



**HAL**  
open science

## On the method of reflections

Philippe Laurent, Guillaume Legendre, Julien Salomon

► **To cite this version:**

Philippe Laurent, Guillaume Legendre, Julien Salomon. On the method of reflections. 2017. hal-01439871v1

**HAL Id: hal-01439871**

**<https://hal.science/hal-01439871v1>**

Preprint submitted on 18 Jan 2017 (v1), last revised 23 Jun 2021 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# ON THE METHOD OF REFLECTIONS

PHILIPPE LAURENT, GUILLAUME LEGENDRE, AND JULIEN SALOMON

**ABSTRACT.** This paper aims at reviewing and analysing the method of reflections. The latter is an iterative procedure designed to linear boundary value problems set in multiply connected domains. Being based on a decomposition of the domain boundary, this method is particularly well-suited to numerical solvers relying on integral representation formulas. For the parallel and sequential forms of the method appearing in the literature, we propose a general abstract formulation in a given Hilbert setting and interpret the procedure in terms of subspace corrections. We then prove the unconditional convergence of the sequential form and propose a modification of the parallel one that makes it unconditionally converging. An alternative proof of convergence is provided in a case which does not fit into the previous framework. We finally present some numerical tests.

## 1. INTRODUCTION

In 1911, Smoluchowski [Smo11] introduced an iterative process based on Stokes' law to compute the hydrodynamic forces exerted on an assemblage of an arbitrary number of spheres falling in an unbounded viscous fluid. Later dubbed the *method of reflections*, this technique has subsequently been featured prominently in reference textbooks on hydrodynamics at low Reynolds number, like those by Happel and Brenner [HB83], Kim and Karrila [KK91] or Dhont [Dho96], and employed in several articles dealing with the motion of particles immersed in a viscous fluid (see for instance [Kyn59, Jon78, CK88, IB01, Wil13]). It may be described as a systematic scheme by which a (generally exterior) linear boundary value problem associated with many “objects” (like particles in the case of a suspension) may be solved by computing and summing solutions of boundary value problems involving only a single object, which are called *reflections*. Physically speaking, one may visualize its very principle by “*supposing an initial disturbance to be reflected from the boundaries involved and to produce succeeding smaller effect with each successive reflection*” [HB83, chapter 8], hence its given name.

In a more formal manner, the method follows a divide-and-conquer strategy, which assumes that each of the one-object problems is, in some way, easier to solve than the many-object one. In this sense, it bears similarities to Schwarz-type domain decomposition methods (see [Gan08] for instance), which attempts to solve a (generally interior) boundary value problem by splitting it into boundary value problems set on different, at most partially overlapping, subdomains and iterating to coordinate the solution between adjacent subdomains. There is, however, an important conceptual difference between these two approaches. Indeed, any one-object problem to be solved in the method of reflections is defined on a (possibly unbounded) domain that is the interior of the complement of the considered object and thus contains the whole domain in which the main problem is set. This technique must therefore rather be seen as a *boundary* decomposition method.

The method of reflections was historically devised with human computers, as well as tractable and explicit formulas for the reflections, in mind, which implied that the objects are few and identical, with a shape such that analytical forms of the Stokes and Faxén laws are available and the velocity field can be approximated by a truncated expansion. With time, these restrictions, coupled with the fact that a large number of terms in the truncated series may be required to obtain an accurate approximation (most notably when the objects are close from each other), made its use less attractive. However, since the method in itself does not rely on the manner in which the boundary value problems are solved, it became clear that, with the advent of electronic computers, methods using a numerical discretisation (more advantageously those based on integral representation) could be employed, and even combined<sup>1</sup> with the aforementioned analytical methods, to efficiently address problems involving numerous objects of arbitrary shape, size or type of imposed boundary condition.

Despite a number of recent papers concerned with applications (see [IB01, ML11, Boy+12, Wil13] for instance), it appears that the method has seldom been studied from a mathematical perspective. One must cite the pioneering work of Luke [Luk89], in which the convergence of the original form of the method applied to the solution of the so-called mobility problem (see Section 2) in a bounded domain is investigated theoretically. By formulating the problem in an appropriate Hilbert space setting and interpreting the method in terms of orthogonal projection

---

*Date:* January 18, 2017.

<sup>1</sup>One can for instance check the article [Boy+12], in which different solution methods are used depending on the nature of the considered one-object problem.

operators, Luke proved that the method always converges and that the convergence rate is linear if a geometric property between the subspaces associated with the projection operators holds. Also, in [Tra06], Traytak studied a variant of the method applied to the solution of a Dirichlet problem in a three-dimensional unbounded domain complementary to a set of spheres. Resorting to analytical techniques, he established necessary and sufficient conditions of convergence with respect to the radii of the spheres and the distances between their centres. These two important contributions notwithstanding, it appears that a fully developed mathematical theory of the method of reflections is still lacking.

Furthermore, we may add that a few missed facts have led to some misunderstandings on the part of the scientific communities in which the method is used. For instance, when Ichiki and Brady state in [IB01] that “*while the convergence of the method of reflections has been proven for the mobility problem [Luk89], the convergence for the resistance problem is an open question.*” and subsequently give a “counter-example” to the convergence of the method for configurations involving more than two objects, they somehow undermine their discovery by failing to notice that the method of reflections considered in their work differs from the one introduced and studied by Luke. It was also empirically observed in [Wil13] that the convergence behaviour of the method may depend significantly on the type of boundary value problem considered, and Luke stated that his convergence proof does not extend to other types of boundary conditions or in the case of an unbounded domain, leaving open a number of questions. In the present work, we show that convergence can indeed be established for several kinds of boundary value problems using the same mathematical tools.

In the present paper, we propose a unified theoretical framework for the analysis of the existing forms of the method of reflections. The outline is the following. Using a well-known analogy between Stokesian hydrodynamics and electrostatics, we review in Section 2 the different presentations of the method of reflections found in the literature on a couple of toy problems involving the Laplace operator. This exposition leads to a more general and abstract definition of two variants of the method. The first, called *parallel* in the present work, is the original one from an historical perspective, whereas the second, which we call *sequential*, was introduced by Luke in [Luk89]. Next, in Section 3, the method is recasted in a loose functional framework, allowing us to derive key algebraic properties and error formulas. Section 4 is devoted to the convergence analysis of the method using a Hilbert space setting. A conditional convergence result, in the spirit of iterative methods for the solution of linear systems, is given for the parallel form, and a modification of the latter, ensuring unconditional convergence when orthogonal projectors are involved, is proposed. For the sequential form, unconditional convergence is obtained (again when orthogonal projectors are involved). Several examples of application, in both orthogonal and non-orthogonal cases, are dealt with. Numerical experiments are finally presented in Section 5.

## 2. TWO EXAMPLES OF APPLICATIONS FROM THE LITERATURE

In this section, we describe the two forms of the method of reflections found in the literature.

The method of reflections is typically considered in the context of hydrodynamics at low Reynolds number to solve two distinct types of problems for flows involving hydrodynamic interactions among particles, respectively called the *resistance problem* and the *mobility problem* (see [KK91] for instance). Both are boundary value problems based on the Stokes equation for an incompressible flow of Newtonian fluid. In the former, the forces and torques are to be determined for specified particle velocities in the ambient fluid, while in the latter the particles’ forces and torques are prescribed in the ambient fluid and the velocities are unknown. However, to simplify the presentation, we exploit an existing analogy between Stokesian hydrodynamics and electrostatics, recalled by Luke in [Luk89], replacing the Stokes equation by the Laplace equation and the fluid velocity by the electrostatic potential.

**2.1. An electrostatic analogue of the resistance problem: the parallel form.** From a mathematical standpoint, the resistance problem amounts to solve a boundary value problem for the Stokes equation with Dirichlet conditions. Its electrostatic analogue is thus to determine the electrostatic potential in a grounded container  $\Omega$ , knowing its values on the surfaces of  $N$  conducting objects  $O_j \subset \Omega$ ,  $j = 1, \dots, N$ , that is: *given  $N$  scalar functions  $U_j$ ,  $j = 1, \dots, N$ , respectively defined on  $\partial O_j$ ,  $j = 1, \dots, N$ , find the scalar function  $u$  satisfying*

$$\begin{aligned} -\Delta u &= 0 \text{ in } \Omega \setminus \cup_{j=1}^N \overline{O_j}, \\ u &= U_j \text{ on } \partial O_j, \quad j = 1, \dots, N. \end{aligned}$$

The container may be bounded or not. In the latter case, one should prescribe an additional condition at infinity for the problem to be well posed (see Section 4.3 for instance). In what follows, we assume that the domain  $\Omega$  is bounded and impose a homogeneous Dirichlet condition on its boundary,

$$u = 0 \text{ on } \partial\Omega.$$

To approximate the solution of this problem, the method of reflections consists in summing the auxiliary fields (the so-called *reflections*)  $u_j^{(k)}$ ,  $j = 1, \dots, N$ ,  $k \in \mathbb{N}$ , defined recursively as the solutions of the respective single object problems

$$(1) \quad \forall i \in \{1, \dots, N\}, \begin{cases} -\Delta u_i^{(k+1)} = 0 \text{ in } \Omega \setminus \overline{O_i}, \\ u_i^{(k+1)} = -\sum_{j=1}^{i-1} u_j^{(k)} - \sum_{j=i+1}^N u_j^{(k)} \text{ on } \partial O_i, \\ u_i^{(k+1)} = 0 \text{ on } \partial \Omega, \end{cases}$$

where each of the subproblems at the initialisation stage ( $k = 0$ ) only involves a single datum  $U_i$ , that is

$$\forall i \in \{1, \dots, N\}, \begin{cases} -\Delta u_i^{(0)} = 0 \text{ in } \Omega \setminus \overline{O_i}, \\ u_i^{(0)} = U_i \text{ on } \partial O_i, \\ u_i^{(0)} = 0 \text{ on } \partial \Omega. \end{cases}$$

The approximation of the solution at the end of the  $\ell$ th “cycle” (or “sweep”) is then defined by

$$(2) \quad u^{(\ell)} = \sum_{k=0}^{\ell} \sum_{i=1}^N u_i^{(k)} \text{ in } \Omega \setminus \cup_{j=1}^N \overline{O_j}.$$

While the presentations and the underlying equation for the boundary value problem may differ, the above form of the method of reflections was originally introduced by Golusin in [Gol34] to solve the Dirichlet problem in multiply connected circular domains and is the most commonly found in the literature, as in the textbook by Kim and Karrila [KK91] and the articles by Traytak [Tra06] or Wilson [Wil13], to name a few. We observe that the datum for each of the single object problems (1) of a given cycle depends solely on quantities computed during the previous cycle (or, during the first cycle, on the data of the boundary value problem). The solutions of these subproblems may thus be carried out simultaneously, like in the Jacobi method for the iterative solution of linear systems of equations, leading us to call this procedure the *parallel form* of the method of reflections.

Concerning the convergence of this form of the method<sup>2</sup>, Happel and Brenner write in [HB83] that: “*it must be pointed out that no rigorous proof exists that the iteration scheme converges to the desired solution.*” As a matter of fact, a numerical example of divergence for a particular configuration is presented by Ichiki and Brady in [IB01], the method being used there to solve the Stokes resistance problem in the presence of rigid spherical particles in an unbounded domain. In this very case, the solutions of the single particle problems are numerically approximated by truncated multipole expansions, but the authors conjecture that the observed divergence is unrelated to the order of truncation of the expansions. More recently, in [Tra06], Traytak analysed the method applied to the solution of a Dirichlet problem for the Laplace equation in the unbounded complement of a set of spheres and obtained necessary and sufficient conditions for convergence, which allowed him to exhibit simple cases of divergence when the number of spheres is greater than or equal to eight. Additionally, it may be seen that the boundary decomposition technique, introduced<sup>3</sup> by Balabane in [Bal04] (see also [GH09, WL13] for practical and numerical applications) for solving a boundary value problem involving the Helmholtz equation in the unbounded complement of a scatterer made of a union of disjoint sub-scatterers, is identical to this form of the method of reflections. Links between the principle of this form of the method of reflections and the approach to the multiple scattering of waves by point-like objects known as the Foldy–Lax model [Fol45, Lax51, Lax52] may also be pointed<sup>4</sup>.

**2.2. An electrostatic analogue of the mobility problem: the sequential form.** Another version of the method of reflections, corresponding to the one initially introduced by Smoluchowski [Smo11] for two objects, was analysed by Luke in [Luk89] when used to solve the mobility problem in hydrodynamics, whose electrostatic analogue is the following: *given  $N$  scalars  $Q_j$ ,  $j = 1, \dots, N$ , find the scalar function  $u$  and the  $N$  scalars  $c_j$ ,*

<sup>2</sup>Let us observe here that the version appearing in Chapter 6 of [HB83] differs slightly from the above procedure by focusing on one specific object. The reflections with respect to this object are then computed as in the sequential form (see the next subsection) of the method, whereas the reflections with respect to the other  $N - 1$  objects are computed as in the parallel form.

<sup>3</sup>In the same work, a sufficient condition for convergence, depending on the frequency, the diameters and the areas of the sub-scatterers, as well as the distances between the sub-scatterers, is established.

<sup>4</sup>To see this, one can compare the hierarchy of different levels of approximation, ranging from the Born approximation to the Foldy–Lax model, given in [CH13] with the sequence of approximate solutions produced by the method of reflections.

$j = 1, \dots, N$ , which verify

$$(3) \quad \begin{cases} -\Delta u = 0 \text{ in } \Omega \setminus \cup_{j=1}^N \overline{O_j}, \\ u = 0 \text{ on } \partial\Omega, \\ u = c_j \text{ on } \partial O_j, \quad j = 1, \dots, N, \\ \int_{\partial O_j} \frac{\partial u}{\partial \mathbf{n}}(y) \, dS(y) = Q_j, \quad j = 1, \dots, N. \end{cases}$$

This boundary value problem involves an equivalued surface boundary condition, itself consisting of two conditions and sometimes said to be of the *fourth type*<sup>5</sup> (see [Sho77, Chapter III] for instance). The first of these conditions states that the value of the electrostatic potential remains constant on each conductor but is to be determined. The second one, which is non-local in nature, indicates that the integral (but not the precise distribution) of the normal derivative of the electrostatic potential on each boundary, that is the surface charge on each conductor, is known.

Reformulated<sup>6</sup> in order to involve reflections, the method introduced by Luke amounts to compute recursively the sequences of functions  $(u_i^{(k)})_{k \in \mathbb{N}}$  and scalars  $(c_i^{(k)})_{k \in \mathbb{N}}$ ,  $i = 1, \dots, N$ , such that,

$$(4) \quad \forall k \in \mathbb{N}, \forall i \in \{1, \dots, N\}, \begin{cases} -\Delta u_i^{(k+1)} = 0 \text{ in } \Omega \setminus \overline{O_i}, \\ u_i^{(k+1)} = 0 \text{ on } \partial\Omega, \\ u_i^{(k+1)} = c_i^{(k+1)} - \sum_{j=1}^{i-1} u_j^{(k+1)} - \sum_{j=i+1}^N u_j^{(k)} \text{ on } \partial O_i, \\ \int_{\partial O_i} \frac{\partial u_i^{(k+1)}}{\partial \mathbf{n}}(y) \, dS(y) = - \sum_{j=1}^{i-1} \int_{\partial O_i} \frac{\partial u_j^{(k+1)}}{\partial \mathbf{n}}(y) \, dS(y) - \sum_{j=i+1}^N \int_{\partial O_i} \frac{\partial u_j^{(k)}}{\partial \mathbf{n}}(y) \, dS(y), \end{cases}$$

the subproblems at the initialisation stage being

$$(5) \quad \forall i \in \{1, \dots, N\}, \begin{cases} -\Delta u_i^{(0)} = 0 \text{ in } \Omega \setminus \overline{O_i}, \\ u_i^{(0)} = 0 \text{ on } \partial\Omega, \\ u_i^{(0)} = c_i^{(0)} - \sum_{j=1}^{i-1} u_j^{(0)} \text{ on } \partial O_i, \\ \int_{\partial O_i} \frac{\partial u_i^{(0)}}{\partial \mathbf{n}}(y) \, dS(y) = Q_i - \sum_{j=1}^{i-1} \int_{\partial O_i} \frac{\partial u_j^{(0)}}{\partial \mathbf{n}}(y) \, dS(y), \end{cases}$$

and the approximation of the solution at the  $\ell$ th cycle being given by (2). Note that, in problems (4) and (5), we have that, for any natural integer  $k$ ,  $\int_{\partial O_i} \frac{\partial u_j^{(k)}}{\partial \mathbf{n}}(y) \, dS(y) = 0$  if  $j \neq i$ , the function  $u_j^{(k)}$  being, by construction, harmonic in the neighbourhood of  $\partial O_i$ .

Observe that the datum for each of the single object problems (4) of a given cycle depends on quantities computed during both the previous and current cycles (as well as the data of the problem for the initialisation), implying a successive computation of their respective solutions, in the spirit of the Gauss–Seidel method to solve linear systems of equations. We hence call this second method the *sequential form* of the method of reflections. An unconditional convergence result for this form of the method is shown in [Luk89] (see Subsection 4.2.2). A similar boundary decomposition method was proposed by Balabane and Tirel [BT97] to solve the Helmholtz equation outside a union of obstacles without trapping rays.

### 3. ABSTRACT FORMULATIONS

We now summarize the application of both forms of the method of reflections to the solution of an abstract boundary value problem. We emphasize that, while we use a single mathematical framework, the principle of the method is very general and we thus remain deliberately vague on the functional context in order to focus on the algebraic aspects. A rigorous setting is provided in Subsection 4.1, and explicit examples are given with the applications presented in Subsection 4.2.

Given a nonzero integer  $N$ , we consider a simply connected regular domain  $\Omega \subseteq \mathbb{R}^d$ , and a family of subsets  $O_j \subset \Omega$ ,  $j = 1, \dots, N$ , which are bounded simply connected regular open sets such that their closures are nonoverlapping. We are interested in solving a boundary value problem of the following form: *find a field  $u$ , in an appropriate*

<sup>5</sup>One may consider the first kind of boundary condition for the Laplace equation to be Dirichlet's, the second, Neumann's, and the third, Robin's.

<sup>6</sup>These auxiliary fields do not appear explicitly in the presentation of the method provided in [Luk89].

functional space, such that

$$(6) \quad \begin{cases} \mathcal{L}u &= 0 \text{ in } \Omega \setminus \cup_{j=1}^N \overline{O_j} \\ \mathcal{B}_j u &= b_j, \quad j = 1, \dots, N, \end{cases}$$

where  $\mathcal{L}$  and  $\mathcal{B}_j$ ,  $j = 1, \dots, N$  are linear operators and each of the functions  $b_j$ ,  $j = 1, \dots, N$ , is given in the range of the respective operator  $\mathcal{B}_j$ . In the applications we have previously mentioned, the function  $u$  physically corresponds to a velocity field or an electric potential, and the operator  $\mathcal{L}$  governing its behaviour, like the Stokes or the Laplace operators, is of elliptic type (but this is not necessarily the case: see the article [Bal04] for instance, in which the Helmholtz equation is considered). Given an integer  $j$  in  $\{1, \dots, N\}$ , the operator  $\mathcal{B}_j$  is a differential ‘‘boundary’’ operator, in the sense that it acts, in a possibly non-local way, on the boundary  $\partial O_j$  of the  $j$ th object (it may be the trace operator  $\gamma|_{\partial O_j}$  for instance), which verifies admissibility conditions with respect to the operator  $\mathcal{L}$ . The function  $b_j$  is a datum associated with the type of boundary condition defined by  $\mathcal{B}_j$  (it may be the value of the said trace for instance). If the domain  $\Omega$  is bounded, a homogeneous boundary condition on  $\partial\Omega$  is added to the above system of equations. In the case where it is unbounded, this condition on the outer boundary is replaced by one set at infinity. Either way, it is assumed that the supplemented system of equations defines a well-posed boundary value problem in the chosen functional space, admitting a unique solution  $u^*$ .

To solve this boundary value problem using the parallel form of the method of reflections, one begins by computing an initialisation  $u^{(0)}$  of the form

$$(7) \quad u^{(0)} = \sum_{i=1}^N u_i^{(0)}|_{\Omega \setminus \cup_{j=1}^N \overline{O_j}},$$

the functions  $u_i^{(0)}$ ,  $i = 1, \dots, N$ , satisfying

$$(8) \quad \forall i \in \{1, \dots, N\}, \quad \begin{cases} \mathcal{L}u_i^{(0)} &= 0 \text{ on } \Omega \setminus \overline{O_i}, \\ \mathcal{B}_i u_i^{(0)} &= b_i. \end{cases}$$

An iteration phase follows, in which one solves recursively the subproblems

$$(9) \quad \forall k \in \mathbb{N}, \quad \forall i \in \{1, \dots, N\}, \quad \begin{cases} \mathcal{L}u_i^{(k+1)} &= 0 \text{ in } \Omega \setminus \overline{O_i}, \\ \mathcal{B}_i u_i^{(k+1)} &= -\mathcal{B}_i \left( \sum_{j=1}^{i-1} u_j^{(k)} + \sum_{j=i+1}^N u_j^{(k)} \right), \end{cases}$$

in order to update the approximation using the formula

$$(10) \quad u^{(k+1)} = u^{(k)} + \sum_{i=1}^N u_i^{(k+1)}|_{\Omega \setminus \cup_{j=1}^N \overline{O_j}}.$$

The approximate solution after the  $\ell$ th cycle is thus defined by a finite double sum

$$(11) \quad \forall \ell \in \mathbb{N}, \quad u^{(\ell)} = \sum_{k=0}^{\ell} \sum_{i=1}^N u_i^{(k)}|_{\Omega \setminus \cup_{j=1}^N \overline{O_j}}.$$

One may observe that the quantity  $\sum_{k=0}^{\ell} u_i^{(k)}|_{\Omega \setminus \cup_{j=1}^N \overline{O_j}}$  represents the contribution of the object  $i$  to the total field after a finite number  $\ell$  of cycles of interactions (the reflections). This entails, should each of these sums converge (in a sense to be given) as  $\ell$  tends to infinity, a (constructive) decomposition result, with respect to the object boundaries, for the solution similar to that of Theorem 1 in [Bal04]. Such a decomposition suggests that the method of reflections is particularly adapted to the use of numerical methods of approximation based on an integral representation of the solution, in which the unknowns are boundary values.

Let us turn to the sequential version of the method of reflections. While the initialisation is still given by (7), the subproblems to be solved at this step are now

$$(12) \quad \forall i \in \{1, \dots, N\}, \quad \begin{cases} \mathcal{L}u_i^{(0)} &= 0 \text{ in } \Omega \setminus \overline{O_i}, \\ \mathcal{B}_i \left( \sum_{j=1}^i u_j^{(0)} \right) &= b_i. \end{cases}$$

while those of the iteration phase are replaced by

$$(13) \quad \forall i \in \{1, \dots, N\}, \quad \begin{cases} \mathcal{L}u_i^{(k+1)} &= 0 \text{ in } \Omega \setminus \overline{O_i}, \\ \mathcal{B}_i \left( \sum_{j=1}^i u_j^{(k+1)} \right) &= -\mathcal{B}_i \left( \sum_{j=i+1}^N u_j^{(k)} \right), \end{cases}$$

the update of the approximate solution being given by (7).

**3.1. Algebraic properties of the reflections.** The sequences of reflections  $(u_i^{(k)})_{k \in \mathbb{N}}$ ,  $i = 1, \dots, N$ , constructed in both forms of the method may be viewed as partial correctors of the sequences of approximate solutions  $(u^{(k)})_{k \in \mathbb{N}}$  in the following sense.

**Lemma 3.1.** *Suppose that the completed boundary value problem (6) admits a unique solution  $u^*$  and that the  $N$  sequences  $(u_i^{(k)})_{k \in \mathbb{N}}$ ,  $i = 1, \dots, N$ , computed either by the parallel or by the sequential form of the method of reflections are well defined. Then, one has*

$$(14) \quad \forall k \in \mathbb{N}, \forall i \in \{1, \dots, N\}, \mathcal{B}_i \left( u^{(k)} + u_i^{(k+1)} \right) = \mathcal{B}_i u^*,$$

for the parallel form, and

$$(15) \quad \forall k \in \mathbb{N}, \forall i \in \{1, \dots, N\}, \mathcal{B}_i \left( u^{(k)} + \sum_{j=1}^i u_j^{(k+1)} \right) = \mathcal{B}_i u^*,$$

for the sequential form, where the sequence  $(u^{(k)})_{k \in \mathbb{N}}$  is given by (11).

*Proof.* Both equalities are easily established by induction. First, for the parallel form, one has, for the base case  $k = 0$  and for any integer  $i$  in  $\{1, \dots, N\}$ ,

$$\mathcal{B}_i \left( u^{(0)} + u_i^{(1)} \right) = \mathcal{B}_i \left( u^{(0)} - \sum_{j=1}^{i-1} u_j^{(0)} - \sum_{j=i+1}^N u_j^{(0)} \right) = \mathcal{B}_i u_i^{(0)} = b_i = \mathcal{B}_i u^*,$$

using the linearity of the operators, (7), (9), (8) and (6). Assuming that the equality

$$\mathcal{B}_i \left( u^{(k-1)} + u_i^{(k)} \right) = \mathcal{B}_i u^*$$

holds for some integer  $k$  in  $\{0, \dots, N-1\}$  and any integer  $i$  in  $\{1, \dots, N\}$ , one finds, according to (10) and (9), that

$$\begin{aligned} \mathcal{B}_i \left( u^{(k)} + u_i^{(k+1)} \right) &= \mathcal{B}_i \left( u^{(k-1)} + \sum_{j=1}^N u_j^{(k)} + u_i^{(k+1)} \right) \\ &= \mathcal{B}_i \left( u^{(k-1)} + \sum_{j=1}^N u_j^{(k)} - \sum_{j=1}^{i-1} u_j^{(k)} - \sum_{j=i+1}^N u_j^{(k)} \right) \\ &= \mathcal{B}_i \left( u^{(k-1)} + u_i^{(k)} \right), \end{aligned}$$

which ends the proof. Likewise, for the sequential form, it stems from (7), (13), (12) that, for any integer  $i$  in  $\{1, \dots, N\}$ ,

$$\mathcal{B}_i \left( u^{(0)} + \sum_{j=1}^i u_j^{(1)} \right) = \mathcal{B}_i \left( \sum_{j=1}^N u_j^{(0)} + \sum_{j=1}^i u_j^{(1)} \right) = \mathcal{B}_i \left( \sum_{j=1}^i u_j^{(0)} \right) = b_i = \mathcal{B}_i u^*,$$

and, assuming that  $\mathcal{B}_i \left( u^{(k-1)} + \sum_{j=1}^i u_j^{(k)} \right) = \mathcal{B}_i u^*$  for some integer  $k$  in  $\{0, \dots, N-1\}$ ,

$$\begin{aligned} \mathcal{B}_i \left( u^{(k)} + \sum_{j=1}^i u_j^{(k+1)} \right) &= \mathcal{B}_i \left( u^{(k-1)} + \sum_{j=1}^N u_j^{(k)} + \sum_{j=1}^i u_j^{(k+1)} \right) \\ &= \mathcal{B}_i \left( u^{(k-1)} + \sum_{j=1}^i u_j^{(k)} + \sum_{j=1}^i u_j^{(k+1)} + \sum_{j=i+1}^N u_j^{(k)} \right) \\ &= \mathcal{B}_i \left( u^{(k-1)} + \sum_{j=1}^i u_j^{(k)} \right). \end{aligned}$$

□

**3.2. Projection setting and error formulas.** Following [Luk89], we formulate the method by means of projection operators and then interpret it as a *subspace correction method*. To this end, we consider the error of the method at any step,

$$\forall k \in \mathbb{N}, e^{(k)} = u^{(k)} - u^*,$$

which is naturally defined on the domain  $\Omega \setminus \cup_{j=1}^N \overline{O_j}$ , and extend it to the whole of  $\Omega$ . This is effectively achieved by requiring the extension  $\tilde{e}^{(k)}$  to satisfy the system

$$(16) \quad \begin{cases} \mathcal{L}\tilde{e}^{(k)} &= 0 \text{ in } \Omega \setminus \cup_{j=1}^N \partial O_j \\ \mathcal{B}_j \tilde{e}^{(k)}|_{\Omega \setminus \cup_{j=1}^N \overline{O_j}} &= \mathcal{B}_j e^{(k)}, j = 1, \dots, N, \end{cases}$$

with additional transmission conditions across the boundaries of the objects, in order to couple the field in the interiors of the objects with that in the exterior. There are generally several ways to choose these conditions, and one should select those leading to a well-posed<sup>7</sup> transmission problem set on  $\Omega$ , for which the solution  $\tilde{e}^{(k)}$  restricted to  $\Omega \setminus \cup_{j=1}^N \overline{O_j}$  coincides with  $e^{(k)}$ . For any positive integer  $k$ , the auxiliary fields  $u_i^{(k)}$ ,  $i = 1, \dots, N$ , solutions to the single-object problems (9) or (13), possess extensions built in a similar manner. Several concrete examples are provided in Subsection 4.2.

We suppose that the solutions to these various extended problems are sought (possibly in a weak sense) in an underlying functional space  $H$ , and introduce, for any integer  $i$  in  $\{1, \dots, N\}$ , the subspace  $V_i$  of  $H$  as the range of the linear operator  $P_i$  on  $H$  such that

$$P_i : v \mapsto w,$$

where  $w$  is the solution to the problem

$$\begin{cases} \mathcal{L}w &= 0 \text{ in } \Omega \setminus \partial O_i, \\ \mathcal{B}_i w &= \mathcal{B}_i v, \end{cases}$$

completed by an adequately chosen transmission condition across  $\partial O_i$ . This operator is well-defined as soon as the underlying single-object problem is well-posed and it is a simple matter to check that it is a projection operator in the sense that it is idempotent, i.e.  $P_i^2 = P_i$ .

The introduction of these projection operators enables the derivation of error propagation formulas on which part of our analysis of the convergence the method is based. Note that, for the sake of simplicity, we will use in the remainder of the paper the same notation for both the error of the method and its extension.

**Theorem 3.2.** *Suppose that the completed boundary value problem (6) admits a unique solution  $u^*$  and assume that the reflections and the error of the parallel form of the method of reflections can be extended, the latter satisfying (16). Then, one has the following error propagation formula for the method*

$$(17) \quad \forall k \in \mathbb{N}, u^{(k+1)} - u^* = \left( Id - \sum_{i=1}^N P_i \right) (u^{(k)} - u^*).$$

*Proof.* Owing to (14), one has

$$\forall k \in \mathbb{N}, \forall i \in \{1, \dots, N\}, \mathcal{B}_i u_i^{(k+1)}|_{\Omega \setminus \cup_{j=1}^N \overline{O_j}} = -\mathcal{B}_i (u^{(k)} - u^*)|_{\Omega \setminus \cup_{j=1}^N \overline{O_j}},$$

which gives, due to the definition of the projection operator  $P_i$  and the extension of (9),

$$\forall k \in \mathbb{N}, \forall i \in \{1, \dots, N\}, u_i^{(k+1)} = -P_i (u^{(k)} - u^*).$$

Substituting into (10), one reaches

$$(18) \quad \forall k \in \mathbb{N}, u^{(k+1)} - u^* = u^{(k)} - u^* + \sum_{i=1}^N u_i^{(k+1)} = u^{(k)} - u^* - \sum_{i=1}^N P_i (u^{(k)} - u^*),$$

yielding (17). □

One may notice that error formula (17) is characteristic (see [XZ02], Subsection 2.2) of the so-called *parallel subspace correction (PSC) methods*, a notion introduced by Xu in [Xu92, Xu01] to regroup diverse iterative procedures used for solving linear systems of equations and which are similar in nature, like the Jacobi method, the Richardson method, the additive Schwarz method or some multigrid methods.

A propagation formula for the error of the sequential version of the method of reflections can also be obtained.

<sup>7</sup>In some particular cases, further constraints may be needed to uniquely define the extension in the interiors of the objects, see for instance the Neumann problem in Subsection 4.2.



**Theorem 3.3.** *Suppose that the completed boundary value problem (6) admits a unique solution  $u^*$  and assume that the reflections and the error of the sequential form of the method of reflections can be extended, the latter satisfying (16). Then, one has the following error propagation formula for the method*

$$(19) \quad \forall k \in \mathbb{N}, \quad u^{(k+1)} - u^* = E_N \left( u^{(k)} - u^* \right),$$

where  $E_N = (Id - P_N) \dots (Id - P_1)$ .

*Proof.* Let us denote  $E_0 = Id$ ,  $E_i = (Id - P_i)E_{i-1}$ ,  $i = 1, \dots, N$ . We shall first prove by induction that

$$\forall k \in \mathbb{N}, \forall i \in \{1, \dots, N\}, \quad u_i^{(k+1)} = -P_i E_{i-1} \left( u^{(k)} - u^* \right).$$

First, it follows from (15) that

$$\forall k \in \mathbb{N}, \forall i \in \{1, \dots, N\}, \quad \mathcal{B}_i \left( u_i^{(k+1)} \right) = -\mathcal{B}_i \left( u^{(k)} - u^* + \sum_{j=1}^{i-1} u_j^{(k+1)} \right),$$

which, due to the definition of the projection operator  $P_i$  and to the extension of (13), translates into

$$\forall k \in \mathbb{N}, \forall i \in \{1, \dots, N\}, \quad u_i^{(k+1)} = -P_i \left( u^{(k)} - u^* + \sum_{j=1}^{i-1} u_j^{(k+1)} \right).$$

For  $i = 1$ , this identity is simply

$$\forall k \in \mathbb{N}, \quad u_1^{(k+1)} = -P_1 \left( u^{(k)} - u^* \right).$$

Next, assume that

$$\forall k \in \mathbb{N}, \forall i \in \{1, \dots, N-1\}, \quad u_i^{(k+1)} = -P_i E_{i-1} \left( u^{(k)} - u^* \right).$$

Then, one has

$$u_{i+1}^{(k+1)} = -P_{i+1} \left( u^{(k)} - u^* + \sum_{j=1}^i u_j^{(k+1)} \right) = -P_{i+1} \left( Id - \sum_{j=1}^i P_j E_{j-1} \right) \left( u^{(k)} - u^* \right) = -P_{i+1} E_i \left( u^{(k)} - u^* \right),$$

which ends the induction. Using definition (10), one finally reaches

$$u^{(k+1)} - u^* = u^{(k)} - u^* + \sum_{i=1}^N u_i^{(k+1)} = \left( Id - \sum_{i=1}^N P_i E_{i-1} \right) \left( u^{(k)} - u^* \right) = E_N \left( u^{(k)} - u^* \right).$$

□

Error formula (19) is typical (see [XZ02], Subsection 2.1) of *successive subspace correction (SSC) methods*, which constitute a class of iterative methods to which belong the Gauss-Seidel method or the alternating Schwarz method.

#### 4. CONVERGENCE ANALYSIS IN A HILBERT SPACE SETTING

We now flesh out the structure of the previous mathematical setting by making a number of additional assumptions. We suppose that the space  $H$  is a Hilbert space and that the problem to solve can be written under an equivalent variational formulation, involving a continuous coercive bilinear form  $a(\cdot, \cdot)$  from  $H \times H$  to  $\mathbb{R}$ , defining an inner product which induces a norm on  $H$ , denoted  $\|\cdot\|_H$ . Naturally, we will say that the method of reflections is convergent if the sequence of extended errors  $(u^{(k)} - u^*)_{k \in \mathbb{N}}$  tends to zero with respect to the norm  $\|\cdot\|_H$ .

In view of the formulas derived in the preceding section, it is clear that the error of the method behaves like a geometric sequence. A first general convergence result can thus be trivially stated.

**Theorem 4.1.** *A necessary and sufficient condition for the convergence of the method of reflections is*

$$\rho \left( Id - \sum_{i=1}^N P_i \right) < 1$$

for its parallel form and

$$\rho(E_N) < 1,$$

for its sequential form, where  $\rho(\cdot)$  denotes the spectral radius.

*Proof.* **???? give short proof citing that  $A^k \rightarrow 0$  when  $k \rightarrow +\infty \Leftrightarrow \rho(A) < 1$ .**

□

A setback for the generality of the previous result is that the given condition is often not readily checkable. In some references (see [Bal04, Tra06] for the parallel form or Section 4.3 for the sequential one), it is traded for a sufficient condition on the summability of a series.

**4.1. The orthogonal case: theoretical results.** In some applications, notably if the operator  $\mathcal{L}$  in the considered boundary value problem is of elliptic type, the operators  $P_i$  may be shown to be *orthogonal* projection operators, and stronger convergence results are then obtained. In the Hilbertian context introduced above, the projection operators  $P_i$ ,  $i = 1, \dots, N$ , are orthogonal if, and only if, they satisfy the condition

$$\forall u \in H, \forall v \in V_i, a(u - P_i u, v) = 0.$$

When it is the case, recall that the orthogonal projector on the subspace  $V_i^\perp$ ,  $i = 1, \dots, N$ , the orthogonal complement of  $V_i$  in  $H$ , is given by  $Id - P_i$ .

We now address the convergence of both forms of the method when the operators are orthogonal projectors. In what follows, we assume that the following decomposition of  $H$  holds

$$(20) \quad H = \overline{\sum_{i=1}^N V_i}.$$

**4.1.1. Parallel form.** As seen previously, the method of reflections will fail to converge if the spectral radius of the operator  $Id - \sum_{i=1}^N P_i$  is larger than one. The fact that the projection operators are orthogonal does not generally improve this result (one may nevertheless consult the article [BM91] for some results on the spectrum of a sum of orthogonal projection operators).

Still, a way of obtaining a unconditionally convergent and properly parallel method is to modify the algorithm so that the resulting error formula reads

$$\forall k \in \mathbb{N}, u^{(k+1)} - u^* = \left( Id - \frac{1}{N} \sum_{i=1}^N P_i \right) (u^{(k)} - u^*) = \left( \frac{1}{N} \sum_{i=1}^N (Id - P_i) \right) (u^{(k)} - u^*),$$

which can be interpreted as a *relaxation* of the recurrence relation (17) of the form

$$\forall k \in \mathbb{N}, u^{(k+1)} - u^* = \omega \left( Id - \sum_{i=1}^N P_i \right) (u^{(k)} - u^*) + (1 - \omega) (u^{(k)} - u^*),$$

with relaxation factor  $\omega = \frac{1}{N}$ . This indeed yields

$$\forall k \in \mathbb{N}, u^{(k)} - u^* = \left( \frac{1}{N} \sum_{i=1}^N (Id - P_i) \right)^k (u^{(0)} - u^*),$$

and it is a well-known fact, since the work of Cimmino [Cim38] on the *method of averaged projections* (a geometrical approach for an iterative method to solve linear systems of equations), and its extension by Auslender [Aus76], that, for any closed (linear) subspaces  $M_1, \dots, M_N$  of a Hilbert space  $H$ , one has the pointwise convergence result<sup>8</sup>

$$\forall v \in H, \lim_{k \rightarrow +\infty} \left\| \left( \frac{1}{N} \sum_{i=1}^N P_{M_i} \right)^k v - P_{\cap_{i=1}^N M_i} v \right\|_H = 0.$$

where  $P_{M_j}$  (resp.  $P_{\cap_{i=1}^N M_i}$ ) denote by the orthogonal projection operator onto  $M_j$  (resp.  $\cap_{i=1}^N M_i$ ).

By choosing  $M_i = V_i^\perp$ ,  $i = 1, \dots, N$ , and noting that  $\cap_{i=1}^N M_i = \cap_{i=1}^N V_i^\perp = (\sum_{i=1}^N V_i)^\perp = \{0_H\}$  owing to assumption (20), we have obtained the following convergence result.

**Theorem 4.2.** *Under the assumptions of Theorem 3.3, suppose that the projection operators  $P_i$ ,  $i = 1, \dots, N$ , are orthogonal and that the decomposition (20) holds. Then, the sequence of approximations  $(u^{(k)})_{k \in \mathbb{N}}$  generated by the averaged version of the parallel form of the method of reflections converges to the solution  $u^*$ .*

In terms of practical changes induced by the modification, the update formula for the approximation of the solution (see equation (10)) becomes

$$u^{(k+1)} = u^{(k)} + \frac{1}{N} \sum_{i=1}^N u_i^{(k+1)},$$

<sup>8</sup>This results remains valid for the sum  $\sum_{i=1}^N \alpha_i P_{M_i}$ , where the scalars  $\alpha_i$ ,  $i = 1, \dots, N$ , are the weights of a convex combination (that is, such that,  $\forall i \in \{1, \dots, N\}$ ,  $\alpha_i > 0$  and  $\sum_{i=1}^N \alpha_i = 1$ ).

the fields  $u_i^{(k+1)}$  being computed by solving the subproblem (9) with the second equation replaced by

$$\mathcal{B}_i(u_i^{(k+1)}) = -\mathcal{B}_i \left( \frac{1}{N} \sum_{j=1}^{i-1} u_j^{(k)} + \frac{1-N}{N} u_i^{(k)} + \frac{1}{N} \sum_{j=i+1}^N u_j^{(k)} \right).$$

As a consequence, the modification may be viewed as a simple relaxation method applied to the parallel form of the method of reflections. One may also remark that the interpretation of the auxiliary fields as “reflections” remains, since it can be shown that (14) still holds. We thus refer to this variant as the *averaged parallel* form of the method of reflections.

**4.1.2. Sequential form.** When the projections are orthogonal, the sequential form of the method of reflections is closely related to the *method of (cyclic) alternating projections* (MAP for short). The latter is a simple iterative procedure for determining the orthogonal projection of an element onto an intersection of closed (linear) subspaces of a Hilbert space using a sequence of orthogonal projections onto the said subspaces. It has numerous applications (the interested reader may check the review by Deutsch [Deu92] for details) and its design relies on the following pointwise convergence result, first proved by von Neumann in 1933 (but not published until 1949) in the  $N = 2$  case while working on the theory of operators [Neu49] and later generalised by Halperin [Hal62] to any integer  $N \geq 2$  (an extension to closed convex sets is due to Bregman [Bre65]), stating that, keeping the notations introduced in Subsection 4.1.1, one has

$$\forall v \in H, \lim_{k \rightarrow +\infty} \|(P_{M_N} P_{M_{N-1}} \dots P_{M_1})^k v - P_{\cap_{i=1}^N M_i} v\|_H = 0.$$

Consequently, the computation of the orthogonal projection of a point  $v$  in  $H$  onto  $\cap_{i=1}^N M_i$  by the MAP consists in building the sequence  $(w^{(k)})_{k \in \mathbb{N}}$  defined by

$$w^{(0)} = v \text{ and, } \forall k \in \mathbb{N}, w^{(k+1)} = P_{M_N} \dots P_{M_1} w^{(k)},$$

that is, by recursively applying the orthogonal projections to the current iterate. This is exactly what error formula (19) does to the error of the sequential version of the method of reflections and a convergence result may hence be stated by setting/choosing  $M_i = V_i^\perp$ ,  $i = 1, \dots, N$ .

**Theorem 4.3.** *Under the assumptions of Theorem 3.3, suppose that the projection operators  $P_i$ ,  $i = 1, \dots, N$ , are orthogonal and that the decomposition (20) holds. Then, the sequence of approximations  $(u^{(k)})_{k \in \mathbb{N}}$  generated by the sequential form of the method of reflections converges to the solution  $u^*$ .*

**4.1.3. Rate of convergence.** The above results do not say anything about the rate of convergence of the method. To deal with this aspect, let us recall the following dichotomy property for the MAP (see Theorem 6.4 in [DH10] and also Theorem 1.4 in [BDH09] for the case  $N = 2$ ):

- If  $\sum_{i=1}^N M_i^\perp$  is closed, the sequence  $((P_{M_N} P_{M_{N-1}} \dots P_{M_1})^k)_{k \in \mathbb{N}}$  converges to  $P_{\cap_{i=1}^N M_i}$  linearly<sup>9</sup>, that is, there exist constants  $C > 0$  and  $\alpha \in [0, 1)$  such that

$$\forall k \in \mathbb{N}, \|(P_{M_N} P_{M_{N-1}} \dots P_{M_1})^k - P_{\cap_{i=1}^N M_i}\|_{\mathcal{L}(H)} \leq C \alpha^k.$$

- If  $\sum_{i=1}^N M_i^\perp$  is not closed, the sequence  $((P_{M_N} P_{M_{N-1}} \dots P_{M_1})^k)_{k \in \mathbb{N}}$  converges to  $P_{\cap_{i=1}^N M_i}$  arbitrarily slowly, that is,

- the sequence  $((P_{M_N} P_{M_{N-1}} \dots P_{M_1})^k)_{k \in \mathbb{N}}$  converges pointwise to  $P_{\cap_{i=1}^N M_i}$ ,
- for each real-valued function  $\phi$  on the positive integers that converges to 0, there exists a point  $v$  in  $H$  such that

$$\forall k \in \mathbb{N}, \|(P_{M_N} P_{M_{N-1}} \dots P_{M_1})^k v - P_{\cap_{i=1}^N M_i} v\|_H \geq \phi(k).$$

The implication of this result for the sequential form of the method of reflections is that it converges linearly as soon as the sum  $\sum_{i=1}^N V_i$  is closed. Equivalent conditions for the above result, based on the notion of angle<sup>10</sup>

<sup>9</sup>Some authors say the sequence converges *uniformly* (see [BGM12]).

<sup>10</sup>According to the definition of Friedrichs [Fri37], if  $M_i$  and  $M_j$  are closed subspaces in a Hilbert space  $H$ , the angle between  $M_i$  and  $M_j$  is the angle in  $[0, \frac{\pi}{2}]$  whose cosine is defined by

$$c(M_i, M_j) = \sup \left\{ |\langle x, y \rangle| \mid x \in M_i \cap (M_i \cap M_j)^\perp, \|x\| \leq 1, y \in M_j \cap (M_i \cap M_j)^\perp, \|y\| \leq 1 \right\}.$$

Another notion is that of the minimal angle between  $M_i$  and  $M_j$ , given by Dixmier in [Dix49], which is the angle in  $[0, \frac{\pi}{2}]$  whose cosine is defined by

$$c_0(M_i, M_j) = \sup \{ |\langle x, y \rangle| \mid x \in M_i, \|x\| \leq 1, y \in M_j, \|y\| \leq 1 \}.$$

between subspaces, can be found in the literature<sup>11</sup>, and it is worth noting that the linear convergence of the (sequential version of the) method of reflections was established in [Luk89] using conditions on the gaps between two collections of the involved subspaces, that is, the sines of the angles between two collections of these subspaces.

In the case of a linear convergence of the MAP, error bounds can be derived. Let us recall some of the existing results on this topic. When  $N = 2$ , one has  $\|(P_{M_2}P_{M_1})^k - P_{M_1 \cap M_2}\|_{\mathcal{L}(H)} \leq c(M_1, M_2)^{2k-1}$ , where  $c(M_1, M_2)$  denotes the cosine of the angle between the subspaces  $M_1$  and  $M_2$ . This result is due to Aronszajn (see Section 12 in [Aro50]) and has been rediscovered several times. It is also sharp (see [KW88, Theorem 2]). For  $N \geq 3$ , upper bounds were given by Smith, Solomon and Wagner [SSW77, Theorem 2.2], Kayalar and Weinert [KW88, Theorem 3], and also Deutsch and Hundal [DH97, Theorem 2.7]. In this case however, note that any error bound depending only on the angles between the various subspaces involved can never be sharp [DH97, Example 3.7]. More recently, using the link between the MAP and the SSC methods, Xu and Zikatanov [XZ02] obtained, under the assumption  $H = \sum_{i=1}^N M_i^\perp$ , the following *equality*

$$\|P_{M_N}P_{M_{N-1}} \cdots P_{M_1} - P_{\cap_{i=1}^N M_i}\|_{\mathcal{L}(H)}^2 = \frac{c_0}{1 + c_0},$$

where

$$c_0 = \sup_{\substack{v \in H \\ \|v\|_H=1}} \inf_{\substack{(v_1, \dots, v_N) \in M_1^\perp \times \cdots \times M_N^\perp \\ \sum_{k=1}^N v_k = v}} \sum_{i=1}^N \|(Id - P_{M_i}) \sum_{j=i+1}^N v_j\|_H^2.$$

Note that various ways of accelerating the MAP, using relaxation or symmetrization for instance, have been proposed and studied (see [BDHP03] and the references therein), and could be directly used on the sequential version of the method of reflections.

Finally, because parallel projections correspond to alternating projections in the product space, adequate variants of the above results exist for the method of averaged projections, as a consequence of results in [BB96, BDH09], and thus apply to the averaged parallel version of the method of reflections.

**4.2. The orthogonal case: practical examples.** In view of the previous results, the orthogonality of the projectors  $P_i$  is a sufficient condition for the convergence of the method, either in its sequential form or in a modified (i.e., averaged) version of its parallel form. As an illustration, we show how to establish this property in several practical cases for which the main operator in the boundary value problem is of elliptic type. More precisely, our attention is focused on examples that involve the Laplace or the Stokes equations.

**4.2.1. The Laplace equation.** In what follows, we assume the domain  $\Omega$  is a bounded, simply connected, open set of  $\mathbb{R}^d$ , with boundary  $\partial\Omega$ , containing  $N$  simply connected open subdomains  $O_j$ , with respective boundaries  $\partial O_j$ ,  $j = 1, \dots, N$ . All these boundaries are supposed to be sufficiently smooth, say twice continuously differentiable.

As in Section 3, we consider the boundary value problem for the Laplace equation, which means the operator  $\mathcal{L}$  is the negative Laplacian operator, set on the “extended” domain  $\Omega \setminus \cup_{j=1}^N \partial O_j$ , comprised of the following system

$$(21) \quad -\Delta w = 0 \text{ in } \Omega \setminus \cup_{j=1}^N \partial O_j,$$

$$(22) \quad w = 0 \text{ on } \partial\Omega,$$

complemented with some boundary and transmission conditions on  $\partial O_j$ ,  $j = 1, \dots, N$ .

A key tool in the forthcoming analysis is the following representation formula, which asserts that any function  $w$  defined on  $\Omega$ , which belongs to  $H^1(\Omega \setminus \cup_{j=1}^N \partial O_j)$  and satisfies the Laplace equation (21) and boundary condition (22) in the appropriate weak sense, can be written as

$$(23) \quad \text{a. e. } x \in \Omega \setminus \cup_{j=1}^N \partial O_j, \quad w(x) = \sum_{j=1}^N \int_{\partial O_j} \left( \left[ \frac{\partial w}{\partial \mathbf{n}} \right] G(x, y) - [w(y)] \frac{\partial G}{\partial \mathbf{n}_y}(x, y) \right) dS(y),$$

where the kernel  $G$  is the Green function of the Laplace equation for the region  $\Omega$ , the braces  $[\cdot]_{\partial O_j}$  denote the jump of the considered quantity across the (hyper)surface  $\partial O_j$ , that is the difference between the interior and

<sup>11</sup>For  $N = 2$ , it is known (see [Deu85] for instance) that  $c(M_1, M_2) < 1$ , where  $c(M_1, M_2)$  denotes the cosine of the angle between the subspaces  $M_1$  and  $M_2$ , if and only if  $M_1 + M_2$  is closed, if and only if  $M_1^\perp + M_2^\perp$  is closed, if and only if  $(M_1 \cap (M_1 \cap M_2)^\perp) + (M_2 \cap (M_1 \cap M_2)^\perp)$  is closed. For  $N \geq 3$ , a generalization of the Friedrichs angle is introduced in [BGM12] and it is shown in the same paper that the method converges linearly if and only if  $c(M_1, \dots, M_N) < 1$ , which is a weaker condition (see Example 4.5 in [BGM12]) than the sufficient one, based on Theorems 2.1 in [DH97] and 4.1 in [BGM12], that one of the cosines  $c_{ij} = c_0(M_i \cap (\cap_{\ell=1}^N M_\ell)^\perp, M_j \cap (\cap_{\ell=1}^N M_\ell)^\perp)$  of the Dixmier angles involving two subspaces  $M_i$  and  $M_j$  is strictly less than one.

exterior traces of the quantity, and the integrals on  $\partial O_j$  are understood in the sense of the duality product between  $H^{-1/2}(\partial O_j)$  and  $H^{1/2}(\partial O_j)$ ,  $j = 1, \dots, N$ . We recall that the Green function  $G$  is such that

$$\forall (x, y) \in \Omega \times \Omega, x \neq y, G(x, y) = \Phi(y - x) - \varphi_x(y),$$

the function  $\Phi$  being the fundamental solution of the Laplace equation, given by

$$\Phi(x) = -\frac{1}{2\pi} \log(|x|) \text{ for } d = 2 \text{ and } \Phi(x) = \frac{1}{d(d-2)V_d(1)} \frac{1}{|x|^{d-2}} \text{ for } d \geq 3,$$

with  $|\cdot|$  the Euclidean norm over  $\mathbb{R}^d$ ,  $V_d(1)$  the volume of a ball of unit radius in  $\mathbb{R}^d$ , and  $\varphi_x$  being a corrector function which, for a fixed  $x$  in  $\Omega$ , satisfies  $\Delta \varphi_x = 0$  in  $\Omega$  and  $\varphi_x = \Phi(\cdot - x)$  on  $\partial\Omega$ . The function  $w$  is also said to be represented by a *simple* (resp. *double*) *layer potential* when  $\left[\frac{\partial w}{\partial \mathbf{n}}\right] = 0$  (resp.  $[w] = 0$ ) across  $\partial O_j$  for any integer  $j$  in  $\{1, \dots, N\}$ .

Another useful formula is a Green identity, which follows from the divergence theorem and states that for any function  $w$  in  $H^1(\Omega \setminus \cup_{j=1}^N \partial O_j)$  satisfying the Laplace equation (21) and boundary condition (22) and any function  $v$  in  $H_0^1(\Omega)$ , one has

$$(24) \quad \int_{\Omega} \nabla w(x) \cdot \nabla v(x) dx = \sum_{j=1}^N \int_{\partial O_j} \left[ \frac{\partial w}{\partial \mathbf{n}}(y) \right] v(y) dS(y).$$

Details on the derivation of these two formulas can be found in various textbooks (see [Eva98] for instance). Finally, note that all the results given in this section still hold for unbounded domains, on the condition one works with a weighted Sobolev space in order to account for the condition at infinity. Such a setting is considered in Section 4.3.

**Dirichlet boundary conditions.** We first deal with the boundary value problem presented in Section 2.1, which corresponds to the case where the boundary operator  $\mathcal{B}_j$  is the trace operator of order  $O$  on  $\partial O_j$ . Observing that the error field satisfies

$$\forall k \in \mathbb{N}, \begin{cases} -\Delta e^{(k)} = 0 \text{ in } \Omega \setminus \cup_{j=1}^N O_j, \\ e^{(k)} = 0 \text{ on } \partial\Omega, \end{cases}$$

we require its extension to have vanishing jumps on the boundaries  $\partial O_j$ ,  $j = 1, \dots, N$ . This is achieved by setting the problem in the space

$$(25) \quad H = \{v \in H_0^1(\Omega) \mid -\Delta v = 0 \text{ in } \Omega \setminus \cup_{j=1}^N \partial O_j\},$$

equipped with the usual  $H_0^1(\Omega)$  scalar product

$$\forall (u, v) \in H^2, (u, v)_H = \int_{\Omega} \nabla u(x) \cdot \nabla v(x) dx.$$

Note that the definition of  $H$  implies that any function in  $H$  may be represented by a double layer potential, that is

$$\forall v \in H, \text{ a. e. } x \in \Omega \setminus \cup_{j=1}^N \partial O_j, v(x) = \sum_{j=1}^N \int_{\partial O_j} G(x, y) \left[ \frac{\partial v}{\partial \mathbf{n}}(y) \right] dS(y).$$

Accordingly, we define the subspaces

$$(26) \quad \forall i \in \{1, \dots, N\}, V_i = \{w \in H \mid -\Delta w = 0 \text{ in } \Omega \setminus \partial O_i\},$$

and characterise the projection operator  $P_i$  from  $H$  to  $V_i$ ,  $i = 1, \dots, N$ , by saying that  $w = P_i v$  solves the following boundary value problem: find  $w$  in  $H$  such that

$$\begin{cases} -\Delta w = 0 \text{ in } \Omega \setminus \partial O_i, \\ w = v \text{ on } \partial O_i. \end{cases}$$

We shall show that  $P_i$  is an orthogonal projector and that we moreover have  $H = \sum_{j=1}^N V_j$ , thus implying the linear convergence of the method.

To do this, we shall first establish a Galerkin-type orthogonality relation. Consider a function  $u$  in the space  $H$  and, for any integer  $i$  in  $\{1, \dots, N\}$ , a function  $v$  in the subspace  $V_i$ . Owing to Green's identity (24) and the respective definitions of  $H$  and  $V_i$ , we have on the one hand that

$$\int_{\Omega} \nabla u(x) \cdot \nabla v(x) dx = \int_{\partial O_i} u(y) \left[ \frac{\partial v}{\partial \mathbf{n}}(y) \right] dS(y).$$

On the other, it follows from the definition of the operator  $P_i$  that

$$\int_{\Omega} \nabla(P_i u)(x) \cdot \nabla v(x) dx = \int_{\partial O_i} P_i u(y) \left[ \frac{\partial v}{\partial \mathbf{n}}(y) \right] dS(y).$$

It ensues that

$$\forall u \in H, \forall v \in V_i, \int_{\Omega} \nabla(u - P_i u)(x) \cdot \nabla v(x) dx = 0,$$

since  $P_i u = u$  on  $\partial O_i$ , again by definition of  $P_i$ .

Next, using that any function  $v$  of  $H$  may be represented by a double layer potential, one has the decomposition  $v = \sum_{j=1}^N v_j$  by simply setting

$$\begin{aligned} \forall i \in \{1, \dots, N\}, v_i(x) &= \int_{\partial O_i} G(x, y) \left[ \frac{\partial v}{\partial \mathbf{n}}(y) \right] dS(y) \\ &= \int_{\partial O_i} \Phi(x - y) \left[ \frac{\partial v}{\partial \mathbf{n}}(y) \right] dS(y) - \int_{\partial O_i} \varphi_x(y) \left[ \frac{\partial v}{\partial \mathbf{n}}(y) \right] dS(y). \end{aligned}$$

In the last equation, the second term is smooth and does not contribute to the jump values of  $v_i$ . On the other hand, the first term corresponds to the unique solution  $w$  of the transmission problem (see Proposition 2 in [DL00, XI, B, §2.2]) on  $\mathbb{R}^d$

$$\begin{cases} -\Delta w = 0 \text{ in } \mathbb{R}^d \setminus \partial O_i, \\ [w] = 0 \text{ on } \partial O_i, \\ \left[ \frac{\partial w}{\partial \mathbf{n}} \right] = \left[ \frac{\partial u}{\partial \mathbf{n}} \right] \text{ on } \partial O_i. \end{cases}$$

so that  $[v_i] = [w] = 0$  on  $\partial O_i$ . The regularity of  $\partial \Omega$  and the definition of  $G$  implying that  $v_i \in H_0^1(\Omega)$ , we conclude that  $v_i$  belongs to  $V_i$ .

**Neumann boundary conditions.** We now treat the case of Neumann boundary conditions, for which the boundary operator  $\mathcal{B}_j$  is the trace operator of order 1 on  $\partial O_j$ . For any of the forms of the method, the error satisfies in this case

$$\forall k \in \mathbb{N}, \begin{cases} -\Delta e^{(k)} = 0 \text{ in } \Omega \setminus \cup_{j=1}^N O_j, \\ \int_{\partial O_j} \frac{\partial e^{(k)}}{\partial \mathbf{n}}(y) dS(y) = 0, \quad j = 1, \dots, N, \\ e^{(k)} = 0 \text{ on } \partial \Omega, \end{cases}$$

and it may be extended to the whole of  $\Omega$  by requiring that the jumps of its normal derivative vanish across the boundaries  $\partial O_j$ ,  $j = 1, \dots, N$ , i.e.

$$\forall k \in \mathbb{N}, \forall j \in \{1, \dots, N\}, \left[ \frac{\partial e^{(k)}}{\partial \mathbf{n}} \right] = 0 \text{ on } \partial O_j.$$

Note that the condition  $\int_{\partial O_i} \frac{\partial e^{(k)}}{\partial \mathbf{n}}(y) dS(y) = 0$ ,  $i = 1, \dots, N$ , corresponds to the usual compatibility condition associated with a pure Neumann problem. The solution to the above system of equations is thus defined up to an additive constant inside each object. The values of the constants can be fixed by imposing, for instance, the mean conditions  $\int_{\partial O_i} v(y) dS(y) = 0$ ,  $i = 1, \dots, N$ , guaranteeing the uniqueness of the solution to the extended problem.

This leads us to introduce the space

$$\begin{aligned} H &= \left\{ v \in H^1(\Omega \setminus \cup_{j=1}^N \partial O_j) \mid -\Delta v = 0 \text{ in } \Omega \setminus \cup_{j=1}^N \partial O_j, v = 0 \text{ on } \partial \Omega, \right. \\ &\quad \left. \left[ \frac{\partial v}{\partial \mathbf{n}} \right] = 0 \text{ on } \partial O_j, \int_{\partial O_j} \frac{\partial v}{\partial \mathbf{n}}(y) dS(y) = 0 \text{ and } \int_{\partial O_j} v(y) dS(y) = 0, \quad j = 1, \dots, N \right\}, \end{aligned}$$

endowed with the scalar product

$$(u, v)_H = \int_{\Omega \setminus \cup_{j=1}^N \partial O_j} \nabla u(x) \cdot \nabla v(x) dx,$$

its subspaces  $V_i$ ,  $i = 1, \dots, N$ , being defined by (26). The projection operator  $P_i$  from  $H$  to  $V_i$ ,  $i = 1, \dots, N$ , is then such that  $w = P_i v$  is the unique solution to the boundary value problem: find  $w$  in  $H$  satisfying

$$\begin{cases} -\Delta w = 0 \text{ in } \Omega \setminus \partial O_i, \\ \frac{\partial w}{\partial \mathbf{n}} = \frac{\partial v}{\partial \mathbf{n}} \text{ on } \partial O_i. \end{cases}$$

By definition, any function in  $H$  can be represented by a simple layer potential, that is

$$\forall v \in H, \text{ a. e. } x \in \Omega \setminus \cup_{j=1}^N \partial O_j, v(x) = - \sum_{j=1}^N \int_{\partial O_j} [v(y)] \frac{\partial G}{\partial \mathbf{n}_y}(x, y) dS(y).$$

We prove the orthogonality of the projector in a similar way to the preceding example. Fixing an integer  $i$  and taking  $u$  in  $H$  and  $v$  in  $V_i$ , one finds, owing to the Green formula and the respective definitions of  $H$  and  $V_i$ , that

$$\begin{aligned} \int_{\Omega \setminus \cup_{j=1}^N \partial O_j} \nabla u(x) \cdot \nabla v(x) \, dx &= - \int_{\Omega \setminus \cup_{j=1}^N \partial O_j} \Delta u(x) v(x) \, dx + \sum_{j=1}^N \int_{\partial O_j} \frac{\partial u}{\partial \mathbf{n}}(y) [v(y)] \, dS(y) \\ &= \int_{\partial O_i} \frac{\partial u}{\partial \mathbf{n}}(y) [v(y)] \, dS(y), \end{aligned}$$

and

$$\begin{aligned} \int_{\Omega \setminus \cup_{j=1}^N \partial O_j} \nabla(P_i u)(x) \cdot \nabla v(x) \, dx &= - \int_{\Omega \setminus \cup_{j=1}^N \partial O_j} \Delta(P_i u)(x) v(x) \, dx + \sum_{j=1}^N \int_{\partial O_j} \frac{\partial(P_i u)}{\partial \mathbf{n}}(y) [v(y)] \, dS(y) \\ &= \int_{\partial O_i} \frac{\partial u}{\partial \mathbf{n}}(y) [v(y)] \, dS(y), \end{aligned}$$

using that  $\frac{\partial(P_i u)}{\partial \mathbf{n}} = \frac{\partial u}{\partial \mathbf{n}}$  on  $\partial O_i$  by definition of the projector  $P_i$ . The orthogonality of the projection operator then follows.

Finally, setting

$$\forall i \in \{1, \dots, N\}, \quad v_i(x) = - \int_{\partial O_i} [v(y)] \frac{\partial G}{\partial \mathbf{n}_y}(x, y) \, dS(y).$$

we obtain the decomposition  $v = \sum_{j=1}^N v_j$  for any given field  $v$  in  $H$ . Noting that any element of the subspace  $V_i$  may be represented by a simple layer potential with moment supported in  $\partial O_i$ , we argue as in the case of Dirichlet boundary conditions to conclude.

**Boundary conditions of the fourth type.** By this name, we refer to the boundary conditions used in the problem presented in Section 2.2. We first observe that the error of the method satisfies

$$\forall k \in \mathbb{N}, \quad \begin{cases} -\Delta e^{(k)} = 0 \text{ in } \Omega \setminus \cup_{j=1}^N O_j, \\ \int_{\partial O_i} \frac{\partial e^{(k)}}{\partial \mathbf{n}}(y) \, dS(y) = 0, \quad i = 1, \dots, N, \\ e^{(k)} = 0 \text{ on } \partial \Omega, \end{cases}$$

and define the space

$$H = \left\{ v \in H_0^1(\Omega) \mid -\Delta v = 0 \text{ in } \Omega \setminus \cup_{j=1}^N \partial O_j \text{ and } \int_{\partial O_i} \frac{\partial v}{\partial \mathbf{n}}(y) \, dS(y) = 0, \quad i = 1, \dots, N \right\},$$

equipped with the usual scalar product on  $H_0^1(\Omega)$ . Any element of  $H$  may be represented by a double layer potential whose moments have a vanishing mean value on any of the boundaries  $\partial O_i$ ,  $i = 1, \dots, N$ .

The subspaces  $V_i$ ,  $i = 1, \dots, N$ , are given by (26) and, in order to characterise their associated projection operators, it is in this case somewhat easier to observe that, for any  $u$  in  $H$ , there exists a constant  $c$  such that the function  $w = (Id - P_i)u$  in  $H$  satisfies

$$w = c \text{ on } \partial O_i.$$

We thus have

$$\begin{aligned} \forall u \in H, \quad \forall v \in V_i, \quad \int_{\Omega} \nabla(u(x) - P_i u(x)) \cdot \nabla v(x) \, dx &= \\ - \int_{\Omega \setminus \partial O_i} (u(x) - P_i u(x)) \Delta v(x) \, dx + \int_{\partial O_i} (u(y) - P_i u(y)) \left[ \frac{\partial v}{\partial \mathbf{n}}(y) \right] \, dS &= c \int_{\partial O_i} \left[ \frac{\partial v}{\partial \mathbf{n}}(y) \right] \, dS = 0, \end{aligned}$$

so that the operator  $P_i$  is an orthogonal projector.

It remains to prove  $H = \sum_{j=1}^N V_j$ . As in the case of the Dirichlet boundary conditions, we have

$$\forall v \in H, \quad v(x) = \sum_{j=1}^N \int_{\partial O_j} G(x, y) \left[ \frac{\partial v}{\partial \mathbf{n}}(y) \right] \, dS(y),$$

so that we obtain the decomposition  $H = \sum_{j=1}^N V_j$  by setting

$$v_i(x) = \int_{\partial O_i} G(x, y) \left[ \frac{\partial v}{\partial \mathbf{n}}(y) \right] \, dS(y).$$

The fact that  $v_i$  belongs to  $V_i$  follows from arguments previously used in the case of Dirichlet boundary conditions.

4.2.2. *The mobility problem for the Stokes equations.* In [Luk89], Luke proposed the variant of the method of reflections (the one we called sequential) and proved its convergence when applied to the solution of a system of equations modelling the motion of a sedimenting suspension, the so-called mobility problem. In this subsection, we recall this problem and show, without providing any proof, how it fits into the general framework previously used.

Denoting by  $\Omega$  the container (a bounded, connected, open set of  $\mathbb{R}^3$  with a smooth boundary), and by  $O_i \subset \Omega$ ,  $i = 1, \dots, N$ , the rigid particles (which are also connected open sets with smooth boundaries, such that their closures are non overlapping), the set  $\cup_{j=1}^N O_j$  is called the solid phase, while the set  $\Omega \setminus \overline{\cup_{j=1}^N O_j}$  is the fluid phase. By extending the fluid flow inside the particles, by requiring that the Stokes equation also holds inside the particles and that the flow field is continuous across the particle boundaries, the inertialess motion of the rigid particles due to externally imposed forces and torques is then described by the fluid velocity field  $\mathbf{u}$  and the pressure field  $p$  satisfying the problem

$$(27) \quad -\mu \Delta \mathbf{u} + \nabla p = \mathbf{0} \text{ in } \Omega \setminus \cup_{j=1}^N \partial O_j,$$

$$(28) \quad \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega \setminus \cup_{j=1}^N \partial O_j,$$

$$(29) \quad \mathbf{u} = \mathbf{0} \text{ on } \partial \Omega,$$

$$(30) \quad [\mathbf{u}] = 0 \text{ on } \partial O_j, \quad j = 1, \dots, N,$$

$$(31) \quad \forall x \in \partial O_j, \quad \mathbf{u}(x) = \mathbf{U}_{O_j} + \boldsymbol{\Omega}_{O_j} \times (x - x_{O_j}), \quad j = 1, \dots, N,$$

$$(32) \quad \int_{\partial O_j} [\boldsymbol{\sigma}(\mathbf{u})(y)] \cdot \mathbf{n}(y) \, dS(y) = \mathbf{F}_{O_j}, \quad j = 1, \dots, N,$$

$$(33) \quad \int_{\partial O_j} (y - x_{O_j}) \times [\boldsymbol{\sigma}(\mathbf{u})(y)] \cdot \mathbf{n}(y) \, dS(y) = \boldsymbol{\tau}_{O_j}, \quad j = 1, \dots, N,$$

where  $\mu$  is the kinematic viscosity coefficient,  $\boldsymbol{\sigma}(\mathbf{u})$  is the stress tensor,  $\boldsymbol{\sigma}(\mathbf{u}) = -p\mathbf{I} + \mu (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$ , and the instantaneous linear and angular velocities  $\mathbf{U}_{O_i}$  and  $\boldsymbol{\Omega}_{O_i}$  of the  $i$ th particle are unknowns to be determined (along with the fluid velocity and pressure), whereas the centres of mass  $x_{O_i}$  of the particles, the total forces  $\mathbf{F}_{O_i}$  and the total torques  $\boldsymbol{\tau}_{O_i}$ ,  $i = 1, \dots, N$ , applied to the particles are given<sup>12</sup>.

The variational formulation of this problem allows to reduce it to that of finding solely the velocity field  $\mathbf{u}$ , as the pressure  $p$  can be recovered (up to a constant) from it. We thus introduce the space

$$F = \{\mathbf{v} \in (H_0^1(\Omega))^3 \mid \operatorname{div} \mathbf{v} = 0 \text{ in } \mathcal{D}'(\Omega)\}$$

and the form

$$\forall (\mathbf{u}, \mathbf{v}) \in F \times F, \quad a(\mathbf{u}, \mathbf{v}) = \mu \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} \, dx,$$

which is bilinear, symmetric, continuous and coercive over  $F \times F$ . One can then define the set of weak solutions to the Stokes equation associated with forcing functions that act on the fluid at the particle boundaries by setting

$$\forall K \subset \Omega, \quad \mathcal{F}(K) = \{\mathbf{f} \in (H^{-1}(\Omega))^3 \mid \operatorname{supp} \mathbf{f} \subset K\},$$

$$H = \{\mathbf{u} \in F \mid \exists \mathbf{f} \in \mathcal{F}(\cup_{j=1}^N \partial O_j) \text{ such that, } \forall \mathbf{v} \in F, \quad a(\mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_{(H^{-1}(\Omega))^3, (H_0^1(\Omega))^3}\},$$

and

$$\forall i \in \{1, \dots, N\}, \quad H_i = \{\mathbf{u} \in F \mid \exists \mathbf{f} \in \mathcal{F}(\partial O_i) \text{ such that, } \forall \mathbf{v} \in F, \quad a(\mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_{(H^{-1}(\Omega))^3, (H_0^1(\Omega))^3}\}.$$

Indeed, it can be seen that the elements of  $H$  are the functions in  $(H^1(\Omega))^3$  which satisfy equations (27) to (29) in a weak sense. Moreover, the projection operator  $P_i$  from  $H$  to

$$V_i = \left\{ \mathbf{u} \in H_i \mid \int_{\partial O_i} [\boldsymbol{\sigma}(\mathbf{u})(y)] \cdot \mathbf{n}(y) \, dS(y) = 0 \text{ and } \int_{\partial O_i} (y - x_{O_i}) \times [\boldsymbol{\sigma}(\mathbf{u})(y)] \cdot \mathbf{n}(y) \, dS(y) = 0 \right\},$$

defined in this case is orthogonal with respect to the inner product on  $H$  defined by the form  $a(\cdot, \cdot)$ , as established by Luke. Using this fact, Luke proves the linear convergence of the method by showing that the sum  $\sum_{i=1}^N M_i^\perp$  is a closed subspace of  $H$ .

<sup>12</sup>For instance, for suspensions sedimenting in a uniform gravitational field, one has  $\mathbf{F}_{O_i} = m_{O_i} \mathbf{g}$ , where  $m_{O_i}$  is the mass of the particle adjusted for buoyancy and  $\mathbf{g}$  is the gravitational acceleration, and  $\boldsymbol{\tau}_{O_i} = \mathbf{0}$ .



**4.3. An example of non-orthogonal case.** We now deal with the Laplace equation in an unbounded domain, the object having either Dirichlet or Neumann conditions imposed on their boundary. For such a configuration, we were unable to prove the orthogonality of the projection operators, and cases of divergence for the sequential form of the method can effectively be observed in numerical tests in two dimensions (see Subsection 5.2). This leads us to conjecture that the convergence theory previously used does not apply in this case and to consider it as “non-orthogonal”.

Nevertheless, one can find sufficient conditions for convergence by following the approach used by Balabane in [Bal04] for the parallel form. We will thus show that, under a certain geometrical condition, the series  $\sum_{k \in \mathbb{N}} u^{(k)}$  computed by the sequential form of the method of reflections converges in an appropriate functional sense, and that its limit is solution to the boundary value problem under consideration.

In what follows, we consider  $N$  objects, that is disjoint compact sets,  $O_i$  in  $\mathbb{R}^3$ ,  $i = 1, \dots, N$ , with respective boundaries  $\partial O_i$  of class  $\mathcal{C}^2$ . Given two positive integers  $N_D$  and  $N_N$  such that  $N = N_D + N_N$ , the objects associated with a Dirichlet boundary condition are numbered from 1 to  $N_D$  and that the ones associated with a Neumann boundary condition are numbered from  $N_D + 1$  to  $N$ . Then, given  $N_D$  functions  $U_i$ ,  $i = 1, \dots, N_D$ , and  $N_N$  functions  $W_i$ ,  $i = 1, \dots, N_N$ , we look for a function  $u$  satisfying

$$(34) \quad \begin{aligned} -\Delta u &= 0 \text{ in } \mathbb{R}^3 \setminus \bigcup_{j=1}^N \overline{O_j}, \\ u &= U_j \text{ on } \partial O_j, \quad j = 1, \dots, N_D, \\ \frac{\partial u}{\partial \mathbf{n}} &= W_j \text{ on } \partial O_{N_D+j}, \quad j = 1, \dots, N_N. \end{aligned}$$

Observe that the former boundary value problem is an exterior one (it is set in the complement of a union of bounded sets), and necessitates the introduction of a weighted Sobolev space to properly define its solution (see [AGG97] or [DL00, XI, B]), namely

$$W(\Omega) = \left\{ v : \Omega \rightarrow \mathbb{R} \mid \int_{\Omega} \frac{v^2(x)}{1 + |x|^2} dx < +\infty, \quad \nabla v \in (L^2(\Omega))^3 \right\}.$$

To prove the convergence of the sequential form of the method of reflections applied to the solution of this problem, we work directly with the reflections in order to establish the following boundary estimate.

**Lemma 4.4.** *Consider the sequence of reflections  $(u_i^{(k)})_{k \in \mathbb{N}, i \in \{1, \dots, N\}}$  generated by the sequential form of the method of reflections applied to problem (34) and define the associated sequence of scalars*

$$\forall k \in \mathbb{N}^*, \quad \forall (i, j) \in \{1, \dots, N\}^2, \quad a_{i,j}^{(k)} = \left\| u_i^{(k)} \right\|_{H^1(\partial O_j)}^2 + \left\| \frac{\partial u_i^{(k)}}{\partial \mathbf{n}} \right\|_{L^2(\partial O_j)}^2.$$

*Then, for any pair of distinct integers  $i$  and  $j$  in  $\{1, \dots, N\}$ , there exists a positive constant  $\kappa_{i,j}$  depending on the geometry of the problem such that*

$$(35) \quad \forall k \in \mathbb{N}^*, \quad a_{i,j}^{(k+1)} \leq N \kappa_{i,j} \left( \sum_{m=1}^{i-1} a_{m,i}^{(k+1)} + \sum_{m=i+1}^N a_{m,i}^{(k)} \right).$$

*Proof.* First, we remark that, for a positive integer  $k$ , the datum of the boundary value problem defining a reflection is the trace (for a reflection associated to an object with a Dirichlet boundary condition) or the trace of the normal derivative (for a reflection associated to an object with a Neumann condition) on an interior curve of a sum of harmonic functions. It follows from Weyl’s lemma on the interior regularity of harmonic functions and from results in Chapter 2 of [Gri85] that these reflections enjoy smoothness properties which, using a trace continuity theorem (see Chapter 1 of [Gri85] for instance), allow to show there exist positive constants  $C_i$  depending only on the geometry such that

$$(36) \quad \forall k \in \mathbb{N}^*, \quad \forall i \in \{1, \dots, N_D\}, \quad \left\| \frac{\partial u_i^{(k)}}{\partial \mathbf{n}} \right\|_{L^2(\partial O_i)} \leq C_i \left\| u_i^{(k)} \right\|_{H^1(\partial O_i)},$$

and

$$(37) \quad \forall k \in \mathbb{N}^*, \quad \forall i \in \{N_D + 1, \dots, N\}, \quad \left\| u_i^{(k)} \right\|_{H^1(\partial O_i)} \leq C_i \left\| \frac{\partial u_i^{(k)}}{\partial \mathbf{n}} \right\|_{L^2(\partial O_i)}.$$

In what follows, we shall assume, without loss of generality, that  $C_i \geq 1$ ,  $i = 1, \dots, N$ .

Next, for any integer  $k$  in  $\mathbb{N}^*$ , we extend the reflections  $u_i^{(k)}$ ,  $i = 1, \dots, N$ , to the whole of  $\mathbb{R}^3$  by requiring their respective extension  $\tilde{u}_i^{(k)}$  to belong to the weighted Sobolev space

$$W(\mathbb{R}^3) = \left\{ v : \mathbb{R}^3 \rightarrow \mathbb{R} \mid \int_{\mathbb{R}^3} \frac{v^2(x)}{1 + |x|^2} dx < +\infty, \nabla v \in (L^2(\mathbb{R}^3))^3 \right\},$$

to satisfy the same equations as  $u_i^{(k)}$  in the complement of  $O_i$  and the Laplace equation in  $O_i$ , and to have a vanishing jump across  $\partial O_i$  if  $1 \leq i \leq N_D$ , or a vanishing jump of its normal derivative across  $\partial O_i$  if  $N_D + 1 \leq i \leq N$ . Note that the interior problem associated with an extended reflection satisfying a Neumann boundary condition is indeed well-posed, since its normal derivative has a zero mean value on  $\partial O_i$  for any  $k$  in  $\mathbb{N}^*$ . We may then define the space

$$H = \left\{ v \in W, \mid -\Delta v = 0 \text{ in } \mathbb{R}^3 \setminus \cup_{j=1}^N \partial O_j, [v] = 0 \text{ on } \partial O_j, j = 1, \dots, N_D, \right. \\ \left. \left[ \frac{\partial v}{\partial \mathbf{n}} \right] = 0 \text{ on } \partial O_j, \int_{\partial O_j} \frac{\partial v}{\partial \mathbf{n}}(y) dS(y) = 0 \text{ and } \int_{\partial O_j} v|_{O_j}(y) dS(y) = 0, j = N_D + 1, \dots, N \right\},$$

and its subspaces

$$\forall i \in \{1, \dots, N\}, V_i = \{w \in H \mid -\Delta w = 0 \text{ in } \Omega \setminus \partial O_i\}.$$

Functions  $v$  in  $H$  have an explicit formulation in terms of their jumps on the boundaries  $\partial O_i$ ,  $i = 1, \dots, N$ , due to the following integral representation

$$(38) \quad \forall x \in \Omega \setminus \cup_{j=1}^N \partial O_j, v(x) = \frac{1}{4\pi} \sum_{j=N_D+1}^N \int_{\partial O_j} \frac{(x-y) \cdot \mathbf{n}(y)}{|x-y|^3} [v(y)] dS(y) - \frac{1}{4\pi} \sum_{j=1}^{N_D} \int_{\partial O_j} \frac{1}{|x-y|} \left[ \frac{\partial v}{\partial \mathbf{n}}(y) \right] dS(y),$$

and the same goes for their gradient,

$$(39) \quad \forall x \in \Omega \setminus \cup_{j=1}^N \partial O_j, \nabla v(x) = \frac{1}{4\pi} \sum_{j=N_D+1}^N \int_{\partial O_j} \frac{1}{|x-y|^3} \left( \mathbf{n}(y) - 3 \frac{(x-y) \cdot \mathbf{n}(y)}{|x-y|^2} (x-y) \right) [v(y)] dS(y) \\ + \frac{1}{4\pi} \sum_{j=1}^{N_D} \int_{\partial O_j} \frac{x-y}{|x-y|^3} \left[ \frac{\partial v}{\partial \mathbf{n}}(y) \right] dS(y).$$

Let us now denote by  $S_i$  the area of  $\partial O_i$ ,  $i = 1, \dots, N$  and set, for any pair of distinct integers  $i$  and  $j$  in  $\{1, \dots, N\}$ ,  $d_{i,j} = \min_{(x,y) \in \partial O_i \times \partial O_j} |x-y|$ . The treatment depends on the type of boundary condition satisfied by the considered reflection.

For an extended reflection  $\tilde{u}_i^{(k)}$  associated with an object with an imposed Dirichlet boundary condition, that is for  $1 \leq i \leq N_D$ , the jump condition on  $\partial O_i$  yields

$$\forall y \in \partial O_i, \left[ \frac{\partial \tilde{u}_i^{(k+1)}}{\partial \mathbf{n}}(y) \right] = \frac{\partial u_i^{(k+1)}}{\partial \mathbf{n}}(y) + \sum_{m=1}^{i-1} \frac{\partial u_m^{(k+1)}}{\partial \mathbf{n}}(y) + \sum_{m=i+1}^N \frac{\partial u_m^{(k)}}{\partial \mathbf{n}}(y).$$

On the other hand, since the function  $\tilde{u}_i^{(k)}$  belongs to  $V_i$ , it follows from the integral representation formula (38) that

$$\forall x \in \mathbb{R}^3 \setminus \partial O_i, \tilde{u}_i^{(k+1)}(x) = -\frac{1}{4\pi} \int_{\partial O_i} \frac{1}{|x-y|} \left[ \frac{\partial \tilde{u}_i^{(k+1)}}{\partial \mathbf{n}}(y) \right] dS(y),$$

hence

$$\forall x \in \mathbb{R}^3 \setminus \partial O_i, \tilde{u}_i^{(k+1)}(x) = -\frac{1}{4\pi} \int_{\partial O_i} \frac{1}{|x-y|} \frac{\partial u_i^{(k+1)}}{\partial \mathbf{n}}(y) dS(y) - \frac{1}{4\pi} \sum_{m=1}^{i-1} \int_{\partial O_i} \frac{1}{|x-y|} \frac{\partial u_m^{(k+1)}}{\partial \mathbf{n}}(y) dS(y) \\ - \frac{1}{4\pi} \sum_{m=i+1}^N \int_{\partial O_i} \frac{1}{|x-y|} \frac{\partial u_m^{(k)}}{\partial \mathbf{n}}(y) dS(y).$$

By means of the Cauchy-Schwarz inequality, one has, for any integer  $j$  in  $\{1, \dots, N\}$  distinct from  $i$ ,

$$\forall x \in \partial O_j, \left| u_i^{(k+1)}(x) \right| \leq \frac{\sqrt{S_i}}{4\pi d_{i,j}} \left( \left\| \frac{\partial u_i^{(k+1)}}{\partial \mathbf{n}} \right\|_{L^2(\partial O_i)} + \sum_{m=1}^{i-1} \left\| \frac{\partial u_m^{(k+1)}}{\partial \mathbf{n}} \right\|_{L^2(\partial O_i)} + \sum_{m=i+1}^N \left\| \frac{\partial u_m^{(k)}}{\partial \mathbf{n}} \right\|_{L^2(\partial O_i)} \right),$$

so that using (36), squaring both sides of the inequality and integrating over  $\partial O_j$ , we obtain

$$(40) \quad \left\| u_i^{(k+1)} \right\|_{L^2(\partial O_j)}^2 \leq \frac{N S_i S_j C_i^2}{(4\pi)^2 d_{i,j}^2} \left( \sum_{m=1}^{i-1} a_{m,i}^{(k+1)} + \sum_{m=i+1}^N a_{m,i}^{(k)} \right).$$

In addition, repeating these computations starting from (39), we get similar estimates the normal and tangential derivative traces,

$$(41) \quad \left\| \frac{\partial u_i^{(k+1)}}{\partial \mathbf{n}} \right\|_{L^2(\partial O_j)}^2 \leq \frac{N S_i S_j C_i^2}{(4\pi)^2 d_{i,j}^4} \left( \sum_{m=1}^{i-1} a_{m,i}^{(k+1)} + \sum_{m=i+1}^N a_{m,i}^{(k)} \right),$$

$$(42) \quad \left\| \frac{\partial u_i^{(k+1)}}{\partial \boldsymbol{\tau}} \right\|_{L^2(\partial O_j)}^2 \leq \frac{N S_i S_j C_i^2}{(4\pi)^2 d_{i,j}^4} \left( \sum_{m=1}^{i-1} a_{m,i}^{(k+1)} + \sum_{m=i+1}^N a_{m,i}^{(k)} \right).$$

Likewise, for an extended reflection  $\tilde{u}_i^{(k)}$  satisfying a Neumann boundary condition on  $\partial O_i$ , that is for  $N_D + 1 \leq i \leq N$ , we have

$$\forall y \in \partial O_i, \left[ \tilde{u}_i^{(k+1)}(y) \right] = u_i^{(k+1)}(y) + \sum_{m=1}^{i-1} u_m^{(k+1)}(y) + \sum_{m=i+1}^N u_m^{(k)}(y) + \lambda_i^{(k+1)},$$

where  $\lambda_i^{(k+1)}$  is a real number chosen in such a way that the condition  $\int_{\partial O_i} \tilde{u}_i^{(k+1)}|_{\partial O_i}(y) \, dS(y) = 0$  is satisfied. The integral representation formula then gives

$$\begin{aligned} \forall x \in \mathbb{R}^3 \setminus \partial O_i, \tilde{u}_i^{(k+1)}(x) &= \frac{1}{4\pi} \int_{\partial O_i} \frac{(x-y) \cdot \mathbf{n}(y)}{|x-y|^3} u_i^{(k+1)}(y) \, dS(y) + \frac{1}{4\pi} \sum_{m=1}^{i-1} \int_{\partial O_i} \frac{(x-y) \cdot \mathbf{n}(y)}{|x-y|^3} u_m^{(k+1)}(y) \, dS(y) \\ &\quad + \frac{1}{4\pi} \sum_{m=i+1}^N \int_{\partial O_i} \frac{(x-y) \cdot \mathbf{n}(y)}{|x-y|^3} u_m^{(k)}(y) \, dS(y) + \frac{\lambda_i^{(k+1)}}{4\pi} \int_{\partial O_i} \frac{(x-y) \cdot \mathbf{n}(y)}{|x-y|^3} \, dS(y). \end{aligned}$$

For any integer  $j$  in  $\{1, \dots, N\}$  distinct from  $i$ , we next have

$$\forall x \in \partial O_j, \left| u_i^{(k+1)}(x) \right| \leq \frac{\sqrt{S_i}}{4\pi d_{i,j}^2} \left( \left\| u_i^{(k+1)} \right\|_{L^2(\partial O_i)} + \sum_{m=1}^{i-1} \left\| u_m^{(k+1)} \right\|_{L^2(\partial O_i)} + \sum_{m=i+1}^N \left\| u_m^{(k)} \right\|_{L^2(\partial O_i)} \right),$$

from which we get

$$(43) \quad \left\| u_i^{(k+1)} \right\|_{L^2(\partial O_j)}^2 \leq \frac{N S_i S_j C_i^2}{(4\pi)^2 d_{i,j}^2} \left( \sum_{m=1}^{i-1} a_{m,i}^{(k+1)} + \sum_{m=i+1}^N a_{m,i}^{(k)} \right).$$

In the same manner, we also obtain

$$(44) \quad \left\| \frac{\partial u_i^{(k+1)}}{\partial \mathbf{n}} \right\|_{L^2(\partial O_j)}^2 \leq \frac{N S_i S_j C_i^2}{\pi^2 d_{i,j}^6} \left( \sum_{m=1}^{i-1} a_{m,i}^{(k+1)} + \sum_{m=i+1}^N a_{m,i}^{(k)} \right),$$

$$(45) \quad \left\| \frac{\partial u_i^{(k+1)}}{\partial \boldsymbol{\tau}} \right\|_{L^2(\partial O_j)}^2 \leq \frac{N S_i S_j C_i^2}{\pi^2 d_{i,j}^6} \left( \sum_{m=1}^{i-1} a_{m,i}^{(k+1)} + \sum_{m=i+1}^N a_{m,i}^{(k)} \right).$$

Finally, summing estimates (40), (41) and (42) on the one hand, estimates (43), (44) and (45) on the other, setting

$$(46) \quad \kappa_{i,j} = \frac{S_i S_j C_i^2}{(4\pi)^2} \min \left\{ \frac{1}{d_{i,j}^2} + \frac{2}{d_{i,j}^4}, \frac{1}{d_{i,j}^4} + \frac{16}{d_{i,j}^6} \right\},$$

and using the fact that the extension  $\tilde{u}_i^{(k)}$  coincides with  $u_i^{(k)}$  outside of  $\bar{O}_i$ , we easily see that the claim holds whatever the type of condition imposed on the object boundary.  $\square$

A convergence criterion can now be stated.

**Theorem 4.5.** *Let  $\kappa(N) = \max\{\kappa_{i,j} \mid (i,j) \in \{1, \dots, N\}^2, i \neq j\}$ ,  $\kappa_{i,j}$  being defined by (46), and assume that*

$$(47) \quad N(N-1)\kappa(N) < 1.$$

Then, the series  $\sum_{k=0}^{+\infty} u_i^{(k)}$ ,  $i = 1, \dots, N$ , where the functions  $u_i^{(k)}$  are the reflections generated by the sequential form of the method of reflections applied to problem (34), converge respectively in  $W(\mathbb{R}^3 \setminus \overline{O_i})$ , and their respective limits  $u_i$ ,  $i = 1, \dots, N$ , are such that the sum  $\sum_{i=1}^N u_i|_{\mathbb{R}^3 \setminus \cup_{j=1}^N \overline{O_j}}$  is the unique solution to problem (34) in  $W(\mathbb{R}^3 \setminus \cup_{j=1}^N \overline{O_j})$ .

*Proof.* Define the sequence  $(\alpha_\ell)_{\ell \in \mathbb{N}^*}$  by

$$\forall k \in \mathbb{N}^*, \forall i \in \{1, \dots, N\}, \alpha_{i+(k-1)N} = \max_{j \in \{1, \dots, N\} \setminus \{i\}} a_{i,j}^{(k)},$$

The coefficients  $a_{i,j}^{(k)}$  being defined in Lemma 4.4. Setting  $\ell = i + kN$  for any positive integer  $k$  and any  $i$  in  $\{1, \dots, N\}$ , one has, owing to (35),

$$\alpha_\ell \leq N(N-1)\kappa(N) \max_{m \in \{1, \dots, N-1\}} \alpha_{\ell-m} \leq N(N-1)\kappa(N) \max_{m \in \{0, \dots, N-1\}} \alpha_{\ell-m}.$$

For  $\ell > 2N$ , denoting  $m_0 = \arg \max_{m \in \{0, \dots, N-1\}} \alpha_{\ell-m}$ , we then have

$$\alpha_{\ell-m_0} \leq N(N-1)\kappa(N) \max_{m \in \{0, \dots, N-1\}} \alpha_{\ell-m_0-m},$$

and, due to condition (47),

$$N(N-1)\kappa(N) \max_{m \in \{0, \dots, N-1-m_0\}} \alpha_{\ell-m_0-m} < \max_{m \in \{m_0, \dots, N-1\}} \alpha_{\ell-m} \leq \alpha_{\ell-m_0}.$$

This implies that

$$\alpha_{\ell-m_0} \leq N(N-1)\kappa(N) \max_{m=N-m_0, \dots, N-1} \alpha_{\ell-m_0-m} \leq N(N-1)\kappa(N) \max_{m=N, \dots, N-1+m_0} \alpha_{\ell-m},$$

so that finally

$$\max_{m \in \{0, \dots, N-1\}} \alpha_{\ell-m} \leq N(N-1)\kappa(N) \max_{m=0, \dots, N-1} \alpha_{\ell-N-m}.$$

As a consequence, each of the series  $\sum_{k \in \mathbb{N}^*} u_i^{(k)}$  is convergent on the boundary of the domain, and thus in  $W(\mathbb{R}^3 \setminus \cup_{j=1}^N \overline{O_j})$ . This convergence then allows to check that the functions  $u_i = \sum_{k \in \mathbb{N}} u_i^{(k)}$ ,  $i = 1, \dots, N$ , satisfy

$$\forall i \in \{1, \dots, N_D\}, \quad \begin{cases} -\Delta u_i = 0 \text{ in } \mathbb{R}^3 \setminus \cup_{j=1}^N \overline{O_j}, \\ u_i = U_i - \sum_{j=1}^{i-1} u_j - \sum_{j=i+1}^N u_j \text{ on } \partial O_i, \end{cases}$$

$$\forall i \in \{N_D + 1, \dots, N\}, \quad \begin{cases} -\Delta u_i = 0 \text{ in } \mathbb{R}^3 \setminus \cup_{j=1}^N \overline{O_j}, \\ \frac{\partial u_i}{\partial \mathbf{n}} = W_{i-N_D} - \sum_{j=1}^{i-1} \frac{\partial u_j}{\partial \mathbf{n}} - \sum_{j=i+1}^N \frac{\partial u_j}{\partial \mathbf{n}} \text{ on } \partial O_i, \end{cases}$$

and thus provide a decomposition of the solution to problem (34).  $\square$

## 5. NUMERICAL TESTS

In this section, we aim at both briefly checking numerically the theoretical results obtained in the paper and investigating cases not handled by the previous analysis.

The problems solved numerically by the method of reflections are for the Laplace equation in interior and exterior domains of  $\mathbb{R}^2$  with both Neumann and Dirichlet boundary conditions. We used a publicly available package<sup>13</sup> of Matlab functions to solve by the Nyström method the integral equations of the problem at collocation points.

For problems set in bounded domains, we also used a finite element code for the simulations and obtained similar results (which are not reprinted here). Note however that to employ this second type of discretisation method with the method of reflections is not a sensible choice. Indeed, the computational effort required to solve any reflection subproblem is always higher than that of the original problem, due to the fact that the meshes for the subproblems being larger than the one for the original one (see Figure 1 for an illustration). This emphasizes the fact that, as a boundary decomposition method, the method of reflections is, in practice, better suited to discretization methods based on boundary integral representation of the solution.

<sup>13</sup>Integral Equation Solver (<http://www.mathworks.com/matlabcentral/fileexchange/34241>) by Alexandre Munnier and Bruno Pinçon, MATLAB Central File Exchange. Retrieved February 15, 2016.

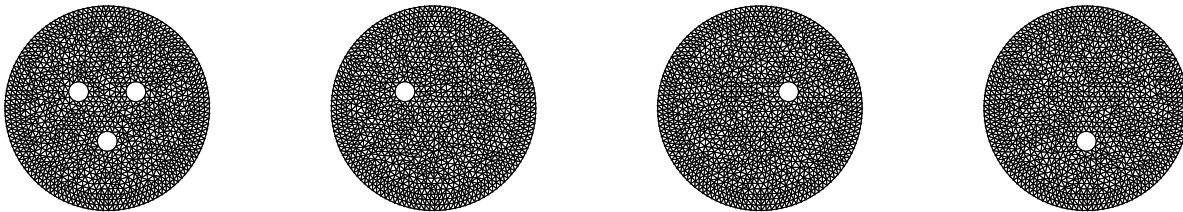


FIGURE 1. From left to right, example of finite element meshes for both the problem dealt with in Subsection 5.1 and its three associated reflection subproblems. We observe that the mesh for the original problem is a subset of the meshes for the subproblems.

**5.1. Rate of convergence for an interior problem.** The first numerical experiment concerns the rate of convergence of the method. It is inspired by a counterexample to the convergence of the parallel form of the method of reflections found in [IB01]. In two dimensions, we consider a bounded domain, a ball centred at the origin and with radius equal to 10, containing three objects, which are balls with identical radii equal to 1 and centres respectively set at the vertices of an equilateral triangle which centroid lies at the origin. Using the length  $l$  of a side of the triangle as a parameter, we investigate the convergence of the different forms of the method for solving a Dirichlet problem for the Laplace equation as  $l$  varies.

Figure 2 presents the relative error of the method as a function of the number of cycles for three distinct values of the parameter  $l$ . This relative error is based on the  $\ell^2$ -norm of the difference between the numerical solution for the full problem with that of the method of reflections after a given number of cycles computed at a finite number of points in the domain  $\Omega \setminus \cup_{i=1}^3 \overline{O}_i$ . One can observe the sequential and the averaged parallel forms of the method are convergent in each case, as predicted by the theoretical results. The parallel form is seen to diverge for the smallest chosen value of  $l$ , but converges for larger values of the parameter.

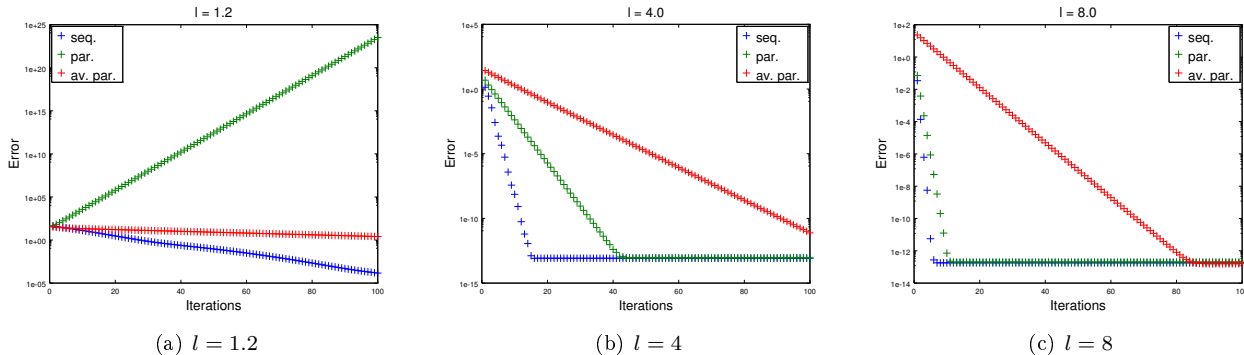


FIGURE 2. Relative quadratic error of the method of reflections as a function of the number of cycles for three chosen values of the distance between the objects.

We note that the convergence is linear and that its rate increases with the value of  $l$ . The sequential form always has the highest rate of convergence. However, from an effective computational perspective, it may not be the fastest method when a large number of objects is involved, since the implementation of the parallel variants can be achieved using parallelisation in practice. A quantitative study of the trade-off between the parallelisation and the rates of convergence of the different forms of the method is out of the scope of the present work.

**5.2. Mixed boundary conditions: a case of divergence of the sequential form.** Since Theorem 4.3 only provides unconditional convergence of the sequential method of reflections in an orthogonal case, cases of divergence are expected outside of this framework, but may not be straightforwardly found. Such a configuration was obtained in two dimensions by considering a ball of radius equal to 2 centred at the origin, on which a constant Dirichlet boundary condition was imposed, and a C-shaped set, partly surrounding it and on which a constant Neumann boundary conditions was imposed, both contained in the bounded domain previously considered. The setting of this example is shown in Figure 3.

In the numerical experiments, none of the three forms of the method converged for such a configuration.

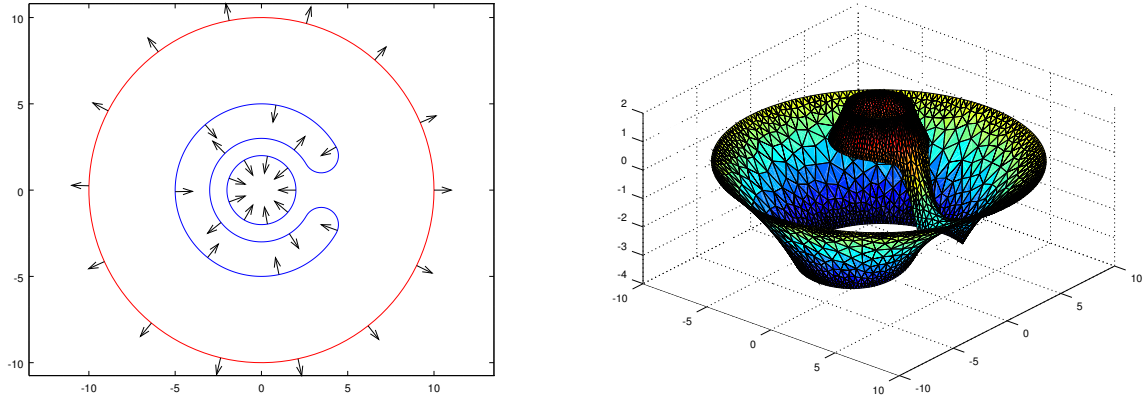


FIGURE 3. Example of configuration giving rise to divergence of the method of reflections for a Laplace problem with mixed boundary conditions. Left subfigure: representation of the domain (red boundary) and the objects (blue boundaries). Right subfigure: representation of the exact solution.

**5.3. Exterior problem and influence of the distance.** The asymptotics of the rate of convergence of the method seen as a function of the distance between the objects is finally investigated. To do this, the Laplace problem dealt with in Subsection 5.1 is recasted as an exterior problem set in  $\mathbb{R}^2$ , considering Neumann boundary conditions satisfying instead of Dirichlet ones, chosen such a way that the solution tends to zero at infinity. The distance  $l$  between the objects being fixed, the contraction coefficient  $c(l)$ , defined by

$$\forall k \in \mathbb{N}, \|u^{(k+1)} - u\| \leq c(l) \|u^{(k)} - u\|,$$

where  $\|\cdot\|$  correspond to the  $\ell^2$ -norm of the function evaluated at a finite number of points on the boundaries of the objects, is estimated by fitting the error as a function of the iteration. The results are presented in Figure 4. The convergence rate of the averaged parallel form appears to be asymptotically independent of the distance between the objects, but the theoretical proof of such a result is an open question.

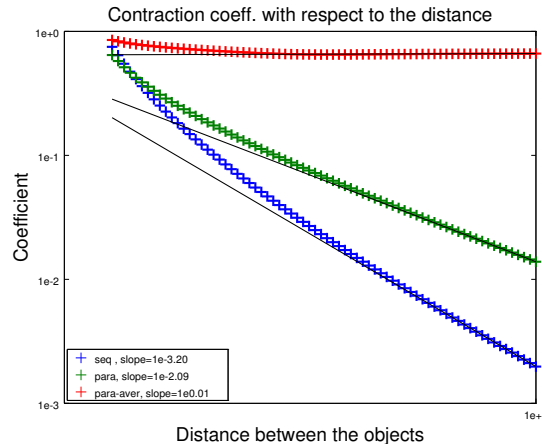


FIGURE 4. Plot of the value of the contraction factor of the method as a function of the distance between the objects. The linear regressions are done using the values associated with the five largest considered distances and are represented by black solid lines.

More generally, scalability issues, that is the analysis of the rate of convergence as a function of the number of the objects involved in the problem, is out of the scope of this paper. Note that settings for which a decomposition method achieves scalability have already been exhibited for Schwarz-type methods, see, e.g., [CG16], but remains an open question in the context of the method of reflections.

**Acknowledgements.** Philippe Laurent would like to thank Frédéric Boyer for introducing him to the method of reflections. Guillaume Legendre would like to thank Christophe Hazard for pointing him to the relevant paper by Mikhael Balabane, and Mikhael Balabane himself for an interesting discussion on the topic. Julien Salomon would like to thank Gabriele Ciaramella, Olivier Glass and Alexandre Munnier for helpful discussions.

## REFERENCES

- [AGG97] C. AMROUCHE, V. GIRAULT, and J. GIROIRE. Dirichlet and Neumann exterior problems for the  $n$ -dimensional Laplace operator. An approach in weighted Sobolev spaces. *J. Math. Pures Appl.*, 76(1):55–81, 1997. DOI: 10.1016/S0021-7824(97)89945-X.
- [Aro50] N. ARONSZAJN. Theory of reproducing kernels. *Trans. Amer. Math. Soc.*, 68(3):337–404, 1950. DOI: 10.1090/S0002-9947-1950-0051437-7.
- [Aus76] A. AUSLENDER. *Optimisation : méthodes numériques*. Masson, 1976.
- [Bal04] M. BALABANE. Boundary decomposition for Helmholtz and Maxwell equations 1: disjoint sub-scatterers. *Asymptotic Anal.*, 38(1):1–10, 2004.
- [BB96] H. H. BAUSCHKE and J. M. BORWEIN. On projection algorithms for solving convex feasibility problems. *SIAM Rev.*, 38(3):367–426, 1996. DOI: 10.1137/S0036144593251710.
- [BDH09] H. H. BAUSCHKE, F. DEUTSCH, and H. HUNDAL. Characterizing arbitrarily slow convergence in the method of alternating projections. *Internat. Trans. Oper. Res.*, 16(4):413–425, 2009. DOI: 10.1111/j.1475-3995.2008.00682.x.
- [BDHP03] H. H. BAUSCHKE, F. DEUTSCH, H. HUNDAL, and S.-H. PARK. Accelerating the convergence of the method of alternating projections. *Trans. Amer. Math. Soc.*, 355(9):3433–3461, 2003. DOI: 10.1090/S0002-9947-03-03136-2.
- [BGM12] C. BADEA, S. GRIVAUX, and V. MÜLLER. The rate of convergence in the method of alternating projections. *St. Petersburg Math. J.*, 23(3):413–434, 2012. DOI: 10.1090/S1061-0022-2012-01202-1.
- [BM91] P. E. BJØRSTAD and J. MANDEL. On the spectra of sums of orthogonal projections with applications to parallel computing. *BIT*, 31(1):76–88, 1991. DOI: 10.1007/BF01952785.
- [Boy+12] F. BOYER, P. B. GOSSIAUX, B. JAWAD, V. LEBASTARD, and M. POREZ. Model for a sensor inspired by electric fish. *IEEE Trans. Robot.*, 28(2):492–505, 2012. DOI: 10.1109/TR0.2011.2175764.
- [Bre65] L. M. BREGMAN. The method of successive projection for finding a common point of convex sets. *Soviet Math. Dokl.*, 6(3):688–692, 1965.
- [BT97] M. BALABANE and V. TIREL. Décomposition de domaine pour un calcul hybride de l'équation de Helmholtz. *C. R. Acad. Sci. Paris Sér. I Math.*, 324(3):281–286, 1997. DOI: 10.1016/S0764-4442(99)80361-9.
- [CG16] G. CIARAMELLA and M. GANDER. Analysis of the parallel Schwarz method for growing chains of fixed-size subdomains: part I. Preprint. To appear in SIAM J. Numer. Anal. 2016.
- [CH13] M. CASSIER and C. HAZARD. Multiple scattering of acoustic waves by small sound-soft obstacles in two dimensions: mathematical justification of the Foldy–Lax model. *Wave Motion*, 50(1):18–28, 2013. DOI: 10.1016/j.wavemoti.2012.06.001.
- [Cim38] G. CIMMINO. Calcolo approssimato per le soluzioni dei sistemi di equazioni lineari. *La Ricerca Scientifica*, 16(9):326–333, 1938.
- [CK88] S. B. CHEN and H. J. KEH. Electrophoresis in a dilute dispersion of colloidal spheres. *AIChE J.*, 34(7):1075–1085, 1988. DOI: 10.1002/aic.690340704.
- [Deu85] F. DEUTSCH. Rate of convergence of the method of alternating projections. In *Parametric Optimization and Approximation*. B. BROSOWSKI and F. DEUTSCH, editors. Volume 72. In International series of numerical mathematics. Birkhäuser-Verlag, 1985, pages 96–107.
- [Deu92] F. DEUTSCH. The method of alternating orthogonal projections. In S. P. SINGH, editor, *Approximation theory, spline functions and applications*. Volume 356, in NATO ASI Series, pages 105–121. Springer Netherlands, 1992. DOI: 10.1007/978-94-011-2634-2\_5.
- [DH10] F. DEUTSCH and H. HUNDAL. Slow convergence of sequences of linear operators II: arbitrarily slow convergence. *J. Approx. Theory*, 162(9):1717–1738, 2010. DOI: 10.1016/j.jat.2010.05.002.
- [DH97] F. DEUTSCH and H. HUNDAL. The rate of convergence for the method of alternating projections, II. *J. Math. Anal. Appl.*, 205(2):381–405, 1997. DOI: 10.1006/jmaa.1997.5202.
- [Dho96] J. K. G. DHONT. *An introduction to dynamics of colloids*. Volume 2 of *Studies in interface science*. Elsevier, 1996.
- [Dix49] J. DIXMIER. Étude sur les variétés et les opérateurs de Julia, avec quelques applications. *Bull. Soc. Math. France*, 77:11–101, 1949.

- [DL00] R. DAUTRAY and J.-L. LIONS. *Mathematical analysis and numerical methods for science and technology. Volume 4 Integral equations and numerical methods*. Springer, 2000. DOI: 10.1007/978-3-642-61531-3.
- [Eva98] L. C. EVANS. *Partial differential equations*. Volume 19 of *Graduate studies in mathematics*. American Mathematical Society, 1998.
- [Fol45] L. L. FOLDY. The multiple scattering of waves. I. General theory of isotropic scattering by randomly distributed scatterers. *Phys. Rev.*, 67(3-4):107–119, 1945. DOI: 10.1103/PhysRev.67.107.
- [Fri37] K. FRIEDRICHS. On certain inequalities and characteristic value problems for analytic functions and for functions of two variables. *Trans. Amer. Math. Soc.*, 41(3):321–364, 1937. DOI: 10.1090/S0002-9947-1937-1501907-0.
- [Gan08] M. J. GANDER. Schwarz methods over the course of time. *Electron. Trans. Numer. Anal.*, 31:228–255, 2008.
- [GH09] M. GANESH and S. C. HAWKINS. A high-order algorithm for multiple electromagnetic scattering in three dimensions. *Numer. Algor.*, 50(4):469–510, 2009. DOI: 10.1007/s11075-008-9238-z.
- [Gol34] G. M. GOLUSIN. Auflösung einiger ebenen Grundaufgaben der mathematischen Physik im Fall der Laplaceschen Gleichung und mehrfachzusammenhängender Gebiete, die durch Kreise begrenzt sind. *Mat. Sb.*, 41(2):246–276, 1934.
- [Gri85] P. GRISVARD. *Elliptic problems in nonsmooth domains*. Pitman, 1985.
- [Hal62] I. HALPERIN. The product of projection operators. *Acta Sci. Math. (Szeged)*, 23(1-2):96–99, 1962.
- [HB83] J. HAPPEL and H. BRENNER. *Low Reynolds number hydrodynamics with special applications to particulate media*. Volume 1 of *Mechanics of fluids and transport processes*. Martinus Nijhoff publishers, 1983. DOI: 10.1007/978-94-009-8352-6.
- [IB01] K. ICHIKI and J. F. BRADY. Many-body effects and matrix inversion in low-Reynolds-number hydrodynamics. *Phys. Fluids*, 13(1):350–353, 2001. DOI: 10.1063/1.1331320.
- [Jon78] R. B. JONES. Hydrodynamic interaction of two permeable spheres I: The method of reflections. *Phys. A*, 92(3-4):545–556, 1978. DOI: 10.1016/0378-4371(78)90150-4.
- [KK91] S. KIM and S. J. KARRILA. *Microhydrodynamics: principles and selected applications*. Butterworth-Heinemann, 1991.
- [KW88] S. KAYALAR and H. L. WEINERT. Error bounds for the method of alternating projections. *Math. Control Signals Systems*, 1(1):43–59, 1988. DOI: 10.1007/BF02551235.
- [Kyn59] G. J. KYNCH. The slow motion of two or more spheres through a viscous fluid. *J. Fluid Mech.*, 5(2):193–208, 1959. DOI: 10.1017/S0022112059000155.
- [Lax51] M. LAX. Multiple scattering of waves. *Rev. Mod. Phys.*, 23(4):287–310, 1951. DOI: 10.1103/RevModPhys.23.287.
- [Lax52] M. LAX. Multiple scattering of waves. II. The effective field in dense systems. *Phys. Rev.*, 85(4):621–629, 1952. DOI: 10.1103/PhysRev.85.621.
- [Luk89] J. H. C. LUKE. Convergence of a multiple reflection method for calculating Stokes flow in a suspension. *SIAM J. Appl. Math.*, 49(6):1635–1651, 1989. DOI: 10.1137/0149099.
- [ML11] C. METTOT and E. LAUGA. Energetics of synchronized states in three-dimensional beating flagella. *Phys. Rev. E*, 84(6):061905, 2011. DOI: 10.1103/PhysRevE.84.061905.
- [Neu49] J. von NEUMANN. On rings of operators. Reduction theory. *Ann. Math. (2)*, 50(2):401–485, 1949. DOI: 10.2307/1969463.
- [Sho77] R. E. SHOWALTER. *Hilbert space methods for partial differential equations*. Volume 1 of *Monographs and studies in mathematics*. Pitman, 1977.
- [Smo11] M. SMOLUCHOWSKI. Über die Wechselwirkung von Kugeln, die sich in einer zähen Flüssigkeit bewegen. *Bull. Int. Acad. Sci. Cracovie, Cl. Sci. Math. Nat., Sér. A Sci. Math.*:28–39, 1911.
- [SSW77] K. T. SMITH, D. C. SOLOMON, and S. L. WAGNER. Practical and mathematical aspects of the problem of reconstructing objects from radiographs. *Bull. Amer. Math. Soc.*, 83(6):1227–1270, 1977. DOI: 10.1090/S0002-9904-1977-14406-6.
- [Tra06] S. D. TRAYTAK. Convergence of a reflection method for diffusion-controlled reactions on static sinks. *Phys. A Statist. Mech. Appl.*, 362(2):240–248, 2006. DOI: 10.1016/j.physa.2005.03.061.
- [Wil13] H. J. WILSON. Stokes flow past three spheres. *J. Comput. Phys.*, 245:302–316, 2013. DOI: 10.1016/j.jcp.2013.03.020.
- [WL13] H. WANG and J. LIU. On decomposition method for acoustic wave scattering by multiple obstacles. *Acta Math. Sci.*, 33(1):1–22, 2013. DOI: 10.1016/S0252-9602(12)60191-X.
- [Xu01] J. XU. The method of subspace corrections. *J. Comput. Appl. Math.*, 128(1-2):335–362, 2001. DOI: 10.1016/S0377-0427(00)00518-5.



- [Xu92] J. XU. Iterative methods by space decomposition and subspace correction. *SIAM Rev.*, 34(4):581–613, 1992. DOI: 10.1137/1034116.
- [XZ02] J. XU and L. ZIKATANOV. The method of alternating projections and the method of subspace corrections in Hilbert space. *J. Amer. Math. Soc.*, 15(3):573–597, 2002. DOI: 10.1090/S0894-0347-02-00398-3.

INSTITUT DE RECHERCHE EN COMMUNICATIONS ET CYBERNÉTIQUE DE NANTES, CNRS, UMR 6597, ÉCOLE DES MINES DE NANTES, 4, RUE ALFRED KASTLER, 44307 NANTES, FRANCE  
*E-mail address:* philippe.laurent@mines-nantes.fr

CEREMADE, UMR CNRS 7534, UNIVERSITÉ PARIS-DAUPHINE, PSL RESEARCH UNIVERSITY, PLACE DU MARÉCHAL DE LATTRE DE TASSIGNY, 75775 PARIS CEDEX 16, FRANCE  
*E-mail address:* guillaume.legendre@dauphine.fr

CEREMADE, UMR CNRS 7534, UNIVERSITÉ PARIS-DAUPHINE, PSL RESEARCH UNIVERSITY, PLACE DU MARÉCHAL DE LATTRE DE TASSIGNY, 75775 PARIS CEDEX 16, FRANCE  
*E-mail address:* julien.salomon@dauphine.fr