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# Homogenization of heat diffusion in a cracked medium

Xavier Blanc, Benjamin-Edouard Peigney

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## Abstract

We develop in this note a homogenization method to tackle the problem of a diffusion process through a cracked medium. We show that the cracked surface of the domain induces a source term in the homogenized equation. We assume that the cracks are orthogonal to the surface of the material, where an incoming heat flux is applied. The cracks are supposed to be of depth 1, of small width, and periodically arranged.

## 1 Motivation and setting of the problem

We consider the propagation of radiation through a cracked medium, made of an optically thick material. The propagation is initiated by an incoming given energy flux imposed on the left boundary of the crack (see figure 1 where the flux is represented in dashed lines).

Physically, the exchange surface between the optically thick medium and the energy source may be greatly modified by the fractures. This may have a significant impact on the energy balance of the considered system. In many situations, the intricacies of the cracked medium are such that it is almost impossible to carry out a direct calculation. Besides, many spatial scales may be involved simultaneously. Full numerical simulations of such multi-scaled media become hence infeasible.

That is why we have been looking for an **average** approach that could capture the effects of cracks in a homogenized medium. The model presented here is simple enough to be coupled to standard FEM codes. The physical idea behind the model developed in this paper, called "MOSAIC" (Method Of Sinks Averaging Inhomogeneous behavior of Cracked media), is to treat the flux enhancement induced by the crack as a volume **source term** in the homogenized energy equation. We will show that this can be rigorously justified by homogenization theory.

Since the medium is assumed to be optically thick, the propagation of radiation follows a diffusion process [21]. For the sake of simplicity, we shall assume a linear behavior law, which means that the energy flux  $\mathcal{F}$  is proportional to the energy gradient  $\nabla u$ .

Besides, we consider the diffusion process on a short time scale, so that the hydrodynamic effects, which are not supposed to play a significant role, are disregarded.

The linear diffusion problem can thus be modeled by:

$$\begin{cases} \partial_t u_\varepsilon - \Delta u_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ \partial_n u_\varepsilon = 0 & \text{on } \Gamma_\varepsilon^0, \\ \partial_n u_\varepsilon = 1 & \text{on } \Gamma_\varepsilon^1, \\ \partial_n u_\varepsilon = \frac{\alpha - \beta}{2} \varepsilon & \text{on } \Gamma_\varepsilon^\alpha, \\ \partial_n u_\varepsilon = \frac{\beta}{\alpha} & \text{on } \Gamma_\varepsilon^\beta. \end{cases} \quad (1.1)$$

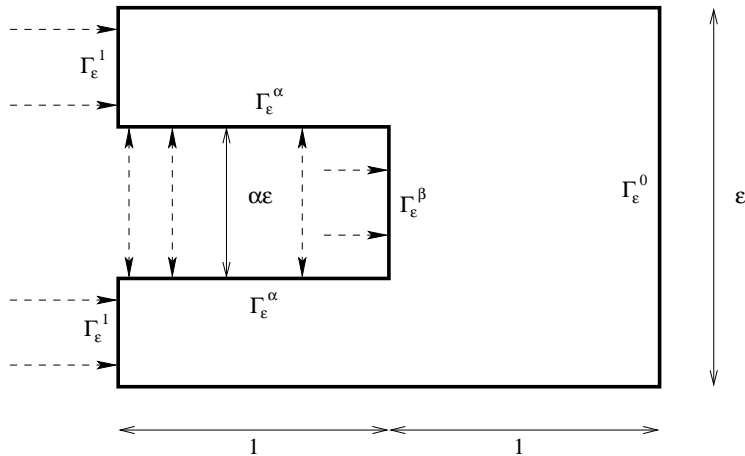


Figure 1: The cracked domain  $\Omega_\varepsilon$ .

We also need an initial condition:

$$u_\varepsilon(x, y, t = 0) = u^0(x, y), \quad (1.2)$$

so that problem (1.1)-(1.2) is well-posed. For the problem at hand,  $u_\varepsilon(x, y, t)$  is the energy density but it could represent any field following a diffusion process.

We impose  $u_\varepsilon(x, y, t)$  to be periodic of period  $\varepsilon$  with respect to the  $y$  direction. The domain  $\Omega_\varepsilon$ , as well as the boundaries  $\Gamma_\varepsilon^0, \Gamma_\varepsilon^1, \Gamma_\varepsilon^\alpha, \Gamma_\varepsilon^\beta$  are defined on figure 1. The right-most boundary  $\Gamma_\varepsilon^0$  is supposed to coincide with the set  $\{x = 1\}$ , while  $\Gamma_\varepsilon^\beta$  is a subset of  $\{x = 0\}$ , and the left-most boundary  $\Gamma_\varepsilon^1$  is a subset of  $\{x = -1\}$ . The parameter  $\varepsilon > 0$  is supposed to be small and will tend to 0, whereas  $\alpha \in [0, 1)$  is a fixed parameter related to the width of the crack. The parameter  $\beta \in [0, \alpha)$  measures the portion of the flux which, coming through the segment  $\{x = -1, -\alpha\varepsilon/2 < y < \alpha\varepsilon/2\}$ , reaches the bottom  $\Gamma_\varepsilon^\beta$  of the crack. The remaining part of the incoming flux is distributed on the horizontal part of the boundary, namely  $\Gamma_\varepsilon^\alpha$ . The parameter  $\beta$  is supposed to be fixed. The boundary conditions in (1.1) are defined in such a way that the total incoming flux is exactly equal to 1, which is the value we impose on the left boundary in the case  $\alpha = 0$  (no crack).

A space-time dependence of the flux applied on the boundaries  $\Gamma_\varepsilon^\alpha$  may be introduced but it does not affect the homogenization process that we describe here.

Let us mention some previous works on similar problems : in [2], the problem of finding an effective boundary condition for stationary Navier-Stokes equation is studied. The domain is bounded, and part of its boundary has oscillations similar to our case, but the corresponding "crack depth" is of size  $\varepsilon$ , whereas in our case it is of size 1. A similar question is addressed in [18]. In [10], the same question was studied in the case of an infinite domain. In [11], the case of Stokes equation with a Coriolis-type term is addressed. The case of Couette flow is studied in [19]. Elliptic equations were studied, in a similar setting, in [7, 19, 22, 24]. In [15], Poisson problem is studied with three variation scales in the boundary :  $1, \varepsilon, \varepsilon^2$ . The stochastic case was addressed in [4]. The only works we have found in which the geometry is similar to our case is [3, 5]. In these works, the equation is of Poisson type, and the indentations of the boundary are of size 1. However, the boundary condition in the indentation is of Dirichlet type, whereas here it is a Neumann boundary condition.

## 2 Changing the scale

To carry out an asymptotic expansion of the solution  $u = u_\varepsilon$  of (1.1) in powers of  $\varepsilon$ , we "scale" the variable  $y$ , in the spirit of [6]. Actually, 2 scales describe the model: the variable  $y$  is the macroscopic one, whereas  $\frac{y}{\varepsilon}$  represents the "microscopic geometry". Thus, we define:

$$u_\varepsilon(x, y, t) = v_\varepsilon\left(x, \frac{y}{\varepsilon}, t\right),$$

so that  $v$  is periodic of period 1.

We notice that:

$$\begin{aligned} \partial_x u_\varepsilon(x, y, t) &= \partial_x v_\varepsilon\left(x, \frac{y}{\varepsilon}, t\right), & \partial_x^2 u_\varepsilon(x, y, t) &= \partial_x^2 v_\varepsilon\left(x, \frac{y}{\varepsilon}, t\right), \\ \partial_y u_\varepsilon(x, y, t) &= \frac{1}{\varepsilon} \partial_y v_\varepsilon\left(x, \frac{y}{\varepsilon}, t\right), & \partial_y^2 u_\varepsilon(x, y, t) &= \frac{1}{\varepsilon^2} \partial_y^2 v_\varepsilon\left(x, \frac{y}{\varepsilon}, t\right). \end{aligned}$$

And  $v_\varepsilon$  is solution of (2.1):

$$\begin{cases} -\partial_x^2 v_\varepsilon - \frac{1}{\varepsilon^2} \partial_y^2 v_\varepsilon + \partial_t v_\varepsilon = 0 & \text{in } \Omega_1, \\ \partial_n v_\varepsilon = 0 & \text{on } \Gamma_1^0, \\ \partial_n v_\varepsilon = 1 & \text{on } \Gamma_1^1, \\ \frac{1}{\varepsilon^2} \partial_n v_\varepsilon = \frac{\alpha - \beta}{2} & \text{on } \Gamma_1^\alpha, \\ \partial_n v_\varepsilon = \frac{\beta}{\alpha} & \text{on } \Gamma_1^\beta \end{cases} \quad (2.1)$$

## 3 Asymptotic expansion

Firstly, we notice that in the system (2.1) the domain  $\Omega_1$  does not depend of  $\varepsilon$  anymore. We have to study an equation depending on  $\varepsilon$  in a fixed domain. Besides, the parameter  $\varepsilon$  appears in the equation (2.1) only as  $\varepsilon^2$ , which means that  $\varepsilon^2$  is a relevant parameter for an asymptotic expansion. Thus, it seems natural to look for  $v_\varepsilon$  as follows:

$$v_\varepsilon(x, y, t) = v_0(x, y, t) + \varepsilon^2 v_1(x, y, t) + \varepsilon^4 v_2(x, y, t) + \dots \quad (3.1)$$

**Remark 3.1** *As we will see below, a boundary layer is present at  $x = 0$ . It is necessary to account for it in order to derive the transmission condition (4.4). It is possible to include this boundary layer in the asymptotic expansion of  $v$ . Equation (3.1) would then read:*

$$\begin{aligned} v_\varepsilon(x, y, t) &= v_0(x, y, t) + \varepsilon^2 v_1(x, y, t) + \varepsilon^4 v_2(x, y, t) + \dots \\ &\quad + v_{BL_0}\left(\frac{x}{\varepsilon}, y, t\right) + \varepsilon^2 v_{BL_1}\left(\frac{x}{\varepsilon}, y, t\right) + \dots \end{aligned} \quad (3.2)$$

*Nevertheless, for the sake of simplicity, we shall deal with the boundary layer separately and consider the asymptotic expansion (3.1) to guess the shape of the homogenized problem.*

Hence, we insert the ansatz (3.1) into the system (2.1) and identify the different powers of  $\varepsilon^2$ . We obtain:

- At the order  $\varepsilon^{-2}$  :

$$\partial_y^2 v_0 = 0,$$

which means that  $v_0$  can be written

$$v_0(x, y, t) = f(x, t) + yg(x, t),$$

where  $f$  and  $g$  are 2 functions independent of the variable  $y$ . The condition of periodicity in  $y$  verified by  $v_\varepsilon$  implies that  $v_0(x, 1/2, t) = v_0(x, -1/2, t)$ , hence:  $g = 0$ . Thus,  $v_0$  does not depend on  $y$  :

$$v_0(x, y, t) = v_0(x, t). \quad (3.3)$$

Besides, the boundary conditions on  $v_0$  are:

$$\partial_n v_0 = 0 \text{ on } \Gamma_1^0, \quad \partial_n v_0 = 1 \text{ on } \Gamma_1^1, \quad \partial_n v_0 = 0 \text{ on } \Gamma_1^\alpha, \quad \partial_n v_0 = \frac{\beta}{\alpha} \text{ on } \Gamma_1^\beta. \quad (3.4)$$

We check that the boundary condition on  $\Gamma_1^\alpha$  is consistent with (3.3). This condition is indeed equivalent to the statement:  $\partial_y v_0 = 0$  on  $\Gamma_1^\alpha$ , which is automatically verified since  $v_0$  does not depend on  $y$ .

- At the order  $\varepsilon^0$  :

$$-\partial_x^2 v_0 - \partial_y^2 v_1 + \partial_t v_0 = 0. \quad (3.5)$$

The boundary conditions on  $v_1$  give

$$\partial_n v_1 = 0 \text{ on } \Gamma_1^0, \quad \partial_n v_1 = 0 \text{ on } \Gamma_1^1, \quad \partial_n v_1 = \frac{\alpha - \beta}{2} \text{ on } \Gamma_1^\alpha, \quad \partial_n v_1 = 0 \text{ on } \Gamma_1^\beta.$$

We now integrate (3.5) with respect to  $y$  : we get

$$-(1 - \alpha)\partial_x^2 v_0 + (1 - \alpha)\partial_t v_0 = \int_{-\frac{1}{2}}^{-\frac{\alpha}{2}} \partial_y^2 v_1 dy + \int_{\frac{\alpha}{2}}^{\frac{1}{2}} \partial_y^2 v_1 dy, \quad (3.6)$$

if  $x < 0$ , and

$$-\partial_x^2 v_0 + \partial_t v_0 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \partial_y^2 v_1 dy,$$

if  $x > 0$ . (recall that we assume that the axis  $x = 0$  contains the border  $\Gamma_1^\beta$ .) In the first case  $x < 0$ , we use the boundary value for  $\partial_n v_1$ , which gives

$$\partial_y v_1 \left( x, -\frac{\alpha}{2}, t \right) = \frac{\alpha - \beta}{2}, \quad \partial_y v_1 \left( x, \frac{\alpha}{2}, t \right) = -\frac{\alpha - \beta}{2}.$$

Besides, the periodicity in  $y$  implies that boundary values in  $y = 1/2$  are exactly compensated by those in  $y = -1/2$ . Thus, the right hand side of equation (3.6) is  $\frac{\alpha - \beta}{2} + \frac{\alpha - \beta}{2} = \alpha - \beta$ , and we obtain

$$\begin{cases} -(1 - \alpha)\partial_x^2 v_0 + (1 - \alpha)\partial_t v_0 = \alpha - \beta & \text{if } x < 0, \\ -\partial_x^2 v_0 + \partial_t v_0 = 0 & \text{if } x > 0. \end{cases}$$

Hence,  $v_0$  satisfies an equation in  $\{x < 0\}$ , and an equation in  $\{x > 0\}$ . In order to define it properly, we need boundary conditions. For  $\Gamma_1^0$  and  $\Gamma_1^1$ , we have (3.4). We are now going to derive boundary conditions on  $\{x = 0\}$ . Since we have assumed that  $\partial_n u = \beta/\alpha$  on  $\Gamma_1^\beta$ , and since  $v_0$  does not depend on  $y$ , it may seem natural to impose that  $\partial_x v_0(x = 0) = -\beta/\alpha$ . However, as we will see below, the flux  $\partial_x v_0$  is not continuous across the interface  $\{x = 0\}$ . Therefore, we need to take into account a boundary layer at this interface. For this purpose, we go back to (2.1), in which we zoom at  $x = 0$ , that is, we define

$$v_\varepsilon(x, y, t) = w_\varepsilon\left(\frac{x}{\varepsilon}, y, t\right).$$

Inserting this into (2.1), we see that,

$$-\partial_x^2 w_\varepsilon - \partial_y^2 w_\varepsilon + \varepsilon^2 \partial_t w_\varepsilon = 0. \quad (3.7)$$

We integrate this equation over the domain  $A_\delta = [-\delta, \delta] \times [-1/2, 1/2] \cap \Omega_1$  (see figure 2), integrate by parts, and find

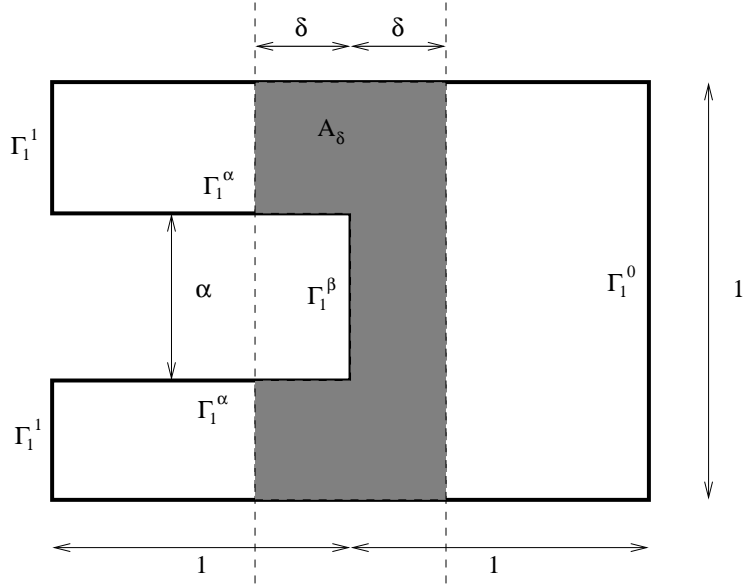


Figure 2: The domain  $A_\delta$  on which we integrate.

$$\begin{aligned} \int_{\{x=\delta\}} \partial_x w_\varepsilon - \int_{\{x=-\delta, \alpha/2 < |y| < 1/2\}} \partial_x w_\varepsilon - \int_{\{x=0, |y| < \alpha/2\}} \partial_x w_\varepsilon \\ - \int_{\{y=\alpha/2, -\delta < x < 0\}} \partial_y w_\varepsilon + \int_{\{y=-\alpha/2, -\delta < x < 0\}} \partial_y w_\varepsilon = \varepsilon^2 \int_{A_\delta} \partial_t w_\varepsilon \end{aligned}$$

The right-hand side vanishes as  $\varepsilon \rightarrow 0$ . In the left-hand side, the fourth and fifth terms are bounded by  $\delta$ , hence, taking  $\varepsilon \rightarrow 0$ , then  $\delta \rightarrow 0$ , we infer

$$\int_{\{x=0^+\}} \partial_x w_\varepsilon = \int_{\{x=0^-, \alpha/2 < |y| < 1/2\}} \partial_x w_\varepsilon - \beta.$$

Therefore, we have, going back to  $v_0$ , and using the fact that it does not depend on  $y$ ,

$$(1 - \alpha)\partial_x v_0(0^-, y, t) = \partial_x v_0(0^+, y, t) + \beta. \quad (3.8)$$

We can also compute the relation between  $v_0(0^+, y, t)$  and  $v_0(0^-, y, t)$  by multiplying (3.7) by  $x$ , and integrating it again over  $A_\delta$ . We then have

$$\begin{aligned} \delta \int_{\{x=\delta\}} \partial_x w_\varepsilon + \delta \int_{\{x=-\delta, \alpha/2 < |y| < 1/2\}} \partial_x w_\varepsilon - \int_{\{y=\alpha/2, -\delta < x < 0\}} x \partial_y w_\varepsilon + \int_{\{y=-\alpha/2, -\delta < x < 0\}} x \partial_y w_\varepsilon \\ - \int_{\{x=\delta\}} w_\varepsilon + \int_{\{x=-\delta, \alpha/2 < |y| < 1/2\}} w_\varepsilon + \int_{\{x=0, |y| < \alpha/2\}} w_\varepsilon = \varepsilon^2 \int_{A_\delta} x \partial_t w_\varepsilon \end{aligned}$$

Here again, the right-hand side vanishes as  $\varepsilon \rightarrow 0$ , while the first line vanishes as  $\delta \rightarrow 0$ . Hence, we find that

$$\int_{\{x=0^+\}} w_\varepsilon = \int_{\{x=0^-, \alpha/2 < |y| < 1/2\}} w_\varepsilon + \int_{\{x=0, |y| < \alpha/2\}} w_\varepsilon.$$

Going back to  $v_0$ , this implies that  $v_0$  is continuous across the interface.

We can thus write the system of equation satisfied by  $v_0$ :

$$\begin{cases} -\partial_x^2 v_0 + \partial_t v_0 = \frac{\alpha - \beta}{1 - \alpha} & \text{in } \Omega_1 \cap \{x < 0\}, \\ \partial_n v_0 = 1 & \text{on } \Gamma_1^1, \\ \partial_n v_0 = 0 & \text{on } \Gamma_1^\alpha, \end{cases} \quad (3.9)$$

and

$$\begin{cases} -\partial_x^2 v_0 + \partial_t v_0 = 0 & \text{in } \Omega_1 \cap \{x > 0\}, \\ \partial_n v_0 = 0 & \text{on } \Gamma_1^0, \end{cases} \quad (3.10)$$

This system is not well-posed, since boundary conditions are missing at the interface  $\{x = 0\}$ . We thus impose the transmission conditions (3.8), together with the fact that  $v_0$  should be continuous across the interface:

$$v_0(x = 0^-) = v_0(x = 0^+), \quad (1 - \alpha)\partial_x v_0(x = 0^-) = \partial_x v_0(x = 0^+) + \beta. \quad (3.11)$$

## 4 Homogenized equation

We have formally shown that

$$u_\varepsilon(x, y, t) \approx v_0\left(x, \frac{y}{\varepsilon}, t\right),$$

with  $v_0$  solution of (3.9)-(3.10). Moreover, if we extend  $u$  by 0 outside  $\Omega_\varepsilon$ , we use the fact that, for any function  $f$  which is 1-periodic with respect to  $y$ , we have

$$f\left(\frac{y}{\varepsilon}\right) \xrightarrow{*} \int_{-1/2}^{1/2} f(y) dy, \quad (4.1)$$

in  $L^\infty$ . Hence,  $u_\varepsilon$  converges to the average of  $v_0$  with respect to  $y$ , that is,  $(1 - \alpha)v_0(x, t)$  if  $x < 0$ , and  $v_0(x, t)$  if  $x > 0$ . In other words, the limit equation on  $u$  is thus (3.9) multiplied by  $(1 - \alpha)$ , and (3.10). Note that the boundary conditions are treated exactly as the equation, using (4.1).

Hence, the system satisfied by  $u$  reads:

$$\begin{cases} -\Delta u + \partial_t u = \alpha - \beta & \text{in } \{-1 < x < 0\}, \\ \partial_n u = 1 - \alpha & \text{on } \{x = -1\}, \\ u \text{ is } 1\text{-periodic in } y. \end{cases} \quad (4.2)$$

$$\begin{cases} -\Delta u + \partial_t u = 0 & \text{in } \{0 < x < 1\}, \\ \partial_n u = 0 & \text{on } \{x = 1\}, \\ u \text{ is } 1\text{-periodic in } y. \end{cases} \quad (4.3)$$

We also have the corresponding transmission condition inherited from (3.11):

$$u(x = 0^-) = (1 - \alpha)u(x = 0^+), \quad \partial_x u(x = 0^-) = \partial_x u(x = 0^+) + \beta. \quad (4.4)$$

(4.2)-(4.3)-(4.4) constitute the homogenized model.

**Remark 4.1** *In the coupling condition (4.4), we have a jump for  $u$  as well as for the flux  $\partial_x u$ . Therefore, it is a priori not possible to recast (4.2)-(4.3)-(4.4) into a single boundary problem in the domain  $\{-1 < x < 1\}$  (however, see Section 7.2 below for a formal formulation of the problem). Nevertheless, when  $\alpha$  is small and  $\beta = 0$ , (4.4) almost amounts to impose continuity of  $u$  and its derivative across the interface  $\{x = 0\}$ . Therefore, in such a case, the approximate problem*

$$\begin{cases} -\Delta u + \partial_t u = \alpha \mathbf{1}_{\{x < 0\}} & \text{in } \{-1 < x < 1\}, \\ \partial_n u = 1 - \alpha & \text{on } \{x = -1\}, \\ \partial_n u = 0 & \text{on } \{x = 1\}, \\ u \text{ is } 1\text{-periodic in } y, \end{cases} \quad (4.5)$$

should give a solution which is close to the solution to (4.2)-(4.3)-(4.4).

**Remark 4.2** *A spatial dependence on the flux imposed on the boundary  $\Gamma_\varepsilon^\alpha$  may be introduced. In this case, and setting  $\beta = 0$  for the sake of simplicity, the formulation of the problem written on the cracked domain  $\Omega_\varepsilon$  reads:*

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega_\varepsilon, \\ \partial_n u = 0 & \text{on } \Gamma_\varepsilon^0, \\ \partial_n u = 1 & \text{on } \Gamma_\varepsilon^1, \\ \partial_n u = f_\alpha(x)\varepsilon & \text{on } \Gamma_\varepsilon^\alpha, \\ \partial_n u = 0 & \text{on } \Gamma_\varepsilon^\beta. \end{cases} \quad (4.6)$$

The function  $f_\alpha(x)$  is such that:

$$\int_{x=-1}^{x=0} f_\alpha(x) dx = \frac{\alpha}{2},$$

ensuring the conservation of energy as  $\varepsilon$  varies.



Furthermore, if  $f_\alpha(x)$  is such that  $f_\alpha(0) = 0$ , then the singularity at  $x = 0$  vanishes and the formulation of the homogenized model is exactly given by:

$$\begin{cases} -\Delta u + \partial_t u = f_\alpha(x) \mathbf{1}_{\{x < 0\}} & \text{in } \{-1 < x < 1\}, \\ \partial_n u = 1 - \alpha & \text{on } \{x = -1\}, \\ \partial_n u = 0 & \text{on } \{x = 1\}, \\ u \text{ is } 1\text{-periodic in } y. \end{cases} \quad (4.7)$$

In such a case, the solution  $u$  and its gradient  $\partial_n u$  are continuous across the interface  $x = 0$ .

## 5 Analysis of the homogenized problem

In this section, we prove that problem (4.2)-(4.3)-(4.4) is well posed. Let us first set the notation:

$$\Omega = \left\{ (x, y) \in \mathbb{R}^2, \quad -\frac{1}{2} < y < \frac{1}{2}, \quad -1 < x < 1 \right\}. \quad (5.1)$$

$$\Omega^+ = \Omega \cap \{x > 0\} = \left\{ (x, y) \in \mathbb{R}^2, \quad -\frac{1}{2} < y < \frac{1}{2}, \quad 0 < x < 1 \right\}. \quad (5.2)$$

$$\Omega^- = \Omega \cap \{x < 0\} = \left\{ (x, y) \in \mathbb{R}^2, \quad -\frac{1}{2} < y < \frac{1}{2}, \quad -1 < x < 0 \right\}. \quad (5.3)$$

$$\Gamma^1 = \left\{ (x, y) \in \mathbb{R}^2, \quad -\frac{1}{2} < y < \frac{1}{2}, \quad x = -1 \right\}. \quad (5.4)$$

$$\Gamma^0 = \left\{ (x, y) \in \mathbb{R}^2, \quad -\frac{1}{2} < y < \frac{1}{2}, \quad x = 1 \right\}. \quad (5.5)$$

$$\Gamma^\beta = \left\{ (x, y) \in \mathbb{R}^2, \quad -\frac{1}{2} < y < \frac{1}{2}, \quad x = 0 \right\}. \quad (5.6)$$

**Lemma 5.1** *Assume that  $\alpha \in (0, 1)$ . Then for any  $T > 0$ , problem (4.2)-(4.3)-(4.4) has a unique solution  $(u_-, u_+) \in X$ , where*

$$X = C([0, T], H^1(\Omega^-)) \cap C^1([0, T], L^2(\Omega^-)) \times C([0, T], H^1(\Omega^+)) \cap C^1([0, T], L^2(\Omega^+))$$

**Proof:** First, let us point out that, for any  $F \in L^2(\Gamma^\beta)$ , the problem

$$\begin{cases} -\Delta u + \partial_t u = 0 & \text{in } \Omega^+, \\ \partial_n u = 0 & \text{on } \Gamma^0, \\ \partial_n u = F & \text{on } \Gamma^\beta, \\ u \text{ is } 1\text{-periodic in } y, \\ u(t = 0) = u_0 & \text{in } \Omega^+. \end{cases} \quad (5.7)$$

admits a unique solution  $u^+ \in C([0, T], H^1(\Omega^+)) \cap C^1([0, T], L^2(\Omega^+))$ , for any  $T > 0$ . This is easily proved using standard tools of the analysis of parabolic PDEs. See for instance [14] for the details.

Next, consider the problem (here,  $g \in L^2(\Gamma^\beta)$ ):

$$\begin{cases} -\Delta u + \partial_t u = \alpha - \beta & \text{in } \Omega^-, \\ \partial_n u = 1 - \alpha & \text{on } \Gamma^1, \\ u = g & \text{on } \Gamma^\beta, \\ u \text{ is 1-periodic in } y, \\ u(t = 0) = u_0 & \text{in } \Omega^-. \end{cases} \quad (5.8)$$

Here again, standard theory of parabolic equations allows to prove that (5.7) has a unique solution  $u^- \in C([0, T], H^1(\Omega^-)) \cap C^1([0, T], L^2(\Omega^-))$ , for any  $T > 0$  (see [14]). We now study the following fixed-point approach: consider an initial guess  $F^0 \in L^2(\Gamma^\beta)$ , to which we associate the solution  $u^{+,0}$  of (5.7). Then, define  $g^0$  as the trace of  $(1 - \alpha)u^{+,0}$  on  $\Gamma^\beta$ , and solve (5.8) with data  $g = g^0$ : this defines  $u^{-,0}$ , and a new flux  $F^1 = \beta - \partial_n u^{-,0}$  in  $L^2(\Gamma^\beta)$ . Repeating this procedure, we build a sequence  $(F^n)_{n \in \mathbb{N}}$  in  $L^2(\Gamma^\beta)$ , together with the corresponding solutions  $u^{\pm,n}$  and the data  $g^n$ . In order to prove that this sequence converges, we are going to prove that the application  $\Phi : L^2([0, T] \times \Gamma^\beta) \rightarrow L^2([0, T] \times \Gamma^\beta)$ , which associates  $F^{n+1}$  to  $F^n$  is a contraction. For this purpose, we define

$$G^n = F^{n+1} - F^n, \quad v^{n,\pm} = u^{n+1,\pm} - u^{n,\pm}, \quad h^n = g^{n+1} - g^n.$$

It is clear that  $v^{n,+}$  satisfies (5.7) with  $F = G^n$  and  $u_0 = 0$ . Similarly,  $v^{n,-}$  is the solution to (5.8) with  $\alpha - \beta = 0$ ,  $1 - \alpha = 0$ ,  $u_0 = 0$ , and  $g = h^n$ . As a consequence, using the uniqueness for problem (5.8), we infer  $v^{n,-}(x, y, t) = (1 - \alpha)v^{n,+}(-x, y, t)$ , whence

$$G^{n+1} = (1 - \alpha)G^n. \quad (5.9)$$

This immediately implies that the sequence  $(F^n)_{n \in \mathbb{N}}$  converges in  $L^2([0, T] \times \Gamma^\beta)$ . Denoting by  $F$  its limit, it is clear that the corresponding solution  $u^\pm$  is a solution to (4.2)-(4.3)-(4.4).

This proves the existence of a solution. Next, considering the uniqueness, we assume that we have two solutions  $u_1$  and  $u_2$  such that  $u_1 \neq u_2$ , we necessarily have  $\partial_x u_1|_{\Gamma^\beta} \neq \partial_x u_2|_{\Gamma^\beta}$ , according to the uniqueness of the solution to (5.7). Hence, defining  $F_1 = \partial_x u_1|_{\Gamma^\beta}$  and  $F_2 = \partial_x u_2|_{\Gamma^\beta}$ , these functions are fixed points of the above application  $\Phi$ . Hence, the above argument implies that  $F_1 = F_2$ , which proves uniqueness.  $\square$

**Remark 5.2** *The above proof is in fact useful for numerical purposes. Indeed, it proves that this fixed-point approach always converges. Hence, it may be used to compute the solution  $u$  to (4.2)-(4.3)-(4.4). However, it should be noted that (5.9) is an equality. Hence, if  $\alpha$  is close to 1, the convergence will be very slow.*

## 6 Proof of convergence

In this section, we give a rigorous proof of the fact that the solution  $u_\varepsilon$  to (1.1) converges to the solution  $u$  to (4.2)-(4.3) as  $\varepsilon \rightarrow 0$ .

As a preliminary remark, let us point out that, using standard results of PDE analysis, one easily proves that (1.1) has a unique solution in  $C([0, T], H^1(\Omega_\varepsilon)) \cap C^1([0, T], L^2(\Omega_\varepsilon))$ , for any  $T > 0$  and any  $\varepsilon > 0$ . See for instance [14] for the details.

**Proposition 6.1** *Let  $u_\varepsilon$  be the unique solution to (1.1). We extend it by 0 outside  $\Omega_\varepsilon$ , and assume that the initial data  $u_\varepsilon(t = 0)$  is such that*

$$u_\varepsilon(t = 0) \rightharpoonup u_0 \text{ in } L^2(\Omega). \quad (6.1)$$

Then, for any  $T > 0$ , we have

$$u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} u \text{ in } L^2(\Omega \times [0, T]), \quad (6.2)$$

where  $u$  is the unique solution to (4.2)-(4.3)-(4.4).

**Remark 6.2** We have set  $u_\varepsilon = 0$  outside  $\Omega_\varepsilon$ . This strategy is physically relevant, since  $u_\varepsilon$  is a temperature, and the heat transfer is only modeled inside  $\Omega_\varepsilon$ : one may think of the outside of  $\Omega_\varepsilon$  as the vacuum, or at least a domain which is transparent to radiation.

**Remark 6.3** In (6.2), we have only a weak convergence. The reason for this is the fact that we have extended  $u_\varepsilon$  by 0 outside  $\Omega_\varepsilon$ , as it is explained in Remark 6.2, whereas it is positive in  $\Omega_\varepsilon$ , due to the imposed incoming flux. Hence,  $u_\varepsilon$  qualitatively behaves like a function which is equal to 1 in  $\Omega_\varepsilon$ , and 0 outside. In the domain  $\Omega_1$ , this function converges weakly to its average, but does not converge strongly in  $L^2$ .

**Remark 6.4** The convergence (6.2) is only local in time ( $T$  cannot be infinite). This is due to the fact that we impose a constant incoming flux. Therefore, integrating (1.1) over  $\Omega_\varepsilon$ , and using an integration by parts, we have

$$\frac{d}{dt} \int_{\Omega_\varepsilon} u = 1,$$

hence  $u_\varepsilon$  cannot be bounded with respect to  $t$ .

Before we prove this result, we need a few technical lemmas:

## 6.1 Technical preliminary results

**Lemma 6.5** Under the hypotheses of Proposition 6.1, for any  $\varepsilon \in (0, 1)$ , we have the following estimate:

$$\int_{\Gamma_\varepsilon^\beta} u_\varepsilon^2 + \int_{\Gamma_\varepsilon^0} u_\varepsilon^2 + \int_{\Gamma_\varepsilon^1} u_\varepsilon^2 + \varepsilon \int_{\Gamma_\varepsilon^\alpha} u_\varepsilon^2 \leq C \left( \int_{\Omega_\varepsilon} u_\varepsilon^2 + \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \right), \quad (6.3)$$

where  $C > 0$  does not depend on  $\varepsilon$ .

**Proof:** We use the same rescaling as in Section 2, and define

$$v(x, y) = u_\varepsilon(x, \varepsilon y). \quad (6.4)$$

Then, the fact that  $u_\varepsilon \in H^1(\Omega_\varepsilon)$  implies that  $v \in H^1(\Omega_1)$ . On this fixed domain, we can apply standard trace theorems [13, 16, 20], which imply that there is a constant  $C > 0$  depending on  $\alpha$  only, such that

$$\int_{\partial\Omega_1} v^2 \leq C \left( \int_{\Omega_1} v^2 + \int_{\Omega_1} |\nabla v|^2 \right).$$

Inserting (6.4) into this equation, we find that

$$\begin{aligned} \int_{\Gamma_1^\beta} u_\varepsilon^2 + \int_{\Gamma_1^0} u_\varepsilon(x, \varepsilon y)^2 + \int_{\Gamma_1^1} u_\varepsilon(x, \varepsilon y)^2 + \int_{\Gamma_1^\alpha} u_\varepsilon(x, \varepsilon y)^2 &\leq C \int_{\Omega_1} u_\varepsilon(x, \varepsilon y)^2 \\ &+ C \int_{\Omega_1} \left( \frac{\partial u_\varepsilon}{\partial x}(x, \varepsilon y) \right)^2 + \varepsilon^2 \left( \frac{\partial u_\varepsilon}{\partial y}(x, \varepsilon y) \right)^2 \end{aligned}$$

Hence, changing variables in these integrals, we have

$$\frac{1}{\varepsilon} \int_{\Gamma_\varepsilon^\beta} u_\varepsilon^2 + \frac{1}{\varepsilon} \int_{\Gamma_\varepsilon^0} u_\varepsilon^2 + \frac{1}{\varepsilon} \int_{\Gamma_\varepsilon^1} u_\varepsilon^2 + \int_{\Gamma_\varepsilon^\alpha} u_\varepsilon^2 \leq C \int_{\Omega_\varepsilon} \frac{1}{\varepsilon} u_\varepsilon^2 + \frac{1}{\varepsilon} \left( \frac{\partial u_\varepsilon}{\partial x} \right)^2 + \varepsilon \left( \frac{\partial u_\varepsilon}{\partial y} \right)^2,$$

which proves the result.  $\square$

**Remark 6.6** *In the above proof, we did not use the fact that  $u_\varepsilon$  satisfies (1.1). Hence, the result of Lemma 6.5 is valid for any  $u_\varepsilon \in H^1(\Omega_\varepsilon)$ .*

**Lemma 6.7** *Under the hypotheses of Proposition 6.1, there exists  $u \in L^2([0, T] \times \Omega)$  such that  $\nabla u \in L^2([0, T] \times \Omega^+)$  and  $\nabla u \in L^2([0, T] \times \Omega^-)$ , and the following convergences hold, up to extracting a subsequence:*

$$u_\varepsilon \rightharpoonup u \text{ in } L^2([0, T] \times \Omega), \quad (6.5)$$

$$\nabla u_\varepsilon \rightharpoonup \nabla u \text{ in } L^2([0, T] \times \Omega^+), \quad (6.6)$$

**Proof:** First note that the function  $u_\varepsilon(t=0)$  converges weakly to  $u(t=0)$ . Hence,

$$\|u_\varepsilon(t=0)\|_{L^2(\Omega)} \leq C, \quad (6.7)$$

for some constant  $C > 0$  independent of  $\varepsilon$ . We consider (1.1), multiply it by  $u_\varepsilon$ , and integrate it over  $\Omega_\varepsilon$ :

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega_\varepsilon} u_\varepsilon^2(x, y, t) dx dy - \int_{\Omega_\varepsilon} \Delta u_\varepsilon(x, y, t) u_\varepsilon(x, y, t) dx dy = 0.$$

Using an integration by parts and the boundary conditions in (1.1), we have

$$\begin{aligned} & - \int_{\Omega_\varepsilon} \Delta u_\varepsilon(x, y, t) u_\varepsilon(x, y, t) dx dy = \int_{\Omega_\varepsilon} |\nabla u_\varepsilon(x, y, t)|^2 dx dy - \int_{\partial\Omega_\varepsilon} u_\varepsilon \partial_n u_\varepsilon \\ & = \int_{\Omega_\varepsilon} |\nabla u_\varepsilon(x, y, t)|^2 dx dy - \int_{\Gamma_\varepsilon^1} u_\varepsilon(x, y, t) dy - (\alpha - \beta) \varepsilon \int_{\Gamma_\varepsilon^\alpha} u_\varepsilon(x, y, t) dx - \frac{\beta}{\alpha} \int_{\Gamma_\varepsilon^\beta} u_\varepsilon(x, y, t) dy. \end{aligned}$$

Thus, applying Cauchy-Schwarz inequality,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( \int_{\Omega_\varepsilon} u_\varepsilon^2(x, y, t) \right) + \int_{\Omega_\varepsilon} |\nabla u_\varepsilon(x, y, t)|^2 = \int_{\Gamma_\varepsilon^1} u_\varepsilon + (\alpha - \beta) \varepsilon \int_{\Gamma_\varepsilon^\alpha} u_\varepsilon + \frac{\beta}{\alpha} \int_{\Gamma_\varepsilon^\beta} u_\varepsilon \\ & \leq \sqrt{\varepsilon(1 - \alpha)} \left( \int_{\Gamma_\varepsilon^1} u_\varepsilon^2 \right)^{1/2} + (\alpha - \beta) \varepsilon \sqrt{2} \left( \int_{\Gamma_\varepsilon^\alpha} u_\varepsilon^2 \right)^{1/2} + \beta \sqrt{\frac{\varepsilon}{\alpha}} \left( \int_{\Gamma_\varepsilon^\beta} u_\varepsilon^2 \right)^{1/2} \end{aligned}$$

We then apply Lemma 6.5, finding

$$\frac{1}{2} \frac{d}{dt} \left( \int_{\Omega_\varepsilon} u_\varepsilon^2(x, y, t) \right) + \int_{\Omega_\varepsilon} |\nabla u_\varepsilon(x, y, t)|^2 \leq C \sqrt{\varepsilon} \left( \int_{\Omega_\varepsilon} u_\varepsilon^2 + \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \right)^{1/2} \quad (6.8)$$

for some constant  $C$  depending only on  $\alpha$ .

It is then possible to find a constant  $C'$  such that:

$$\frac{1}{2} \frac{d}{dt} \left( \int_{\Omega_\varepsilon} u_\varepsilon^2(x, y, t) \right) + \int_{\Omega_\varepsilon} |\nabla u_\varepsilon(x, y, t)|^2 \leq C' \varepsilon + \int_{\Omega_\varepsilon} u_\varepsilon^2 + \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2$$

As a consequence, there exists a constant  $C$  (possibly different from the preceding one), for which we have

$$\frac{1}{2} \frac{d}{dt} \left( \int_{\Omega_\varepsilon} u_\varepsilon^2(x, y, t) dx \right) - \int_{\Omega_\varepsilon} u_\varepsilon^2 \leq C\varepsilon.$$

Then, applying Gronwall lemma to  $\int_{\Omega_\varepsilon} u_\varepsilon^2 + C\varepsilon$ , we have:

$$\int_{\Omega_\varepsilon} u_\varepsilon^2(x, y, t) dx \leq 2 \left( \int_{\Omega_\varepsilon} u_\varepsilon^2(x, y, 0) dx + C\varepsilon \right) e^t$$

Then, we split  $\Omega$  into  $1/\varepsilon$  domains of size  $\varepsilon$  in the direction  $y$ , and apply this inequality to each of these domains. Since  $u_\varepsilon = 0$  outside  $\Omega_\varepsilon$ , this immediately implies

$$\int_{\Omega} u_\varepsilon^2(x, y, t) dx dy \leq 2 \left( \int_{\Omega} u_\varepsilon^2(x, y, 0) dx dy + C \right) e^t \leq C e^t,$$

for some constant  $C > 0$ . The last inequality uses (6.7). We next integrate with respect to  $t$ , finding that the sequence  $u_\varepsilon$  is bounded independently of  $\varepsilon$  in  $L^2(\Omega \times [0, T])$ . Hence, up to extracting a subsequence, it converges weakly to some  $u \in L^2(\Omega \times [0, T])$ .

Next, going back to (6.8), and integrating with respect to time, we have

$$\begin{aligned} \int_0^T \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2(x, y, t) dx dy dt &\leq C\varepsilon T + \frac{1}{2} \int_{\Omega_\varepsilon} u_\varepsilon^2(x, y, 0) dx dy + \frac{1}{2} \int_0^T \int_{\Omega_\varepsilon} u_\varepsilon^2(x, y, t) dx dy dt \\ &\leq C\varepsilon T + C \left( \int_{\Omega_\varepsilon} u_\varepsilon^2(x, y, 0) dx dy \right) e^T. \end{aligned} \quad (6.9)$$

This immediately implies that

$$\int_0^T \int_{\Omega^+} |\nabla u_\varepsilon|^2(x, y, t) dx dy dt \leq CT + C \left( \int_{\Omega} u_\varepsilon^2(x, y, 0) dx dy \right) e^T.$$

Hence,  $u_\varepsilon$  is bounded in  $L^2([0, T], H^1(\Omega^+))$ . Extracting a subsequence if necessary, we thus have (6.6).  $\square$

**Lemma 6.8** *Under the hypotheses of Proposition 6.1, there exists a constant  $C > 0$  independent on  $\varepsilon$  and  $T$  such that:*

$$\|u_\varepsilon\|_{L^2([0, T] \times \Gamma_\varepsilon^o)} \leq C e^T \quad (6.10)$$

up to the extraction of a subsequence.

**Proof:** We go back to (6.9), which implies that

$$\int_0^T \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \leq C\varepsilon e^T,$$

where  $C$  does not depend on  $\varepsilon$  nor on  $T$ . Indeed,  $u_\varepsilon(t = 0)$  satisfies (6.7), and is  $\varepsilon$ -periodic with respect to  $y$ . Hence,

$$\int_{\Omega_\varepsilon} u_\varepsilon^2(x, y, 0) dx dy \leq C\varepsilon.$$

Here again, we use the scaling (6.4), namely

$$v_\varepsilon(x, y, t) = u_\varepsilon(x, \varepsilon y, t),$$

and find that

$$\int_0^T \int_{\Omega_1} (\partial_x v_\varepsilon)^2 + \frac{1}{\varepsilon^2} (\partial_y v_\varepsilon)^2 \leq C e^T.$$

In particular,  $v_\varepsilon$  is bounded in  $L^2([0, T], H^1(\Omega_1))$ . Using trace theorems [13, 16, 20], we infer that  $v_\varepsilon$  is bounded in  $L^2([0, T], H^{1/2}(\Gamma_1^\alpha))$ , where

$$\Gamma_1^\alpha = \left\{ \left( x, \frac{\alpha}{2} \right), -1 < x < 0 \right\} \cup \left\{ \left( x, -\frac{\alpha}{2} \right), -1 < x < 0 \right\}.$$

In particular,  $v_\varepsilon$  is bounded in  $L^2([0, T] \times \Gamma_1^\alpha)$ . We finally point out that

$$\|u_\varepsilon\|_{L^2([0, T] \times \Gamma_\varepsilon^\alpha)} = \|v_\varepsilon\|_{L^2([0, T] \times \Gamma_1^\alpha)},$$

which completes the proof.  $\square$

**Lemma 6.9** *Under the hypotheses of Proposition 6.1, we have the following convergences, up to the extraction of a subsequence:*

$$u_\varepsilon \rightharpoonup u \text{ in } L^2([0, T] \times \Gamma^1), \quad (6.11)$$

$$u_\varepsilon \rightharpoonup u \text{ in } L^2([0, T] \times \Gamma^0), \quad (6.12)$$

where  $u$  is defined in Lemma 6.7. Moreover, for any  $\delta \in (0, 1)$ , we have the following convergences, up to the extraction of a subsequence:

$$u_\varepsilon \rightharpoonup u \text{ in } L^2([0, T] \times \Omega \cap \{x = -\delta\}), \quad (6.13)$$

$$u_\varepsilon \rightharpoonup u \text{ in } L^2([0, T] \times \Omega \cap \{x = \delta\}), \quad (6.14)$$

where  $u$  is defined in Lemma 6.7.

**Proof:** We already know that  $u_\varepsilon$  converges weakly to  $u$  in  $L^2([0, T], H^1(\Omega^+))$ . Using trace theorems, we infer that we have weak convergence in  $L^2([0, T], H^{1/2}(\Gamma^0))$ , hence strong convergence in  $L^2([0, T], L^2(\Gamma^0)) = L^2([0, T] \times \Gamma^0)$ . This proves (6.12).

Next, we prove (6.11). In view of (6.3), we already know that  $u_\varepsilon$  is bounded in  $L^2([0, T] \times \Gamma^1)$ , thus it converges weakly, up to extracting a subsequence, to some limit. We are now going to prove that this limit is  $u$ . For this purpose, we use here again the scaling (6.4), namely

$$v_\varepsilon(x, y, t) = u_\varepsilon(x, \varepsilon y, t).$$

We use (6.9), which implies that  $v_\varepsilon$  is bounded in  $L^2([0, T], H^1(\Omega_1))$ . Hence, we have the following convergence:

$$v_\varepsilon \rightharpoonup v \text{ in } L^2([0, T], H^1(\Omega_1)),$$

for some  $v \in L^2([0, T], H^1(\Omega_1))$ . Using the link between  $u_\varepsilon$  and  $v_\varepsilon$ , one easily proves using (6.5), that

$$u(x, y, t) = \int_{-1/2}^{1/2} v(x, z, t) dz.$$

Now, using trace theorems [13, 16, 20], we also have weak convergence of  $v_\varepsilon$  to  $v$  in  $L^2([0, T] \times H^{1/2}(\Gamma_1^1))$ , hence in  $L^2([0, T] \times \Gamma_1^1)$ . Now, let  $\varphi$  be a test function in  $C^\infty([0, T] \times \Gamma^1)$ , and let us compute the integral of  $u_\varepsilon \varphi$  on  $[0, T] \times \Gamma^1$ :

$$\begin{aligned}
\int_0^T \int_{\Gamma^1} u_\varepsilon \varphi &= \sum_{k \in \mathbb{Z}, \varepsilon|k| < 1/2} \int_0^T \int_{\Gamma_\varepsilon^1 + \varepsilon k e_2} u_\varepsilon \varphi \\
&= \sum_{k \in \mathbb{Z}, \varepsilon|k| < 1/2} \int_0^T \int_{-\varepsilon/2}^{\varepsilon/2} u_\varepsilon(-1, y, t) \varphi(-1, y + k\varepsilon, t) dy dt \\
&= \sum_{k \in \mathbb{Z}, \varepsilon|k| < 1/2} \int_0^T \int_{-\varepsilon/2}^{\varepsilon/2} v_\varepsilon\left(-1, \frac{y}{\varepsilon}, t\right) \varphi(-1, y + k\varepsilon, t) dy dt \\
&= \sum_{k \in \mathbb{Z}, \varepsilon|k| < 1/2} \varepsilon \int_0^T \int_{-1/2}^{1/2} v_\varepsilon(-1, z, t) \varphi(-1, \varepsilon z + k\varepsilon, t) dz dt,
\end{aligned}$$

where we have used the fact that  $u_\varepsilon$  is extended by 0 outside  $\Omega_\varepsilon$ , the fact that  $u_\varepsilon$  is  $\varepsilon$ -periodic in  $y$ , and the link between  $u_\varepsilon$  and  $v_\varepsilon$ . Since  $\varphi$  is smooth, one easily proves that

$$\sum_{k \in \mathbb{Z}, \varepsilon|k| < 1/2} \varepsilon \varphi(-1, \varepsilon z + k\varepsilon, t) = \int_{-1/2}^{1/2} \varphi(-1, y, t) dy + O(\varepsilon),$$

where the remainder does not depend on  $z$  nor on  $v_\varepsilon$ . Hence, since  $v_\varepsilon$  converges to  $v$  in  $L^2([0, T] \times \Gamma_1^1)$ , we infer

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Gamma^1} u_\varepsilon \varphi &= \int_0^T \left( \int_{-1/2}^{1/2} v(-1, z, t) dz \int_{-1/2}^{1/2} \varphi(-1, y, t) dy \right) dt \\
&= \int_0^T \int_{-1/2}^{1/2} u(-1, y, t) \varphi(-1, y, t) dy dt.
\end{aligned}$$

We thus have proved (6.11).

The convergence (6.13) follows exactly the same pattern. The proof of (6.14) is a direct consequence of (6.6) and of trace theorems [13, 16, 20].  $\square$

**Lemma 6.10** *Under the hypotheses of Proposition 6.1, we have the following convergence, up to extraction of a subsequence:*

$$\frac{1}{\varepsilon} \int_0^T \int_{\Gamma_\varepsilon^\beta} u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \alpha \int_0^T \int_{\Gamma^\beta} u. \tag{6.15}$$

The function  $u$  is defined in Lemma 6.7, and in the right-hand side,  $u$  is the trace on  $\Gamma_\beta$  of  $u|_{\Omega^+}$ .

**Proof:** We already know that  $u_\varepsilon$  is bounded in  $L^2([0, T], H^1(\Omega^+))$ . Hence, using trace theorems [13, 16, 20],  $u_\varepsilon$  is bounded in  $L^2([0, T], H^{1/2}(\{x=0\} \cap \Omega)) = L^2([0, T], H^{1/2}(\Gamma^\beta))$ . Hence, up to extracting a subsequence,  $u_\varepsilon$  converges in  $L^2([0, T] \times \Gamma^\beta)$ . Since the trace operator is continuous, its limit must be the trace of  $u|_{\Omega^+}$ . Now, since  $u_\varepsilon$  is  $\varepsilon$ -periodic in  $y$ , we have

$$\frac{1}{\varepsilon} \int_0^T \int_{\Gamma_\varepsilon^\beta} u_\varepsilon = \sum_{k \in \mathbb{Z}, \varepsilon|k| < 1/2} \int_0^T \int_{\Gamma_\varepsilon^\beta + k\varepsilon e_2} u_\varepsilon = \int_0^T \int_{\Gamma^\beta} u_\varepsilon \sum_{k \in \mathbb{Z}, \varepsilon|k| < 1/2} \mathbf{1}_{|y - k\varepsilon| < \alpha/2}.$$

In this integral, we have strong convergence on  $u_\varepsilon$ , whereas

$$\sum_{k \in \mathbb{Z}, \varepsilon|k| < 1/2} \mathbf{1}_{|y-k\varepsilon| < \alpha/2} \xrightarrow{\varepsilon \rightarrow 0} \alpha \text{ in } L^2([0, T] \times \Gamma^\beta).$$

Hence, we may pass to the limit and obtain (6.15).  $\square$

## 6.2 Proof of Proposition 6.1

We are now in position to give the

**Proof of Proposition 6.1:** We first apply Lemmas 6.7, 6.8 and 6.9, getting the convergences (6.5), (6.6), (6.10) and (6.11). We next prove that this limit  $u$  is a solution to (4.2)-(4.3)-(4.4). For this purpose, we define

$$\Omega_\varepsilon^- = \Omega_\varepsilon \cap \{x < 0\}, \quad (6.16)$$

and

$$\Omega_\varepsilon^+ = \Omega_\varepsilon \cap \{x > 0\}. \quad (6.17)$$

We assume that  $\varphi \in C^\infty(\Omega \times [0, T])$ , is periodic of period 1 with respect to  $y$  and is such that

$$\forall (x, y) \in \Omega, \varphi(x, y, T) = 0, \quad \text{and} \quad \forall y \in \left(-\frac{1}{2}, \frac{1}{2}\right), \forall t \in [0, T], \varphi(0, y, t) = \partial_x \varphi(0, y, t) = 0.$$

We multiply the equation satisfied by  $u_\varepsilon$  by  $\varphi$ , and integrate over  $\Omega_\varepsilon^+ \times [0, T]$ :

$$\int_0^T \int_{\Omega_\varepsilon^+} \partial_t u_\varepsilon \varphi dx dy dt - \int_0^T \int_{\Omega_\varepsilon^+} \Delta u_\varepsilon \varphi dx dy dt = 0. \quad (6.18)$$

In the first term, we integrate by parts with respect to  $t$ :

$$\begin{aligned} \int_0^T \int_{\Omega_\varepsilon^+} \partial_t u_\varepsilon \varphi dx dy dt &= - \int_0^T \int_{\Omega_\varepsilon^+} u_\varepsilon \partial_t \varphi dx dy dt \\ &\quad - \int_{\Omega_\varepsilon^+} u_\varepsilon(x, y, 0) \varphi(x, y, 0) dx dy \end{aligned} \quad (6.19)$$

Repeating this argument in each set  $\Omega_\varepsilon^+ + k\varepsilon e_2$ , where  $k \in \mathbb{Z}$  and  $\varepsilon|k| < 1/2$ , and using the fact that the union of these sets is  $\Omega^+$  (if  $\varepsilon^{-1}$  is an integer, which we may assume), we have

$$\begin{aligned} \int_0^T \int_{\Omega^+} \partial_t u_\varepsilon \varphi dx dy dt &= - \int_0^T \int_{\Omega^+} u_\varepsilon \partial_t \varphi dx dy dt \\ &\quad - \int_{\Omega^+} u_\varepsilon(x, y, 0) \varphi(x, y, 0) dx dy \end{aligned} \quad (6.20)$$

The weak convergence of  $u_\varepsilon$  in  $L^2(\Omega \times [0, T])$  and of  $u_\varepsilon(t=0)$  in  $L^2(\Omega)$  allows to pass to the limit in the right-hand side, finding

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \sum_{k \in \mathbb{Z}, \varepsilon|k| < \frac{1}{2}} \int_0^T \int_{\Omega_\varepsilon^+ + k\varepsilon e_2} \partial_t u_\varepsilon \varphi dx dy dt &= \\ &= - \int_0^T \int_{\Omega^+} u \partial_t \varphi dx dy dt - \int_{\Omega^+} u(x, y, 0) \varphi(x, y, 0) dx dy \end{aligned} \quad (6.21)$$



The same argument allows to prove this convergence in  $\Omega^-$  (recall that we have extended  $u_\varepsilon$  by 0 outside  $\Omega_\varepsilon$ ):

$$\lim_{\varepsilon \rightarrow 0} \sum_{k \in \mathbb{Z}, \varepsilon|k| < \frac{1}{2}} \int_0^T \int_{\Omega_\varepsilon^- + k\varepsilon e_2} \partial_t u_\varepsilon \varphi dx dy dt = - \int_0^T \int_{\Omega^-} u \partial_t \varphi dx dy dt - \int_{\Omega^-} u(x, y, 0) \varphi(x, y, 0) dx dy \quad (6.22)$$

We next deal with the second term in (6.18), in which we integrate by parts with respect to  $(x, y)$ . Using the boundary conditions we have on  $u_\varepsilon$ , we infer

$$\begin{aligned} \int_0^T \int_{\Omega_\varepsilon^+} \Delta u_\varepsilon \varphi dx dy dt &= \int_0^T \int_{\partial \Omega_\varepsilon^+} \partial_n u_\varepsilon \varphi - \int_0^T \int_{\Omega_\varepsilon^+} \nabla u_\varepsilon \cdot \nabla \varphi \\ &= - \int_0^T \int_{\Omega_\varepsilon^+} \nabla u_\varepsilon \cdot \nabla \varphi. \end{aligned} \quad (6.23)$$

Using (6.6), we thus have

$$\lim_{\varepsilon \rightarrow 0} \sum_{k \in \mathbb{Z}, \varepsilon|k| < \frac{1}{2}} \int_0^T \int_{\Omega_\varepsilon^+ + k\varepsilon e_2} \Delta u_\varepsilon \varphi dx dy dt = - \int_0^T \int_{\Omega^+} \nabla u \cdot \nabla \varphi.$$

This and (6.21) implies

$$- \int_0^T \int_{\Omega^+} u \partial_t \varphi dx dy dt - \int_{\Omega^+} u(x, y, 0) \varphi(x, y, 0) dx dy = - \int_0^T \int_{\Omega^+} \nabla u \cdot \nabla \varphi, \quad (6.24)$$

which is the weak formulation of (4.3).

We are now going to apply the same strategy on the set  $\Omega_\varepsilon^-$ , but the situation is more delicate here. Integrating over  $\Omega_\varepsilon^-$  instead of  $\Omega_\varepsilon^+$ , we already know that (6.22) holds. Moreover, we have

$$\begin{aligned} \int_0^T \int_{\Omega_\varepsilon^-} \Delta u_\varepsilon \varphi dx dy dt &= \int_0^T \int_{\partial \Omega_\varepsilon^-} \partial_n u_\varepsilon \varphi - \int_0^T \int_{\partial \Omega_\varepsilon^-} u_\varepsilon \partial_n \varphi + \int_0^T \int_{\Omega_\varepsilon^-} u_\varepsilon \Delta \varphi \\ &= \int_0^T \int_{\Gamma_\varepsilon^1} \varphi + \int_0^T (\alpha - \beta) \frac{\varepsilon}{2} \int_{\Gamma_\varepsilon^\alpha} \varphi - \int_0^T \int_{\partial \Omega_\varepsilon^-} u_\varepsilon \partial_n \varphi + \int_0^T \int_{\Omega_\varepsilon^-} u_\varepsilon \Delta \varphi. \end{aligned} \quad (6.25)$$

In order to recover an integral over the domain  $\Omega^-$ , we repeat this operation for the domain  $\Omega_\varepsilon^- + \varepsilon k$ , for  $k \in \mathbb{Z}$  and  $\varepsilon|k| \leq 1/2$ . We thus have (6.25) for this domain. Summing these equalities with respect to  $k$ , and setting

$$\tilde{\Omega}_\varepsilon^- = \bigcup_{k \in \mathbb{Z}, \varepsilon|k| < 1/2} (\Omega_\varepsilon^- + \varepsilon k e_2), \quad (6.26)$$

we find

$$\begin{aligned} \int_0^T \int_{\tilde{\Omega}_\varepsilon^-} \Delta u_\varepsilon \varphi dx dy dt &= \sum_{k \in \mathbb{Z}, \varepsilon|k| < 1/2} \int_0^T \int_{\Gamma_\varepsilon^1 + \varepsilon k e_2} \varphi \\ &+ \sum_{k \in \mathbb{Z}, \varepsilon|k| < 1/2} \int_0^T (\alpha - \beta) \frac{\varepsilon}{2} \int_{\Gamma_\varepsilon^\alpha + \varepsilon k e_2} \varphi - \sum_{k \in \mathbb{Z}, \varepsilon|k| < 1/2} \int_0^T \int_{\partial(\Omega_\varepsilon^- + \varepsilon k e_2)} u_\varepsilon \partial_n \varphi \\ &+ \int_0^T \int_{\Omega^-} u_\varepsilon \Delta \varphi. \end{aligned} \quad (6.27)$$

We deal with each term of the right-hand side of (6.27) separately: for the first term, we have

$$\sum_{k \in \mathbb{Z}, \varepsilon|k| < 1/2} \int_0^T \int_{\Gamma_\varepsilon^1 + \varepsilon k e_2} \varphi \xrightarrow{\varepsilon \rightarrow 0} (1 - \alpha) \int_0^T \int_{\Gamma^1} \varphi. \quad (6.28)$$

Next, the second term of the right-hand side of (6.27) is interpreted as a Riemann sum with respect to  $k$  for the map  $y \mapsto \int_{\Gamma_\varepsilon^1 + y e_2} \varphi$ . Since  $\varphi$  is smooth, we thus have

$$\sum_{k \in \mathbb{Z}, \varepsilon|k| < 1/2} \int_0^T (\alpha - \beta) \frac{\varepsilon}{2} \int_{\Gamma_\varepsilon^\alpha + \varepsilon k e_2} \varphi \xrightarrow{\varepsilon \rightarrow 0} (\alpha - \beta) \int_0^T \int_{\Omega^-} \varphi. \quad (6.29)$$

Finally, we deal with the third term of the right-hand side of (6.27). We first note that, due to (6.26), this term is equal to

$$\begin{aligned} \sum_{k \in \mathbb{Z}, \varepsilon|k| < 1/2} \int_0^T \int_{\partial(\Omega_\varepsilon^- + \varepsilon k e_2)} u_\varepsilon \partial_n \varphi &= \int_0^T \int_{\partial\tilde{\Omega}_\varepsilon^-} u_\varepsilon \partial_n \varphi \\ &= - \sum_{\varepsilon|k| < 1/2} \int_0^T \int_{\Gamma_\varepsilon^1 + \varepsilon k e_2} u_\varepsilon \partial_x \varphi + \sum_{\varepsilon|k| < 1/2} \int_0^T \int_{\Gamma_\varepsilon^\alpha + \varepsilon k e_2} u_\varepsilon \partial_n \varphi \end{aligned} \quad (6.30)$$

Here again, each term of the right-hand side of (6.30) is dealt with separately. The first term reads

$$- \sum_{\varepsilon|k| < 1/2} \int_0^T \int_{\Gamma_\varepsilon^1 + \varepsilon k e_2} u_\varepsilon \partial_x \varphi = - \int_0^T \int_{\Gamma^1} u_\varepsilon \partial_x \varphi.$$

Hence, applying (6.11), we find

$$- \sum_{\varepsilon|k| < 1/2} \int_0^T \int_{\Gamma_\varepsilon^1 + \varepsilon k e_2} u_\varepsilon \partial_x \varphi \xrightarrow{\varepsilon \rightarrow 0} - \int_{\Gamma^1} u \partial_x \varphi. \quad (6.31)$$

Turning to the second term of the right-hand side of (6.30), we note that the value of  $u_\varepsilon$  on this boundary does not depend on  $k$  since  $u_\varepsilon$  is periodic of period  $\varepsilon$  with respect to  $y$ . Using:

$$u_\varepsilon(x, \pm\alpha\varepsilon/2 + k\varepsilon, t) = u_\varepsilon(x, \pm\alpha\varepsilon/2, t),$$

we write

$$\begin{aligned} \sum_{\varepsilon|k| < 1/2} \int_0^T \int_{\Gamma_\varepsilon^\alpha + \varepsilon k e_2} u_\varepsilon \partial_n \varphi &= \\ &= \sum_{\varepsilon|k| < 1/2} \int_0^T \int_{-1}^0 u_\varepsilon\left(x, -\frac{\alpha}{2}\varepsilon, t\right) \partial_y \varphi\left(x, \varepsilon k - \frac{\alpha}{2}\varepsilon, t\right) dx dt \\ &\quad - \sum_{\varepsilon|k| < 1/2} \int_0^T \int_{-1}^0 u_\varepsilon\left(x, \frac{\alpha}{2}\varepsilon, t\right) \partial_y \varphi\left(x, \varepsilon k + \frac{\alpha}{2}\varepsilon, t\right) dx dt. \end{aligned}$$

We use a Taylor expansion of  $\varphi$  with respect to the  $y$  variable, around the point  $y = \varepsilon k$ :

$$\partial_y \varphi\left(x, \varepsilon k \pm \frac{\alpha}{2}\varepsilon, t\right) = \partial_y \varphi(x, \varepsilon k, t) \pm \varepsilon \frac{\alpha}{2} \partial_y^2 \varphi(x, \varepsilon k, t) + O(\varepsilon^2).$$

The remainder  $O(\varepsilon^2)$  depends only on  $\varphi$ , and may be chosen bounded independently of  $x$  and  $t$ . Hence, we have

$$\begin{aligned} & \sum_{\varepsilon|k|<1/2} \int_0^T \int_{\Gamma_\varepsilon^\alpha + \varepsilon k e_2} u_\varepsilon \partial_y \varphi = O(\varepsilon) \\ & + \sum_{\varepsilon|k|<1/2} \int_0^T \int_{-1}^0 \left( u_\varepsilon \left( x, -\frac{\alpha}{2} \varepsilon, t \right) - u_\varepsilon \left( x, \frac{\alpha}{2} \varepsilon, t \right) \right) \partial_y \varphi(x, \varepsilon k, t) dx dt \\ & - \varepsilon \frac{\alpha}{2} \sum_{\varepsilon|k|<1/2} \int_0^T \int_{-1}^0 \left( u_\varepsilon \left( x, -\frac{\alpha}{2} \varepsilon, t \right) + u_\varepsilon \left( x, \frac{\alpha}{2} \varepsilon, t \right) \right) \partial_y^2 \varphi(x, \varepsilon k, t) dx dt. \end{aligned}$$

The last term is a Riemann sum with respect to  $k$ . The corresponding integral is

$$\int_{-1/2}^{1/2} \partial_y^2 \varphi(x, y, t) dy = 0,$$

since  $\varphi$  is assumed to be periodic of period 1 with respect to  $y$ .

Hence, using Lemma 6.8, one easily proves that this last term converges to 0. For the first term, we use the same strategy, but we need to prove a little more: we write the sum over  $k$  as follows:

$$\begin{aligned} \sum_{\varepsilon|k|<1/2} \partial_y \varphi(x, \varepsilon k, t) &= \sum_{\varepsilon|k|<1/2} \left( \partial_y \varphi(x, \varepsilon k, t) - \frac{1}{\varepsilon} \int_{\varepsilon k - \varepsilon/2}^{\varepsilon k + \varepsilon/2} \partial_y \varphi(x, y, t) dy \right) \\ &= \sum_{\varepsilon|k|<1/2} \int_{\varepsilon k - \varepsilon/2}^{\varepsilon k + \varepsilon/2} \frac{\partial_y \varphi(x, \varepsilon k, t) - \partial_y \varphi(x, y, t)}{\varepsilon} dy \\ &= \sum_{\varepsilon|k|<1/2} \int_{\varepsilon k - \varepsilon/2}^{\varepsilon k + \varepsilon/2} (y - \varepsilon k) \partial_y^2 \varphi(x, \varepsilon k, t) dy \\ & \quad + \varepsilon \sum_{\varepsilon|k|<1/2} \int_{\varepsilon k - \varepsilon/2}^{\varepsilon k + \varepsilon/2} \frac{1}{2} (y - \varepsilon k)^2 \partial_y^3 \varphi(x, \varepsilon k, t) dy + O(\varepsilon), \end{aligned}$$

where the remainder  $O(\varepsilon)$  is uniform with respect to  $x$  and  $t$ . The first term vanishes because the integrand is even with respect to the variable  $y - \varepsilon k$ . The second term is easily shown to be of order  $\varepsilon$  by computing the integrals explicitly. Finally, we thus have

$$\sum_{\varepsilon|k|<1/2} \int_0^T \int_{\Gamma_\varepsilon^\alpha + \varepsilon k e_2} u_\varepsilon \partial_n \varphi \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (6.32)$$

Inserting (6.31) and (6.32) into (6.30), we infer

$$\sum_{k \in \mathbb{Z}, \varepsilon|k|<1/2} \int_0^T \int_{\partial(\Omega_\varepsilon + \varepsilon k e_2)} u_\varepsilon \partial_n \varphi \xrightarrow{\varepsilon \rightarrow 0} - \int_{\Gamma^1} u \partial_x \varphi. \quad (6.33)$$

Then, we collect (6.28), (6.29), (6.33), and insert them into (6.27). Hence,

$$\int_0^T \int_{\tilde{\Omega}_\varepsilon^-} \Delta u_\varepsilon \varphi dx dy dt \xrightarrow{\varepsilon \rightarrow 0} (1 - \alpha) \int_0^T \int_{\Gamma^1} \varphi + (\alpha - \beta) \int_0^T \int_{\Omega^-} \varphi - \int_{\Gamma^1} u \partial_x \varphi - \int_0^T \int_{\Omega^-} u \Delta \varphi.$$

Finally, we use this convergence and (6.22), and insert it into the equation

$$\int_0^T \int_{\Omega_\varepsilon^-} \partial_t u_\varepsilon \varphi dx dy dt - \int_0^T \int_{\Omega_\varepsilon^-} \Delta u_\varepsilon \varphi dx dy dt = 0,$$

finding

$$\begin{aligned} - \int_0^T \int_{\Omega^-} u \partial_t \varphi dx dy dt - \int_{\Omega^-} u(x, y, 0) \varphi(x, y, 0) dx dy dt = \\ (1 - \alpha) \int_0^T \int_{\Gamma^1} \varphi + (\alpha - \beta) \int_0^T \int_{\Omega^-} \varphi + \int_0^T \int_{\Omega^-} \nabla u \cdot \nabla \varphi, \end{aligned} \quad (6.34)$$

which is a weak formulation of (4.2).

To end the proof, we need to show that the transmission conditions (4.4) hold. For this purpose, we first point out that, since  $u_\varepsilon \rightharpoonup u$  in  $L^2([0, T] \times \Omega)$  and since  $u_\varepsilon$  is  $\varepsilon$ -periodic in  $y$ ,  $u$  must be independent of  $y$ . Next, we define a test function  $\varphi$  which depends only on  $t$ , and has compact support in  $(0, T)$ . We multiply the first line of (1.1) by  $\varphi$  and integrate over  $[0, T] \times \tilde{\Omega}_{\varepsilon, \delta}$ , where

$$\tilde{\Omega}_{\varepsilon, \delta} = \tilde{\Omega}_\varepsilon \cap \{|x| < \delta\}.$$

(Recall that  $\tilde{\Omega}_\varepsilon$  is defined by (6.26).) Integrating by parts, we have

$$\begin{aligned} 0 &= \int_0^T \int_{\tilde{\Omega}_{\varepsilon, \delta}} (\partial_t u_\varepsilon - \Delta u_\varepsilon) \varphi(t) dx dy dt \\ &= - \int_0^T \int_{\tilde{\Omega}_{\varepsilon, \delta}} u_\varepsilon \partial_t \varphi - \int_0^T \int_{\partial(\tilde{\Omega}_{\varepsilon, \delta})} \partial_n u_\varepsilon \varphi \end{aligned}$$

Computing the boundary term, we get

$$\begin{aligned} 0 &= - \int_0^T \int_{\tilde{\Omega}_{\varepsilon, \delta}} u_\varepsilon \partial_t \varphi - \int_0^T \int_{\Omega \cap \{x=\delta\}} \partial_x u_\varepsilon \varphi + \int_0^T \int_{\tilde{\Omega}_\varepsilon \cap \{x=-\delta\}} \partial_x u_\varepsilon \varphi \\ &\quad - (\alpha - \beta) \frac{\varepsilon}{2} \sum_{k \in \mathbb{Z}, \varepsilon|k| < 1/2} \int_0^T \int_{-\delta}^0 2\varphi(t) dx dt \\ &\quad - \frac{\beta}{\alpha} \sum_{k \in \mathbb{Z}, \varepsilon|k| < 1/2} \int_0^T \int_{\Gamma_\varepsilon^\beta + k\varepsilon e_2} \varphi. \end{aligned} \quad (6.35)$$

It is easy to pass to the limit in each of the above terms, except for the second and third one. We thus deal with them separately: we multiply the equation by  $\varphi$  and integrate on  $\tilde{\Omega}_\varepsilon \cap \{x < -\delta\}$  instead of  $\tilde{\Omega}_{\varepsilon, \delta}$ . Integrating by parts, we find

$$\begin{aligned} 0 &= - \int_0^T \int_{\tilde{\Omega}_\varepsilon \cap \{x < -\delta\}} u_\varepsilon \partial_t \varphi - \int_0^T \int_{\tilde{\Omega}_\varepsilon \cap \{x=-\delta\}} \partial_x u_\varepsilon \varphi \\ &\quad - (\alpha - \beta) \frac{\varepsilon}{2} \sum_{k \in \mathbb{Z}, \varepsilon|k| < 1/2} \int_0^T \int_{-1}^{-\delta} 2\varphi(t) dx dt. \end{aligned}$$

Passing to the limit in the first and third term, we find that

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\tilde{\Omega}_\varepsilon \cap \{x=-\delta\}} \partial_x u_\varepsilon \varphi = - \int_0^T \int_{\tilde{\Omega}_\varepsilon \cap \{x < -\delta\}} u \partial_t \varphi + (\alpha - \beta) \int_0^T \int_{\Omega \cap \{x < -\delta\}} \varphi.$$

Now, we go back to the equation verified by  $u$ , namely (4.2). We multiply it by  $\varphi$  and integrate by parts. We infer

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\tilde{\Omega}_\varepsilon \cap \{x = -\delta\}} \partial_x u_\varepsilon \varphi = \int_0^T \int_{\Omega \cap \{x = -\delta\}} \partial_x u \varphi. \quad (6.36)$$

Likewise, integrating over  $\tilde{\Omega}_\varepsilon \cap \{x > \delta\}$ , we have

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\tilde{\Omega}_\varepsilon \cap \{x = \delta\}} \partial_x u_\varepsilon \varphi = \int_0^T \int_{\Omega \cap \{x = \delta\}} \partial_x u \varphi. \quad (6.37)$$

We insert (6.36) and (6.37) into (6.35), and interpreting the last term of (6.35) as Riemann sum, we get

$$\begin{aligned} 0 = & - \int_0^T \int_{\Omega \cap \{|x| < \delta\}} u \partial_t \varphi - \int_0^T \int_{\Omega \cap \{x = \delta\}} \partial_x u \varphi + \int_0^T \int_{\Omega \cap \{x = -\delta\}} \partial_x u \varphi \\ & - (\alpha - \beta) \delta \int_0^T \varphi(t) dt - \beta \int_0^T \int_{\Gamma^\beta} \varphi. \end{aligned}$$

Letting  $\delta \rightarrow 0$ , and using the fact that  $u$  does not depend on  $y$ , we thus find

$$\partial_x u^- = \partial_x u^+ + \beta \quad \text{if } x = 0. \quad (6.38)$$

Then, we repeat the same argument with  $x\varphi(t)$  instead of  $\varphi(t)$ , and we find

$$0 = - \int_0^T \int_{\Omega \cap \{|x| < \delta\}} x u_\varepsilon \partial_t \varphi + \int_0^T \int_{\partial(\tilde{\Omega}_\varepsilon, \delta)} u_\varepsilon \partial_n (x\varphi) - x\varphi \partial_n u_\varepsilon.$$

Since  $x\varphi$  does not depend on  $y$ , the first boundary term contains only terms on the vertical boundaries  $\{x = \delta\}$ ,  $\tilde{\Omega}_\varepsilon \cap \{x = -\delta\}$ , and  $\Gamma_\varepsilon^\beta + k\varepsilon e_2$ , for  $k \in \mathbb{Z}$ ,  $\varepsilon|k| < 1/2$ . We thus have

$$\begin{aligned} 0 = & - \int_0^T \int_{\Omega \cap \{|x| < \delta\}} x u_\varepsilon \partial_t \varphi \\ & + \int_0^T \int_{\Omega \cap \{x = \delta\}} u_\varepsilon \varphi - \int_0^T \int_{\tilde{\Omega}_\varepsilon \cap \{x = -\delta\}} u_\varepsilon \varphi - \sum_{k \in \mathbb{Z}, \varepsilon|k| < 1/2} \int_0^T \int_{\Gamma_\varepsilon^\beta + k\varepsilon e_2} u_\varepsilon \varphi \\ & - \int_0^T \int_{\Omega \cap \{x = \delta\}} \partial_x u_\varepsilon x \varphi + \int_0^T \int_{\tilde{\Omega}_\varepsilon \cap \{x = -\delta\}} \partial_x u_\varepsilon x \varphi \\ & - (\alpha - \beta) \frac{\varepsilon}{2} \sum_{k \in \mathbb{Z}, \varepsilon|k| < 1/2} \int_0^T \int_{-\delta}^0 2x\varphi(t) dx dt \end{aligned}$$

Here again, we use (6.13), (6.14), (6.36) and (6.37), to pass to the limit in the above equation, finding

$$\begin{aligned} 0 = & - \int_0^T \int_{\Omega \cap \{|x| < \delta\}} x u \partial_t \varphi + \int_0^T \int_{\Omega \cap \{x = \delta\}} u \varphi - \int_0^T \int_{\Omega \cap \{x = -\delta\}} u \varphi - \alpha \int_0^T \int_{\Gamma^\beta} u^+ \varphi \\ & - \int_0^T \int_{\Omega \cap \{x = \delta\}} \partial_x u x \varphi + \int_0^T \int_{\tilde{\Omega}_\varepsilon \cap \{x = -\delta\}} \partial_x u_\varepsilon x \varphi - (\alpha - \beta) \frac{\delta^2}{2} \int_0^T \varphi \end{aligned}$$

Hence, letting  $\delta \rightarrow 0$ , we have

$$u^+ - u^- - \alpha u^+ = 0 \quad \text{if } x = 0. \quad (6.39)$$

Collecting (6.38) and (6.39), we find (4.4).

Thus, we have proved that, up to extracting a subsequence, the sequence  $u_\varepsilon$  converges to  $u$ , weakly in  $L^2$ , and that  $u \in H^1(\Omega)$  is solution to (4.2)-(4.3)-(4.4). Next, we point out that, according to Lemma 5.1, such a solution is unique. Hence, the whole sequence converges, without any need to extract a subsequence.  $\square$

## 7 A numerical illustration

In this section, we consider a numerical application of MOSAIC on a given crack geometry, which is represented on figure 1. We can thus compare the solution of the diffusion problem (1.1) solved in the cracked domain  $\Omega_\varepsilon$  with the solution of the homogenized model (4.2)-(4.3) solved in the average domain  $\Omega$ . This domain represents the limit of  $\Omega_\varepsilon$  as  $\varepsilon \rightarrow 0$ . Note that  $\Omega$  does not contain any crack and is invariant in the  $y$  direction. Its length is 2 in the  $x$  direction. It is thus of the same length as the cracked domain  $\Omega_\varepsilon$ .

To calculate the solution of the homogenized problem, we actually use two independent methods that we describe in details:

- a weak formulation in the whole homogenized model  $\Omega$ .
- a fixed-point approach connecting the 2 sub-domains  $\Omega^+$  and  $\Omega^-$ .

Computations presented here are performed using a  $P^1$  finite element approximation on triangular meshes. It has been implemented using the software FreeFem++ [17].

### 7.1 Direct simulation of the cracked domain

We start by solving the problem corresponding to the diffusion model (1.1) on the cracked domain  $\Omega_\varepsilon$  (see figure 1). We recall that the domain  $\Omega_\varepsilon$  is periodic of period  $\varepsilon$  with respect to the  $y$  direction. The width of the crack  $\alpha$  is fixed: we take  $\alpha = 0.1$  for the simulations corresponding to figure 3. In what follows, we always take  $\beta = 0$ . We represent the field  $u(x, y, t_1)$  solution of (1.1) at a given time  $t_1 = 0.5$  for different periods  $\varepsilon$ . The time  $t_1$  is such that the normalized spatial profile of the field has reached a stationary state. We carry out direct calculations for the periods  $\varepsilon = 1$ ,  $\varepsilon = 0.5$ ,  $\varepsilon = 0.2$ ,  $\varepsilon = 0.02$ . We make sure the simulations have reached convergence with respect to the mesh size as well as the time step. The fields calculated for each period  $\varepsilon$  are shown on figure 3a. We note that the convergence in  $\varepsilon$  is quite fast: for  $\varepsilon \leq 0.2$ , the field does not depend on the variable  $y$  anymore and the limit in  $\varepsilon$  appears to be reached. This is not true for  $\varepsilon = 1$  where the field still depends on  $y$ , especially in the vicinity of the crack. Besides, since  $\alpha$  is rather small here, we make use of remark 4.1 and compare the field solution of the direct calculation with that of the approximated homogenized model (4.5). Equation (4.5) is solved in the domain  $\Omega$  where the crack is not described but its effect on the diffusion process is modeled by a source term of the form  $\alpha \mathbf{1}_{\{x < 0\}}$  in the right hand side of equation (4.5). The incoming flux applied on the left boundary located at  $x = -1$  is now  $1 - \alpha$  (see figure 3b). It is consistent with the fact that the indentation of width  $\alpha$  is no more described in the homogenized domain  $\Omega$ . A fraction of the incoming flux is thus "converted" into a source term in the homogenized equation. This accounts for the name given to our approach: MOSAIC, as Model Of Sinks

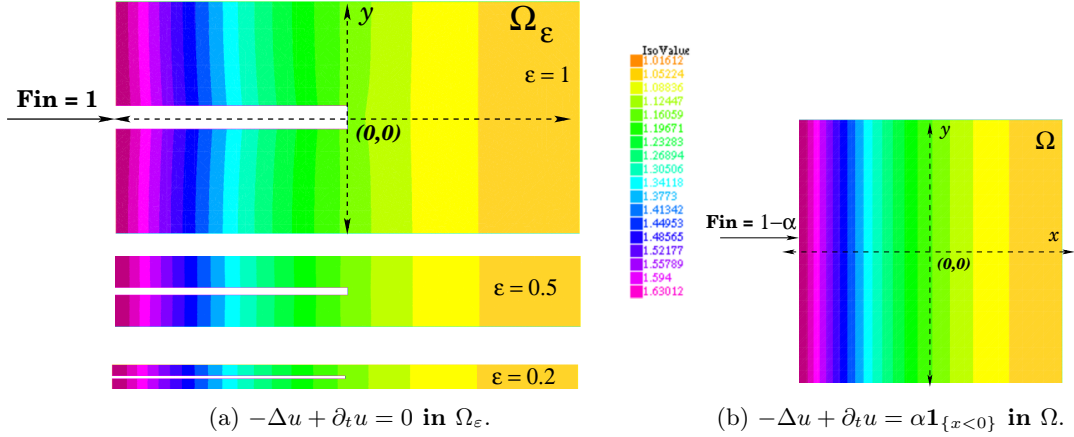


Figure 3: **Direct calculation on the periodic cracked domain  $\Omega_\varepsilon$  for different periods  $\varepsilon \rightarrow 0$ .** The crack occupies the fraction  $\alpha = 0.1$ . The field solution of (1.1) is plotted at time  $t_1 = 0.5$ . We compare the result with that of the homogenized approach approximated by equation (4.5) where the singularity at  $x = 0$  is disregarded. This approximation makes sense here because  $\alpha$  is rather small.

Averaging Inhomogeneous behavior of Cracked media. Note that the homogenized model (4.5) does not take the singularity at  $x = 0$  into account, which should play a significant role as  $\alpha$  increases. We are now going to devise two approaches to treat this singularity in the homogenized model for any given  $\alpha$ .

## 7.2 Weak formulation of the homogenized model

The first strategy to compute the solution of (4.2)-(4.3) is to find an equivalent weak formulation of the problem and then solve it numerically using standard finite element methods.

We recall the equations that have to be solved:

$$\begin{cases} -\Delta u + \partial_t u = \alpha - \beta & \text{in } \Omega^- = \{-1 < x < 0\}, \\ \partial_n u = 1 - \alpha & \text{on } \{x = -1\}, \\ u \text{ is } 1\text{-periodic in } y. \end{cases} \quad (7.1)$$

$$\begin{cases} -\Delta u + \partial_t u = 0 & \text{in } \Omega^+ = \{0 < x < 1\}, \\ \partial_n u = 0 & \text{on } \{x = 1\}, \\ u \text{ is } 1\text{-periodic in } y. \end{cases} \quad (7.2)$$

with the so-called transmission conditions at the bottom of the crack located at  $x = 0$ :

$$u(x = 0^-) = (1 - \alpha)u(x = 0^+), \quad \partial_x u(x = 0^-) = \partial_x u(x = 0^+) + \beta. \quad (7.3)$$

Multiplying (7.1) by a test function  $v$  and integrating by parts on the domain  $\Omega^-$ , we get:

$$\int_{\Omega^-} \partial_t u v - \Delta v u + \int_{\partial\Omega^-} u \partial_n v - v \partial_n u = (\alpha - \beta) \int_{\Omega^-} v.$$

Using the boundary conditions on  $\Omega^-$ , this becomes:

$$\begin{aligned} \int_{\Omega^-} \partial_t uv - \Delta vu + \int_{x=-1} u \partial_n v - v(1-\alpha) + \int_{x=0} u(x=0^-) \partial_x v - v \partial_x u(0^-) \\ = (\alpha - \beta) \int_{\Omega^-} v. \end{aligned} \quad (7.4)$$

Then, applying the same processes (7.2) and integrating on  $\Omega^+$ , we obtain :

$$\int_{\Omega^+} \partial_t uv - \Delta vu + \int_{\partial\Omega^+} u \partial_n v - v \partial_n u = 0.$$

The boundary conditions lead to:

$$\int_{\Omega^+} \partial_t uv - \Delta vu - \int_{x=1} u \partial_x v + \int_{x=0} \partial_x u(0^+) v - u(x=0^+) \partial_x v = 0. \quad (7.5)$$

We sum (7.4) and (7.5) :

$$\begin{aligned} \int_{\Omega} \partial_t uv - \Delta vu - \int_{x=1} u \partial_x v + \int_{x=-1} u \partial_n v - v(1-\alpha) \\ + \int_{x=0} v (\partial_x u(0^+) - \partial_x u(0^-)) + (u(x=0^-) - u(x=0^+)) \partial_x v = (\alpha - \beta) \int_{\Omega^-} v. \end{aligned}$$

We now make use of the transmission conditions (7.3) at  $x = 0$ , leading to

$$\begin{aligned} \int_{\Omega} \partial_t uv - \Delta vu - \int_{x=1} u \partial_x v + \int_{x=-1} u \partial_n v - v(1-\alpha) \\ - \int_{x=0} \beta v - \int_{x=0} \alpha u(x=0^+) \partial_x v = (\alpha - \beta) \int_{\Omega^-} v. \end{aligned}$$

Integrating by parts again, we finally obtain:

$$\begin{aligned} \int_{\Omega} \partial_t uv + \nabla v \nabla u - \int_{x=-1} v(1-\alpha) \\ - \int_{x=0} \beta v - \int_{x=0} \alpha u(x=0^+) \partial_x v - (\alpha - \beta) \int_{\Omega^-} v = 0, \end{aligned} \quad (7.6)$$

which is the weak formulation of our problem.

The corresponding equation is:

$$\begin{cases} -\Delta u + \partial_t u = (\alpha - \beta) \mathbf{1}_{x < 0} \\ -\alpha \partial_x (u(x=0^+) \delta_{x=0}) + \beta \delta_{x=0} & \text{in } \Omega = \{-1 < x < 1\}, \\ \partial_n u = 1 - \alpha & \text{on } \{x = -1\}, \\ \partial_n u = 0 & \text{on } \{x = -1\}, \\ u \text{ is } 1\text{-periodic in } y. \end{cases} \quad (7.7)$$

**Remark 7.1** Taking  $v = 1$  in (7.6) leads to the energy conservation equation:

$$\frac{d}{dt} \int_{\Omega} u = 1.$$



We compute the solution of (7.6) using the numerical approximation of the Dirac mass at  $x = 0$ :

$$\int_{x=0} \alpha u(x = 0^+) \partial_x v \approx \int_{\Omega} \frac{\alpha}{\delta} \mathbf{1}_{0 < x < \delta} u \partial_x v,$$

for  $\delta$  arbitrary small. Note that this approximation may not be satisfactory as  $\alpha$  increases. Taking a piecewise linear approximation of the Dirac mass around  $x = 0$  may improve the numerical estimate. This approach nevertheless diverges from the direct calculation for bigger  $\alpha$  (see figure 5), especially in the vicinity of the crack interface at  $x = 0$ .

### 7.3 Fixed-point approach

Another numerical approach to solve (4.2)-(4.3) that might be more accurate is to use the fixed-point approach, in the spirit of section 5 and remark 5.2. We apply the following iterative process:

- Starting with an initial guess  $F^0$  corresponding the flux imposed on  $\Gamma^\beta$ , we solve the problem (5.8), and get the unique solution  $u^{+,0}$ .
- Then, we compute the trace  $g^0$  of  $(1 - \alpha)u^{+,0}$  on  $\Gamma^\beta$ , and solve (5.7) with the data  $g = g^0$ . This gives us the unique solution  $u^{-,0}$ .
- We compute a new flux on  $\Gamma^\beta$   $F^1 = \beta - \partial_n u^{-,0}$  and we repeat the first step with  $F^1$ .

This procedure builds a converging sequence  $(F^n)_{n \in \mathbb{N}}$  in  $L^2(\Gamma^\beta)$ , together with the corresponding solutions  $u^{\pm,n}$  and the data  $g^n$ . We assess the convergence of  $(F^n)_{n \in \mathbb{N}}$  when the relative change between two successive iterations  $\left\| \frac{F^{n+1} - F^n}{F^n} \right\|$  is smaller than a small fixed parameter. This has to be carried out at each time step of the simulation.

### 7.4 Synthesis

We thus compute the results given by each method:

- direct calculation on the cracked domain  $\Omega_\varepsilon$ ,
- weak formulation on the homogenized domain  $\Omega$ ,
- fixed point approach on  $\Omega$ ,

for two crack configurations:  $\alpha = 0.1$  (small crack) and  $\alpha = 0.6$  (big crack). We fix  $\beta = 0$  in the simulations for the sake of simplicity. We compare the different simulations by plotting the time evolution of the solution at  $x = 0.5$  (that is to say, in the core of the intact part of the cracked material) and the spatial profile of the solution at final time  $t_1 = 0.5$ . Results are shown in figures 4 and 5. We can note that the fixed-point method appears to be more accurate than the approach based on the weak formulation (7.6) especially as  $\alpha$  increases. This is due to the fact that (7.6) involves a Dirac mass at  $x = 0$ , proportional to the width of the crack  $\alpha$ . The corresponding term is only approximately calculated in our finite element simulations and leads to more significant errors for greater  $\alpha$ . In some way, the fixed-point method amounts to treat the Dirac mass at  $x = 0$  exactly.

We conclude this numerical illustration by studying the error associated with the homogenized model (4.2)-(4.3) with respect to the period  $\varepsilon$  of the cracked domain.

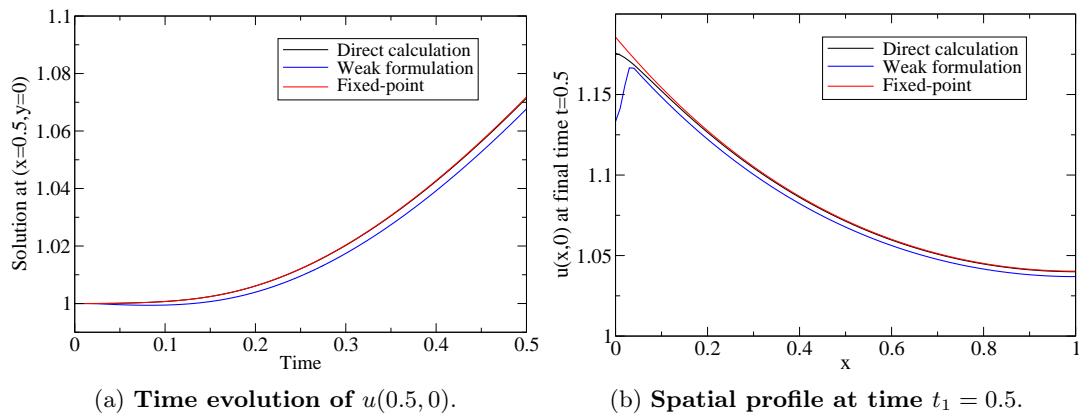


Figure 4: **Comparison of the direct and homogenized approaches for  $\alpha = 0.1$ .** The black curve is the direct calculation of the crack, i.e the limit solution of (1.1) as  $\varepsilon \rightarrow 0$ . The two other curves correspond to the solution of the homogenized problem (4.2)-(4.3) computed by two approaches: in red, it is the fixed-point method whereas the blue curve represents the numerical solution of the weak formulation (7.6).

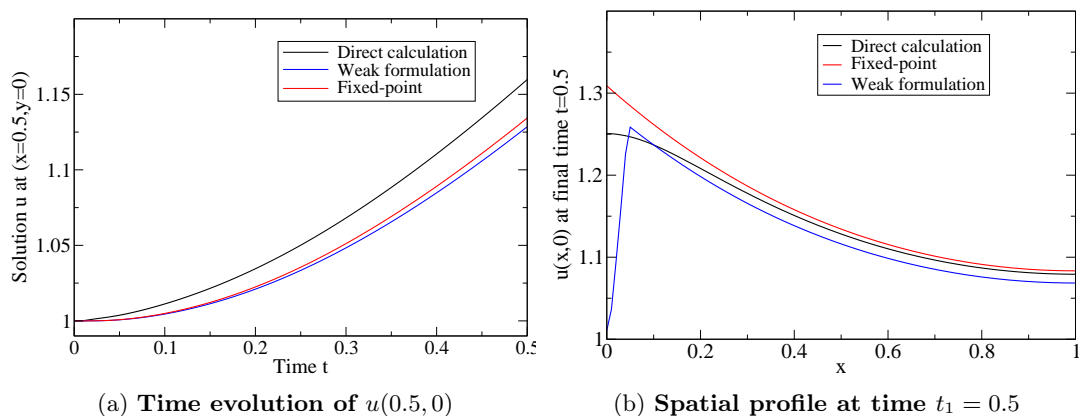


Figure 5: **Comparison of the direct and homogenized approaches for  $\alpha = 0.6$ .** We note that when we get close to  $x = 0$ , i.e the bottom of the crack, the agreement between the different approaches is not so good. This is due to the singularity represented by a Dirac mass at  $x = 0$  in the weak formulation of the homogenized problem.

More precisely, we calculate the error defined by:

$$\text{err} = \frac{\|\tilde{u}_\varepsilon - u_\varepsilon\|_{L^2(\Omega_\varepsilon^+)}}{\|u_\varepsilon\|_{L^2(\Omega_\varepsilon^+)}} \tag{7.8}$$

where  $u_\varepsilon$  is the solution of the exact model (1.1) in the cracked domain  $\Omega_\varepsilon$  and  $\tilde{u}_\varepsilon$  is the projection of the solution  $u$  of the homogenized model (4.2)-(4.3) on the sub-domain  $\Omega_\varepsilon^+$ . The error is plotted as a function of  $\varepsilon$  on figure (6). It shows that the error depends linearly on the period  $\varepsilon$ . Besides, as  $\varepsilon \rightarrow 0$ , the error tends to the residual error linked to the mesh size used in the finite element calculations.

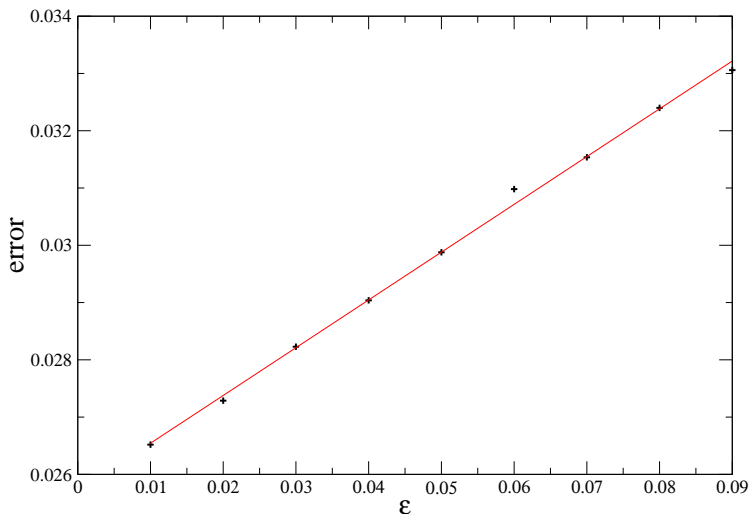


Figure 6: The error associated with the homogenized model as a function of  $\varepsilon$ . The error depends linearly on the period  $\varepsilon$ . As  $\varepsilon \rightarrow 0$ , the error tends to the residual error linked to the mesh size used in the finite element calculations,  $\sim 0.02$  here.

## 8 Conclusion and future work

We have shown that a diffusion process initiated by an incoming flux through a periodic cracked medium can be modeled by a volume source term in the diffusion equation solved in the homogenized domain. The crack induces also a singularity at  $x = 0$  giving rise to some complications in the formulation of the homogenized problem. We have introduced a boundary layer around  $x = 0$  to treat it properly. This leads to a Dirac mass located at  $x = 0$  in the weak formulation of the homogenized problem. Note that this singularity might be interpreted physically by considering that the temperature field in the vicinity of the crack is well described by a point source located at  $x = 0$ . This remains true as long as we observe the temperature sufficiently far from the crack, where we are not too sensitive to the details of the fracture profile. This can be shown rigorously in a very particular configuration for which the shape of the fracture is smooth enough to be described analytically using conformal mappings [8, 9].

Besides, we have developed our method in a very particular setting, where the cracks are supposed to be orthogonal to the surface of the material and periodically arranged. This is an elementary case on which we have tested our approach. We may now have in view to address much more general situations on which MOSAIC could be applied. For example, we may consider the case where  $\alpha$  becomes a **stochastic** variable. The periodicity of the structure may then disappear. It would be interesting to see how the model proved on a simple geometry may be extended to more intricate stochastic configurations. Even more generally, the homogenization process could be studied in a fractal cracked medium, such as the one given in [1].

Another extension of the present work may be to consider that the cracked material is not homogeneous but filled with radiation-free micro-cracks (see figure 7 for an illustration). Provided the size of such micro-cracks is much smaller than  $\varepsilon$ , the effective behavior of the micro-cracked medium can be estimated - and in some cases rigorously bounded - using general methods from the theory of composites materials, possibly extended in a non-linear regime ([28], [27], [26]). Those methods make use of the so-called "translation method" devised by Murat and Tartar [23]. Combining such approaches with the MOSAIC method is the object of ongoing work [25].

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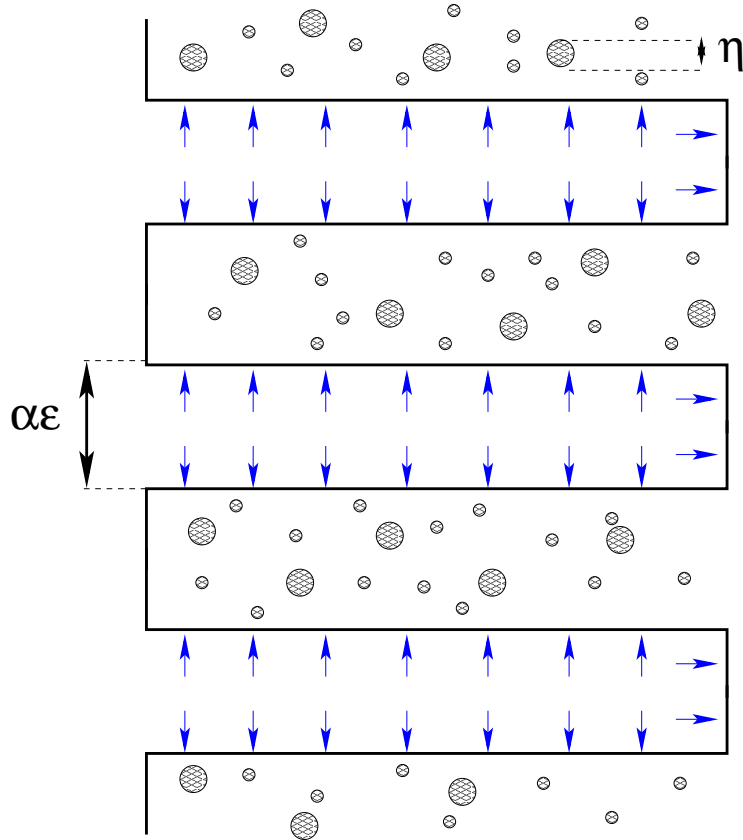


Figure 7: An example of a cracked heterogeneous medium. *The medium is characterized by two length scales:  $\alpha\varepsilon$  corresponding to the width of the fractures on which an incoming flux is imposed (symbolized by the blue arrows), and  $\eta \ll \varepsilon$  which is the scale of smaller heterogeneities. The fractures of width  $\alpha\varepsilon$  can be treated by the MOSAIC approach whereas the smaller structures may be described by an effective approach.*

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