Computer Visualization of the Riemann Zeta Function

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Abstract: *Various representations of the multi-dimensional Riemann zeta function $\zeta(s)$ are provided herein. The representations include 3D plots in the xyz space as well as contour maps where the third dimension is represented by color or iso-line contours. Some polar projections of the zeta function are shown near some non-trivial zeros of the zeta function. 3D polar contour maps can be a powerful tool in the analysis of the Riemann zeta function. For example, shape orientation in the 3D polar contour map projections can be used to provide a location of consecutive zeros.*

Keywords: *Riemann Zeta Function, Number Theory, 3D polar contour maps*

The Riemann zeta function $\zeta(s)$ has been explored for over a century and it is still investigated and continues to be the subject of numerous studies. The Riemann zeta function was constructed by Bernhard Riemann and published in his 1859 famous paper entitled "*On the Number of Primes Less Than a Given Magnitude.*"[1] The Riemann zeta function $\zeta(s)$ is a function of a complex variable $s = \sigma + it$. The notation s, σ, and t has been adopted in the study of the *zeta* function following Riemann's notation[2].

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

where $s = \sigma + it$, σ and $t \in R$

However, it is Leonard Euler who first introduced and studied the zeta function in 1740 without using the complex variable which was not known at that time. Euler studied the zeta function as a function of a real variable (i.e., with "σ only"). Riemann extended Euler's definition of the zeta function to the complex space by introducing the complex variable $s = \sigma + it$. Euler further provided a relationship between the zeta function and the prime numbers. An elegant derivation of this relationship known as the Euler Product is provided by William Dunham in "*Euler, The Master of Us All*."[3]

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_p \left(\frac{1}{1 - p^{-s}}\right)$$

Although this relationship was proven by Euler in the case where s is a real variable, this relationship holds true in the complex space, where s is a complex variable.

The Riemann zeta function has a number of zeros. There are zeros located at σ = -2, -4, -6, …and t = 0, and are known as the trivial zeros. The Riemann zeta function also has the so-called non-trivial zeros that are located within the critical strip $0 \le \sigma \le 1$. The Riemann hypothesis states that *all* non-trivial zeros of the zeta function are located along the line $\sigma=1/2$. A number of zeros have been computed and found to lie on the line $\sigma=1/2$. Andrew Odlyzko provides an extensive list of non-trivial zeros of the zeta function[4]. For example, the first three zeros are located at, approximately, t=14.13, 21.02, 25.01. However, at the present time, no proof is available to show that *all* non-trivial zeros lie on the line $\sigma=1/2$. Proving that *all* non-trivial zeros lie on the line $\sigma=1/2$ is part of Hilbert's eighth problem in David Hilbert's list of 23 unsolved problems. It is also one of the Clay Mathematics Institute's Millennium Prize Problems.

This paper is not intended to develop any new mathematical proof but simply provide various graphical visualizations of the zeta function to show its various facets depending on the selected real variables or dimensions. The zeta function being a complex function of a complex variable is necessarily a function in a four-dimensional space. However, since we cannot visualize a four-dimensional space, the zeta function can be "projected" in the three-dimensional space. In the following paragraphs, instead of the traditional notation $s = \sigma + it$, the notation $s = ReS + i \cdot ImS$ is used. The zeta function $\zeta(s)$ can then be expressed as follows:

$$ReZ + i \cdot ImZ = \zeta(ReS + i \cdot ImS)$$

In this case, the 4 variables or dimensions are *ReS*, *ImS*, *ReZ* and *ImZ*, where *ReS* and *ImS* are, respectively, the real part and the imaginary part of the variable s, and *ReZ* and *ImZ* are, respectively, the real part and the imaginary part of the zeta function.

Fig. 1 shows a contour map of the real and imaginary parts of the zeta function as a function of the real and imaginary parts of the s variable. The contour map is plotted for ReS in the range [-13, 0.9] and for ImS in the range [-5, 27]. The real part ReZ at value 0 is shown as contours with solid thicker lines. The imaginary part ImZ at value 0 is shown as contours with solid thinner lines. The position of the three first non-trivial zeros can be seen at the intersection of a thicker line corresponding to ReZ = 0 and a thinner line corresponding to ImZ = 0, at ReS = 0.5 and ImS = 14.13, 21.02, and 25.01. This contour plot is similar to a contour plot provided by J. Arias-de-Reyna in "*X-Ray of Riemann's Zeta-Function*"[5].

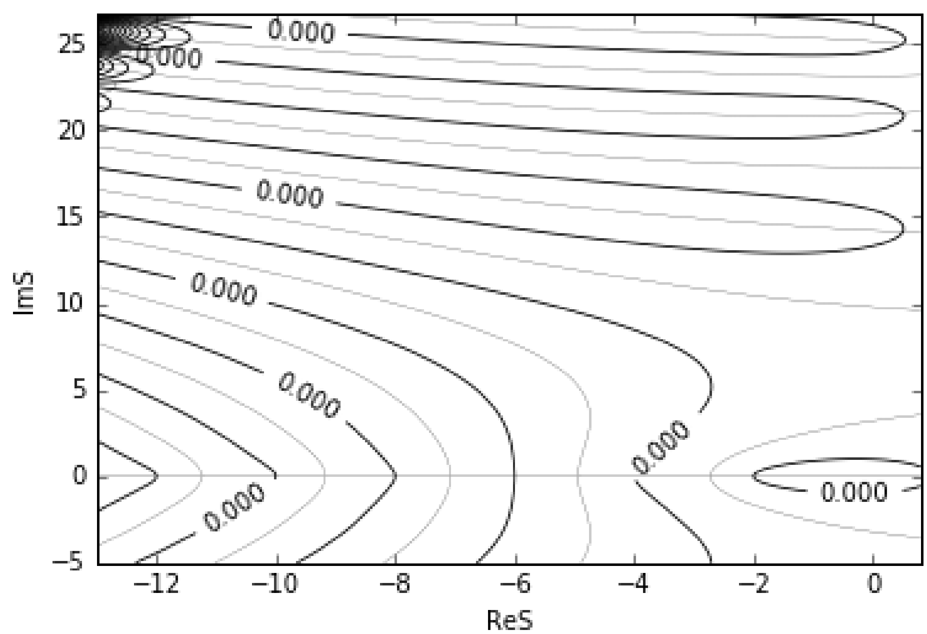

Fig. 1: Contour map of real part ReZ and imaginary part ImZ of the zeta function versus ReS and ImS. Thicker lines correspond to contour lines at ReZ=0, and thinner lines correspond to contour lines at ImZ = 0.

Fig. 2a and 2b show contour maps of the real and imaginary parts of the zeta function as a function of the real and imaginary parts of the s variable. The contour map is plotted around the line ½, for ReS in the range [0.4, 0.6], and for ImS in the range [0, 27]. The values of the real part ReZ are shown in various colors with the color cyan corresponding to the zero values and the red color corresponding to the higher value 2.5, as shown on the color-coded strip. The values of the imaginary part ImZ are shown as various colors with the color blue/purple corresponding to the negative value -1.6, the zero values corresponding to the green color, and the red color corresponding to the higher value 1.6, as shown on the color-coded strip.

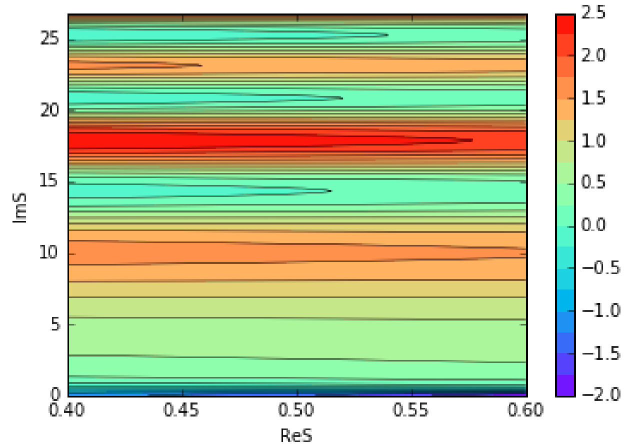

Fig. 2a: Contour map of real part ReZ of the zeta function as a function of ReS and ImS.

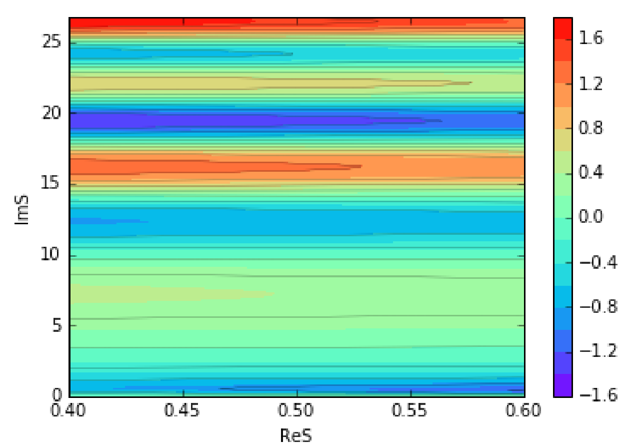

Fig. 2b: Contour map of imaginary part ImZ of the zeta function as a function of ReS and ImS.

The non-trivial zeros can be seen at ReZ = 0 and ImZ = 0, corresponding to the cyan areas for ReZ and green areas for ImZ. A superposition of these two maps provides the location of the three first zeros at ImS equal to about 14.13, 21.02, and 25.01.

A more precise location of the zeros can be seen when performing a cross-cut through the above contour maps at exactly ReS = 0.5. Fig. 3 is a plot of ReS and ImZ versus ImS at ReS = 0.5. ReZ is plotted as a blue line and ImZ is plotted as a red line. The intersection of these two lines at ReZ (blue) equal to 0 and ImZ (red) equal to 0 indicates the position of first three non-trivial zeros which are highlighted by green triangles in the interval ImS=[0,27], at ImS = 14.13, ImS = 21.02, and ImS = 25.01, respectively.

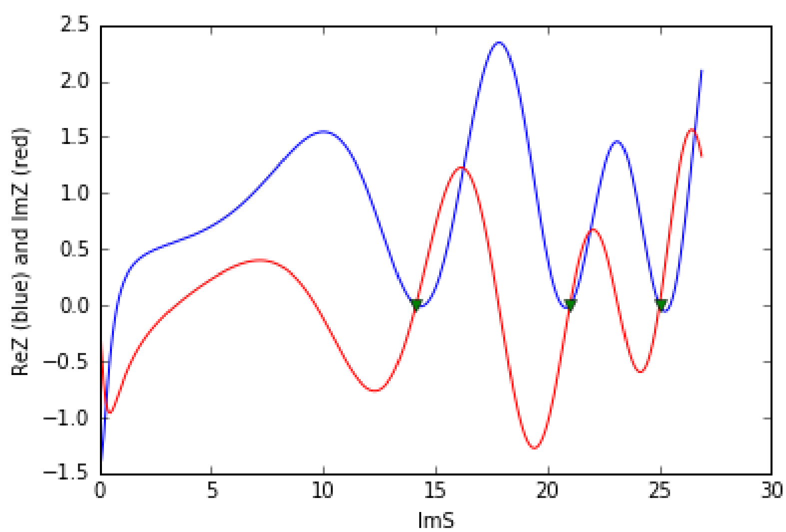

Fig. 3: Plot of ReZ and ImZ versus ImS in the the interval [0,27].
The position of the first three non-trivial zeros is indicated by 3 green triangles.

The position of the first three non-trivial zero can also be seen when plotting the magnitude of the zeta function, i.e., MagZ = sqrt(ReZ^2+ImZ^2) versus ImS in the interval [0, 27], as shown in Fig.4. The zeros correspond to MagZ = 0.

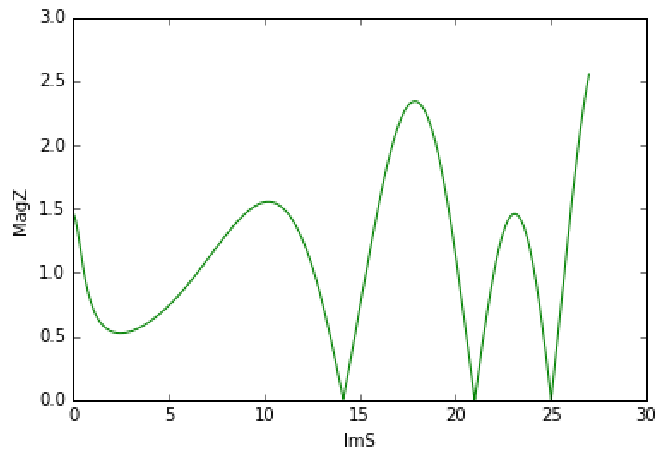

Fig. 4: Plot of Magnitude of the zeta function as a function of ImS.

An interesting three-dimensional graphical representation of the zeta function is a 3D representation in which the x and y axes are, respectively, the real part ReZ and the imaginary part ImZ of the zeta function, and the z axis is the real part ReS or the imaginary part ImS of the complex variable s. Fig. 5 shows such a 3D plot of the zeta function where the x-axis corresponds to ReZ, the y-axis corresponds to ImZ, and the z-axis corresponds to either ReS or ImS. This 3D plot is obtained for a fixed value ReS = 0.5 (in blue) and for ImS in the range [0, 27] (in red). The position of the zeros of the zeta function is at ReZ = 0 and ImZ = 0. The zeros are located at the intersection point of the blue curve with itself, and at ImS=14.13, ImS = 21.02, and ImS = 25.01, for the red curve. For example, the blue line intersects three times with itself at ReZ=0 and ImZ=0, which correspond to the three non-trivial zeros in the interval ImS = [0, 27].

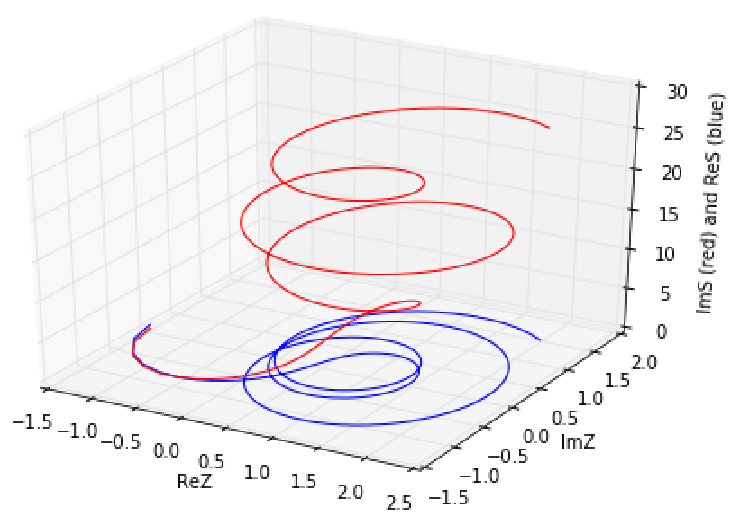

Fig. 5: 3D plot of the zeta function where the x-axis corresponds to ReZ,
the y-axis corresponds to ImZ and the z-axis corresponds to ReS (blue) or ImS (red).

The above 3D representation can be projected onto the 2D plane to plot the argand diagram of the zeta function where the real part of the zeta function ReZ is plotted along the x-axis and the imaginary part of the zeta function ImZ is plotted along the y-axis, for ReS = 0.5 and ImS in the interval [0, 27], as shown in Fig. 6. The argand diagram shows closed loops of the zeta function where the intersection points, at ReZ = 0 and ImZ = 0, correspond to the zeros of the zeta function.

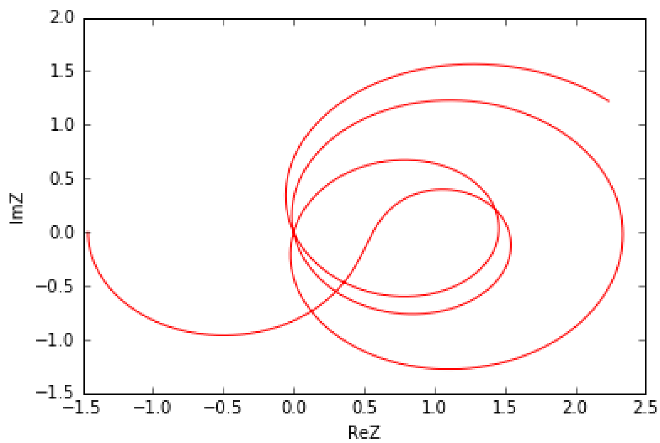

Fig. 6: Argand diagram of the zeta function

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Similar plots representing the relationship between ReZ, ImZ and ReS and ImS can be found in various publications. However, it is worthwhile to provide a representation of the magnitude and phase of the zeta function $\zeta(s)$ as a function of ReS and ImS, but most importantly as a function of the magnitude of the complex variable s and the phase of the complex variable s. Indeed, the zeta function $\zeta(s)$ can also be expressed in the polar form as follows:

$$|z| \cdot e^{i \cdot \varphi_z} = \zeta\left(|s| \cdot e^{i \cdot \varphi_s}\right)$$

In this case, the 4 variables or dimensions are $|s|, \varphi_s, |z|, \varphi_z$, where $|s|$ is the magnitude of the complex variable s, φ_s is the phase of the complex variable s, $|z|$ is the magnitude of the zeta function ζ, and φ_z is the phase of the of the zeta function ζ. These 4 variables are expressed as follows:

$$|s| = \sqrt{ReS^2 + ImS^2}$$

$$\varphi_s = \arctan\left(\frac{ImS}{ReS}\right) = \text{atan2}(ImS, ReS)$$

$$|z| = \sqrt{ReZ^2 + ImZ^2}$$

$$\varphi_z = \arctan\left(\frac{ImZ}{ReZ}\right) = \text{atan2}(ImZ, ReZ)$$

In the following paragraphs, the following notation will be used instead of the above notation.

$|s|, \varphi_s, |z|, \varphi_z$ → MagS, PhS, MagZ, PhZ

It may be worthwhile to show the behavior of the phase of the zeta function and/or the phase of variable s. In computing the phase PhS or PhZ, “atan2” is used instead or “atan” to take into account the sign of ImZ, ReZ, ImS and ReS. The phase is computed in the interval [-π, π] for both the complex variable s and the zeta function. Phase plots can provide novel ideas[6]. Fig. 7 below is a contour map of the phase PhZ of the zeta function versus the real part ReS of the complex variable s (the x-axis) and the imaginary part ImS of the complex variable s (the y-axis). The third dimension which is represented by color corresponds to the value of the phase angle PhZ. The color-coded strip shows the corresponding values of the phase angle PhZ in the interval [-π,π] or [-180°, 180°]. The phase contour map is plotted for values of ReS in the interval [-27, 27] and values of ImS in the interval [-27, 27]. As shown in Fig. 7, for certain values of ReS, ImS, the phase PhZ jumps from negative to positive. In fact, the phase shifts along the "ribs" of the zeta function (the term "ribs" being borrowed from reference[5]). Along the critical line, every time the zeta function changes sign, a discontinuous jump by π in the phase is introduced[7]. As can be seen in Fig. 7, the phase contour is symmetric relative to the zero axis ReS. The negative values of the phase of the zeta function on negative ImS side are the "mirror image" of the positive values of the phase of the zeta function on positive ImS side.

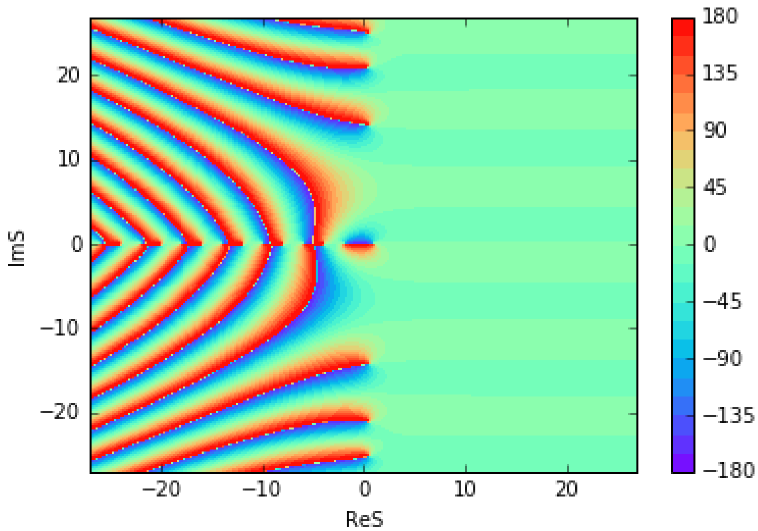

Fig. 7: Contour map of the phase PhZ of the zeta function versus ReS and ImS.

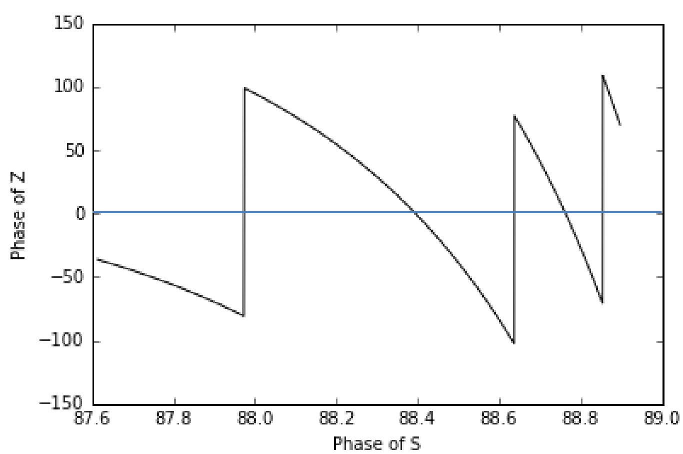

Fig. 8: Plot of phase PhZ of the zeta function versus the phase PhS of complex variable s.

Fig. 8 shows the variation of phase of the zeta function as a function of the phase of the complex variable s, for ReS = 0.5 and ImS in the interval [12, 26]. At the zeros, the phase of the zeta function shifts from a negative value to a positive value by π, as shown by the vertical lines in Fig. 8.

Fig. 9 below is a polar contour map of the phase PhZ of the zeta function as a function of the magnitude MagS of the complex variable s and the phase PhS of the complex variable s. The magnitude MagS is represented by the radius on the polar plot, from 0 to about 38, and the phase PhS is represented by the angle in deg. from 0° to 360° on the polar plot. The third dimension which is represented by levels of color corresponds to the value of the phase angle PhZ. The color-coded strip shows the corresponding values of the phase angle PhZ in the interval [$-\pi,\pi$] or [-180°, 180°]. The phase PhS and the magnitude MagS are computed using values of ReS in the interval [-27, 27] and ImS in the interval [-27, 27]. As shown in Fig. 9, similar to Fig. 7, for certain values of ReS, ImS, the phase PhZ jumps from negative values to positive values (blue to red in the contour map). Fig. 9 can be seen as a projection of a sphere wherein the radial distance (radius) of the sphere represents the magnitude of s (MagS), the azimuth angle represents the phase angle of s (PhS), and the inclination or elevation angle represents the phase of the zeta function (PhZ).

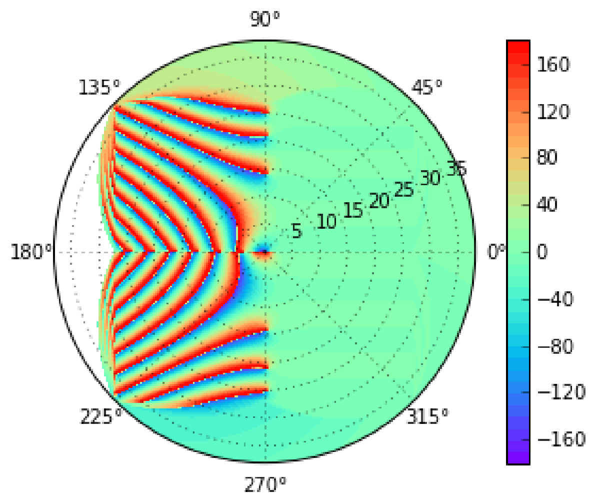

Fig. 9: Polar contour map of the phase PhZ of the zeta function as a function of the magnitude MagS of the complex variable s and the phase PhS of the complex variable s.

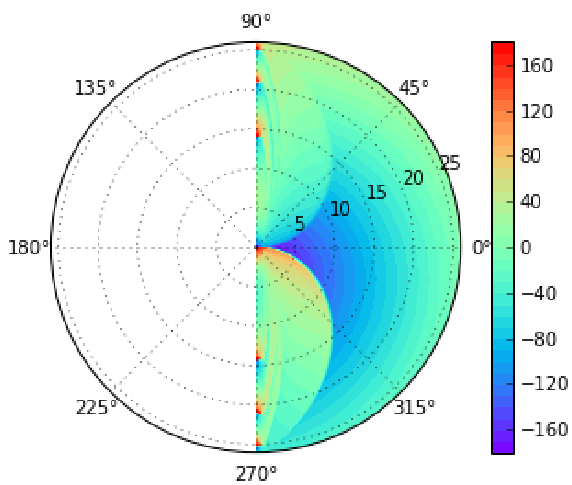

Fig. 10: Polar contour map of the phase PhZ of the zeta function as a function of the magnitude MagS of the complex variable s and the phase PhS of the complex variable s.

Fig. 10 above is also a polar contour map of the phase PhZ of the zeta function as a function of the magnitude MagS of the complex variable s and the phase PhS of the complex variable s. However, in this plot, the phase PhS and the magnitude MagS are computed using values of ReS in the interval [0.0, 0.9] within the critical strip, and ImS in the interval [-26, 26]. The zeros can be seen at PhS around 90° with a magnitude of about 14, 21and 25, and with a phase of the zeta function (PhZ) shifting from negative values (blue) to positive values (red).

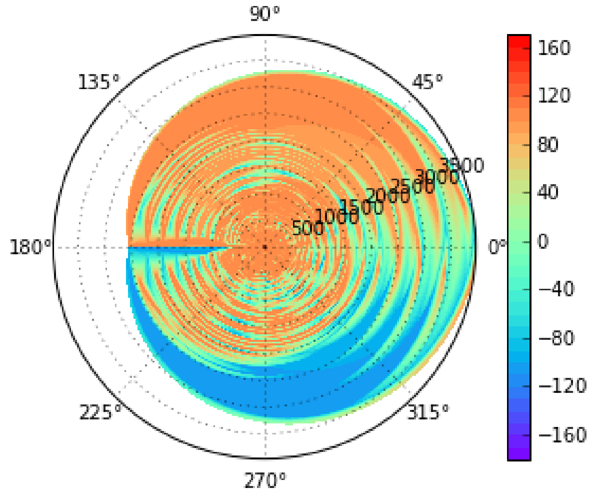

Fig. 11: Polar contour map of the phase of the complex variable s, the radius representing the magnitude of the zeta function, the azimuthal angle representing the phase angle PhZ of the zeta function, and the color representing the phase angle PhS of the complex variable s.

Fig. 11 above is a polar contour map of the phase of variable s as a function of the magnitude MagZ of the zeta function and the phase PhZ of the zeta function, where the radius represents the magnitude of the zeta function, i.e, MagZ, and the azimuthal angle represents the phase angle PhZ of the zeta function. The various levels of color represent the value of the phase angle PhS of the complex variable s. The phase angle PhS varies from -90° (blue color) to +90° (orange color). This plot is obtained using values of ReS in the interval [-5, 5] and ImS in the interval [-28, 28]. The zeros are obviously located at the origin where the magnitude of the zeta function is equal to zero.

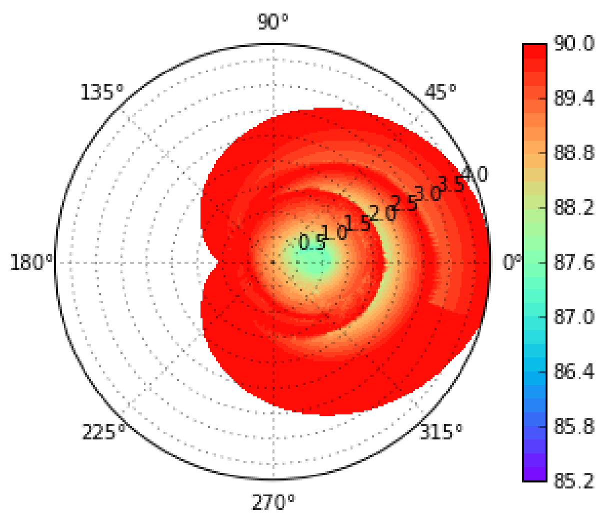

Fig. 12: Polar contour map of the phase of the complex variable s, the radius representing the magnitude of the zeta function, the azimuthal angle representing the phase angle PhZ of the zeta function, and the color representing the phase angle of the complex variable s.

Fig. 12 above is a polar contour map of the phase of the complex variable s, where the radius represents the magnitude of the zeta function, i.e, MagZ, and the azimuthal angle represents the phase angle PhZ of the zeta function. The various levels of color represent the value of the phase angle PhS of the complex variable s. This plot is similar to the plot in Fig. 11 but obtained when using values of ReS in the narrower interval [0, 1] (with a step of 0.01) and ImS in the narrower interval [12, 28] (with a step of 0.01). For a phase of s (PhS) around 87° to 88° (yellow-green color), a forward shape is generated in the zeta function with a phase angle (PhZ) -45° to 45°. A lobe can be seen for MagZ between 0 and 1.5, a semi-ring can be seen for MagZ of about 2.5, and a further outer semi-ring can be seen for MagZ of about 3.5.

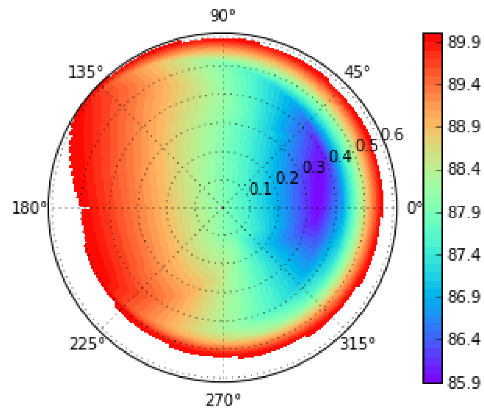

Fig. 13a: ReS=[0,1] and ImS=[14.0, 14.5]

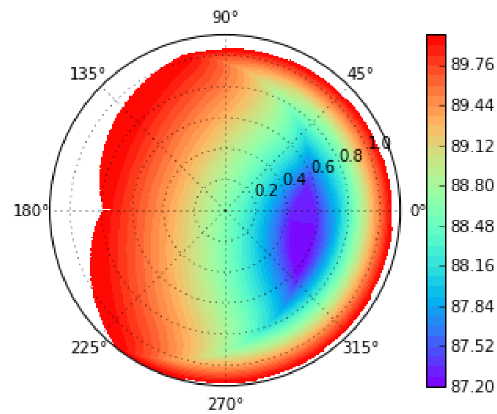

Fig. 13b: ReS=[0,1] and ImS=[20.5, 21.5]

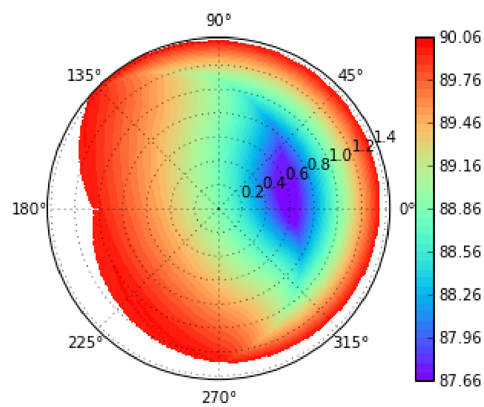

Fig. 13c: ReS=[0,1] and ImS=[24.5, 25.5]

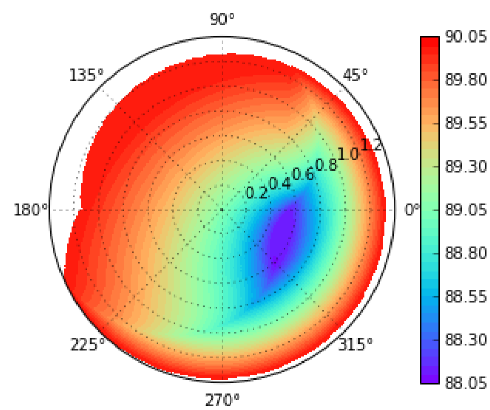

Fig. 13d: ReS=[0,1] and ImS=[30.0, 31.0]

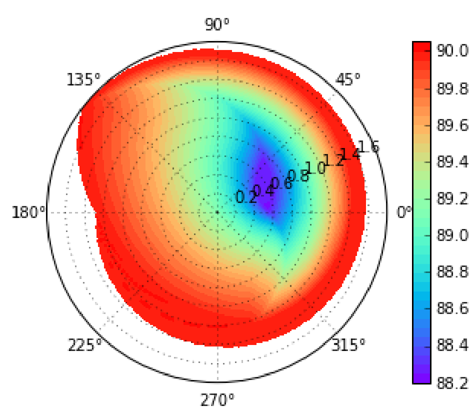

Fig. 13e: ReS=[0,1] and ImS=[32.5, 33.5]

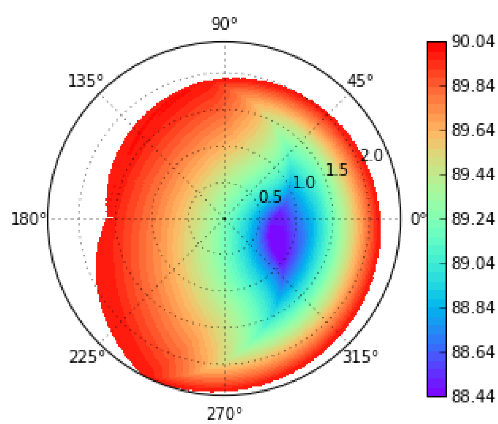

Fig. 13f: ReS=[0,1] and ImS=[37.0, 38.0]

Figs. 13a-f are similar to Fig. 12. However, Figs. 13a-f are obtained at specific ImS ranges around the respective ImS values corresponding to the zeros of the zeta function. It appears that for the smallest value for phase PhS (in blue) the phase PhZ of the zeta function alternates from positive to negative at consecutive zeros. For the first zero (at ImS = 14.13), the phase PhZ is located between 0° and 45° in the first quadrant (Fig. 13a). For the second zero (at ImS = 21.02), the phase PhZ is located between 0° and -45° (315°) in the fourth quadrant. Thereafter, the phase alternates from positive to negative values at consecutive zeros.

In Figs. 14a-d, the ImS range is further narrowed around the ImS value corresponding to the zeros of the zeta function. The ReS interval is not modified and is maintained as [0, 1]. These plots show a reversal of direction in the obtained "snail" shape at consecutive zeros, from clockwise to counter-clockwise and from counter-clockwise to clockwise, etc.

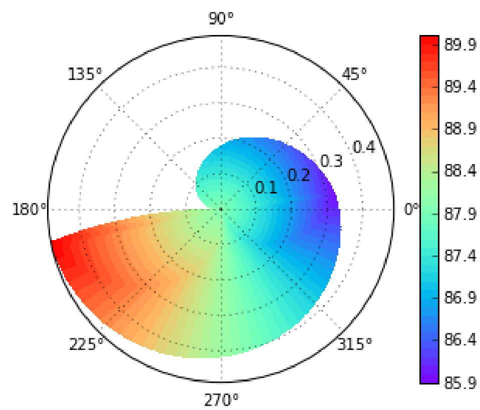

Fig. 14a: ReS=[0,1] and ImS=[14.13, 14.14]

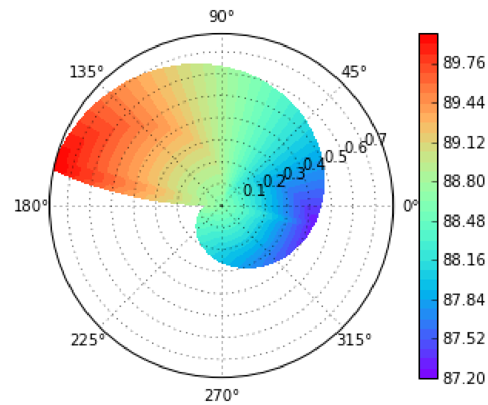

Fig. 14b: ReS=[0,1] and ImS=[21.01, 21.03]

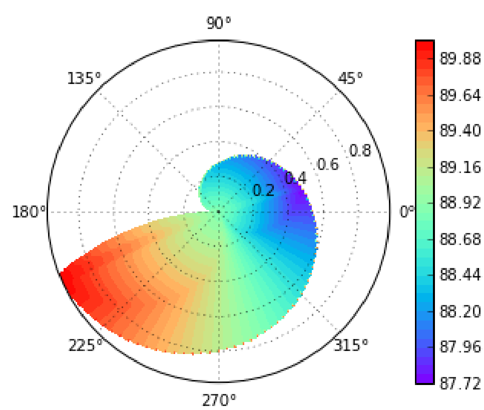

Fig. 14c: ReS=[0,1] and ImS=[25.00, 25.02]

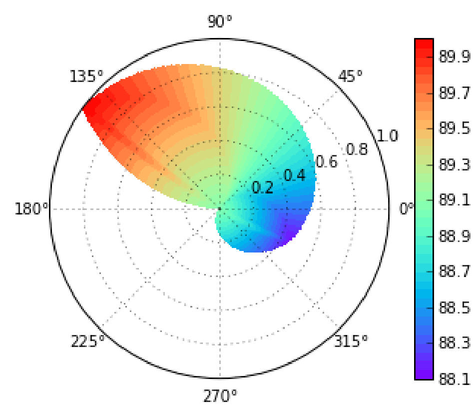

Fig. 14d: ReS=[0,1] and ImS=[30.42, 30.43]

Figs. 15a and 15b are two plots obtained for ReS in the interval [0,1] for a zero at two ImS intervals centered around two consecutive larger ImS values that generate non-trivial zeros, respectively, 10000.92 and 10002.28. Similarly, the "snail-like" form reverses direction for consecutive zeros. Hence, shape orientation in the 3D polar contour map projections can be used to provide a location of consecutive zeros.

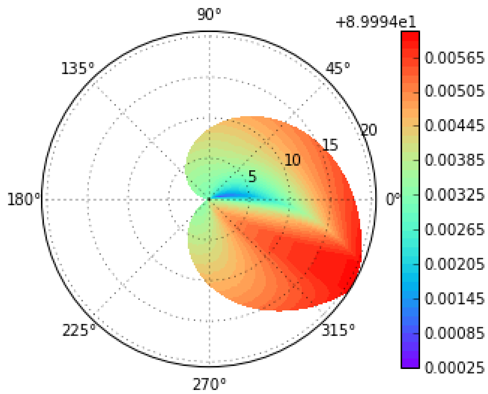

Fig. 15a: ReS=[0,1] and ImS=[10000.91, 10000.93]

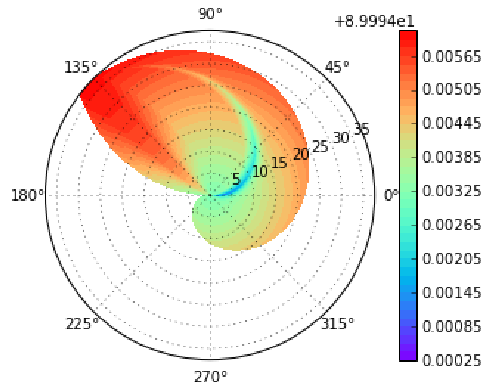

Fig. 15b: ReS=[0,1] and ImS=[10002.27, 10002.29]

The next logical step would then be to look into the variation of the phase of the zeta function as a function of the phase of the complex variable s, around some non-trivial zeros. Figs.16a-f show plots of the phase of the zeta function vs. the phase of the variable s around consecutive zeros. The phase of the zeta function reverses its shifting direction for consecutive zeros. This feature can be used to locate consecutive zeros of the zeta function.

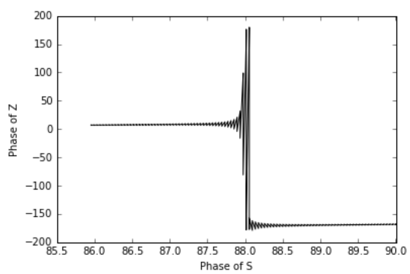

Fig. 16a: ReS=[0,1] and ImS=[14.13, 14.14]

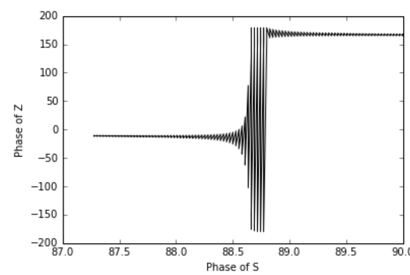

Fig. 16b: ReS=[0,1] and ImS=[21.01, 21.03]

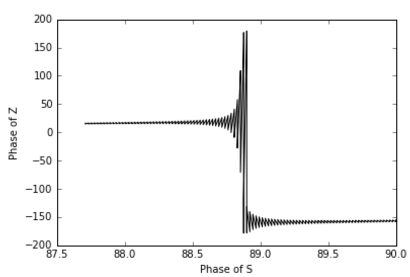

Fig. 16c: ReS=[0,1] and ImS=[25.00, 25.02]

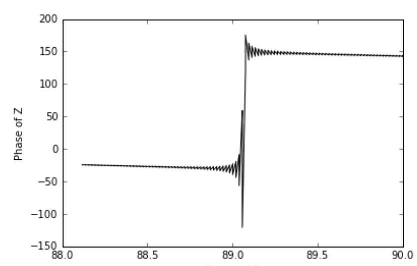

Fig. 16d: ReS=[0,1] and ImS=[30.42, 30.43]

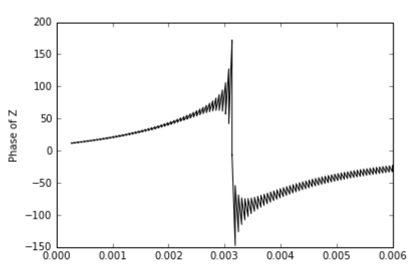

Fig. 16e: ReS=[0,1] and ImS=[10000.91, 10000.93]

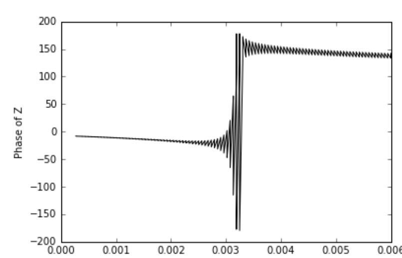

Fig. 16f: ReS=[0,1] and ImS=[10002.27, 10002.29]

In conclusion, there are various ways to "slice" or project the four-dimensional zeta function. For example, graphical visualization on a polar contour map provides another perspective of analyzing the zeta function. It may be worthwhile to explore this type of polar projection to further analyze the zeta function at various real and imaginary values of the complex variable s (for example, at higher values). The above described projections can also be applied to any complex function in the four-dimensional space. All the above plots were obtained using a Python code written by the author.

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