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DICTATOR FUNCTIONS MAXIMIZE MUTUAL INFORMATION

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Let $(\mathbf{X}, \mathbf{Y})$ denote n independent, identically distributed copies of two arbitrarily correlated Rademacher random variables (X, Y) . We prove that the inequality $I(f(\mathbf{X}); g(\mathbf{Y})) \leq I(X; Y)$ holds for any two Boolean functions: $f, g: \{-1, 1\}^n \rightarrow \{-1, 1\}$ ($I(\cdot; \cdot)$ denotes mutual information). We further show that equality in general is achieved only by the dictator functions $f(\mathbf{x}) = \pm g(\mathbf{x}) = \pm x_i, i \in \{1, 2, \dots, n\}$.

1. Introduction and Main Results. Let (X, Y) be two dependent Rademacher random variables on $\{-1, 1\}$, with correlation coefficient $\rho := \mathbb{E}[XY] \in [-1, 1]$. For given $n \in \mathbb{N}$, let $(\mathbf{X}, \mathbf{Y}) = (X, Y)^n$ be n independent, identically distributed copies of (X, Y) . We will use the notation from [3] for information-theoretic quantities. In particular, $\mathbb{E}[X]$, $H(X)$, and $I(X; Y)$ denote expectation, entropy, and mutual information, respectively. Motivated by problems in computational biology [4], Kumar and Courtade formulated the following conjecture [5, Conjecture 1].

CONJECTURE 1. *For any Boolean function $f: \{-1, 1\}^n \rightarrow \{-1, 1\}$,*

$$(1) \quad I(f(\mathbf{X}); \mathbf{Y}) \leq I(X; Y).$$

This claim – while seemingly innocent at first sight – has received significant interest and resisted several efforts to find a proof (see the discussion in [2, Section IV]). Note that $f = \chi_i$ for any dictator function [6, Definition 2.3] $\chi_i(\mathbf{x}) := x_i, i \in \{1, 2, \dots, n\}$ achieves equality in (1).

We next state the main result of this paper, which is a relaxed version of Conjecture 1, involving two Boolean functions.

THEOREM 1. *For any two Boolean functions $f, g: \{-1, 1\}^n \rightarrow \{-1, 1\}$,*

$$(2) \quad I(f(\mathbf{X}); g(\mathbf{Y})) \leq I(X; Y).$$

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If (1) were true, this statement would readily follow from the data processing inequality [3, Theorem 2.8.1]. Theorem 1 was stated as an open problem in [2] and [5, Section IV], and separately investigated in [1]. A proof of (2) was previously available only under the additional restrictive assumptions that f and g are equally biased (i.e., $\mathbb{E}[f(\mathbf{X})] = \mathbb{E}[g(\mathbf{X})]$) and satisfy the condition

$$(3) \quad \mathbb{P}\{f(\mathbf{X}) = 1, g(\mathbf{X}) = 1\} \geq \mathbb{P}\{f(\mathbf{X}) = 1\}\mathbb{P}\{g(\mathbf{X}) = 1\}.$$

The reader is invited to see [2, Section IV] for further details. In this paper, we use Fourier-analytic tools to prove Theorem 1 without any additional restrictions on f and g . We suitably bound the Fourier coefficients of f and g , and thereby reduce (2) to an elementary inequality, which is subsequently established.

A careful inspection of the proof of Theorem 1 reveals that in general, up to sign changes, the dictator functions χ_i , $i \in \{1, 2, \dots, n\}$ are the unique maximizers of $I(f(\mathbf{X}); g(\mathbf{Y}))$.

PROPOSITION 1. *If $0 < |\rho| < 1$, equality in (2) is achieved if and only if $f = \pm g = \pm \chi_i$ for some $i \in \{1, 2, \dots, n\}$.*

2. Proof of Theorem 1. Define $[n] := \{1, 2, \dots, n\}$ and let f, g be two Boolean functions on the Boolean hypercube, i.e., $f, g: \{-1, 1\}^n \rightarrow \{-1, 1\}$. Denote their Fourier expansions (cf. [6, (1.6)]) $f(\mathbf{x}) = \sum_{S \subseteq [n]} \hat{f}_S \chi_S(\mathbf{x})$ and $g(\mathbf{x}) = \sum_{S \subseteq [n]} \hat{g}_S \chi_S(\mathbf{x})$, using the basis $\chi_S(\mathbf{x}) := \prod_{i \in S} x_i$ for $S \subseteq [n]$. Define $a := \frac{1 + \hat{f}_\emptyset}{2} = \mathbb{P}\{f(\mathbf{X}) = 1\}$, $b := \frac{1 + \hat{g}_\emptyset}{2} = \mathbb{P}\{g(\mathbf{X}) = 1\}$ and $\theta_\rho := \frac{1}{4} \sum_{S: |S| \geq 1} \hat{f}_S \hat{g}_S \rho^{|S|}$. Without loss of generality, we may assume $\frac{1}{2} \leq a \leq b \leq 1$ and $\rho \in [0, 1]$, as mutual information is symmetric and we have, with $\mathbf{Y}^* := \text{sgn}(\rho)\mathbf{Y}$,

$$(4) \quad I(f(\mathbf{X}); g(\mathbf{Y})) = I(\text{sgn}(\hat{f}_\emptyset)f(\mathbf{X}); \text{sgn}(\hat{g}_\emptyset)g(\text{sgn}(\rho)\mathbf{Y}^*)).$$

In analogy to [6, Proposition 1.9], the inner product satisfies

$$(5) \quad \langle f, T_\rho g \rangle = \mathbb{E}[f(\mathbf{X})g(\mathbf{Y})] = \hat{f}_\emptyset \hat{g}_\emptyset + 4\theta_\rho = 1 - 2\mathbb{P}\{f(\mathbf{X}) \neq g(\mathbf{Y})\},$$

where T_ρ is the noise operator [6, Definition 2.46]. Defining $\bar{t} := 1 - t$ for a generic t , we can express the probabilities

$$(6) \quad \mathbb{P}\{f(\mathbf{X}) = 1, g(\mathbf{Y}) = -1\} = a\bar{b} - \theta_\rho, \quad \mathbb{P}\{f(\mathbf{X}) = g(\mathbf{Y}) = 1\} = ab + \theta_\rho,$$

$$(7) \quad \mathbb{P}\{f(\mathbf{X}) = -1, g(\mathbf{Y}) = 1\} = \bar{a}b - \theta_\rho, \quad \mathbb{P}\{f(\mathbf{X}) = g(\mathbf{Y}) = -1\} = \bar{a}\bar{b} + \theta_\rho.$$

Using (6), (7) and fundamental properties of mutual information [3, Section 2.4], we obtain $I(f(\mathbf{X}); g(\mathbf{Y})) = \xi(\theta_\rho, a, b)$ with

$$(8) \quad \xi(\theta, a, b) := H(a) + H(b) - H(ab + \theta, a\bar{b} - \theta, \bar{a}b - \theta, \bar{a}\bar{b} + \theta),$$

where, slightly abusing notation, we defined the binary entropy function $H(p) := H(p, \bar{p})$ and $H((p_i)_{i \in \mathcal{I}}) := -\sum_{i \in \mathcal{I}} p_i \log_2 p_i$ for $|\mathcal{I}| > 1$. By the non-negativity of probabilities (6) and (7), for any $\rho \in [0, 1]$,

$$(9) \quad -\bar{a}\bar{b} \leq \theta_\rho \leq a\bar{b}.$$

With $\mathcal{P} := \{\mathcal{S} \subseteq [n] : \hat{f}_\mathcal{S} \hat{g}_\mathcal{S} > 0\} \setminus \{\emptyset\}$ and $\mathcal{N} := \{\mathcal{S} \subseteq [n] : \hat{f}_\mathcal{S} \hat{g}_\mathcal{S} < 0\}$, we define

$$(10) \quad \tau^+ := \frac{1}{4} \sum_{\mathcal{S} \in \mathcal{P}} \hat{f}_\mathcal{S} \hat{g}_\mathcal{S}, \quad \tau^- := \frac{1}{4} \sum_{\mathcal{S} \in \mathcal{N}} \hat{f}_\mathcal{S} \hat{g}_\mathcal{S}$$

and apply the Schwarz inequality to show

$$(11) \quad \tau^+ - \tau^- = \frac{1}{4} \sum_{\mathcal{S}: |\mathcal{S}| \geq 1} |\hat{f}_\mathcal{S}| |\hat{g}_\mathcal{S}|$$

$$(12) \quad \leq \frac{1}{4} \sqrt{(1 - \hat{f}_\emptyset^2)(1 - \hat{g}_\emptyset^2)} = \sqrt{a\bar{a}b\bar{b}}.$$

As $\theta_1 = \tau^+ + \tau^-$, we combine (9) and (12) to obtain

$$(13) \quad \tau^+ \leq \frac{a\bar{b} + \sqrt{a\bar{a}b\bar{b}}}{2}, \quad \tau^- \geq -\frac{\bar{a}\bar{b} + \sqrt{a\bar{a}b\bar{b}}}{2}.$$

By definition, $\rho\tau^- \leq \theta_\rho \leq \rho\tau^+$ and hence, $\theta_\rho \in [\theta_\rho^-, \theta_\rho^+]$, where

$$(14) \quad \theta_\rho^- := \max \left\{ -\bar{a}\bar{b}, -\rho \frac{\bar{a}\bar{b} + \sqrt{a\bar{a}b\bar{b}}}{2} \right\}, \quad \theta_\rho^+ := \min \left\{ a\bar{b}, \rho \frac{a\bar{b} + \sqrt{a\bar{a}b\bar{b}}}{2} \right\}.$$

The function $\xi(\theta, \alpha, \beta)$ is convex in θ by the concavity of entropy [3, Theorem 2.7.3] and consequently, $I(f(\mathbf{X}); g(\mathbf{Y})) \leq \max_{\theta \in \{\theta_\rho^+, \theta_\rho^-\}} \xi(\theta, a, b)$. Thus, Theorem 1 can be proved by establishing $1 - H\left(\frac{\rho+1}{2}\right) - \xi(\theta, a, b) \geq 0$ for $\theta \in \{\theta_\rho^+, \theta_\rho^-\}$. Furthermore, it suffices to consider $\frac{1}{2} < a < b < 1$ by continuity of ξ .

Define $C_{a,b} := \frac{a\bar{b} + \sqrt{a\bar{a}b\bar{b}}}{2}$, $\rho^+ := \min \left\{ \rho, \frac{a\bar{b}}{C_{a,b}} \right\}$, $\rho^- := \min \left\{ \rho, \frac{\bar{a}\bar{b}}{C_{a,b}} \right\}$, and

$$(15) \quad \phi(\rho, a, b) := 1 - H\left(\frac{\rho+1}{2}\right) - \xi(\rho C_{a,b}, a, b).$$

Note that

$$(16) \quad \phi(\rho^+, a, b) = 1 - H\left(\frac{\rho^+ + 1}{2}\right) - \xi(\theta_\rho^+, a, b)$$

$$(17) \quad \leq 1 - H\left(\frac{\rho + 1}{2}\right) - \xi(\theta_\rho^+, a, b)$$

by the monotonicity of the binary entropy function and accordingly we also have $\phi(\rho^-, \bar{a}, b) \leq 1 - H\left(\frac{\rho+1}{2}\right) - \xi(\theta_\rho^-, a, b)$. Theorem 1 thus follows from the following lemma.

LEMMA 1. *For $0 < \alpha < \beta < 1$ and $\rho \in \left[0, \frac{\alpha\bar{\beta}}{C_{\alpha,\beta}}\right]$, we have $\phi(\rho, \alpha, \beta) \geq 0$ with equality if and only if $\rho = 0$.*

Before proving Lemma 1, we note the following facts.

LEMMA 2. *For $x \in (0, 1)$, we have*

$$(18) \quad \frac{1}{x^{-1} - 1} + \log(1 - x) > 0.$$

PROOF. Using Taylor series expansion, we immediately obtain

$$(19) \quad -\log(1 - x) = \sum_{n=1}^{\infty} \frac{x^n}{n} < \sum_{n=1}^{\infty} x^n = \frac{x}{1 - x}.$$

□

The following lemma collects elementary facts about convex/concave functions and follows from elementary properties of convex functions on the real line (see, e.g., [7, Chapter I]).

LEMMA 3. *Let $f: U \rightarrow \mathbb{R}$ be a continuous function, defined on the compact interval $U := [u_1, u_2] \subset \mathbb{R}$. Assuming that f is twice differentiable on V , where $(u_1, u_2) \subseteq V \subseteq U$, the following properties hold.*

1. *If $f''(u) \geq 0$ for all $u \in (u_1, u_2)$ and $f'(u^*) = 0$ for some $u^* \in V$, then $f(u) \geq f(u^*)$ for all $u \in U$. Furthermore, if additionally $f''(u) > 0$ for all $u \in (u_1, u_2)$, then $f(u) > f(u^*)$ for all $u \in U \setminus \{u^*\}$.*
2. *If $f''(u) \leq 0$ for all $u \in (u_1, u_2)$, then $f(u) \geq \min\{f(u_1), f(u_2)\}$ for all $u \in U$. Furthermore, if $f''(u) < 0$ for all $u \in (u_1, u_2)$, then $f(u) > \min\{f(u_1), f(u_2)\}$ for all $u \in (u_1, u_2)$.*

PROOF OF LEMMA 1. Let $I := \{(\alpha, \beta) \in \mathbb{R}^2 : 0 < \alpha < \beta < 1\}$, fix arbitrary $(\alpha, \beta) \in I$ and define

$$(20) \quad \rho_- := \frac{\max\{\alpha\beta, \bar{\alpha}\bar{\beta}\}}{C_{\alpha,\beta}}, \quad \rho_o := \frac{\min\{\alpha\beta, \bar{\alpha}\bar{\beta}\}}{C_{\alpha,\beta}}, \quad \rho_+ := \frac{\alpha\bar{\beta}}{C_{\alpha,\beta}}.$$

We shall adopt the simplified notation $\phi(\rho) := \phi(\rho, \alpha, \beta)$, suppressing the fixed parameters (α, β) . For $\rho \in [0, \rho_+)$, we have the derivatives

$$(21) \quad \phi'(\rho) = \frac{1}{2} \log_2 \left(\frac{1+\rho}{1-\rho} \right) + C_{\alpha,\beta} \log_2 \left(\frac{(\bar{\alpha}\beta - C_{\alpha,\beta}\rho)(\alpha\bar{\beta} - C_{\alpha,\beta}\rho)}{(\alpha\beta + C_{\alpha,\beta}\rho)(\bar{\alpha}\bar{\beta} + C_{\alpha,\beta}\rho)} \right),$$

$$(22) \quad \phi''(\rho) = \frac{C_{\alpha,\beta}^2}{\log 2} \left(\frac{1}{C_{\alpha,\beta}^2(1-\rho^2)} - \frac{1}{\bar{\alpha}\beta - C_{\alpha,\beta}\rho} - \frac{1}{\alpha\bar{\beta} - C_{\alpha,\beta}\rho} - \frac{1}{\alpha\beta + C_{\alpha,\beta}\rho} \right).$$

We write $\phi''(\rho) = \frac{p(\rho)}{q(\rho)}$, where both p and q are polynomials in ρ , and choose

$$(23) \quad q(\rho) = \log(2)(1-\rho^2)(\bar{\alpha}\beta - C_{\alpha,\beta}\rho) \times (\alpha\bar{\beta} - C_{\alpha,\beta}\rho)(\bar{\alpha}\bar{\beta} + C_{\alpha,\beta}\rho)(\alpha\beta + C_{\alpha,\beta}\rho),$$

such that $q(\rho) > 0$ for $\rho \in [0, \rho_+)$. By (22), $p(\rho)$ is given by

$$(24) \quad \begin{aligned} p(\rho) = & (\bar{\alpha}\beta - C_{\alpha,\beta}\rho)(\alpha\bar{\beta} - C_{\alpha,\beta}\rho)(\bar{\alpha}\bar{\beta} + C_{\alpha,\beta}\rho)(\alpha\beta + C_{\alpha,\beta}\rho) \\ & - C_{\alpha,\beta}^2(1-\rho^2) \left((\alpha\bar{\beta} - C_{\alpha,\beta}\rho)(\bar{\alpha}\bar{\beta} + C_{\alpha,\beta}\rho)(\alpha\beta + C_{\alpha,\beta}\rho) \right. \\ & + (\bar{\alpha}\beta - C_{\alpha,\beta}\rho)(\bar{\alpha}\bar{\beta} + C_{\alpha,\beta}\rho)(\alpha\beta + C_{\alpha,\beta}\rho) \\ & + (\bar{\alpha}\beta - C_{\alpha,\beta}\rho)(\alpha\bar{\beta} - C_{\alpha,\beta}\rho)(\alpha\beta + C_{\alpha,\beta}\rho) \\ & \left. + (\bar{\alpha}\beta - C_{\alpha,\beta}\rho)(\alpha\bar{\beta} - C_{\alpha,\beta}\rho)(\bar{\alpha}\bar{\beta} + C_{\alpha,\beta}\rho) \right). \end{aligned}$$

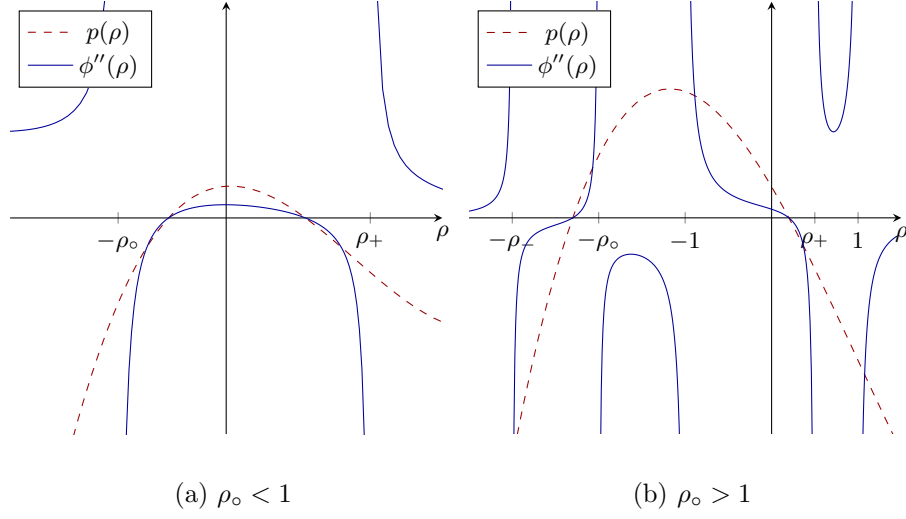
This entails $\deg(p) \leq 5$ and a careful calculation of the coefficients reveals $\deg(p) \leq 3$.

We will now demonstrate that there is a unique point $\rho^* \in (0, \rho_+)$, such that $p(\rho^*) = 0$. To this end, reinterpret $\phi''(\rho)$ as a rational function of ρ on $\mathbb{R}$. We evaluate (24) and use $\alpha < \beta$ to obtain the two inequalities

$$(25) \quad p(0) = \alpha\bar{\alpha}\beta\bar{\beta}(\alpha\bar{\alpha}\beta\bar{\beta} - C_{\alpha,\beta}^2) > 0, \text{ and}$$

$$(26) \quad p(\rho_+) = -(C_{\alpha,\beta}^2 - (\alpha\bar{\beta})^2)(\beta - \alpha)\bar{\beta}\alpha < 0.$$

The number of roots of p in $(0, \rho_+)$ is thus odd and at most equal to its degree, i.e., either one or three. If we have $\rho_o \leq 1$, then evaluation of (24)

Fig 1: Sketch of $p(\rho)$ and $\phi''(\rho)$.

readily yields $p(-\rho_0) \leq 0$. If, on the other hand, $\rho_0 > 1$, we obtain $p(-\rho_-) \leq 0$ from (24). Thus, p has at least one negative root and a unique root $\rho^* \in (0, \rho_+)$. Figure 1 qualitatively illustrate the behavior of $p(\rho)$ and $\phi''(\rho)$.

Consequently, $\phi''(\rho) > 0$ for $\rho \in (0, \rho^*)$. By part 1 of Lemma 3, $\phi(\rho) > \phi(0) = 0$ for $\rho \in (0, \rho^*]$ as $\phi'(0) = 0$. Since $\phi''(\rho) < 0$ for $\rho \in (\rho^*, \rho_+)$, we have $\phi(\rho) > \min\{\phi(\rho^*), \phi(\rho_+)\}$ for all $\rho \in (\rho^*, \rho_+)$, by part 2 of Lemma 3. In total, $\phi(\rho) > \min\{0, \phi(\rho_+)\}$ for $\rho \in (0, \rho_+)$.

As $\phi(0) = 0$, it remains to show that $\phi(\rho_+, \alpha, \beta) > 0$ for $(\alpha, \beta) \in I$. To this end, we introduce the transformation

$$(27) \quad (\alpha, \beta) \mapsto (c, x) := \left(\frac{\log \frac{\alpha}{\beta}}{\log \frac{\alpha\bar{\beta}}{\bar{\alpha}\beta}}, \sqrt{\frac{\alpha\bar{\beta}}{\bar{\alpha}\beta}} \right),$$

a bijective mapping from I to $(0, 1)^2$ with the inverse

$$(28) \quad (c, x) \mapsto (\alpha, \beta) = \left(\frac{x^{2c} - x^2}{1 - x^2}, \frac{1 - x^{2-2c}}{1 - x^2} \right).$$

In terms of c and x , we have $\phi(\rho_+, \alpha, \beta) = \psi(c, x)$, where

$$(29) \quad \psi(c, x) := 1 - H\left(\frac{1}{2} + \frac{x}{1+x}\right) - H\left(\frac{x^{2c} - x^2}{1 - x^2}\right) + \frac{1 - x^{2-2c}}{1 - x^2} H(x^{2c})$$

$$(30) \quad = 1 - H\left(\frac{1+3x}{2+2x}\right) + \frac{H(x^2)}{1-x^2} + \frac{x^{2c}H(x^{2-2c}) + x^{2-2c}H(x^{2c})}{x^2 - 1}.$$

We fix a particular $x \in (0, 1)$ and use the simplified notation $\psi(c) := \psi(c, x)$, obtaining the derivatives

$$(31) \quad \psi'(c) = \frac{2 \log(x)}{(x^2 - 1) \log(2)} \left[2x^{2c} c \log(x) + x^{2(1-c)} \log(1 - x^{2c}) - x^{2c} \log(x^{2c} - x^2) \right],$$

$$(32) \quad \psi''(c) = \frac{4 \log(x)^2 x^{2c}}{(1 - x^2) \log(2)} \left[\left(\frac{1}{x^{-2(1-c)} - 1} + \log(1 - x^{2(1-c)}) \right) + \frac{x^2}{x^{4c}} \left(\log(1 - x^{2c}) + \frac{1}{x^{-2c} - 1} \right) \right].$$

By applying Lemma 2 twice, we obtain $\psi''(c) > 0$. Thus, $\psi(c) > \psi(\frac{1}{2})$ by part 1 of Lemma 3 as $\psi'(\frac{1}{2}) = 0$. It remains to show that $\gamma(x) := \psi(\frac{1}{2}, x) > 0$. Note that $\gamma(0) = \gamma(1) = 0$ and

$$(33) \quad \gamma'(x) = \frac{1}{(1+x)^2} \log_2 [(1+3x)(1-x)],$$

for $x \in [0, 1]$. If $\gamma(x) \leq 0$ for any $x \in (0, 1)$ then f necessarily attains its minimum in $(0, 1)$ and there exists $x^* \in (0, 1)$ with $\gamma(x^*) \leq 0$ and $\gamma'(x^*) = 0$. As $x^* = \frac{2}{3}$ is the only point in $(0, 1)$ with $\gamma'(x^*) = 0$ and $\gamma(\frac{2}{3}) = \log_2(\frac{27}{25}) > 0$, this concludes the proof. $\square$

3. Proof of Proposition 1. We may assume $0 < \rho < 1$ and $\frac{1}{2} \leq a \leq b \leq 1$ by virtue of (4). Clearly, $g = \pm f = \pm \chi_i$ for some $i \in [n]$ is a sufficient condition to maximize $I(f(\mathbf{X}); g(\mathbf{Y}))$. A careful inspection of the proof of Theorem 1 shows that this condition is also necessary.

In the following, we will use the notation of Section 2. As $b = 1$ implies $I(f(\mathbf{X}); g(\mathbf{Y})) = 0$, we assume $\frac{1}{2} \leq a \leq b < 1$. For equality in Theorem 1, we need either $\phi(\rho^+, a, b) = 0$ or $\phi(\rho^-, \bar{a}, b) = 0$. By Lemma 1, $\phi(\rho^-, \bar{a}, b) > 0$ unless $\bar{a} = a = \frac{1}{2}$, which in turn implies $\phi(\rho^-, \bar{a}, b) = \phi(\rho^+, a, b)$. The equality $\phi(\rho^+, a, b) = 0$ can only occur for $b = a$, implying $\rho^+ = \rho$. We want to show that $\phi(\rho, a, a) = 0$ implies $a = \frac{1}{2}$. For $a \neq \frac{1}{2}$ we have

$$(34) \quad \frac{\partial \phi}{\partial \rho}(\rho, a, a) = \frac{1}{2} \log_2 \left(\frac{1+\rho}{1-\rho} \right) - a\bar{a} \log_2 \left(\frac{\rho}{a\bar{a}\rho^2} + 1 \right),$$

$$(35) \quad \frac{\partial^2 \phi}{\partial \rho^2}(\rho, a, a) = \frac{\rho(1-2a)^2}{\log(2)(a+\rho\bar{a})(1-a\bar{\rho})(1-\rho^2)} > 0.$$

Part 1 of Lemma 3 now yields $0 = \phi(0, a, a) < \phi(\rho, a, a)$ as $\frac{\partial \phi}{\partial \rho}(0, a, a) = 0$. By the strict convexity of $\xi(\theta, \frac{1}{2}, \frac{1}{2})$ in θ , necessarily $\theta_\rho = \frac{\langle f, T_\rho g \rangle}{4} \in \{\theta_\rho^+, \theta_\rho^-\} = \pm \frac{\rho}{4}$. The Cauchy-Schwarz inequality together with [6, Proposition 2.50] yields $\rho^2 = \langle f, T_\rho g \rangle^2 = \langle T_{\sqrt{\rho}} f, T_{\sqrt{\rho}} g \rangle^2 \leq \langle f, T_\rho f \rangle \langle g, T_\rho g \rangle \leq \rho^2$. Thus, necessarily $g = \pm f = \pm \chi_i$ for some $i \in [n]$ by [6, Proposition 2.50].

4. Discussion. The key idea underlying the proof of Theorem 1 is to split $\theta_1 = \tau^+ + \tau^-$ into its positive and negative part (see Section 2). After reducing the problem to the inequality in Lemma 1, the remaining proof is routine analysis. However, Lemma 1 might turn out to be useful in the context of other converse proofs, in particular for the optimization of rate regions with binary random variables.

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References.

- [1] ANANTHARAM, V., GOHARI, A. A., KAMATH, S. and NAIR, C. (2013). On Hypercontractivity and the Mutual Information between Boolean Functions. In *Proc. 51st Annual Allerton Conference on Communication, Control, and Computing* 13–19.
- [2] COURTADE, T. A. and KUMAR, G. R. (2014). Which Boolean Functions Maximize Mutual Information on Noisy Inputs? *IEEE Trans. Inf. Theory* **60** 4515–4525.
- [3] COVER, T. M. and THOMAS, J. A. (2006). *Elements of Information Theory*. John Wiley & Sons.
- [4] KLOTZ, J. G., KRACHT, D., BOSSERT, M. and SCHÖBER, S. (2014). Canalizing Boolean Functions Maximize Mutual Information. *IEEE Trans. Inf. Theory* **60** 2139–2147.
- [5] KUMAR, G. R. and COURTADE, T. A. (2013). Which Boolean Functions are Most Informative? In *Proc. IEEE Int. Symp. on Inform. Theory* 226–230.
- [6] O’DONNELL, R. (2014). *Analysis of Boolean Functions*. Cambridge University Press.
- [7] ROBERTS, A. W. and VARBERG, D. E. (1973). *Convex Functions*. Academic Press.

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