A Compactly Generated Pseudo-Group which is Not Realizable

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A COMPACTLY GENERATED PSEUDOGROUP WHICH IS NOT REALIZABLE

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ABSTRACT. We exhibit a pseudogroup of smooth local transformations of the real line, which is compactly generated, but not realizable as the holonomy pseudogroup of a foliation of codimension 1 on a compact manifold. The proof relies on a description of all foliations with the same dynamic as the Reeb component.

1. INTRODUCTION

To every foliated manifold $(M, \mathcal{F})$ of arbitrary dimension and codimension, one associates, following Ehresmann, a *pseudogroup* $Hol(\mathcal{F})$ of local transformations, called the *holonomy pseudogroup,* that represents its "dynamic" or "transverse structure". The holonomy pseudogroup is well-defined up to some natural equivalence between pseudogroups: *Haefliger equivalence.*

The inverse *realization* problem has been raised: make a foliation with prescribed dynamic, the ambiant manifold not being prescribed (but it must be compact.) More precisely, given a pseudogroup G, make if possible a *compact* foliated manifold $(M, \mathcal{F})$ such that $Hol(\mathcal{F})$ is Haefliger-equivalent to G .

Note that if one drops the compactness condition, the question vanishes : every pseudogroup is easily seen to represent the dynamic of some foliated *open* manifold.

André Haefliger made the realization problem precise by exhibiting a necessary condition, *compact generation.* Is it sufficient? Some partial positive answers have been given for rather rigid species of pseudogroups [1][3][6][7].

The object of this paper is to answer negatively in general. We give a counterexample among pseudogroups of smooth local transformations of the real line.

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2. Pseudogroups

We recall briefly the definitions and the basic properties, now classical. See also [5].

An arbitrary differentiability class is understood. Let T be a manifold, not necessarily compact. A boundary is allowed.

A *local transformation* of T is a diffeomorphism between two nonempty open subsets $Dom(\gamma)$, $Im(\gamma)$ of T . The compose $\gamma'\gamma$ is defined whenever $Im(\gamma)$ meets $Dom(\gamma')$, and its domain is $\gamma^{-1}(Dom(\gamma'))$. If A , B denote two sets of local transformations, then as usual AB denotes the set of their composes. If $U \subset T$, then $A|U$ denotes the set of the elements of A whose domains and images are both contained in U .

Note that ∂T is necessarily preserved by every local transformation.

Definition 2.1. [9] *A* pseudogroup *on T is a set G of local transformations such that :*

1. *For every nonempty open $U \subset T$ the identity map 1_U belongs to G ;*
2. *$GG = G^{-1} = G$;*
3. *For every local transformation γ of T , if $Dom(\gamma)$ admits an open cover (U_i) such that every restriction $\gamma|U_i$ belongs to G , then γ belongs to G .*

Then, by (1) and (2), G is also stable by restrictions: if γ belongs to G and if $U \subset Dom(\gamma)$ is nonempty open, then $\gamma|U$ belongs to G .

For example, every set S of local transformations of T is contained in a smallest pseudogroup $< S >$ containing S . Call $< S >$ the pseudogroup *generated* by S .

Every point t in T has under a pseudogroup G :

1. An *orbit*: the set of the images $\gamma(t)$ through the local transformations $\gamma \in G$ defined at t ;
2. An *isotropy group*: the group of the germs at t of the local transformations $\gamma \in G$ defined at t and fixing t .

Let $(M, \mathcal{F})$ be a manifold foliated in codimension q . By a *transversal* one means a q-manifold T immersed into M transversely to $\mathcal{F}$, not necessarily compact, and such that $\partial T = T \cap \partial M$. One calls T *exhaustive* (or *total*) if it meets every leaf.

Definition 2.2. [2] *The holonomy pseudogroup $Hol(\mathcal{F}, T)$ of a foliation $\mathcal{F}$ on an exhaustive transversal T is the pseudogroup* generated *by the local transformations of T of the form $f(x, 0) \mapsto f(x, 1)$, where :*

1. *D^q is the open q-disk ;*
2. *$f : D^q \times [0, 1] \to M$ is a map transverse to $\mathcal{F}$;*
3. *$f^*\mathcal{F}$ is the foliation on $D^q \times [0, 1]$ by the first projection;*
4. *f embeds $D^q \times 0$ and $D^q \times 1$ into T .*

This holonomy pseudogroup does represent the dynamic of the foliation in the sense that there is a one-to-one correspondance $L \mapsto L \cap T$ between the leaves of $\mathcal{F}$ and the orbits of $Hol(\mathcal{F}, T)$. A topologically closed orbit corresponds to a closed leaf. The isotropy group of $Hol(\mathcal{F}, T)$ at any point is isomorphic with the holonomy group of the corresponding leaf. Etc.

DEFINITION 2.3. [3] *A* Haefliger equivalence *between two pseudogroups* (T_i, G_i) *(*$i = 0, 1$*) is a pseudogroup* G *on the disjoint union of* T_0 *with* T_1 *, such that* $G|T_i = G_i$ *(*$i = 0, 1$*) and that no orbit of* G *in entirely contained in* T_0 *or in* T_1 *.*

For example, obviously, the two holonomy pseudogroups of a same foliation on two exhaustive transversals are Haefliger equivalent.

A Haefliger equivalence between (T_1, G_1) and (T_2, G_2) induces a one-to-one correspondance between the orbit spaces T_i/G_i $(i = 0, 1)$; a closed orbit corresponds to a closed orbit; the isotropy groups at points on corresponding orbits are isomorphic; etc.

Let (T, G) be a pseudogroup. Call a subset $T' \subset T$ *exhaustive* if it meets every orbit. Call $\gamma \in G$ *extendable* if it is the restriction to $Dom(\gamma)$ of some $\bar{\gamma} \in G$ such that $Dom(\gamma)$ is relatively compact in $Dom(\bar{\gamma})$.

DEFINITION 2.4. [4] *A pseudogroup* (T, G) *is* compactly generated *if there are an exhaustive, relatively compact, open subset* $T' \subset T$ *, and finitely many elements of* $G|T'$ *which are extendable in* G *and which generate* $G|T'$ *.*

This property is invariant by Haefliger equivalence [4][5]. The holonomy pseudogroup of every foliated compact manifold is compactly generated [4]. Also, recently N. Raimbaud has given a natural generalization of compact generation in the realm of Lie groupoids, where it is a Morita-equivalence invariant [8].

3. The example

In this paper, to fix ideas one works in the smooth (C^∞) differentiability class; all foliations and pseudogroups are transversely orientable; all diffeomorphisms are orientation-preserving.

In the realization problem, one may allow that M have some boundary components transverse to $\mathcal{F}$, or not. This has no influence on the answer. Indeed assume that some pseudogroup G is realized by $(M, \mathcal{F})$ which has some transverse boundary components $\partial_{tr} M$. Let $\mathbf{D}^2$ denote the compact 2-disk. Then G is also realized by $(M', pr_1^* \mathcal{F})$ where M' is, in $\partial(M \times \mathbf{D}^2)$, the union of $M \times \mathbf{S}^1$ with $\partial_{tr} M \times \mathbf{D}^2$.

The counterexample to realizability is as follows. Let α , β be two global diffeomorphisms of the real line such that :

(1) α is a contraction fixing 0 , that is, $|\alpha(t)| < |t|$ for every $t \neq 0$;
(2) The support of β is compact and contained in $(-1, 0]$;
(3) The germs of α and β at 0 generate a nonabelian free group.

The third condition is in some sense generically fulfilled; one can also make an explicit example by the following classical method.

Let $A, B \in PSL(2, \mathbf{R})$ generate a free group. Lift them into two diffeomorphisms $\tilde{A}$, $\tilde{B}$ of the real line commuting with the unit translation. Composing if necessary $\tilde{A}$ with some integral translation, $\tilde{A}(t) > t$ for every t . After a conjugation by the exponential map, one has two diffeomorphisms a , b of $(0, +\infty)$ generating a free group. Moreover they verify the tameness property:

$$Ct \leq a(t), b(t) \leq C't$$

for some constants $0 < C < C'$. After a new conjugation by $\phi : t \mapsto \exp(-\exp(1/t))$, one has two germs of diffeomorphisms $f := \phi^{-1} a \phi$ and $g := \phi^{-1} b \phi$ on the right-hand side of 0 . It is easily verified that :

$$|\frac{\phi^{-1}(Ct)}{\phi^{-1}(t)} - 1| = o(\phi^{-1}(t)^n)$$

for every $C > 0$, n , and $t \to 0$. Thus f and g are flat on the identity at 0; and it remains only to change the orientation on the line, and to extend both germs in an obvious way, to get to diffeomorphisms α , β with the prescribed properties.

THEOREM 3.1. *The pseudogroup $G := < \alpha, \beta >$ generated by the above diffeomorphisms is compactly generated and is not realizable .*

The first affirmation is actually easy:

LEMMA 3.2. *G is compactly generated.*

Proof — Take $T' := (-1, 1)$ and $\alpha' := \alpha | T'$ and $\beta' := \beta | T'$. Then obviously T' is exhaustive and α' , β' are extendable in G . It remains to verify that they do generate $G | T'$.

Let be given the germ, denoted $\gamma(t) \stackrel{\gamma}{\leftarrow} t$, of some element γ of $G | T'$ at some point t in its domain. Thus t , $\gamma(t) \in T'$. We have to write this germ as a compose of germs of the diffeomorphisms $\alpha^{\pm 1}$ and $\beta^{\pm 1}$ *all taken at points of T'* — and this is the marrow of bone of compact generation.

But here it is easy: since G is generated by α and β , by definition the given germ decomposes as a composable sequence :

$$(\gamma(t) \stackrel{\gamma}{\leftarrow} t) = \big(t_n \stackrel{\gamma_n}{\leftarrow} t_{n-1} \stackrel{\gamma_{n-1}}{\leftarrow} \dots t_1 \stackrel{\gamma_1}{\leftarrow} t_0\big)$$

of germs of $\alpha^{\pm 1}$ and $\beta^{\pm 1}$ at some points $t_0 = t$, ..., $t_n \in \mathbf{R}$.

Take such a decomposition of minimal length n . Then we claim that t_0 ,..., $t_n \in T'$. Indeed, if not, one has for example

$$t_\ell := \sup\{t_0, \dots, t_n\} \geq 1$$

By maximality of t_ℓ , and since $\alpha(t_\ell) < t_\ell$, one has either $\gamma_\ell = \beta^{\pm 1}$ or $\gamma_{\ell+1} = \beta^{\pm 1}$ or ${\gamma_\ell}^{-1} = \alpha = \gamma_{\ell+1}$, contrarily to the minimality of the length of the decomposition. •

Observe that the halfline $[0, +\infty)$ is saturated for G , and that the restriction $G|[0, +\infty)$ is actually the transverse structure of a Reeb component. The proof that G is not realizable will rely on a precise description of all the foliations with the same transverse structure as a Reeb component, from which it will then follow that the boundary leaf cannot present such a free holonomy group on the side exterior to the component.

4. Generalized Reeb components

Fix a contraction η of the halfline $\mathbf{R}_+ := [0, +\infty)$, and consider the generated pseudogroup $< \eta >$.

Of course, this pseudogroup has a canonical realization in dimension 3 : the classical Reeb foliation on $\mathbf{D}^2 \times \mathbf{S}^1$, obtained as follows. Having foliated $(\mathbf{R}^2 \times \mathbf{R}_+) \setminus 0$ by its projection onto the halfline, one passes to the quotient by the foliation-preserving diffeomorphism $(x, t) \mapsto (x/2, \eta(t))$.

This obvious construction has a natural generalization (Alcalde-Cuesta--Hector-Schweitzer, unpublished). One is given a compact connected $(n-1)$-manifold C with smooth connected boundary and a self-embedding

$$\phi : C \to Int(C)$$

(In the classical case, C would be $\mathbf{D}^2$; and $\phi(x) = x/2$). From these data, one makes a *generalized Reeb component* as follows.

Consider the projective limit :

$$P := \cap_{i \in \mathbf{N}} \phi^i(C)$$

and the inductive limit :

$$I := (C \times \mathbf{Z})/((x, i+1) \sim (\phi(x), i))$$

Denote $[x, i]$ the class of the pair (x, i) . One has a diffeomorphism :

$$\Phi : I \to I : [x, i] \mapsto [\phi(x), i]$$

Identify C with a subset of I through the embedding $x \mapsto [x, 0]$. Thus $\Phi | C = \phi$. It is also convenient to fix a smooth function f_0 on $C \setminus Int(\phi(C))$ such that ${f_0}^{-1}(0) = \partial C$ and that ${f_0}^{-1}(1) = \phi(\partial C)$. It extends uniquely into a function f on $I \setminus P$ such that $f \circ \Phi = f + 1$. Obviously, f is proper. Set $f = +\infty$ on P .

Also, let g be a function on $(0, +\infty)$ such that $g(\eta(t)) = g(t) + 1$. Set $g(0) = +\infty$.

Define :

$$\tilde{R} := (I \times \mathbf{R}_+) \setminus (P \times 0)$$

Foliate it by its projection onto $\mathbf{R}_+$. Also endow it with the foliation-preserving diffeomorphism :

$$\gamma : \tilde{R} \to \tilde{R} : (x, t) \mapsto (\Phi(x), \eta(t))$$

and with the function :

$$F(x, t) := \min\{f(x), g(t)\}$$

It is immediately verified that F is finite and proper, and that $F \circ \gamma = F + 1$. It follows that γ acts freely, properly discontinuously and cocompactly on $\tilde{R}$. Thus the quotient is a foliated, compact, Hausdorff manifold $(R, \mathcal{R})$.

DEFINITION 4.1. *Call* $(R, \mathcal{R})$ *the generalized Reeb component associated to the self-embedding* (C, ϕ) *.*

Obviously $(R, \mathcal{R})$ realizes $< \eta >$. Conversely :

THEOREM 4.2. *Every realization of the pseudogroup generated by a contraction of the halfline is diffeomorphic to some generalized Reeb component in the sense of Alcalde-Cuesta-Hector-Schweitzer.*

Here "realization" is understood without transverse boundary components.

Proof of theorem 4.2 — Given a realization $(M, \mathcal{F})$ of the contraction η ,

LEMMA 4.3. *The foliation* $\mathcal{F}$ *is developable over* $\mathbf{R}_+$ *and complete ...*

... which means the following : there is an infinite cyclic covering :

$$\pi : \tilde{M} \to M$$

a generator γ of the deck transformation group, and a "developing map" :

$$D : \tilde{M} \to \mathbf{R}_+$$

such that :

(1) The map D is a surjective submersion;
(2) The fibres of D are connected and are the leaves of $\pi^* \mathcal{F}$;
(3) One has $\eta \circ D = D \circ \gamma$.

Proof of lemma 4.3 — One can either call to the general theory of transversely affine foliations, or deduce these properties from the corresponding ones observed on an explicit classifying space, as follows.

Changing 3 and 2 into $+\infty$ in the above construction of the classical Reeb component, one gets an infinite-dimensional Reeb component $(R^\infty, \mathcal{R})$. The holonomy covering of each leaf is weakly contractible. That foliation thus being the classifying space of its pseudogroup $< \eta >$ [3], there exists a classifying map $c : M \to R^\infty$ transverse to $\mathcal{R}$ such that $\mathcal{F} = c^* \mathcal{R}$, and that c induces a Haefliger equivalence between the holonomy pseudogroups of $\mathcal{F}$ and of $\mathcal{R}$.

In particular c induces a bijection of the leaf spaces ; and, for every leaf L of $\mathcal{R}$, the map c also induces a group isomorphism from the holonomy group of the leaf $c^{-1}(L)$ onto the holonomy group of L .

Thence c maps the holonomy group of ∂M onto the holonomy group of ∂R^∞ . Thus c maps the fundamental group $\pi_1 M$ onto $\pi_1 R^\infty \cong \mathbf{Z}$, hence an infinite cyclic covering $\tilde{M}$ and a lifting $\tilde{c} : \tilde{M} \to \tilde{R}^\infty$. Define D as $\tilde{c}$ followed by the projection to $\mathbf{R}_+$. The above properties of c immediately translate into the demanded properties for D . •

(Continuation of the proof of theorem 4.2) Fix in M an arbitrary smooth foliation $\mathcal{N}$ of dimension 1 transverse to $\mathcal{F}$. In particular $\mathcal{N}$ is transverse to ∂M . Lift it into a foliation $\tilde{\mathcal{N}}$ of the covering $\tilde{M}$. Consider the canonical projection onto the space of orbits:

$$pr : \tilde{M} \to I := \tilde{M}/\tilde{\mathcal{N}}$$

and the homeomorphism $\Phi : I \to I$ such that:

$$pr \circ \gamma = \Phi \circ pr$$

and the Φ-invariant, topologically closed subset:

$$P := I \setminus pr(\partial\tilde{M})$$

LEMMA 4.4. *The space of orbits I is a connected Hausdorff manifold. Moreover, there is a diffeomorphism :*

$$\tilde{M} \cong (I \times \mathbf{R}_+) \setminus (P \times 0)$$

through which $\gamma(x,t) = (\Phi(x), \eta(t))$ and $D(x,t) = t$ and $pr(x,t) = x$.

Proof — The halfline bears an η-invariant vector field $u(t)\partial/\partial t$, smooth and nonsingular in $(0,+\infty)$, null at 0. It needs not be differentiable at 0. Clearly it is complete. Let $(\eta^s)_{s\in\mathbf{R}}$ be the associated 1-parameter group of homeomorphisms of the halfline. Consider the unique vector field $\tilde{X}$ in $\tilde{M}$ tangent to $\tilde{\mathcal{N}}$ and projecting onto $u(t)\partial/\partial t$ through D . Since $\tilde{\mathcal{N}}$ is γ-invariant, since $u(t)\partial/\partial t$ is η-invariant and since D is equivariant, $\tilde{X}$ is γ-invariant. In other words $\tilde{X}$ is the pullback into $\tilde{M}$ of some vector field X on the compact manifold M , which is smooth in the interior of M and null on ∂M . The vector field $u(t)\partial/\partial t$ being complete, one concludes easily that X is complete. Thus $\tilde{X}$ is complete. Let $(\xi^s)_{s\in\mathbf{R}}$ be the associated 1-parameter group of homeomorphisms of $\tilde{M}$. Then $D \circ \xi^s = \eta^s \circ D$. From this equivariance follows easily that the following map is one-to-one and onto :

$$\psi : D^{-1}(1) \times \mathbf{R}_+^* \to Int(\tilde{M}) : (x,t) \mapsto \xi^{\int_1^t \frac{d\tau}{u(\tau)}}(x)$$

Being obviously etale, it is a diffeomorphism.

In particular I is diffeomorphic to $D^{-1}(1)$, thus a connected Hausdorff manifold.

It remains to extend ψ to the boundary. For every $x \in I \setminus P$, set $\psi(x,0) := \lim_{s\to-\infty} \xi^s(x) \in \partial\tilde{M}$. Obviously this extends ψ into a global diffeomorphism from $(I \times \mathbf{R}_+) \setminus (P \times 0)$ onto $\tilde{M}$, through which $\gamma(x,t) = (\Phi(x), \eta(t))$ and $D(x,t) = t$ and $pr(x,t) = x$. •

It seems that a little more work is necessary to make the dynamic of Φ , and its relation to P , precise; and thus to achieve the proof of theorem 4.2. For example, at this point it is not obvious that P is compact.

One identifies $\tilde{M}$ with $(I \times \mathbf{R}_+) \setminus (P \times 0)$.

It is a well-known property of infinite cyclic coverings that $\tilde{M}$ admits a proper smooth function F such that $F \circ \gamma = F + 1$. To fix ideas, one can arrange that 0 is a regular value of F and of $F|\partial\tilde{M}$. Also, by 4.3, $\partial\tilde{M}$ is connected. Thence one can arrange also that $\partial(F^{-1}(0))$ is connected.

LEMMA 4.5. *(i) For every $x \in I$, one has :*

$$\lim_{t \to +\infty} F(x,t) = -\infty$$

(ii) For every $p \in P$, one has :

$$\lim_{t \to 0} F(p,t) = +\infty$$

(iii) More precisely, for every $p \in P$, one has :

$$\lim_{(x,t) \in \tilde{M}, (x,t) \to (p,0)} F(x,t) = +\infty$$

Proof — (i) Let T be the maximum of $D(x,t) = t$ on the compact fundamental domain $F^{-1}([-1,0])$. Let g be a decreasing function on $(0,+\infty)$ such that $g(T) = 0$ and $g \circ \eta = g + 1$. Then $F(x,t) \leq g(t)$ at every point (x,t) of $\tilde{M}$. Thus $F(x,t) \to -\infty$ for $t \to +\infty$.

(ii) The halfline $p \times (0,1]$ being properly embedded in $\tilde{M}$, the limit exists, either $-\infty$ or $+\infty$. By contradiction, assume that it is $-\infty$. For every i large enough :

$$F(\gamma^i(p,1)) = F(p,1) + i > 0$$

thus the halfline $\gamma^i(p \times (0,1]) = \Phi^i(p) \times (0, \eta^i(1)]$ would meet $F^{-1}(0)$ in at least one point $(\Phi^i(p), t_i)$. The level set $F^{-1}(0)$ being compact, some subsequence of the sequence $(\Phi^i(p), t_i)$ converges to some $(q,t) \in F^{-1}(0)$. Since $t_i \leq \eta^i(1)$, one has $t = 0$. Since P is Φ-invariant and topologically closed in I , one has $q \in P$. Thus $(q,t) \in P \times 0$, the desired contradiction.

(iii) Consider a fundamental sequence (V_i) of connected neighborhoods of p in I , and :

$$W_i := (V_i \times [0, 1/i]) \cap \tilde{M}$$

and fix a large positive T . Since $F^{-1}[-T,+T]$ is compact and does not contain $(p,0)$, it is disjoint from W_i for every i large enough. Since W_i is connected, either $F > T$ on W_i or $F < -T$ on W_i . The second possibility being ruled out by (ii), the lemma is proved. •

On $I \setminus P$ one has the proper function $f(x) := F(x,0)$ and one defines :

$$C := P \cup f^{-1}[0,+\infty) \subset I$$

COROLLARY 4.6. *The subset $C \subset I$ is a compact submanifold of codimension 0 with smooth boundary and P is contained in its interior. Both C and ∂C are connected.*

Proof — By lemma 4.5, firstly *C is relatively compact in I* . Indeed, for every $x \in C$, by (i) and (ii) the halfline $x \times \mathbf{R}_+$ meets $F^{-1}(0)$. That is, C is contained in $pr(F^{-1}(0))$ which is compact in I .

Secondly, P is contained in the topological interior of C . This follows at once from (iii). In particular, the topological boundary of C in I is $f^{-1}(0) = \partial F^{-1}(0)$, a smooth compact connected $(n-2)$-manifold. Since I and ∂C are connected, C is connected. •

Now, recalling that one has the diffeomorphism Φ of I such that $\Phi(P) = P$ and that $f \circ \Phi = f + 1$ on $I \setminus P$, one gets easily :

$$\Phi(C) \subset Int(C) \text{ and } \mathrm{P} = \cap_{\mathrm{i} \in \mathbf{Z}} \Phi^{\mathrm{i}}(\mathrm{C}) \text{ and } \mathrm{I} = \cup_{\mathrm{i} \in \mathbf{Z}} \Phi^{\mathrm{i}}(\mathrm{C})$$

By lemma 4.4 the foliated manifold $(M, \mathcal{F})$ is diffeomorphic to the generalized Reeb component associated with $(C, \Phi|C)$ according to definition 4.1; and the theorem 4.2 is proved. •

In general, the cobordism $C \setminus Int(\phi(C))$ is of course not trivial. Accordingly, the boundary leaf ∂R of an arbitrary generalized Reeb component (definition 4.1) is not necessarily fibred over the circle. However, we always have the following finiteness property, well-known e.g. in the classical study of knots :

LEMMA 4.7. *Let $\partial \tilde{R}$ be the holonomy covering of the boundary leaf of a generalized Reeb component. Then the homology groups of $\partial \tilde{R}$ with coefficients in any field k are of finite rank over k .*

Proof — (All homology groups are with coefficients in k .) Let C , ϕ , I , P , Φ , R be as in definition 4.1. For every positive i , write $C_i = \Phi^{-i}(C)$ and $W_i = C_i \setminus Int(C_{-i})$. In the following commutative diagram (where all arrows are induced by inclusions) :

$$\begin{array}{ccc} H_*(C_i) & \longrightarrow & H_*(C_i, C_i \setminus Int(W_i)) \\ \uparrow & & \rho \uparrow \\ H_*(W_i) & \xrightarrow{\beta} & H_*(W_i, \partial W_i) \end{array}$$

the right-hand vertical arrow ρ is invertible by the excision theorem, thus :

$$\mathrm{rank}(\beta) \leq \mathrm{rank} H_*(C_i)$$

On the other hand, the long exact relative homology sequence for the couple $(W_i, \partial W_i)$ gives :

$$\mathrm{rank} H_*(W_i) \leq \mathrm{rank}(\beta) + \mathrm{rank} H_*(\partial W_i)$$

But C_i is diffeomorphic to C and ∂W_i is diffeomorphic to two copies of ∂C , thus:

$$\mathrm{rank} H_*(W_i) \leq \mathrm{rank} H_*(C) + 2\mathrm{rank} H_*(\partial C)$$

An upper bound independant on i . The covering space $\partial \tilde{R}$ being the inductive limit of the sequence :

$$W_1 \subset W_2 \subset \cdots \subset W_i \subset \ldots$$

the rank of $H_*(\partial \tilde{R})$ admits the same majoration. •

5. Proof of theorem 3.1

Consider again the pseudogroup $G =< \alpha, \beta >$, where α is a contraction of the real line $\mathbf{R}$ fixing 0 and where β is a diffeomorphism of $\mathbf{R}$ with compact support contained in $\mathbf{R}_-$, and such that their germs at 0 generate a nonabelian free group. In the pseudogroup $(\mathbf{R}, G)$ one may call $\mathbf{R}_+$ a *paradoxical Reeb component:* a saturated domain with the same dynamic as a Reeb component, but whose boundary 0 has a complicated isotropy group outside.

On the contrary, the preceding section has shown us that the corresponding paradoxical Reeb components cannot exist among foliations, and so G is not realizable.

More precisely, in the isotropy group $Iso(G, 0)$ of G at point 0 , one has the subgroup $ExtIso(G, 0)$ consisting of the germs which are the identity on the right-hand side of 0 . Clearly $ExtIso(G, 0)$ is the normal subgroup generated by β , and thus a nonabelian free group of infinite rank. Consider its abelianization (quotient by the derived subgroup) $ExtIso(G, 0)_{ab}$. Then the vector space

$$\mathbf{Q} \otimes ExtIso(G, 0)_{ab}$$

is of infinite rank over $\mathbf{Q}$.

On the other hand, assume by contradiction that G has some realization $(M, \mathcal{F})$. That is, $(M, \mathcal{F})$ would be a foliated compact manifold whose holonomy pseudogroup would be Haefliger-equivalent to G . As aforesaid, one can assume moreover that M is closed. The halfline $\mathbf{R}_+$ being G-invariant, M would contain a compact saturated domain R that would realize the pseudogroup $G|\mathbf{R}_+$, that is, the pseudogroup on the half line generated by the contraction α . After theorem 4.2, R would be a generalized Reeb component. Let $Hol(\mathcal{F}, \partial R)$ denote the holonomy group of the leaf ∂R , and $ExtHol(\mathcal{F}, \partial R)$ denote the subgroup of germs which are the identity inside R . Let also $\tilde{\partial} R$ be the infinite cyclic covering corresponding to the holonomy inside R . So, $\pi_1 \tilde{\partial} R$ is mapped onto $ExtHol(\mathcal{F}, \partial R)$. In consequence, the vector space

$$\mathbf{Q} \otimes ExtHol(\mathcal{F}, \partial R)_{ab}$$

being a quotient of $H_1(\tilde{R}; \mathbf{Q})$, which is of finite rank after lemma 4.7, is also of finite rank over $\mathbf{Q}$.

But, since the holonomy pseudogroup of $\mathcal{F}$ is Haefliger-equivalent to G , the groups $ExtHol(\mathcal{F}, \partial R)$ and $ExtIso(G, 0)$ are of course isomorphic, a contradiction. •

6. Questions

Haefliger has introduced an interesting stronger notion of *compact presentation* for pseudogroups [5]. The holonomy pseudogroup of any foliated compact manifold is compactly presented, and any compactly presented pseudogroup is compactly generated. Unfortunately, compact presentation seems difficult to decide on explicit examples such as ours.

Question — Is the above pseudogroup $< \alpha, \beta >$ compactly presented ?

Presently, I know no pseudogroup which is compactly generated but not compactly presented.

In a direction complementary to the present paper, in a forthcoming one I will show that actually many compactly generated pseudogroups of codimension 1 are realizable, and even realizable on manifolds of small dimension. The result is as follows.

Let (T, G) be a compactly generated pseudogroup, with $\dim T = 1$. The notion of "dead end component", well-known for codimension one foliations, has an obvious analogue for pseudogroups. Those components are bounded by closed orbits, of which we consider the isotropy groups. One can show:

1. If every dead end boundary isotropy group is solvable, then (T, G) is realizable in a 4-manifold.

2. (T, G) is realizable in a 3-manifold if and only if every dead end boundary isotropy group is abelian of rank ≤ 2 .

In particular, if G has no closed orbit, or more generally no dead end component, then it is realizable in dimension 3. If G is PL , or projective (local transformations of the type $t \mapsto (at+b)/(ct+d)$) then it is realizable in dimension 4.

Question — Is every real-analytic compactly generated pseudogroup of codimension 1 realizable?

So, one has seen in the present paper a sufficent condition for *not* being realizable (some Reeb component boundary isotropy group is nonabelian free) and one will see also a sufficent condition for being realizable (every dead end boundary isotropy group is solvable). These two conditions are not exactly complementary, there remains a little gap. Maybe a good understanding of *compact presentation* in codimension 1 would fill the gap.

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