^1\}, \{z_k^2\} \subset \mathbb{B}^n$ converging to ξ such that either $\lim_{k\to\infty} F(z_k^j) = p_j \in \partial D$,

$j = 1, 2$, for some $p_1 \neq p_2$, or $\lim_{k\to\infty} F(z_k^1) = p_1 \in \partial D$ and $\lim_{k\to\infty} \|F(z_k^2)\| = \infty$. In the second case, since the cluster set of F at ξ is connected, we can find another point $p_2 \in \partial D$, $p_2 \neq p_1$ and another sequence $\{w_k\} \subset \mathbb{B}^n$ converging to ξ such that $\lim_{k\to\infty} F(w_k) = p_2$. Therefore, we only need to consider the first case. Let $p_k^j := F(z_k^j)$, $k \in \mathbb{N}$ and $j = 1, 2$.

Let $x_0 \in D$. For every $k \in \mathbb{N}$, let $\gamma_k^j : [0, R_k^j] \to D$ be the unique real (Kobayashi) geodesic such that $\gamma_k^j(0) = x_0$ and $\gamma_k^j(R_k^j) = p_k^j$, $k \in \mathbb{N}$ and $j = 1, 2$. For $j = 1, 2$, let

$$V_j := \bigcup_{k\in\mathbb{N}} [x_0, p_k^j].$$

Up to replace $\{z_k^j\}_{k\in\mathbb{N}}$ with $\{z_k^j\}_{k\geq k_0}$ for some large $k_0 \in \mathbb{N}$ in such a way that $\{p_k^j\}$ is all contained in a small neighborhood of p_j, $j = 1, 2$, by Lemma 4.2 there exists $M > 0$ such that for every $k \in \mathbb{N}$, for every $s \in [0, R_k^j]$ and $j = 1, 2$, it holds

$$\gamma_k^j(s) \in \mathcal{N}_M(V_j). \tag{4.1}$$

Since F is an isometry for the Kobayashi distance, it follows that $F^{-1} \circ \gamma_k^j : [0, R_k^j] \to \mathbb{B}^n$ is the unique real geodesic joining $F^{-1}(x_0)$ and z_k^j, $j = 1, 2$. Since $\{z_k^j\}$ converges to ξ as $k \to \infty$, $j = 1, 2$, it follows that $F^{-1} \circ \gamma_k^j$ converges uniformly on compacta of $[0, \infty)$ to the unique real geodesic $\tilde{\gamma}$ in $\mathbb{B}^n$ joining $F^{-1}(x_0)$ with ξ. Therefore, if we let $\gamma := F \circ \tilde{\gamma}$, since F is an isometry for the Kobayashi distance, it follows that for every $R > 0$ there exists $k_R \in \mathbb{N}$ such that for every $s \in [0, R]$ and for every $k \geq k_R$ it holds for $j = 1, 2$,

$$K_D(\gamma_k^j(s), \gamma(s)) < M. \tag{4.2}$$

By the triangle inequality, (4.1) and (4.2) imply that

$$\gamma(s) \in \mathcal{N}_M(V_1) \cap \mathcal{N}_M(V_2), \quad \forall s \in [0, \infty).$$

In particular, this implies that there exist two sequences $\{q_k^j\} \subset V_j$, $j = 1, 2$, such that for every $k \in \mathbb{N}$ and $j = 1, 2$ it holds

$$K_D(\gamma(k), q_k^j) < M. \tag{4.3}$$

Since the sequence $\{\gamma(k)\}_{k\in\mathbb{N}}$ is not relatively compact in D and D is complete hyperbolic, it follows that $\{q_k^j\}$ is not relatively compact in D, $j = 1, 2$. Using the convexity of D it is not hard to see that the only possibility is that $\lim_{k\to\infty} q_k^j = p_j$, $j = 1, 2$. But, by the triangle inequality, (4.3) implies that

$$\lim_{k\to\infty} K_D(q_k^1, q_k^2) < 2M,$$

contradicting Lemma 4.1. □

5. Infinite discontinuities

Definition 5.1. Let $F : \mathbb{B}^n \to \mathbb{C}^n$ be holomorphic. We say that $\xi \in \partial\mathbb{B}^n$ is an *infinite discontinuity* of F if $\lim_{z\to\xi} \|F(z)\| = \infty$.

Proposition 5.2. *Let $F : \mathbb{B}^n \to D$ be a biholomorphism. Suppose D is convex and unbounded. Then F has either one or two infinite discontinuities. More precisely,*

(1) *if D has one connected component at ∞ then F has one infinite discontinuity,*
(2) *if D has two connected components at ∞ then F has two infinite discontinuities.*

Proof. By Lemma 2.2, there exists $v \in \mathbb{C}^n$ a direction at ∞ for D. Then for every $z \in D$ and $t \geq 0$, it holds $z + tv \in D$. Therefore, for every $t \geq 0$, the map $\phi_t : \mathbb{B}^n \to \mathbb{B}^n$, given by

$$\phi_t(\zeta) := F^{-1}(F(\zeta) + tv),$$

is well defined and univalent. Moreover, it is easy to see that $\phi_0 = \mathsf{id}_{\mathbb{B}^n}$, that $\phi_{t+s} = \phi_t \circ \phi_s$ for every $t, s \geq 0$ and that $[0, \infty) \ni t \mapsto \phi_t$ is continuous with respect to the Euclidean topology in $[0, \infty)$ and the topology of uniform convergence on compacta of $\mathsf{Hol}(\mathbb{B}^n, \mathbb{B}^n)$. In other words, $(\phi_t)_{t\geq 0}$ is a continuous one-parameter semigroup of $\mathbb{B}^n$. Moreover, by definition, ϕ_t has no fixed points in $\mathbb{B}^n$ for $t > 0$. By the continuous version of the Denjoy-Wolff theorem for continuous one-parameter semigroups of $\mathbb{B}^n$ (see [1, 2, 4]), there exists a unique point $\xi \in \partial\mathbb{B}^n$ (which is in fact the Denjoy-Wolff point of ϕ_t for every $t > 0$) such that for every $\zeta \in \mathbb{B}^n$ it holds $\lim_{t\to\infty} \phi_t(\zeta) = \xi$. Unrolling the definition of ϕ_t, this means that for every $z \in D$ we have

$$\lim_{t\to\infty} F^{-1}(z + tv) = \xi.$$

Now, let us assume that there exists another direction at ∞ for D, say w, which is $\mathbb{R}$-linearly independent of v. Then we can define another continuous one-parameter semigroup of $\mathbb{B}^n$, call it $(\psi_s)_{s\geq 0}$, by setting

$$\psi_s(\zeta) := F^{-1}(F(\zeta) + sw).$$

Let $\xi' := \lim_{s\to\infty} F^{-1}(z + sw)$ for some–and hence, as we just proved, for every–$z \in D$. We claim that $\xi = \xi'$.

Suppose by contradiction that $\xi \neq \xi'$. It is easy to see that for every $s, t \geq 0$ the maps ϕ_t and ψ_s commute, *i.e.*, $\phi_t \circ \psi_s = \psi_s \circ \phi_t$. Let $\Delta := (\mathbb{C}(\xi - \xi') + \xi) \cap \mathbb{B}^n$. Since the Denjoy-Wolff point of ϕ_t is ξ for all $t > 0$ and the Denjoy-Wolff point of ψ_s is ξ' for all $s > 0$, and we are supposing $\xi \neq \xi'$, by Theorem 3.2, it follows that for all $s, t \geq 0$,

$$\phi_t(\Delta) = \Delta, \quad \psi_s(\Delta) = \Delta$$

and the restriction of ϕ_t and ψ_s to Δ are hyperbolic automorphisms of Δ. Let $\gamma : (-\infty, +\infty) \to \Delta$ be a parameterization of the real geodesic in Δ whose closure contains ξ and ξ'. Then, it follows that $\phi_t \circ \gamma : (-\infty, \infty) \to \Delta$ and $\psi_s \circ \gamma : (-\infty, \infty) \to \Delta$ are other parameterizations for the same real Kobayashi geodesic whose closure contains ξ and ξ', for all $s, t \geq 0$. Moreover, since ξ is the Denjoy-Wolff point of $\phi_t|_\Delta$ and ξ' is

the Denjoy-Wolff point of $\psi_s|_\Delta$, it follows that (see Remark 3.3), letting $\zeta_0 := \gamma(0)$, there exists $s_0 \in (0,\infty)$ such that $\psi_{s_0}(\phi_1(\zeta_0)) = \zeta_0$. Thus,

$$F(\zeta_0) = F(\psi_{s_0}(\phi_1(\zeta_0))) = F(\zeta_0) + s_0 w + v,$$

that is, $s_0 w + v = 0$, which is a contradiction since w and v are assumed to be $\mathbb{R}$-linearly independent.

Summing up we proved that for every two directions v, w at ∞ for D which are $\mathbb{R}$-linearly independent it holds

$$\lim_{t\to\infty} F^{-1}(z_0 + tw) = \lim_{t\to\infty} F^{-1}(z_1 + tv), \quad \forall z_0, z_1 \in D. \tag{5.1}$$

Now, let $\{p_k\} \subset D$ be a sequence converging to ∞, and assume that $\lim_{k\to\infty} \frac{p_k}{\|p_k\|} = w$, for some $w \in \mathbb{C}^n$ direction at ∞ for D. Let $z_0 \in D$. We claim that

$$\lim_{k\to\infty} F^{-1}(p_k) = \lim_{t\to\infty} F^{-1}(z_0 + tw). \tag{5.2}$$

Assume this is not the case and, up to subsequences, $\lim_{k\to\infty} F^{-1}(p_k) = \xi_0 \in \partial\mathbb{B}^n$ and $\lim_{t\to\infty} F^{-1}(z_0 + tw) = \xi_1 \in \partial\mathbb{B}^n$ with $\xi_0 \neq \xi_1$.

Let fix $R > 0$ and let $E := E^{\mathbb{B}^n}(\xi_0, R)$ be the horosphere of center ξ_0 and radius R in $\mathbb{B}^n$. Then $F(E) = E^D(\{p_k\}, R)$ is convex by Lemma 3.1. Also, there exists $t_0 \geq 0$ such that $z_0 + tw \notin F(E)$ for all $t \geq t_0$, because $F^{-1}(z_0 + tw)$ does not converge to ξ_0 as $t \to \infty$ and hence it is eventually outside E. Moreover, $F(E)$ is unbounded because by Proposition 4.4, $\lim_{z\to\xi_0} \|F(z)\| = \lim_{k\to\infty} \|F(F^{-1}(p_k))\| = \infty$. By Lemma 2.2, there exists a direction at ∞ for $F(E)$, call it u, that is, $z + tu \in F(E)$ for every $z \in F(E)$ and for every $t \geq 0$. Since $\overline{E} \cap \partial\mathbb{B}^n = \{\xi_0\}$, it follows that $\lim_{t\to\infty} F^{-1}(z + tu) = \xi_0$.

Hence, there are two cases: either u is $\mathbb{R}$-linearly independent of w, or $u = -w$. In the first case, we contradict (5.1). Therefore, $u = -w$. In this case, note that for every $R > 0$ there exist $k_R \in \mathbb{N}$ and $t_R \geq 0$ such that p_k and $z_0 + tw$ are in the same unbounded connected component of $D \setminus \overline{B(0,R)}$ for every $k > k_R$, and $t > t_R$, call U_R such an unbounded component. Hence, $\bigcap_{R>0} \overline{F^{-1}(U_R)}$ is compact and connected in $\partial\mathbb{B}^n$ and contains ξ_0 and ξ_1. Since $\xi_0 \neq \xi_1$, this implies that there exists a sequence $\{q_k\} \subset D$ converging to ∞ which is eventually contained in U_R for all $R > 0$ such that $\{F^{-1}(q_k)\}$ converges to a point $\xi_2 \in \partial\mathbb{B}^n$ with $\xi_2 \neq \xi_0$ and $\xi_2 \neq \xi_1$. We can then repeat the previous argument considering ξ_2 instead of ξ_1 and $\{q_k\}$ instead of $\{p_k\}$. But this time, the direction u at ∞ for $F(E)$ can not be w nor $-w$, since, by construction, both $F^{-1}(z_0 + tw)$ and $F^{-1}(z_0 - tw)$ are not eventually contained in $E = E^{\mathbb{B}^n}(\xi_2, R)$ for $t \geq 0$, hence, we are back to the first case and again we contradict (5.1). Therefore, (5.2) holds.

Suppose now that D has one connected component at ∞. If for every direction v at ∞ for D, the vector $-v$ is not a direction at ∞ for D, then it follows immediately from (5.2) and (5.1) that F has only one infinite singularity. In case v and $-v$ are directions at ∞ for D, we claim that

$$\lim_{t\to\infty} F^{-1}(z_0 + tv) = \lim_{t\to\infty} F^{-1}(z_0 - tv).$$

Indeed, by Lemma 2.2, since D has one connected component at ∞, there exists some other direction w at ∞ for D such that $w \neq \pm v$. Therefore, w is $\mathbb{R}$-linearly independent of v (and then of $-v$). Hence by (5.1)

$$\lim_{t\to\infty} F^{-1}(z_0+tv) = \lim_{t\to\infty} F^{-1}(z_0+tw) = \lim_{t\to\infty} F^{-1}(z_0-tv).$$

From this and from (5.2) and (5.1) it follows again that F has a unique infinite discontinuity.

On the other hand, if D has two connected components at ∞, the only directions at ∞ for D are v and $-v$. We claim that in this case

$$\lim_{t\to\infty} F^{-1}(z_0+tv) \neq \lim_{t\to\infty} F^{-1}(z_0-tv). \tag{5.3}$$

Once we proved the claim (5.3), (5.2) implies at once that F has exactly two infinite discontinuities.

In order to prove the claim, assume by contradiction that $\xi := \lim_{t\to\infty} F^{-1}(z_0+tv) = \lim_{t\to\infty} F^{-1}(z_0-tv)$. By Lemma 2.2, there exists R_0 such that for every $R \geq R_0$ the open set $D \setminus \overline{B(0,R)}$ has two unbounded connected components, say U_1, U_2. Up to relabelling, it is clear that z_0+tv is eventually contained in U_1 and z_0-tv is eventually contained in U_2, for t large. Let $K := \overline{F^{-1}(D \cap \overline{B(0,R)})}$. Since $\lim_{z\to\xi} \|F(z)\| = \infty$ by Proposition 4.4, it follows that $\xi \notin K$, while $\zeta_t^+ := F^{-1}(z_0+tv)$ and $\zeta_t^- := F^{-1}(z_0-tv)$ are close to ξ for t large. Therefore, the real (Kobayashi) geodesic γ_t joining ζ_t^+ and ζ_t^- satisfies $\gamma_t \cap K = \emptyset$ for t sufficiently large. But then $F(\gamma_t)$ is a continuous path in $D \setminus \overline{B(0,R)}$ which joins z_0+tv and z_0-tv, against the fact that $z_0+tv \in U_1$ and $z_0-tv \in U_2$. Thus, (5.3) holds. □

6. Homeomorphic extension of convex maps

In this section we collect the previous results and prove our main theorem, from which Theorem 1.2 follows at once:

Theorem 6.1. *Let $F : \mathbb{B}^n \to D$ be a biholomorphism. Suppose that D is convex. Then*

1. *D is bounded and F extends as a homeomorphism $\tilde{F} : \overline{\mathbb{B}^n} \to \overline{D}$, or*
2. *D is unbounded and has one connected component at ∞, F extends as a homeomorphism $\tilde{F} : \overline{\mathbb{B}^n} \to \overline{D}^*$. In particular, F has only one infinite discontinuity $\xi \in \partial\mathbb{B}^n$ such that $\lim_{z\to\xi} \|F(z)\| = \infty$ and F extends as a homeomorphism $\tilde{F} : \overline{\mathbb{B}^n} \setminus \{\xi\} \to \overline{D}$. Or,*
3. *D is unbounded and has two connected components at ∞, F has two infinite discontinuities $\xi_1, \xi_2 \in \partial\mathbb{B}^n$, $\xi_1 \neq \xi_2$, such that $\lim_{z\to\xi_1} \|F(z)\| = \lim_{z\to\xi_2} \|F(z)\| = \infty$ and F extends as a homeomorphism $\tilde{F} : \overline{\mathbb{B}^n} \setminus \{\xi_1, \xi_2\} \to \overline{D}$.*

Proof. By Proposition 4.4, F has a continuous extension $\tilde{F} : \overline{\mathbb{B}^n} \to \overline{D}^*$. If D is unbounded, by Proposition 5.2, $\tilde{F}^{-1}(\infty)$ contains either one point (in case D has one connected component at ∞) or two points (in case D has two connected components at ∞).

We show that, for every $x_1, x_2 \in \partial\mathbb{B}^n$, $x_1 \neq x_2$, such that $\tilde{F}(x_j) \neq \infty$, $j = 1, 2$, it follows $\tilde{F}(x_1) \neq \tilde{F}(x_2)$. The proof of this is similar to that of [5, Corollary 8.3], but we sketch it here for the reader convenience.

Assume by contradiction that there exist $x_1, x_2 \in \partial\mathbb{B}^n$ such that $p = \tilde{F}(x_1) = \tilde{F}(x_2) \in \partial D$. Let $R > 0$ be such that $V := E^{\mathbb{B}^n}(x_1, R) \cap E^{\mathbb{B}^n}(x_2, R) \neq \emptyset$. Then V is an open, convex, relatively compact subset of $\mathbb{B}^n$. Since $E_j := F(E^{\mathbb{B}^n}(x_j, R))$, $j = 1, 2$, is a horosphere in D, hence it is convex by Lemma 3.1, it follows that $F(V)$ is an open, convex, relatively compact subset of D.

Now, let $\{\zeta_k^j\}_{k\in\mathbb{N}} \subset E^{\mathbb{B}^n}(x_j, R)$, be a sequence converging to x_j, $j = 1, 2$. Hence by hypothesis, $p = \lim_{k\to\infty} F(\zeta_k^j)$, $j = 1, 2$, showing that $p \in \overline{E_1} \cap \overline{E_2}$. Therefore, given any $z_0 \in V$, it follows that $[z_0, p) \subset E_j$, $j = 1, 2$. Hence, $[z_0, p) \subset F(V)$, which is not relatively compact in D, a contradiction.

Thus, if either D is relatively compact in $\mathbb{C}^n$, or if D has one connected component at ∞, it follows that $\tilde{F} : \overline{\mathbb{B}^n} \to \overline{D}^*$ is a bijective continuous map from a compact space to a Hausdorff space, hence, a homeomorphism.

In case D has two connected components at ∞, let $\xi_1, \xi_2 \in \partial\mathbb{B}^n$ be the two infinite discontinuities of F. Since every closed subset $C \subset \overline{\mathbb{B}^n} \setminus \{\xi_1, \xi_2\}$ is also compact in $\overline{\mathbb{B}^n} \setminus \{\xi_1, \xi_2\}$, it follows that $\tilde{F}(C)$ is compact in $\overline{D}$, and hence closed. Therefore, $\tilde{F} : \overline{\mathbb{B}^n} \setminus \{\xi_1, \xi_2\} \to \overline{D}$ is a closed, bijective, continuous map, thus it is a homeomorphism. □

References

[1] M. Abate, *Iteration theory of holomorphic maps on taut manifolds*, Mediterranean Press, Rende, 1989.

[2] L. Aizenberg, D. Shoikhet, *Boundary behavior of semigroups of holomorphic mappings on the unit ball in* $\mathbb{C}^n$, Complex Variables Theory Appl. 47 (2002), 109-121.

[3] F. Bracci, *Common fixed points of commuting holomorphic maps in the unit ball of* $\mathbb{C}^n$, Proc. Amer. Math. Soc. 127 (1999), 1133-1141.

[4] F. Bracci, M. D. Contreras, S. Díaz-Madrigal, *Classification of semigroups of linear fractional maps in the unit ball.* Adv. Math., 208, (2007), 318-350.

[5] F. Bracci, H. Gaussier, *Horosphere topology*, ArXiv.1605.04119v4

[6] E. Ghys, P. de La Harpe, *Sur les groupes hyperboliques d'après Mikhael Gromov.* Progress in Mathematics **83**, Birkhauser.

[7] M. Gromov, *Hyperbolic groups. Essays in group theory.* Math. Sci. Res. Inst. Publ., 8, Springer, New York (1987), 75-263.

[8] M. Hervé, *Quelques propriétés des applications analytiques d'une boule à m dimensions dans elle-même.* J. Math. Pures Appl. 42 (1963), 117-147.

[9] S. Kobayashi, *Hyperbolic Complex Spaces*, Springer-Verlag, Grundlehren der mathematischen Wissenschaften 318, 1998.

[10] L. Lempert, *La métrique de Kobayashi et la réprésentation des domaines sur la boule.* Bull. Soc. Math. Fr. 109 (1981), 427-474.

[11] B. D. MacCluer, *Iterates of holomorphic self-maps of the unit ball in* $\mathbb{C}^N$. Mich. Math. J. 30 (1983), 97-106.

[12] J. R. Muir, Jr., T. J. Suffridge, *Unbounded convex mappings of the ball in* $\mathbb{C}^n$, Proc. Amer. Math. Soc., 129 (2001), no. 11, pp. 3389-3393.
[13] J. R. Muir, Jr., T. J. Suffridge, *A generalization of half-plane mappings to the ball in* $\mathbb{C}^N$. Trans. Amer. Math. Soc. 359, (2007), 1485-1498.
[14] D. Shoikhet, *Semigroups in Geometrical Function Theory*, Kluwer, Dordrecht, NL, 2001.
[15] A. M. Zimmer, *Gromov hyperbolicity and the Kobayashi metric on convex domains of finite type.* Math. Ann. 365 (2016), 1425-1498.

F. Bracci: Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica 1, 00133, Roma, Italia.
E-mail address: fbracci@mat.uniroma2.it

H. Gaussier: Univ. Grenoble Alpes, CNRS, IF, F-38000 Grenoble, France
E-mail address: herve.haussier@univ-grenoble-alpes.fr