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# On the non-homogeneous boundary value problem for Schrödinger equations

Corentin Audiard <sup>\*†</sup>

## Abstract

In this paper we study the initial boundary value problem for the Schrödinger equation with non-homogeneous Dirichlet boundary conditions. Special care is devoted to the space where the boundary data belong. When  $\Omega$  is the complement of a non-trapping obstacle, well-posedness for boundary data of optimal regularity is obtained by transposition arguments. If  $\Omega^c$  is convex, a local smoothing property (similar to the one for the Cauchy problem) is proved, and used to obtain Strichartz estimates. As an application local well-posedness for a class of subcritical non-linear Schrödinger equations is derived.

**Keywords :** Schrödinger equations, non homogeneous boundary value problem, non-trapping and convex obstacles, dispersive estimates.

## Introduction

The purpose of this article is to study the initial boundary value problem (IBVP)

$$\begin{cases} i\partial_t u + \Delta_D u = f, & (x, t) \in \Omega \times [0, T], \\ u|_{t=0} = u_0, & x \in \Omega, \\ u|_{\Sigma} = g, & (x, t) \in \Sigma := \partial\Omega \times [0, T], \end{cases} \quad (\text{IBVP})$$

where  $f$  may be a forcing term or a nonlinearity depending on  $u$  (but not its gradient), typically behaving like a power of  $u$ . We recall that an homogeneous boundary value problem (BVP) corresponds to  $g = 0$ , while a pure boundary value problem would be  $u_0 = 0$ .

The homogeneous BVP for the Schrödinger equation in non trivial geometrical settings has received a lot of interest over the last years. The first results that were not consequences of semi-groups arguments were obtained in dimension 2 for  $u_0 \in H_0^1$  (see Brezis-Gallouët [5], Tsutsumi [29]) which is precisely the level of regularity where the semi-groups arguments do not work anymore. They received a number of significant extensions, until the work of Burq-Gerard-Tzvetkov [6] who obtained the first results (to our knowledge) of global well-posedness for large dimensions and data when the equation is posed on the complement of a compact “non trapping” obstacle. An important idea was to separate the solution in two parts localized near and far from the obstacle, it had some similarity with the method of Staffilani-Tataru in [25] who dealt with the Cauchy problem, but with a variable coefficient Laplacian.

Since then, there has been important developments on the link between the geometry of  $\Omega$  and the existence of dispersive estimates (and of course related well-posedness results). Ivanovici [12] proved that the full range of Strichartz estimates holds for the homogeneous BVP posed outside a strictly convex set, and obtained with Planchon [13] the well-posedness of the energy critical quintic Schrödinger equation on general non-trapping domains. Surprisingly, dispersive estimates were

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obtained even for bounded domains [2][3]. Morawetz and virial identities also proved to be powerful tools even for boundary value problems (at least for simple geometries, e.g. if  $\Omega$  is star shaped), see for example [23].

Nevertheless, all the results cited above concern homogeneous BVP. For non-homogeneous boundary data results are more scarce, since even the question of the natural regularity of Dirichlet data is not absolutely clear yet. Bu and Strauss [26] proved a global existence result for defocusing nonlinear Schrödinger equations with smooth boundary conditions. The boundary controllability of the Schrödinger equation has also received a lot of interest, in linear settings [17][16][21][27], and more recently for nonlinear problems [24][22], however works on controllability are done for  $\Omega$  bounded, and to our knowledge trace smoothness and dispersive estimates is still a largely open subject in both bounded and unbounded domains.

It should be underlined that the local-well posedness of the nonlinear IBVP in the special case of dimension one is quite well understood since the work of Holmer [10], in particular he clarified the fact that for an initial data  $u_0 \in H^s(\mathbb{R}^+)$ , the boundary data should belong to  $H^{(s+1/2)/2}(\mathbb{R}_t)$ . The numerology of regularity indices may be understood as follows : local smoothing allows the trace of the solution to gain 1/2 derivative in space, then from the scaling of the equation one expects the solution to be twice less regular in time than in space. In larger dimension however, the situation remains largely open.

The natural space for boundary data can be somehow circumvented by scaling arguments. We note that in any dimension  $d$ , if  $u(x, t)$  is solution of the linear Schrödinger equation,  $u(\lambda x, \lambda^2 t)$  is also a solution, with initial data  $u_0(\lambda x)$ . Now  $\|u_0(\lambda \cdot)\|_{H^s} = \lambda^{s-d/2} \|u_0\|_{H^s}$ , while

$$\|u(\lambda \cdot, \lambda^2 \cdot)\|_{L^2(\mathbb{R}_t, H^{s+1/2}(\partial\Omega))} = \lambda^{s+1/2-(d-1)/2-2/2} \|u\|_{L^2 H^{s+1/2}} = \lambda^{s-d/2} \|u\|_{L^2 H^{s+1/2}},$$

$$\|u(\lambda \cdot, \lambda^2 \cdot)\|_{H^{(s+1/2)/2}(\mathbb{R}_t, L^2(\partial\Omega))} = \lambda^{s-d/2} \|u\|_{H^{(s+1/2)/2} L^2}.$$

From this little computation, it appears that the boundary data  $g$  should belong to some anisotropic Sobolev space of the kind  $H^{(s+1/2)/2}(\mathbb{R}_t, L^2(\partial\Omega)) \cap L^2(\mathbb{R}_t, H^{s+1/2}(\partial\Omega))$ . Another argument in the same direction is the following estimate (which originates at least to [15], see theorem 4.3 in [18] for a simple proof),

$$\sup_{x_j} \iint | |D_{x_j}|^{1/2} e^{it\Delta} u_0 |^2 dx_1 \cdots dx_{j-1} dx_{j+1} \cdots dx_d dt \lesssim \|u_0\|_{L^2}^2,$$

this strengthens the idea that the boundary data should gain 1/2 derivative with respect to the Cauchy data.

Actually, in the flat case  $\Omega = \mathbb{R}^{d-1} \times \mathbb{R}^+$ , we proved in [4] that the IBVP for the linear Schrödinger equation is well posed for  $u_0 \in H_0^1$ ,  $g \in L^2([0, T], H^{3/2}(\partial\Omega)) \cap H_0^{3/4}([0, T], L^2)$ . The proof used direct methods (Fourier-Laplace transform), but relied on a relatively complex construction of a Fourier multiplier to derive a priori estimate, whose adaptation for general curved boundaries is certainly not possible in general, since it is expected that the geometry of the domain plays a crucial role in well-posedness issues (and in particular boundary regularity).

This article aims at giving similar (and more precise) results for less simple geometries, by relying on very different tools. Our main result is Theorem 16, which states a local well-posedness result for the nonlinear Schrödinger equation in dimension 2 and 3 with a nonlinearity satisfying  $|F(z)| \lesssim |z|(1+|z|)^\alpha$ ,  $\alpha < 2/(d-1)$  and some other standard assumptions. The main difficulties are of course

1. Low regularity of the initial data, namely  $u_0 \in H^{1/2}$  (the corresponding *critical power* would be  $\alpha = 4/(d-1)$ ),
2. Consistant numerology of the boundary data  $g \in L^2([0, T], H^1) \cap H^{1/2}([0, T], L^2)$  (actually a slight loss will be necessary for the nonlinear problem, but not for the linear one).

We give in section 2 a well-posedness result (without dispersive estimates) for the linear IBVP that is only obtained by classical duality/representation arguments and a local smoothing property for the homogeneous BVP. Since local smoothing occurs as soon as  $\Omega$  is non trapping (see [6]), we do not need further geometric assumptions for this part. In section 3, we use a virial identity to derive a local smoothing property for the nonhomogeneous BVP. Contrarily to section 2, we assume that  $\Omega$  is the complement of a convex set to have good multipliers so that the virial identity gives useful estimates. With arguments very similar to [6], we then deduce Strichartz estimates with loss of  $1/2$  derivative for the nonhomogeneous BVP.

To summarize, the structure of the paper is as follows :

- Section 1 sets up some notations, defines the spaces used and their embedding or interpolation relations.
- Section 2 prove the well-posedness of the initial boundary value problem with Dirichlet boundary condition of optimal regularity, this is done by classical duality arguments in the spirit of those used in control theory, but we need to carefully use the local smoothing properties of the homogeneous BVP in order to include appropriately the boundary data and forcing term,
- In section 3 we use a virial identity to prove a local smoothing property, which then implies (most likely non-optimal) Strichartz estimates,
- Section 4 makes use of these estimates to prove a local well-posedness result for a class of subcritical nonlinear IBVPs in dimension 2 and 3, without smallness assumptions.

Some comments about directions of further investigation are included in the end.

## 1 Notations and a reminder on Sobolev spaces

$L^p(\Omega)$  denotes the usual Lebesgue spaces on the open set  $\Omega$ . If  $X$  is a Banach space, we use the compact notation

$$L^p([0, T]; X) = L_T^p X.$$

If  $a, b$  depend on a number of parameter we mean by  $|a| \lesssim |b|$  that there exists some constant  $C > 0$  independant of those parameters such that  $|a| \leq C|b|$ . For two Banach spaces  $X, Y$ , the notation  $X \hookrightarrow Y$  means that  $X$  is continuously embedded in  $Y$ .

For  $m$  a positive integer, the space  $W^{m,p}(\Omega)$  is defined as

$$\{u \in L^p(\Omega), \forall |\alpha| \leq m, \partial^\alpha u \in L^p, \|u\|_{W^{m,p}} = \sum_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^p}\},$$

We also use the quasi norm associated to the homogeneous spaces  $\dot{W}^{m,p}$

$$\|u\|_{\dot{W}^{m,p}} = \sum_{|\alpha|=m} \|\partial^\alpha u\|_{L^p}.$$

If  $p = 2$ , we follow the usual notation  $W^{m,2} = H^m$ . For  $\Omega = \mathbb{R}^d$ ,  $H^m(\mathbb{R}^d)$  is equivalently defined as

$$\int_{\mathbb{R}^d} (1 + |\xi|^2)^m |\widehat{u}|^2 d\xi < \infty,$$

moreover this is also the definition used for  $H^s$ ,  $s \in \mathbb{R}$ . The  $C_c^\infty(\mathbb{R}^d)$  functions are a dense subset of  $H^s(\mathbb{R}^d)$ , the  $C_c^\infty(\overline{\Omega})$  functions are a dense subset of  $H^s(\Omega)$  under geometric assumptions that we assume to be satisfied (for details, see Adams [1]).

For  $s \in \mathbb{R}^+ \setminus \mathbb{N}$ , the  $W^{s,p}(\mathbb{R}^d)$  spaces are defined by complex interpolation between  $L^p(\mathbb{R}^d)$  and

$W^{m,p}(\mathbb{R}^d)$ ,  $m > s$  (this gives the family of Bessel potential spaces, sometimes noted  $H^{s,p}$ , as described in [1] chapter 7 or [28]). If  $\Omega$  is a smooth open set,  $s > 0$ , the functions of  $W^{s,p}(\Omega)$  are defined as restriction on  $\Omega$  of functions in  $W^{s,p}(\mathbb{R}^d)$ . If  $\partial\Omega$  is a smooth compact submanifold,  $H^s(\partial\Omega)$  is defined by local maps. The spaces  $\dot{W}^{s,p}$  are defined in the same way. Following Lions and Magenes [20], we will also use anisotropic Sobolev spaces. For  $\mathcal{M} = \Omega$  or  $\partial\Omega$ , we set

$$H^{s,2}(\mathcal{M} \times [0, T]) := L^2([0, T]; H^s(\mathcal{M})) \cap H^{s/2}([0, T]; L^2(\mathcal{M})),$$

(for the reader familiar with Lions-Magenes notations, those are the  $H^{s,s/2}(\Sigma)$  spaces).

Given  $X^s$  a space of functions where  $s$  is the level of regularity, if  $s$  is an integer we define  $X_0^s$  as the closure of  $C_c^\infty(U)$  in  $X$ , and for  $s \geq 0$  it is defined by interpolation. Negative order Sobolev spaces are defined by duality :  $\forall s \geq 0$ ,  $H^{-s} := (H_0^s)'$ .

Note that this definition **does not always coincide** with the usual one where  $X_0^s := \overline{C_c^\infty(U)}^{X^s}$ . For example the closure of  $C_c^\infty$  in  $H^{1/2}$  is  $H^{1/2}$ , while the interpolated space  $[L^2, H_0^1]_{1/2}$  is not  $H^{1/2}$  (it is the space usually called  $H_{00}^{1/2}$ , see [19]), and as a direct consequence  $H^{-1/2}$  is not  $(H^{1/2})'$ . We make this non conventional choice because it will allow us to give much more compact statements, where we will not have to separate the case  $s \equiv 1/2[\mathbb{Z}]$ .

We will use the following results on traces and liftings of functions in Sobolev spaces.

**Theorem 1.** (Lions-Magenes [19] for usual Sobolev spaces, [20] for anisotropic)

Given  $\Omega$  a smooth open set, for any  $s > 1/2$ ,  $\alpha$  multi-index such that  $|\alpha| < s - 1/2$ , the application

$$\begin{aligned} \text{tr}^\alpha : H^s(\Omega) &\rightarrow H^{s-|\alpha|-1/2}(\partial\Omega), \\ u &\rightarrow \partial^\alpha u|_{\partial\Omega}, \end{aligned}$$

is a continuous linear operator. In the special case  $\partial\Omega \times [0, T]$  or  $\Omega \times [0, T]$ ,  $s > 1/2$ , the application

$$\begin{aligned} [0, T] &\rightarrow H^{s-1/2}(\partial\Omega) \text{ or } H^{s-1/2}(\Omega), \\ t &\rightarrow u(\cdot, t), \end{aligned}$$

depends continuously of  $t$ . Conversely, for each of these operators there exists a lifting, namely a right inverse  $(\text{tr}^\alpha)^{-1} : H^{s-|\alpha|-1/2}(\partial\Omega) \rightarrow H^s(\Omega)$  such that  $\text{tr}^\alpha \circ (\text{tr}^\alpha)^{-1} = \text{Id}$ .

Similarly, for any fixed  $t \in [0, T]$  the trace

$$\text{tr}_t : H^{s,2}([0, T] \times \Omega) \rightarrow H^{s-1}(\{t\} \times \Omega) \times H^{s-1/2,2}([0, T] \times \partial\Omega),$$

is well defined, continuous and moreover its first component is continuous in  $t$ . The operator  $\text{tr}_t$  is generally not a surjection (there are usually compatibility conditions), but the projections on the first and second coordinate define surjections. For  $s > 1$ , the trace

$$H^{s,2}([0, T] \times \partial\Omega) \rightarrow H^{s-1,2}(\{t\} \times \partial\Omega),$$

is a continuous surjection for fixed  $t$ , and continuously depends on  $t$ .

We also give a list of interpolation and injection relations between Sobolev spaces.

**Theorem 2.** (Lions-Magenes [19][20], Triebel [28])

Let  $\Omega$  be a smooth open set (or manifold) with compact boundary.

For  $0 \leq s < 1/2$ ,  $H_0^s(\Omega) = H^s(\Omega)$ , so that  $H^{-s} = (H^s)'$ , and  $\overline{C_c^\infty(\Omega)}^{H^r} = H^r$  for  $0 \leq r \leq 1/2$ .

For  $0 \leq \theta \leq 1$ ,  $s_1, s_2 \geq 0$ , the Sobolev spaces satisfy the following (complex) interpolation relations

$$[H^{s_1}, H^{s_2}]_\theta = H^{(1-\theta)s_1 + \theta s_2},$$

$$[H^{s_1}, (H^{s_2})']_\theta = \begin{cases} H^{(1-\theta)s_1 - \theta s_2} & \text{if } (1-\theta)s_1 - \theta s_2 \geq 0, \\ (H^{-(1-\theta)s_1 + \theta s_2})' & \text{if } (1-\theta)s_1 - \theta s_2 < 0. \end{cases}$$

$$[H_0^{s_1}, H_0^{s_2}]_\theta = H_0^{(1-\theta)s_1 + \theta s_2}, [H^{-s_1}, H^{-s_2}]_\theta = H^{-(1-\theta)s_1 - \theta s_2}, [H_0^{s_1,2}, H_0^{s_2,2}]_\theta = H_0^{\theta s_2 + (1-\theta)s_1,2},$$

$$[H^{-s_1}, H_0^{s_2}]_\theta = \begin{cases} H_0^{\theta s_2 - (1-\theta)s_1} & \text{if } \theta s_2 - (1-\theta)s_1 \geq 0, \\ H^{\theta s_2 - (1-\theta)s_1} & \text{if } \theta s_2 - (1-\theta)s_1 < 0, \end{cases}$$

$$[H^{-s_1}, H^{s_2}]_\theta = \begin{cases} H^{\theta s_2 - (1-\theta)s_1} & \text{if } \theta s_2 - (1-\theta)s_1 \geq 0, \\ H^{\theta s_2 - (1-\theta)s_1} & \text{if } \theta s_2 - (1-\theta)s_1 < 0, \end{cases}$$

Given any  $p \geq 1$ , the following Sobolev embeddings hold :

- For  $s < d/p$ ,  $W^{s,p} \hookrightarrow L^q$ ,  $q = dp/(d - sp)$ ,
- For  $d/p + 1 > s > d/p$ ,  $W^{s,p} \hookrightarrow C^{0,\alpha}$ ,  $\alpha = s - d/p$ .

*Proof.* These results are mostly standard, yet some special parameters of the interpolation identities do not seem to be covered in our references. We give a sketch of proof for the most significant case

$$[H^{-1/2}, H_0^{3/2}]_\theta = \begin{cases} H_0^{2\theta - 1/2} & \text{if } \theta \geq 1/4, \\ H_0^{2\theta - 1/2} & \text{if } \theta < 1/4. \end{cases}$$

According to [19] chapter 1 Theorem 12.3,

$$H^{-1/2} = [H^{-1}, H_0^2]_{1/6}, H_0^{3/2} = [H^{-1}, H_0^2]_{5/6},$$

thus, using the reiteration theorem (chapter 1, Theorem 6.1) and the interpolation theorem 12.4 in [19]

$$[H^{-1/2}, H_0^{3/2}]_\theta = [H^{-1}, H_0^2]_{1/6 + 2\theta/3} = \begin{cases} H_0^{2\theta - 1/2} & \text{if } \theta \geq 1/4, \\ H_0^{2\theta - 1/2} & \text{if } \theta < 1/4. \end{cases}$$

The other cases are either classical or direct adaptations of this argument.  $\square$

## 2 The linear boundary value problem

In this section we only assume that  $\Omega$  is a non-trapping open set of compact complement, with smooth boundary (meaning that we have local maps as differentiable as needed). Non trapping means roughly that any ray reflecting on the boundary according to the laws of geometric optics goes to infinity. The mathematical definition is actually more involved since a ray may touch the boundary tangentially, and then “glide” on it, see [11] section 24.3 for a precise definition.

For  $0 < T < \infty$ , we set  $\Sigma = [0, T] \times \partial\Omega$ ,  $\Delta_D$  is the Dirichlet Laplacian on  $\Omega$ , of domain  $H^2 \cap H_0^1$ . It is a self adjoint operator, for which  $e^{it\Delta_D}$  is defined by the functional calculus.

We first recall a few results on the homogeneous boundary value problem.

**Lemma 1.** For  $0 \leq s \leq 1$ , the solution of the boundary value problem

$$\begin{cases} i\partial_t u + \Delta_D u = f \in L^1([0, T], H_0^s(\Omega)), \\ u|_{t=0} = u_0 \in H_0^s, \\ u|_\Sigma = 0, \end{cases} \quad (2.1)$$

is  $u(t) = e^{it\Delta_D} u_0 + \int_0^t e^{i(t-s)\Delta_D} f(s) ds$ , it satisfies the semigroup estimate

$$\|e^{it\Delta_D} u_0\|_{L^\infty([0, T], H_0^s)} = \|u_0\|_{H_0^s}, \quad \left\| \int_0^t e^{i(t-s)\Delta_D} f(s) ds \right\|_{L^\infty([0, T], H_0^s)} \leq \|f\|_{L^1([0, T], H_0^s)}, \quad (2.2)$$

and the local smoothing property

$$\forall \chi \in C_c^\infty(\mathbb{R}^d), \forall T > 0, \|\chi \nabla u\|_{L^2([0,T], L_x^2)} \leq C_T (\|u_0\|_{H_0^{1/2}} + T \|f\|_{L^1([0,T], H_0^{1/2})}). \quad (2.3)$$

*Proof.* The local smoothing without forcing term is Prop 2.7 in [6]

$$\|\chi e^{it\Delta_D} u_0\|_{L_T^2 H^1} \leq C_T \|u_0\|_{H_0^{1/2}},$$

the estimate with a forcing term  $f$  is then a direct consequence of Minkowski's integral inequality

$$\begin{aligned} \left\| \int_0^t \chi e^{i(t-s)\Delta_D} f(s) ds \right\|_{L_T^2 H^1} &\leq \left\| \int_0^T \mathbf{1}_{s \leq t} \|\chi e^{i(t-s)\Delta_D} f(s)\|_{H^1} ds \right\|_{L_T^2} \\ &\lesssim \int_0^T \|\mathbf{1}_{s \leq t} \|\chi e^{i(t-s)\Delta_D} f(s)\|_{H^1}\|_{L_T^2} ds \leq TC_T \|f\|_{L_T^1 H_0^{1/2}}. \end{aligned}$$

□

The local smoothing can be used to derive a (well-known) trace smoothing.

**Proposition 3.** *Let  $u_0 \in H_0^{1/2}(\Omega)$ ,  $u \in C_t H^{1/2} \cap L_T^2 H_{0,loc}^1$  be the unique solution of the boundary value problem (2.1). Its normal derivative on  $\partial\Omega$  is well defined and satisfies the estimate*

$$\|\partial u / \partial n\|_{L^2([0,T] \times \partial\Omega)} \lesssim \|u_0\|_{H_0^{1/2}} + \|f\|_{L_T^1 H_0^{1/2}}.$$

*Proof.* We follow the multiplier method from [21] combined with the local smoothing. (this is also done with a slightly different method, comparably simple but without an  $f$  term, by Planchon-Vega in [23]). By a density argument, we may assume that  $u$  is smooth enough so that all the computations are rigorous. For  $q$  in  $C_c^2(\mathbb{R}^d)$ , and denoting  $n$  the normal on  $\partial\Omega$ , we have the identity

$$\begin{aligned} \frac{1}{2} \int_\Sigma q \cdot n \left| \frac{\partial u}{\partial n} \right|^2 dS &= \frac{1}{2} \left[ \text{Im} \int_\Omega u q \nabla \bar{u} dx \right]_0^T + \frac{1}{2} \text{Re} \int_{[0,T] \times \Omega} u (\nabla \text{div} q) \cdot \nabla \bar{u} dx dt \\ &\quad + \text{Re} \int_{[0,T] \times \Omega} f q \cdot \nabla \bar{u} dx dt + \text{Re} \int_{[0,T] \times \Omega} \nabla \bar{u}^t \nabla q \nabla u dx dt \\ &\quad + \frac{1}{2} \text{Re} \int f \bar{u} \text{div} q dx dt. \end{aligned}$$

We choose  $q$  such that  $q \cdot n > 0$  on  $\partial\Omega$ , this gives

$$\begin{aligned} \|\partial_n u\|_{L^2}^2 &\lesssim \|u\|_{L_T^\infty H_0^{1/2}}^2 + \|\nabla \text{div} q \cdot \nabla u\|_{L_{x,T}^2}^2 + \|f\|_{L^1 H_0^{1/2}} \|u\|_{L^\infty H_0^{1/2}} \\ &\quad + \|\sqrt{|\nabla q|} |\nabla u|\|_{L_{x,T}^2}^2 \\ &\lesssim \|u_0\|_{H_0^{1/2}}^2 + \|f\|_{L_T^1 H_0^{1/2}}^2. \end{aligned}$$

□

As a consequence we prove by duality the well-posedness of the non-homogeneous boundary value problem with a compactly supported forcing term.

**Definition 1.** *Let  $\chi \in C_c^\infty(\mathbb{R}^d)$ ,  $f \in L_T^2 H^{-1}(\Omega)$ . We say that  $u$  is a transposition solution of the problem*

$$\begin{cases} i\partial_t u + \Delta u = \chi f \in L_T^2 H^{-1}, \\ u|_{t=0} = 0, \\ u|_\Sigma = g \in L^2([0, T] \times \Omega), \end{cases} \quad (2.4)$$

when  $u \in C_T H^{-1/2}$ , and for any  $f_1 \in L_T^1 H_0^{1/2}$ , if  $v$  is the solution of

$$\begin{cases} i\partial_t v + \Delta v = f_1 \in L_T^1 H_0^{1/2}, \\ v|_{t=T} = 0, \\ v|_{\Sigma} = 0, \end{cases} \quad (2.5)$$

we have the identity

$$\int_0^T \langle u, f_1 \rangle_{H^{-1/2}, H_0^{1/2}} dt = \int_0^T \langle f, \chi v \rangle_{H^{-1}, H_0^1} dt + \int_0^T (g, \partial_n v)_{L^2(\partial\Omega)} dt, \quad (2.6)$$

(here  $\langle \cdot, \cdot \rangle_{X, X'}$  denotes the usual duality product).

*Remark 4.* It is clear that any smooth solution (say  $C^2$ ) of (2.4) is a transposition solution. Indeed by density of  $C_c^\infty$  in  $L_T^1 H_0^{1/2}$  it is sufficient to check the identity (2.6) for smooth  $v$ , which is simply an integration by parts.

*Remark 5.* For technical reasons that are clarified hereafter, the construction of a  $C_T H^{3/2}$  solution requires that the forcing term  $f$  satisfies  $f|_{\partial\Omega} \in L_T^2 L^2(\partial\Omega)$ , therefore it is preferable to chose  $f$  in the dual smoothing space  $L_T^2 H^1$  rather than the natural space  $L_T^1 H^{3/2}$ .

**Proposition 6.** *For any  $(g, f) \in L^2(\Sigma) \times L^2([0, T], H^{-1}(\Omega))$ , the problem (2.4) has an unique solution  $u \in C_T H^{-1/2}(\Omega)$ . If moreover  $g \in H_0^{s, 2}$ ,  $\chi f \in L_T^2 H^{s-1}$  for  $0 < s \leq 2$ , then  $u \in C_T H^{s-1/2}$ , with the estimate*

$$\|u\|_{C_T H^{s-1/2}} \lesssim \|g\|_{H^{s, 2}} + \|f\|_{L_T^2 H^{s-1}}.$$

In particular, if  $(g_n, f_n) \in C_c^\infty(\Sigma) \times C_c^\infty([0, T] \times \bar{\Omega})$  converges to  $(g, f)$ , then  $u$  is the strong limit of the corresponding sequence of solutions  $(u_n)$ .

*Proof.* The uniqueness is clear since  $(L_T^1 H_0^{1/2})' = L_T^\infty H^{-1/2}$  (actually duality gives uniqueness in  $L_T^\infty H^{-1/2}$ ). For  $f_1 \in L_T^1 H_0^{1/2}(\Omega)$ , we denote by  $v$  the solution of (2.5). The linear form  $L$  is defined as

$$\begin{aligned} L : L_T^1 H_0^{1/2} &\rightarrow \mathbb{R} \\ f_1 &\rightarrow \int_0^T \langle f, \chi v \rangle_{H^{-1}, H_0^1} dt + \int_0^T (g, \partial_n v) dt. \end{aligned}$$

Using Cauchy-Schwarz's inequality, duality, and the local smoothing we get

$$|Lf_1| \leq \|f\|_{L_T^2 H^{-1}} \|\chi v\|_{L_T^2 H_0^1} + \|g\|_{L^2(\Sigma)} \|\partial_n v\|_{L^2(\Sigma)} \lesssim (\|g\|_{L^2(\Sigma)} + \|f\|_{L_T^2 H^{-1}}) \|f_1\|_{L_T^1 H_0^{1/2}},$$

so that  $L$  is continuous on  $L_T^1 H_0^{1/2}$ . Thus it can be represented by some  $u \in L_T^\infty H^{-1/2}$  satisfying

$$\forall f_1 \in L_T^1 H_0^{1/2}, \quad \int_0^T \langle \chi f, v \rangle_{H^{-1}, H_0^1} dt + \int_0^T (g, \partial_n v) dt = \int_0^T \langle u, f_1 \rangle_{H^{-1/2}, H_0^{1/2}} dt,$$

and the continuity of  $L$  reads as

$$\|u\|_{L_T^\infty H^{-1/2}} \lesssim \|g\|_{L^2(\Sigma)} + \|\chi f\|_{L_T^2 H^{-1}}. \quad (2.7)$$

It remains to prove that  $u$  actually belongs to  $C_T H^{-1/2}$ . We take two sequences of smooth approximations  $g_n \rightarrow g$  ( $L^2(\Sigma)$ ),  $f_n \rightarrow f$  ( $L_T^2 H^{-1}$ ). The problem (2.4) with data  $(g_n, f_n)$  admits a smooth solution  $u_n$  (this is a consequence of simple trace arguments and the theory for the homogeneous problem, see for example [20]), which is also a transposition solution. In particular,  $u_n \in C_T H^{-1/2}$



and the estimate (2.7) implies that  $u$  is the limit of  $u_n$  in  $L_T^\infty H^{-1/2}$ . Since  $C_T H^{-1/2}$  is closed in  $L_T^\infty H^{-1/2}$  this implies  $u \in C_T H^{-1/2}$ .

For higher regularities, we first prove that if  $g \in H_0^{2,2}$ ,  $f \in L_T^2 H^1$  then  $u \in C_T H^{3/2}$ . Up to assuming  $g, f$  smooth and doing again a density argument the function  $u$  can be assumed as smooth as needed and we are only reduced to obtain a priori estimates (note that it is essential here that  $g \in H_0^{2,2}$ , and not simply  $H^{2,2}$  else standard compatibility conditions at  $t = 0$ ,  $x \in \partial\Omega$  would not be satisfied, preventing  $u$  from being smooth). Since  $\chi f \in L_T^2 H^1(\Omega)$ , we have  $\chi f|_\Sigma \in L_T^2 L^2(\partial\Omega)$  (actually  $L_T^2 H^{1/2}(\partial\Omega)$ , but this is not needed here). The function  $\Delta u$  is the transposition solution of

$$\begin{cases} i\partial_t u' + \Delta u' = \Delta(\chi f), \\ u'|_{t=0} = \Delta u_0 = 0, \\ u'|_\Sigma = f|_\Sigma - i\partial_t g \in L^2([0, T] \times \partial\Omega), \end{cases} \quad (2.8)$$

in particular (2.7) gives

$$\begin{aligned} \Delta u \in C_T H^{-1/2} \text{ with } \|\Delta u\|_{C_T H^{-1/2}} &\lesssim \|\partial_t g\|_{L^2} + \|f|_{\partial\Omega}\|_{L^2(\Sigma)} + \|\Delta(\chi f)\|_{L_T^2 H^{-1}} \\ &\lesssim \|g\|_{H^{2,2}} + \|\chi f\|_{L_T^2 H^1}. \end{aligned}$$

We can reformulate this as :  $u(\cdot, t)$  is solution of the elliptic boundary value problem

$$\begin{cases} -\Delta u = \varphi, \\ u|_{\partial\Omega} = g, \end{cases}$$

where  $\|\varphi\|_{C_T H^{-1/2}} \lesssim \|g\|_{H^{2,2}} + \|\chi f\|_{L_T^2 H^1}$ . According to Theorem 1,  $H^{2,2}([0, T] \times \partial\Omega) \hookrightarrow C_T H^1(\partial\Omega)$ , so that by lifting (same theorem) there exists  $u_1 \in C_T H^{3/2}$  such that  $u_1|_{\partial\Omega} = g$ ,  $\|u_1\|_{C_T H^{3/2}} \lesssim \|g\|_{H^{2,2}}$ . The function  $w = u - u_1$  satisfies

$$\begin{cases} -\Delta w = \varphi + \Delta u_1, \\ w|_{\partial\Omega} = 0, \end{cases}$$

by elliptic regularity we have  $w \in C_T H^{3/2}$ ,  $\|w\|_{C_T H^{3/2}} \lesssim \|g\|_{H^{2,2}} + \|\chi f\|_{L_T^2 H^1}$  then

$$\|u\|_{C_T H^{3/2}} \leq \|u_1\|_{C_T H^{3/2}} + \|w\|_{C_T H^{3/2}} \lesssim \|g\|_{H^{2,2}} + \|\chi f\|_{L_T^2 H^1},$$

(see Gilbarg and Trudinger [8] Theorem 8.8 for regularity with a forcing term in  $L^2$ , the  $H^{-1}$  case is an application of Lax Milgram theorem, finally the  $H^{-1/2}$  case follows from the relations  $[H^{-1}, L^2]_{1/2} = H^{-1/2}$ , and  $[H_0^1, H^2]_{1/2} \subset H^{3/2}$  in Theorem 2).

The result for  $0 \leq s \leq 2$  is obtained by interpolation.  $\square$

A simple consequence is the well-posedness of the full initial boundary value problem.

**Definition 2.** Let  $f \in L_T^1 H^{-1/2} \cap L_T^2 H_{loc}^{-1}$ ,  $u_0 \in H^{-1/2}(\Omega)$ ,  $g \in L^2(\Sigma)$ . We say that  $u$  is a transposition solution of the problem

$$\begin{cases} i\partial_t u + \Delta u = f, \\ u|_{t=0} = u_0, \\ u|_\Sigma = g, \end{cases} \quad (2.9)$$

when  $u \in C_T H^{-1/2}$ , and for any  $f_1 \in L_T^1 H_0^{1/2}$ , if  $v$  is the solution of

$$\begin{cases} i\partial_t v + \Delta v = f_1 \in L^1 H_0^{1/2}([0, T] \times \Omega), \\ v|_{t=T} = 0, \\ v|_\Sigma = 0, \end{cases}$$

we have the identity for some fixed  $\chi \in C_c^\infty(\mathbb{R}^d)$ ,  $\chi = 1$  near  $\partial\Omega$

$$\begin{aligned} \int_0^T \langle u, f_1 \rangle_{H^{-1/2}, H^{1/2}} dt &= \int_0^T \langle f, \chi v \rangle_{H^{-1}, H_0^1} + \langle f, (1-\chi)v \rangle_{H^{-1/2}, H_0^{1/2}} dt \\ &\quad + \int_0^T (g, \partial_n v)_{L^2(\partial\Omega)} dt + i \langle u_0, v(0) \rangle_{H^{-1/2}, H_0^{1/2}}. \end{aligned}$$

**Theorem 7.** For  $f \in L_T^2 H_{loc}^{s-1/2} \cap L_T^1 H^s$ ,  $g \in H^{s+1/2, 2}(\Sigma)$ ,  $u_0 \in H_0^s$ ,  $-1/2 \leq s \leq 3/2$ , the initial boundary value problem (2.9) has a unique transposition solution. It satisfies

$$\|u\|_{C_T H^s} \lesssim \|f\|_{L_T^2 H_{loc}^{s-1/2} \cap L_T^1 H^s} + \|g\|_{H^{s+1/2, 2}} + \|u_0\|_{H_0^s}.$$

Moreover the assumption  $f \in L_T^1 H^s$  can be dropped if  $f$  is compactly supported.

*Proof.* The uniqueness is again a direct duality argument.

For the existence, by linearity we simply fix  $\chi \in C_c^\infty(\mathbb{R}^d)$ ,  $\chi = 1$  on a neighbourhood of  $\Omega$ , and split (2.9) as two problems, one which is a pure BVP like (2.4)

$$\begin{cases} i\partial_t u_b + \Delta u_b = \chi f \in L_T^2 H^{s-1/2}(\Omega), \\ u_b|_{t=0} = 0, \\ u_b|_\Sigma = g \in H^{s+1/2, 2}(\Sigma), \end{cases}$$

and an other homogeneous one like (2.1)

$$\begin{cases} i\partial_t u_c + \Delta u_c = (1-\chi)f \in L_T^1 H_0^s(\Omega), \\ u_c|_{t=0} = u_0 \in H_0^s, \\ u_c|_\Sigma = 0. \end{cases}$$

It is then sufficient to check that the solution  $u_c$  given by Lemma 1 is also a transposition solution. This is the consequence of an integration by parts if  $u_c$  is smooth enough, and the general case is obtained by a density argument.

If  $f$  is compactly supported it suffices to chose  $\chi$  such that  $(1-\chi)f = 0$ .  $\square$

### 3 A smoothing estimate and application

This section is devoted to the proof of a local smoothing estimate by direct methods and its application : Strichartz estimates with losses. We consider the solution  $u$  of the IBVP (2.9). As a first step we recall the virial identity, which does not seem to be really standard for non zero boundary data. If  $h$  is a nonnegative measurable function on  $\Omega$ , we set

$$M_h(t) = \int_\Omega h(x) |u(x, t)|^2 dx. \quad (3.1)$$

For specific  $h$ , it is often possible to check that  $M_h''$  is signed (usually with quantitative estimates  $\pm M_h'' \geq \|u\|_X^2$ , with  $X$  some Banach space). This has been used in several applications, two notable ones being the explosion of solutions in finite time (the sign of  $M_h''$  implies an impossible change of sign for some quantity) and dispersive estimates, obtained simply by using

$$\|u\|_{L^2([0, T], X)}^2 \lesssim \int_0^T M_h''(T) = M_h'(T) - M_h'(0).$$

This is the second approach that we follow here, with the technical addition that  $\Omega \neq \mathbb{R}^d$  and the boundary data are not 0.

**Proposition 8.** *Let  $h$  belong to  $C^4(\Omega)$  such that  $h|_{\partial\Omega} = 0$ ,  $\nabla^k h$  is bounded for  $1 \leq k \leq 4$ . If  $u$  is a smooth solution of (2.9) we have the identities*

$$\begin{aligned} \frac{dM_h}{dt} &= 2\text{Im} \int_{\Omega} \nabla h \cdot \nabla u \bar{u} + h \bar{u} f dx, & (3.2) \\ \frac{d^2 M_h}{dt^2} - 2\text{Im} \frac{d}{dt} \int_{\Omega} h \bar{u} f dx &= 4\text{Re} \int_{\Omega} \text{Hess}(h)(\nabla u, \nabla \bar{u}) - \frac{1}{4}|u|^2 \Delta^2 h + \nabla h \cdot \nabla u \bar{f} + \frac{1}{2} \bar{u} \Delta h f dx \\ &\quad + \text{Re} \int_{\partial\Omega} 2\partial_n h |\nabla_{\tau} u|^2 - 2\partial_n h |\partial_n u|^2 - 2i\partial_n h \partial_t u \bar{u} dS \\ &\quad + \text{Re} \int_{\partial\Omega} -2\bar{u} \Delta h \partial_n u + |u|^2 \partial_n \Delta h dS, & (3.3) \end{aligned}$$

where we denoted  $\nabla_{\tau}$  the gradient along the tangent plane to  $\partial\Omega$ , or equivalently if  $n$  is the normal derivative pointing outside  $\Omega$ ,  $\nabla_{\tau} = \nabla - n\partial_n$ .

*Proof.* A direct computation gives, using that  $h|_{\partial\Omega} = 0$  :

$$\begin{aligned} \frac{dM_h}{dt} &= \int_{\Omega} h(u\partial_t \bar{u} + \bar{u}\partial_t u) dx = \int_{\Omega} ih(\Delta u \bar{u} - \Delta \bar{u} u - h f \bar{u} + h \bar{f} u) dx \\ &= -2\text{Im} \int_{\Omega} h \bar{u} \Delta u - h \bar{u} f dx \\ &= 2\text{Im} \int_{\Omega} |\nabla u|^2 h + h \bar{u} \nabla u \cdot \nabla h + \bar{u} f dx = 2\text{Im} \int_{\Omega} \bar{u} \nabla u \cdot \nabla h + h \bar{u} f dx. \end{aligned}$$

Differentiating again in  $t$  we get

$$\begin{aligned} \frac{d^2 M_h}{dt^2} - 2\text{Im} \frac{d}{dt} \int_{\Omega} h \bar{u} f dx &= 2\text{Im} \int_{\Omega} \overline{\partial_t u} \nabla u \cdot \nabla h + \bar{u} \nabla \partial_t u \cdot \nabla h dx \\ &= 2\text{Re} \int_{\Omega} \bar{u} \nabla h \cdot \nabla \Delta u - \Delta \bar{u} \nabla h \cdot \nabla u + \bar{f} \nabla u \cdot \nabla h - \bar{u} \nabla f \cdot \nabla h dx \\ &= 2\text{Re} \left( \int_{\Omega} -\nabla \bar{u} \cdot \nabla h \Delta u - \nabla h \cdot \nabla u \Delta \bar{u} - \bar{u} \Delta h \Delta u \right. \\ &\quad \left. + \int_{\Omega} 2\nabla h \cdot \nabla u \bar{f} + \bar{u} \Delta h f dx + \int_{\partial\Omega} \partial_n h \Delta u \bar{u} - \bar{u} f \partial_n h dS \right) \\ &= 2\text{Re} \left( \int_{\Omega} -2\nabla \bar{u} \cdot \nabla h \Delta u dx - \bar{u} \Delta h \Delta u + 2\nabla h \cdot \nabla u \bar{f} \right. \\ &\quad \left. + \bar{u} \Delta h f dx + \int_{\partial\Omega} -i\partial_n h \partial_t u \bar{u} dS \right). \end{aligned}$$

We focus on the integral terms on  $\Omega$  where  $f$  does not appear :

$$\begin{aligned}
\operatorname{Re} \int_{\Omega} -4\nabla\bar{u} \cdot \nabla h \Delta u - 2\bar{u} \Delta h \Delta u dx &= 4 \int_{\Omega} \operatorname{Hess}(h)(\nabla u, \nabla\bar{u}) + \frac{1}{2} \nabla h \cdot \nabla |\nabla u|^2 dx \\
&+ 2\operatorname{Re} \int_{\Omega} \Delta h |\nabla u|^2 + \bar{u} \nabla \Delta h \cdot \nabla u dx \\
&- 2\operatorname{Re} \int_{\partial\Omega} 2\nabla\bar{u} \cdot \nabla h \partial_n u + \bar{u} \Delta h \partial_n u dS \\
&= 4 \int_{\Omega} \operatorname{Hess}(h)(\nabla u, \nabla\bar{u}) - \frac{1}{2} \Delta h |\nabla u|^2 dx \\
&+ 2\operatorname{Re} \int_{\Omega} \Delta h |\nabla u|^2 + \frac{1}{2} \nabla \Delta h \cdot \nabla |u|^2 dx \\
&- 2\operatorname{Re} \int_{\partial\Omega} 2\nabla\bar{u} \cdot \nabla h \partial_n u + \bar{u} \Delta h \partial_n u dS \\
&+ 2 \int_{\partial\Omega} \partial_n h |\nabla u|^2 dS \\
&= 4 \int_{\Omega} \operatorname{Hess}(h)(\nabla u, \nabla\bar{u}) - \frac{1}{4} \Delta^2 h |u|^2 dx \\
&- 2\operatorname{Re} \int_{\partial\Omega} 2\nabla\bar{u} \cdot \nabla h \partial_n u + \bar{u} \Delta h \partial_n u dS \\
&+ 2 \int_{\partial\Omega} \partial_n h |\nabla u|^2 + \frac{1}{2} |u|^2 \partial_n \Delta h dS .
\end{aligned}$$

Finally, the boundary terms are treated by using the fact that  $h = 0$  on  $\partial\Omega$ , so that  $\nabla h = (\partial_n h)n$  :

$$\begin{aligned}
\operatorname{Re} \int_{\partial\Omega} -4\nabla\bar{u} \cdot \nabla h \partial_n u - 2\bar{u} \Delta h \partial_n u + 2\partial_n h |\nabla u|^2 + |u|^2 \partial_n \Delta h dS \\
&= \operatorname{Re} \int_{\partial\Omega} -4|\partial_n u|^2 \partial_n h - 2\bar{u} \Delta h \partial_n u + 2\partial_n h (|\partial_n u|^2 + |\nabla_{\tau} u|^2) + |u|^2 \partial_n \Delta h dS \\
&= \operatorname{Re} \int_{\partial\Omega} -2|\partial_n u|^2 \partial_n h - 2\bar{u} \Delta h \partial_n u + 2\partial_n h |\nabla_{\tau} u|^2 + |u|^2 \partial_n \Delta h dS .
\end{aligned}$$

We obtain as expected

$$\begin{aligned}
\frac{d^2 M_h}{dt^2} &= 4\operatorname{Re} \int_{\Omega} \operatorname{Hess}(h)(\nabla u, \nabla\bar{u}) - \frac{1}{4} \Delta^2 h |u|^2 dx + \nabla h \cdot \nabla u \bar{f} + \frac{1}{2} \bar{u} \Delta h f dx + 2\operatorname{Im} \frac{d}{dt} \int_{\Omega} h \bar{u} f dx \\
&+ \operatorname{Re} \int_{\partial\Omega} -2i \partial_n h \partial_t u \bar{u} - 2|\partial_n u|^2 \partial_n h - 2\bar{u} \Delta h \partial_n u + 2\partial_n h |\nabla_{\tau} u|^2 + |u|^2 \partial_n \Delta h dS .
\end{aligned}$$

□

A cautious look at identities (3.2) and (3.3) indicates that the equation

$$M'_h(T) - M'_h(0) = \int_0^T M''_h(t) dt,$$

can be turned into useful estimates if there exists  $h$  such that

1.  $h|_{\partial\Omega} = 0$ ,
2.  $\operatorname{Hess}(h)(x) \geq \alpha(x)I_d > 0$  (in the sense of quadratic forms), with possibly  $\alpha(x) \rightarrow_{x \rightarrow \infty} 0$ ,

3.  $\partial_n h|_{\partial\Omega} < 0$  or equivalently if  $n_i$  is the normal pointing inside  $\Omega$ ,  $\partial_{n_i} h > 0$ .

As was pointed out by the reviewer, there can be no such function if  $\Omega^c$  is not convex : indeed by assumption 2 the set  $\{h \leq 0\}$  must be convex and thus contains the convex hull of  $\partial\Omega$ , while assumption 3 implies that on a neighbourhood of  $\partial\Omega$ ,  $h > 0$ , so that  $\text{conv}(\partial\Omega) \subset \Omega^c$ , and the converse inclusion is obvious (as  $\Omega^c$  is bounded).

Since conditions 2. and 3. only involve derivatives of  $h$ , we start by looking for a function which is constant on  $\partial\Omega$ . In the case where  $\Omega = K^c$ ,  $K$  convex,  $0 \in \text{int}(K)$  (this last assumption can obviously always be made, up to a translation), there is a natural candidate : the gauge of  $K$  - sometimes called Minkowski's functional-

$$j(x) = \inf\{\lambda > 0 : x/\lambda \in K\}. \quad (3.4)$$

Indeed, by definition  $j = 1$  on  $\partial K$  and it is well known that  $j$  is convex if  $K$  is. We also point out that  $j$  is homogeneous of degree 1 so that  $\forall x \in \Omega$ ,  $\nabla j(x) \neq 0$ .

The next lemma quantifies how positive  $\text{Hess}(j)$  is depending on the geometry of  $\partial\Omega$ . The basic tools of geometry used may be found for example in [9], chapter 10 (where the presentation is limited to dimension 3, but with clear extension to any dimension).

**Lemma 2.** *Assume that  $\partial\Omega$  is a  $C^2$  submanifold. It is defined by the implicit equation  $j(x) = 1$ . For  $x_0 \in \partial\Omega$  we denote by  $T_{x_0}$  the hyperplane tangent to  $\partial\Omega$  at  $x_0$ ,  $II(x_0)$  the second fundamental form at  $x_0$  (this is a quadratic form on  $(T_{x_0})^2$ ), and  $\text{Hess}(j)|_{T_{x_0}}$  is understood as the quadratic form defined by restriction of the bilinear application  $\text{Hess}(j)$  on  $(T_{x_0})^2$ . Then there exists  $c(K) > 0$  such that*

$$\text{Hess}(j)(x_0)|_{T_{x_0}} \geq cII(x_0).$$

*In particular,  $\text{Hess}(j)|_{T_{x_0}}$  is defined positive if and only if  $II(x_0)$  is.*

*Proof.* For conciseness and simplicity, we give a proof which is slightly formal, but that can be made rigorous by using local maps. Since  $j = 1$  on  $\partial\Omega$ , the normal pointing outside  $K$  is  $n = \nabla j / \|\nabla j\|$ . Note that this defines a vector field  $n$  defined smoothly not only on  $\partial\Omega$ , but on  $\mathbb{R}^d \setminus \{0\}$ . The second fundamental form is usually defined as the gradient of  $n$  in local coordinates in  $\partial\Omega$ , we will rather use the fact that it also coincides with the restriction of  $\nabla n(x_0)$  (seen as a quadratic form, and where  $\nabla$  is the "full" gradient on  $\mathbb{R}^d$ ) to  $(T_{x_0})^2$ . This rewrites

$$\nabla n = \text{Hess}(j) / \|\nabla j\| - \nabla j (\nabla \|\nabla j\|)^t / \|\nabla j\|^2,$$

so that for any tangential vector  $\tau \in T_{x_0}$ , using  $\tau \cdot n = 0$  :

$$II(\tau, \tau) = \tau^t \frac{\text{Hess}(j)}{\|\nabla j\|} \tau - \tau^t \frac{\nabla j (\nabla \|\nabla j\|)^t}{\|\nabla j\|^2} \tau = \tau^t \frac{\text{Hess}(j)}{\|\nabla j\|} \tau,$$

using that  $\partial\Omega$  is compact, we have  $\inf_{x \in \partial\Omega} \|\nabla j(x)\| \geq c > 0$ , this concludes the proof.  $\square$

*Remark 9.* It is not true that  $\text{Hess}(j) > 0$  as a quadratic form on  $\mathbb{R}^d$ , actually since  $j$  is homogeneous of order one we always have  $\text{Hess}(j)(x)(x, x) = 0$ . However the non negativity  $\text{Hess}(j) \geq 0$  is true since  $j$  is convex.

In view of the lemma, and since identity (3.3) involves fourth order derivatives of  $h$ , **we assume from now on that  $\partial\Omega$  is a  $C^4$  submanifold of positive principal curvatures**, so that  $II(x) > 0$  at any  $x \in \partial\Omega$ , or equivalently  $\text{Hess}(j)(x)|_{T_x} > 0$ .

**Proposition 10.** *The function  $h(x) = \sqrt{1 + j^2} - \sqrt{2}$  satisfies the three required assumptions.*

*Proof.* The identity  $h|_{\partial\Omega} = 0$  is obvious.

The positivity of  $\partial_n h$  is obtained as follows : let  $x_0 \in \partial K$ , by homogeneity we have  $\left(\frac{\partial j}{\partial x_0}\right)(x_0) = j(x_0) = 1 > 0$ . On the other hand  $j$  is constant on  $\partial\Omega$ , thus  $\nabla j(x_0) = (\partial j / \partial n(x_0))n(x_0)$ ,

$$\left(\frac{\partial j}{\partial x_0}\right)(x_0) = \nabla j(x_0) \cdot x_0 = (\partial j / \partial n(x_0))(x_0 \cdot n(x_0)).$$

Since  $K$  is star shaped with respect to 0,  $x_0 \cdot n(x_0) > 0$ , thus  $\left(\frac{\partial j}{\partial n(x_0)}\right)(x_0) \geq 1/(x_0 \cdot n) \geq 1/\|x\| \geq 1/\text{diam}(K)$ . The positivity of  $\partial_n h$  then directly follows from the identity

$$\partial_n h = j \partial_n j / \sqrt{1+j^2} = \partial_n j / \sqrt{2} \geq 1/(\sqrt{2} \text{diam}(K)).$$

By homogeneity, it is sufficient to prove the positivity of  $\text{Hess}(h)$  for  $x \in \partial K$ . By direct computations

$$\begin{aligned} \nabla h &= \frac{j \nabla j}{\sqrt{1+j^2}}, \quad \text{Hess}(h) = \frac{\nabla j \nabla j^t}{\sqrt{1+j^2}} + \frac{j \text{Hess}(j)}{\sqrt{1+j^2}} - \frac{j^2 \nabla j \nabla j^t}{\sqrt{1+j^2}^3} \\ &= \frac{\nabla j \nabla j^t}{\sqrt{1+j^2}^3} + \frac{j \text{Hess}(j)}{\sqrt{1+j^2}}. \end{aligned}$$

For  $x_0 \in \partial\Omega$ ,  $X \in \mathbb{R}^d$ , we split  $X = X_\tau + X_n$  where  $X_\tau$  belongs to  $T_{x_0}$  and  $X_n$  is parallel to  $n(x_0)$ . Using  $X_\tau \cdot \nabla j = 0$  we get

$$X^t \text{Hess}(j)(x) X = \frac{|\nabla j \cdot X_n|^2}{\sqrt{1+j^2}^3} + \frac{j X^t \text{Hess}(j) X}{\sqrt{1+j^2}}.$$

If  $X_n \neq 0$ , the fact that  $\text{Hess}(j) \geq 0$  gives

$$X^t \text{Hess}(j)(x) X \geq \frac{|\nabla j \cdot X_n|^2}{\sqrt{1+j^2}^3} > 0,$$

while if  $X \perp n$ ,  $X = X_\tau$ , Lemma 2 implies

$$X^t \text{Hess}(j)(x) X = \frac{j X_\tau^t \text{Hess}(j) X_\tau}{\sqrt{1+j^2}} = \frac{j}{\sqrt{1+j^2}} \text{Hess}(j)|_{T_x}(X_\tau, X_\tau) > 0.$$

In either case we have  $\text{Hess}(h)(x_0)(X, X) > 0$ , by compactness of  $\partial\Omega$  and homogeneity we can conclude

$$\exists c > 0 : \forall x \in \Omega, \text{Hess}(h)(x) \geq \frac{c}{\|x\|} I_d.$$

□

We may now state the local smoothing property :

**Proposition 11.** *Let  $u$  be the transposition solution of (2.9). If  $f$  is compactly supported, for any  $\varepsilon > 0$ ,*

$$\begin{aligned} \iint_{[0,T] \times \Omega} \frac{|\nabla u|^2}{\sqrt{1+|x|^2}} dx dt + \iint_{[0,T] \times \partial\Omega} |\partial_n u|^2 dS dt &\leq C_{\varepsilon,T} \left( \|g\|_{H_0^{1+\varepsilon,2}}^2 + \|u_0\|_{H_0^{1/2}}^2 \right. \\ &\quad \left. + \|f\|_{L_T^2 L^2}^2 \right). \end{aligned}$$

The proof requires a preliminary lemma due to the special structure of  $H^{1/2}$ .

**Lemma 3.** *For any  $\varepsilon > 0$ , let  $u \in C_T H^{1/2}$  be the solution of the Schrödinger equation (2.9) with  $g \in H_0^{1+\varepsilon,2}$ ,  $f \in L_T^2 L^2$  with compact support in space,  $u_0 \in H_0^{1/2}$ . Then  $\nabla u \in C_T(H^{1/2})'$  and*

$$\|\nabla u\|_{C_T(H^{1/2})'} \lesssim \|g\|_{H_0^{1+\varepsilon,2}} + \|u_0\|_{H_0^{1/2}} + \|f\|_{L_T^2 L^2}.$$

*Proof.* If  $(u_0, f, g) \in H_0^{1/2+\varepsilon} \times L_T^2 H^\varepsilon \times H_0^{1+\varepsilon,2}$ ,  $u \in C_T H^{1/2+\varepsilon}$ , so that  $\nabla u \in C_T H^{-1/2+\varepsilon} = C_T(H^{1/2-\varepsilon})'$  with obvious norm control.

If  $(u_0, f, g) \in L^2 \times L_T^2 H^{-1/2} \times H_0^{1+\varepsilon,2}$ , the embedding  $H_0^{1+\varepsilon,2} \hookrightarrow C_T H^\varepsilon$  gives the following (formal) identity

$$\forall v \in H^1, j = 1 \cdots d, t \in [0, T], \int_{\Omega} \partial_j u(t) v = - \int_{\Omega} u(t) \partial_j v dx + \int_{\partial\Omega} g(t) n_j v |_{\partial\Omega} dS,$$

which implies for smooth solutions

$$\|\nabla u\|_{C_T(H^1)'} \lesssim \|u\|_{C_T L^2} + \|g\|_{C_T L^2} \lesssim \|g\|_{H_0^{1+\varepsilon,2}} + \|f\|_{L_T^2 H^{-1}} + \|u_0\|_{H^{-1/2}},$$

the gradient is then extended as a continuous operator  $L^2 \times L_T^2 H^{-1/2} \times H_0^{1+\varepsilon,2} \rightarrow C_T(H^1)'$ .

We obtain the lemma by interpolation with interpolation parameter  $\theta = 1/(1+\varepsilon)$ .  $\square$

*Proof of Proposition 11.* By density, it is sufficient to establish the estimate for smooth solutions. We start with (3.2), according to lemma 3

$$\begin{aligned} |M'_h(t) - 2\text{Im} \int_{\Omega} h \bar{u} f dx| &= |2\text{Im} \int_{\Omega} \nabla h \cdot \nabla u \bar{u} dx| \\ &\lesssim \|u_0\|_{H_0^{1/2}(\Omega)}^2 + \|g\|_{H^{1+\varepsilon,2}}^2 + \|f\|_{L_{x,T}^2}. \end{aligned}$$

On the other hand, the identity (3.3) combined with Prop 10 gives

$$\begin{aligned} \frac{d^2 M_h}{dt^2} - 2\text{Im} \frac{d}{dt} \int_{\Omega} h \bar{u} f dx &= 4 \int_{\Omega} \text{Hess}(h)(\nabla u, \nabla \bar{u}) - |u|^2 \Delta^2 h + \nabla h \cdot \nabla u \bar{f} + \frac{1}{2} \bar{u} \Delta h f dx \\ &\quad + \text{Re} \int_{\partial\Omega} 2\partial_n h |\nabla_{\tau} u|^2 - 2\partial_n h |\partial_n u|^2 + 2i\partial_n h \partial_t u \bar{u} dS \\ &\quad + \text{Re} \int_{\partial\Omega} -2\bar{u} \Delta h \partial_n u + |u|^2 \partial_n \Delta h dS \\ &\geq c \left( \int_{\Omega} |\nabla u|^2 / \sqrt{1+|x|^2} dx + \int_{\partial\Omega} |\partial_n u|^2 dS \right) \\ &\quad - C \left( \|u\|_{L^2(\Omega)}^2 + \|\nabla_{\tau} u\|_{\partial\Omega}^2 + \delta \|\partial_n u\|_{\partial\Omega}^2 + C(\delta) \|u\|_{\partial\Omega}^2 \right) \\ &\quad + \|\nabla u\|_{L^2(B(0,R))} \|f\|_{L^2(B(0,R))} + \text{Re} \int_{\partial\Omega} 2i\partial_n h \partial_t u \bar{u} dS \\ &\geq c \left( \int_{\Omega} |\nabla u|^2 / \sqrt{1+|x|^2} dx + \int_{\partial\Omega} \frac{|\partial_n u|^2}{2} dS \right) \\ &\quad - C' \left( \|u\|_{L^2(\Omega)}^2 + \|u\|_{\partial\Omega} \|H_0^{1,2}\| + \frac{\gamma}{2C'} \|\nabla u\|_{L^2(B_R)}^2 + \frac{C'}{2\gamma} \|f\|_{L^2(B_R)}^2 \right) \\ &\quad + \text{Re} \int_{\partial\Omega} 2i\partial_n h \partial_t u \bar{u} dS, \end{aligned}$$

by choosing  $\delta \leq c/2C$ ,  $R$  such that  $\text{supp}(f) \subset B(0, R)$ .

Use next

$$|\text{Re} \int_0^T \int_{\partial\Omega} 2i\partial_n h \partial_t u \bar{u} dS| \lesssim \|\partial_t g|_{\partial\Omega}\|_{H^{-1,2}} \|g\|_{H_0^{1,2}} \lesssim \|g|_{\partial\Omega}\|_{H_0^{1,2}}^2,$$

and fix  $\gamma \leq c/(\sqrt{1+R^2})$ , the identity  $M'_h(T) - M'_h(0) = \int_0^T M''_h(t) dt$  gives as expected

$$\begin{aligned} & c \left( \int_0^T \int_{\Omega} |\nabla u|^2 / \sqrt{1+|x|^2} dx + \int_0^T \int_{\partial\Omega} \frac{|\partial_n u|^2}{2} dS \right) \\ & \leq C(T \|u\|_{L_T^\infty L^2(\Omega)}^2 + \|g\|_{H_0^{1+\varepsilon,2}}^2 + \|f\|_{L_T^2 L^2(\Omega)}^2) \\ & \quad + \frac{c}{2\sqrt{1+R^2}} \|\nabla u\|_{L^2(B_R)}^2 + C(\delta) \|u\|_{L_T^\infty L^2}^2 \\ & \quad + C \|u_0\|_{H_0^{1/2}}^2, \\ \Rightarrow & \frac{c}{2} \left( \int_0^T \int_{\Omega} |\nabla u|^2 / \sqrt{1+|x|^2} dx + \int_0^T \int_{\partial\Omega} |\partial_n u|^2 dS \right) \\ & \leq C_T (\|u_0\|_{H_0^{1/2}}^2 + \|g\|_{H_0^{1+\varepsilon,2}}^2 + \|f\|_{L_T^2 L^2(\Omega)}^2). \end{aligned}$$

□

*Remark 12.* As can be seen in the proof above, the constant  $C_{\varepsilon,T}$  blows up as  $T \rightarrow \infty$  because of the term  $-|u|^2 \Delta^2 h$ , as the sign of  $\Delta^2 h$  is unknown. It can be positive, even in very simple cases : if  $\Omega^c = B(0, 1)$  in dimension 2,  $h(x) = \sqrt{1+x_1^2+x_2^2}$ ,  $\Delta^2 h = 1/h^3 + 6/h^5 - 15/h^7 > 0$  for large  $x$ .

**Corollary 1.** *We say that  $(p, q)$ ,  $p > 2$  is a weakly admissible pair when*

$$\frac{1}{p} + \frac{d}{q} = \frac{d}{2}, \quad (3.5)$$

*Let  $u$  be a transposition solution of (2.9). Assume that  $f \in L_T^2 L^2(\Omega)$  and for some  $\chi$  compactly supported, such that  $\chi = 1$  near  $\partial\Omega$ , we have  $\chi f = f$ . Then*

$$\|u\|_{L_T^p W^{1/2,q}} \lesssim \|u_0\|_{H_0^{1/2}} + \|f\|_{L_T^2 L^2} + \|g\|_{H_0^{1+\varepsilon,2}}.$$

*Remark 13.* The usual Strichartz admissible pairs are couple  $(p, q)$  such that  $(p/2, q)$  is weakly admissible. Scaling wise, they correspond to a gain of one derivative, while the weak pairs correspond to a gain of 1/2 derivative.

The constant hidden in the  $\lesssim$  of Corollary 1 is unbounded as  $T \rightarrow \infty$ , however it is not a concern in this article since we only deal with local well-posedness.

*Proof.* The proof follows closely the method from [6], which also points to ideas of [25]. It consists in splitting  $u = \chi u + (1 - \chi)u$  in one part supported near the obstacle, and the other one away from the obstacle. Near the obstacle local smoothing is used in combination with Sobolev embeddings while the other part satisfies a Schrödinger equation on  $\mathbb{R}^d$  for which better Strichartz estimates are available.

We fix  $\chi \in C_c^\infty(\mathbb{R}^d)$  as in the statement of the corollary. Prop 11 implies

$$\|\nabla \chi u\|_{L_T^2 L^2(\Omega)} \lesssim \|u\|_{L_T^\infty L^2} + \|u/\sqrt{1+|x|^2}\|_{L_T^2 L^2} \lesssim \|u_0\|_{H_0^{1/2}} + \|g\|_{H_0^{1+\varepsilon,2}} + \|f\|_{L_T^2 L^2}.$$

As a consequence,  $\chi u \in L_T^2 H^1 \cap L_T^\infty H^{1/2} \hookrightarrow L_T^p H^{1/2+1/p}$  for any  $p \geq 2$  (the injection is a consequence of a Gagliardo-Nirenberg's inequality on the Sobolev part and Hölder's inequality on the  $L^p$  part). The Sobolev embedding implies  $\chi u \in L_T^p W^{1/2, 2dp/(dp-2)}$ , and indeed the pair  $(p, 2dp/(dp-2))$



satisfies (3.5).

We now turn to the estimate of  $(1 - \chi)u$ : since  $(1 - \chi)u = 0$  near  $\partial\Omega$  we may extend it by 0 on  $\Omega^c$ , and the extension (abusively still denoted  $(1 - \chi)u$ ) is the solution of the following Cauchy problem

$$\begin{cases} i\partial_t(1 - \chi)u + \Delta(1 - \chi)u = (1 - \chi)f + [\Delta, 1 - \chi]u, & (x, t) \in \mathbb{R}^d \times [0, T], \\ (1 - \chi)u(t = 0) = (1 - \chi)u_0, & x \in \mathbb{R}^d. \end{cases}$$

We now use that  $(1 - \chi)f = 0$ , according to the Duhamel formula

$$(1 - \chi)u(t) = e^{it\Delta}(1 - \chi)u_0 + \int_0^t e^{i(t-s)\Delta}[\Delta, 1 - \chi]u ds,$$

where  $\Delta$  is the laplacian on  $\mathbb{R}^d$ . Standard Strichartz estimates (e.g. corollary 2.3.9 in [7]) imply

$$\|e^{it\Delta}((1 - \chi)u_0)\|_{L_T^{2p}W^{1/2,q}} \lesssim \|(1 - \chi)u_0\|_{H_0^{1/2}},$$

and according to (the proof of) Prop 2.10 in [6],

$$\left\| \int_0^t e^{i(t-s)\Delta}[\Delta, (1 - \chi)]u(s) ds \right\|_{L_T^{2p}W^{1/2,q}} \lesssim \|[\Delta, 1 - \chi]u\|_{L_T^2L^2} \lesssim \|u_0\|_{H_0^{1/2}} + \|f\|_{L_T^2L^2} + \|g\|_{H_0^{1,2}}.$$

The embedding  $L_T^{2p}W^{1/2,q} \hookrightarrow L_T^pW^{1/2,q}$  ends the estimate for  $(1 - \chi)u$ , and thus the proof.  $\square$

*Remark 14.* All estimates in this section are done for compactly supported  $f \in L_T^2L^2$ . We did not include any results for the more classical forcing  $f \in L_T^1H^{1/2}$  since it did not prove to be useful for section 4. However it is worth noting that up to a few more computations, the estimate in Prop 11 may be brought to include  $f \in L_T^1H^{1/2}$  such that for some  $\chi$  compactly supported,  $\chi f \in L_T^2L^2$  (the only thing to do would be to split  $f = \chi f + (1 - \chi)f$  in the proof of Prop 11 and use again the duality inequality  $|\int \nabla h \nabla u(1 - \chi)f dx| \lesssim \|u\|_{L_T^\infty H^{1/2}} \|f\|_{L_T^1 H^{1/2}}$ ).

## 4 Local Well-posedness of non-linear boundary value problems

In this section we consider the non-linear boundary value problem

$$\begin{cases} i\partial_t u + \Delta u = F(u), & (x, t) \in \Omega \times [0, T], \\ u|_{t=0} = u_0 \in H_0^{1/2}(\Omega), \\ u|_\Sigma = g \in H_0^{1+\varepsilon, 2}(\Sigma). \end{cases} \quad (4.1)$$

We will use the same technical assumptions sufficient for local well-posedness in [6], namely

$$\exists \alpha > 0, |F(z)| \lesssim |z|(1 + |z|^\alpha), \quad (4.2)$$

$$|F(z_1) - F(z_2)| \lesssim |z_1 - z_2|(1 + |z_1|^\alpha + |z_2|^\alpha) \quad (4.3)$$

$$|\nabla_{z, \bar{z}} F(z_1) - \nabla_{z, \bar{z}} F(z_2)| \lesssim |z_1 - z_2|(1 + |z_1|^{\max(0, \alpha-1)} + |z_2|^{\max(0, \alpha-1)}). \quad (4.4)$$

Since there is no trace operator on  $H^{1/2}(\Omega)$ , we should clarify what we call a solution of the nonlinear boundary value problem.

**Definition 3.** We say that  $u$  is a solution of (4.1) if given  $\tilde{g} \in H^{3/2+\varepsilon, 2}(\Omega)$  a lifting of  $g$ , and if we denote by  $v \in C_T H^{1/2}$  the transposition solution of

$$\begin{cases} i\partial_t v + \Delta v = F(\tilde{g}), & (x, t) \in \Omega \times [0, T], \\ v|_{t=0} = 0 \in H^{1/2}(\Omega), \\ v|_\Sigma = g \in H_0^{1, 2}(\Sigma), \end{cases}$$

then  $F(u) - F(\tilde{g}) \in L_T^1 H_0^{1/2}$  and the following equality holds in  $C_T H_0^{1/2}$  :

$$u(t) - v(t) = e^{it\Delta_D} u_0 + \int_0^t e^{i(t-s)\Delta_D} (F(u) - F(\tilde{g})) ds.$$

Let us first recall some Strichartz estimates for the homogeneous boundary value problem.

**Proposition 15.** (Strichartz estimates, Prop. 2.14 in [6])

If  $(p, q)$  is a weakly admissible pair (see (3.5) ), the operator  $e^{it\Delta_D}$  satisfies the following dispersive estimates :

$$\forall 0 \leq s \leq 1, \forall u_0 \in H_0^s, \|e^{it\Delta_D} u_0\|_{L_T^p W^{s,q}} \lesssim \|u_0\|_{H_0^s}, \quad (4.5)$$

$$\forall f \in L_T^1 H_0^s, \left\| \int_0^t e^{i(t-r)\Delta_D} f(r) dr \right\|_{L_T^p W^{s,q}} \lesssim \|f\|_{L_T^1 H_0^s}. \quad (4.6)$$

In order to use these estimates for the nonlinear problem we recall a number of “rules” on fractional derivatives.

**Lemma 4.** (Kato [14], lemma A2 and A4)

Let  $F \in C^1(\mathbb{C}, \mathbb{C})$  such that  $F(0) = 0$ ,  $F'(z) \lesssim |z|^{k-1}$ ,  $k \geq 1$ . Then for  $0 \leq s \leq 1$ ,  $1/r = 1/p + (k-1)/q$ ,

$$\|F(u)\|_{\dot{W}^{s,r}(\mathbb{R}^d)} \lesssim \|u\|_{L^q(\mathbb{R}^d)}^{k-1} \|u\|_{\dot{W}^{s,p}(\mathbb{R}^d)}.$$

For  $s \geq 0$ ,  $1/r = 1/p_1 + 1/p_2$ ,

$$\|u_1 u_2\|_{\dot{W}^{s,r}} \lesssim \|u_1\|_{\dot{W}^{s,p_1}} \|u_2\|_{L^{p_2}} + \|u_1\|_{L^{p_1}} \|u_2\|_{\dot{W}^{s,p_2}}.$$

**Theorem 16.** For  $d = 2, 3$ ,  $0 < \alpha < 2/(d-1)$ ,  $\varepsilon > 0$  and any  $(u_0, g) \in H_0^{1/2} \times H_0^{1+\varepsilon,2}$ , there exists  $T(\|(u_0, g)\|_{H_0^{1/2} \times H_0^{1+\varepsilon,2}})$  such that the problem (4.1) has a unique solution

$$u \in C([0, T], H^{1/2}) \cap L_T^{(d+1)/(d-1)} W^{1/2,q}, \quad q = \frac{2d(d+1)}{d^2 - d + 2}.$$

The flow map is locally Lipschitz continuous if  $d = 2$ , meaning that given any  $(g, u_0)$ , up to decreasing  $T$  there exists a neighbourhood of this point on which the solution map is Lipschitz continuous  $H_0^{1+\varepsilon,2} \times H_0^{1/2} \rightarrow C_T H^{1/2}$ .

*Proof.* For  $d = 2, 3$ , set  $X_T = C_T H^{1/2}(\Omega) \cap L_T^{(d+1)/(d-1)} W^{1/2,q}(\Omega)$ ,  $q = 3$  (so that  $(\frac{d+1}{d-1}, 3)$  is a weakly admissible pair). Let  $\tilde{g} \in H^{3/2,2}(\Omega \times [0, T])$  be a compactly supported lifting of  $g$ . We define  $u_g$  as the solution of

$$\begin{cases} i\partial_t u_g + \Delta u_g = F(\tilde{g}), & (x, t) \in \Omega \times [0, T] \\ u_g|_{t=0} = 0, \\ u_g|_{\Sigma} = g \in H_0^{1,2}(\Sigma), \end{cases}$$

as a first step we check that  $u_g \in X_T$ . According to Prop. 6 and Corollary 1 it is sufficient to prove that  $F(\tilde{g}) \in L_T^2 L^2$ . We use  $H^{3/2,2} \hookrightarrow H_T^{3/(4+2d)} H^{3d/(2(2+d))} \hookrightarrow L^{2(d+2)/(d-1)}([0, T] \times \Omega)$ , and of course  $H^{3/2,2} \hookrightarrow L^2([0, T] \times \Omega)$ . Since  $2 < 2(\alpha+1) < 2(d+2)/(d-1)$ , the assumption  $|F(u)| \lesssim |u| + |u|^{\alpha+1}$  and Hölder's inequality imply

$$\|F(\tilde{g})\|_{L^2([0,T] \times \Omega)} \lesssim \|\tilde{g}\|_{L^2} + \|\tilde{g}\|_{L^{2(\alpha+1)}}^{\alpha+1} \lesssim \|g\|_{H^{1,2}} (\|g\|_{H^{1,2}}^{\alpha} + 1).$$

Setting  $w = u - u_g$ ,  $u \in X_T$  iff  $w \in X_T$ , and we are reduced to finding a solution  $w \in X_T$  to

$$\begin{cases} i\partial_t w + \Delta w = F(w + u_g) - F(\tilde{g}), & (x, t) \in \Omega \times [0, T], \\ w|_{t=0} = u_0, \\ w|_{\Sigma} = 0, \end{cases} \quad (4.7)$$

which is now a homogeneous boundary value problem. To our knowledge no proof of local well-posedness of this problem was provided yet (the results of [6] are done at  $L^2$  and  $H^1$  regularity), however this can be tackled by methods quite similar to the classical ones for the initial value problem in fractional Sobolev spaces (as done for example by Kato [14]). We set  $X_{0,T} = C_T H_0^{1/2}(\Omega) \cap L_T^{(d+1)/(d-1)} W_0^{1/2,q}(\Omega)$ , and define  $S : X_{0,T} \rightarrow X_{0,T}$  which associates to  $w \in X_{0,T}$  the solution of

$$\begin{cases} i\partial_t v + \Delta v = F(w + u_g) - F(\tilde{g}), & (x, t) \in \Omega \times [0, T], \\ v|_{t=0} = u_0, \\ v|_{\Sigma} = 0, \end{cases} \quad (4.8)$$

the well-posedness of (4.7) will be a consequence of the fact that  $S$  sends  $B_{X_{0,T}}(0, R)$  to  $B_{X_{0,T}}(0, R)$  for  $R$  large enough,  $T$  small enough, and is a contraction - only in a weaker space if  $d = 3$ . We recall that if  $\Delta_D$  is the Dirichlet laplacian,  $S(w)$  writes as

$$S(w)(t) = e^{it\Delta_D} u_0 + \int_0^t e^{i(t-s)\Delta_D} (F(w + \tilde{g}) - F(\tilde{g})) ds.$$

Estimate (4.5) implies

$$\|e^{it\Delta_D} u_0\|_{X_{0,T}} \lesssim \|u_0\|_{H_0^{1/2}}, \quad (4.9)$$

while (4.6) gives

$$\left\| \int_0^t e^{i(t-s)\Delta_D} (F(w + u_g) - F(\tilde{g})) ds \right\|_{X_{0,T}} \lesssim \|F(w + u_g) - F(\tilde{g})\|_{L^1 H_0^{1/2}},$$

and we are left to estimate  $\|F(w + u_g) - F(\tilde{g})\|_{L^1 H_0^{1/2}}$ .

Special care is required since the Sobolev norm  $H_0^{1/2}$  is not equivalent to the  $H^{1/2}$  norm, but it can be split as  $\|v\|_{H^{1/2}} + \|v/d^{1/2}\|_{L^2}$  where  $d(x)$  is the distance of  $x$  to  $\Omega^c$  ([19] chapter 1 section 11). This supplementary term is handled by using that  $u_g - \tilde{g} \in X_{0,T}$  ( $\tilde{g} \in X_T$  by basic Sobolev embeddings) and the assumption  $w \in X_{0,T}$  : we fix  $\beta$  such that  $\min(2/(d+1), \alpha) < \beta < 2/(d-1)$ ,

$$\begin{aligned} \|(F(w + u_g) - F(\tilde{g}))/d^{1/2}\|_{L_T^1 L^2} &\lesssim \|(w + u_g - \tilde{g})(1 + |u_g|^\beta + |\tilde{g}|^\beta + |w|^\beta)/d^{1/2}\|_{L_T^1 L^2} \\ &\leq \|(w + u_g - \tilde{g})/d^{1/2}\|_{L_T^{\frac{d+1}{d-1}} L^3} \|(u_g, \tilde{g}, w)\|_{L_T^{\frac{\beta(d+1)}{2}} L^{6\beta}}^\beta \\ &\quad + \|w + u_g - \tilde{g}\|_{L_T^1 H_0^{1/2}} \\ &\lesssim T^\theta \|w + u_g - \tilde{g}\|_{L_T^{\frac{d+1}{d-1}} W_0^{1/2,3}} \|(u_g, \tilde{g}, w)\|_{L_T^{\frac{d+1}{2}} L^2 \cap L^{12/(d-1)}}^\beta \\ &\quad + T \|w + u_g - \tilde{g}\|_{X_{0,T}}, \end{aligned}$$

where  $\theta = (2 - \beta(d-1))/(d+1) > 0$ . The estimate above is closed by noting that in dimension 2 and 3,  $W^{1/2,3} \hookrightarrow L^{6d/(2d-3)} = L^{12/(d-1)}$ , finally

$$\|(F(w + u_g) - F(\tilde{g}))/d^{1/2}\|_{L_T^1 L^2} \lesssim (T^\theta + T)(1 + \|(u_g - \tilde{g}, w, u_g, \tilde{g})\|_{(X_{0,T})^2 \times (X_T)^2})^{\beta+1} \quad (4.10)$$

To handle the  $H^{1/2}$  part of the norm, we take some  $\psi \in C_c^\infty(\mathbb{R}^+)$ ,  $\psi = 1$  near 0, and split  $F(x) = \psi(|x|)F(x) + (1 - \psi)(|x|)F(x) = F_1 + F_2$ . Since  $F_1$  is clearly a Lipschitz function, we have  $\|F_1(w)\|_{H^{1/2}} \lesssim \|w\|_{H^{1/2}}$ , thus

$$\|F_1(w)\|_{L_T^\infty H^{1/2}} \lesssim T \|w\|_{X_T}, \quad (4.11)$$

(and similarly for  $\tilde{g}, u_g$ ).

For  $F_2$ , we choose  $w_e, \tilde{g}_e, u_{g,e}$  continuous extensions of  $w, \tilde{g}$  on the whole space, and we will only

estimate the  $\dot{H}^{1/2}$  part of the  $H^{1/2}$  norm, the  $L^2$  part being easier. Since  $|F_2(x)| = O(|x|^{\alpha+1})$  near 0, the rules of fractional differentiation imply

$$\begin{aligned} \|F_2(w + u_g) - F_2(\tilde{g})\|_{L_T^1 \dot{H}^{1/2}} &\lesssim \|F_2(w_e + u_{g,e}) - F_2(\tilde{g}_e)\|_{L_T^1 \dot{H}^{1/2}} \\ &\lesssim \|(w_e, g_e, u_{g,e})\|_{L^{(d+1)/(d-1)} L^r}^\alpha \|(w_e, g_e, u_{g,e})\|_{L^{\frac{d+1}{d+1-\alpha(d-1)}} \dot{W}^{1/2, r_1}} \\ &\lesssim \|(w, g, u_g)\|_{L^{(d+1)/(d-1)} L^r}^\alpha \|(w, g, u_g)\|_{L^{\frac{d+1}{d+1-\alpha(d-1)}} \dot{W}^{1/2, r_1}} \end{aligned}$$

where  $\frac{\alpha}{r} + \frac{1}{r_1} = \frac{1}{2}$ . We choose  $r$  to be the exponent of the critical Sobolev embedding  $W^{1/2, q} \hookrightarrow L^r$  (namely  $r = 2d(d+1)/(d-1)^2$ ), so that  $r_1 = 2d(d+1)/(d(d+1) - \alpha(d-1)^2)$ . Since  $\alpha < 2/(d-1)$ , we have  $2 < r_1 < q$ , and we use the inequality  $\|\varphi\|_{W^{1/2, r_1}} \lesssim \|\varphi\|_{H^{1/2}}^\theta \|\varphi\|_{W^{1/2, q}}^{1-\theta}$ , with  $\theta/2 + (1-\theta)/q = 1/r_1$  (this last inequality is a direct combination of the expression of fractional Sobolev norms and Hölder's inequality).

Set  $p_1 = (d+1)(d+1 - \alpha(d-1))$ , using Hölder's inequality in time, we get

$$\begin{aligned} \|F(w + u_g) - F(\tilde{g})\|_{L_T^1 \dot{H}^{1/2}} &\lesssim \|(w, g, u_g)\|_{X_T}^\alpha \|(w, g, u_g)\|_{L^{p_1} H^{1/2}}^\theta \|(w, g, u_g)\|_{L^{p_1} W^{1/2, q}}^{1-\theta} \\ &\leq T^{\theta/p_1 + (1-\theta)(1/p_1 - 1/p)} \|(w, g, u_g)\|_{X_T}^{\alpha+1} \\ &= T^{1-\alpha(d-1)/2} \|(w, g, u_g)\|_{X_T}^{\alpha+1}. \end{aligned} \quad (4.12)$$

Gluing (4.9, 4.10, 4.11, 4.12), we get for some  $\gamma > 0$ ,  $\alpha \leq \beta < 2/(d-1)$ ,

$$\|S(w)\|_{X_{0,T}} \leq C(\|u_0\|_{H^{1/2}} + T^\gamma(1 + \|(w, \tilde{g}, u_g)\|_{X_T}^{\beta+1})),$$

and it is clear that for  $R$  large enough,  $T$  small enough,  $S$  maps  $B_{X_{0,T}}(0, R)$  to  $B_{X_{0,T}}(0, R)$ .

In the special case  $d = 2$ , we now prove that it is a contraction, so that the standard Picard-Banach fixed point theory directly implies existence, uniqueness and smoothness of the solution map. Following the same argument as previously, contractivity reduces to checking that there exists  $\theta > 0$  such that

$$\exists 0 \leq \alpha < 1 : \|F(u_g + w_1) - F(u_g + w_2)\|_{L_T^1 H^{1/2}} \leq T^\theta C(\|(w_1, w_2, u_g)\|_{X_T}) \|w_1 - w_2\|_{X_{0,T}}. \quad (4.13)$$

For the  $L_T^1 L^2$  part of the estimate we fix some  $\beta$  satisfying  $\max(1, \alpha) \leq \beta < 2$ ,

$$\begin{aligned} \|F(u_g + w_1) - F(u_g + w_2)\|_{L_T^1 L^2} &\lesssim \|(1 + |u_g| + |w_1| + |w_2|)^\beta |w_1 - w_2|\|_{L_T^1 L^2} \\ &\leq \|(u_g, w_1, w_2)\|_{L_T^{3\beta/2} L^{4\beta}}^\beta \|w_1 - w_2\|_{L_T^3 L^4} \\ &\quad + T \|w_1 - w_2\|_{L_T^\infty L^2} \\ &\leq (T + T^{(2-\beta)/3})(1 + \|(u_g, w_1, w_2)\|_{X_T})^\beta \|w_1 - w_2\|_{X_T}. \end{aligned} \quad (4.14)$$

The term  $\|(F(u_g + w_1) - F(u_g + w_2))/d^{1/2}\|_{L_T^1 L^2}$  can be estimated similarly, and we now turn to the  $L_T^1 \dot{H}^{1/2}$  part. Let us write

$$F(u_g + w_1) - F(u_g + w_2) = \int_0^1 \nabla F(u_g + w_2 + t(w_1 - w_2)) \cdot (w_1 - w_2) dt,$$

and apply Minkowski's integral inequality,

$$\begin{aligned} & \left\| \int_0^1 \nabla F(u_g + w_2 + t(w_1 - w_2)) \cdot (w_1 - w_2) dt \right\|_{L^1 H^{1/2}} \\ & \lesssim \int_0^1 \left\| \nabla F(u_g + w_2 + t(w_1 - w_2)) \cdot (w_1 - w_2) \right\|_{L_T^1 H^{1/2}} dt \\ & \leq \int_0^1 \left\| (\psi \nabla F)(u_g + w_2 + t(w_1 - w_2)) \cdot (w_1 - w_2) \right\|_{L_T^1 H^{1/2}} dt \\ & \quad + \int_0^1 \left\| ((1 - \psi) \nabla F)(u_g + w_2 + t(w_1 - w_2)) \cdot (w_1 - w_2) \right\|_{L_T^1 H^{1/2}} dt \end{aligned}$$

We only detail the estimate of the first term (the second is simpler). Set  $G(x) = ((1 - \psi) \nabla F)(u_g + w_2 + t(w_1 - w_2))(x)$ ,  $y(x) = (u_g + w_2 + t(w_1 - w_2))(x)$ . Up to using extensions on the whole space as previously, we may use the rules of fractional derivatives (see lemma 4) to get :

$$\|G(x) \cdot (w_1 - w_2)\|_{\dot{H}^{1/2}} \lesssim \|G\|_{L^6} \|w_1 - w_2\|_{\dot{W}^{1/2,3}} + \|G\|_{\dot{W}^{1/2,12/5}} \|w_1 - w_2\|_{L^{12}}$$

We recall that  $G \equiv 0$  on a neighbourhood of 0, thus  $|G(y)| \lesssim |y|^\beta$  for any  $\beta \geq \alpha$ , in particular if we choose  $\max(7/6, \alpha) < \beta < 2$ , we get

$$\|G \cdot (w_1 - w_2)\|_{\dot{H}^{1/2}} \lesssim \|y\|_{L^{6\beta}}^\beta \|w_1 - w_2\|_{\dot{W}^{1/2,3}} + \|y\|_{L^{12(\beta-1)}}^{\beta-1} \|y\|_{W^{1/2,3}} \|w_1 - w_2\|_{L^{12}}. \quad (4.15)$$

As  $W^{1/2,3} \hookrightarrow L^{12}$ ,  $2 \leq 12(\beta - 1)$ ,  $6\beta \leq 12$ , Hölder's inequalities imply

$$\|y\|_{L^{12(\beta-1)}} \lesssim \|y\|_{L^2}^{\theta_1} \|y\|_{W^{1/2,3}}^{1-\theta_1}, \quad \|y\|_{L^{6\beta}} \leq \|y\|_{L^2}^{\theta_2} \|y\|_{W^{1/2,3}}^{1-\theta_2}.$$

The integration in time of (4.15) and similar basic Hölder inequalities finally give

$$\exists \theta > 0 : \|G \cdot (w_1 - w_2)\|_{L_T^1 \dot{H}^{1/2}} \lesssim T^\theta \|y\|_{X_T}^\beta \|w_1 - w_2\|_{X_T}, \quad (4.16)$$

so that (4.14, 4.16) gives (4.13), this ends the proof for  $d = 2$ .

In dimension 3, this argument can not be applied, essentially because we may not choose some  $\beta$  strictly between 1 and  $2/(d - 1) = 1$ . Instead we prove that the map  $S : B_{X_{0,T}}(0, R) \rightarrow B_{X_{0,T}}(0, R)$  is a contraction for the weaker topology associated to  $Y_T = L_T^\infty L^2 \cap L_T^2 L^3$ . Fix  $\beta$  such that  $\max(1/3, \alpha) \leq \beta < 1$ , the inequality (4.6) with  $s = 0$  implies

$$\begin{aligned} \|S(w_1) - S(w_2)\|_{Y_T} & \lesssim \|F(w_1) - F(w_2)\|_{L_T^1 L^2} \lesssim \|(1 + (|w_1| + |w_2|)^\beta) |w_1 - w_2|\|_{L_T^1 L^2} \\ & \leq T \|w_1 - w_2\|_{L_T^\infty L^2} \\ & \quad + \|w_1 - w_2\|_{L_T^{2\beta} L^{6\beta}}^\beta \|w_1 - w_2\|_{L_T^2 L^3} \\ & \leq T^{(1-\beta)/2} (1 + \|(w_1, w_2)\|_{X_T})^\beta \|w_1 - w_2\|_{Y_T}. \end{aligned}$$

Up to decreasing  $T$ ,  $S$  is thus a contraction, whose fixed point is in  $X_T$  since  $S$  maps  $X_T$  to  $X_T$ . The uniqueness in dimension 3 is an obvious repetition of the argument above. The case of dimension 2 is similar : let  $u_1, u_2$  be two solutions, then by definition

$$\text{for } j = 1, 2, \quad u_j(t) - v_j(t) = e^{it\Delta_D} u_0 + \int_0^t e^{i(t-s)\Delta_D} (F(u_j) - F(\tilde{g}_j)) ds,$$

but since  $v_1 - v_2 = \int_0^t e^{i(t-s)\Delta_D} (F(\tilde{g}_1) - F(\tilde{g}_2)) ds$ , we get

$$u_1 - u_2 = \int_0^t e^{i(t-s)\Delta_D} (F(u_1) - F(u_2)) ds,$$

In other words,  $w = u_1 - u_2$  satisfies

$$\begin{cases} i\partial_t w + \Delta w = F(u_1) - F(u_2), & (x, t) \in \Omega \times [0, T], \\ w|_{t=0} = 0, \\ w|_{\Sigma} = 0, \end{cases}$$

independently of the choice of the liftings  $\tilde{g}_1, \tilde{g}_2$ . We note that (3, 3) is a weakly admissible pair in dimension 2 and set  $Y_T = L_T^\infty L^2 \cap L_T^3 L^3$ . For any  $\beta \geq \alpha$ , the Strichartz estimate (4.6) implies

$$\begin{aligned} \|w\|_{Y_T} &\lesssim \|F(u_1) - F(u_2)\|_{L_T^1 L^2} \\ &\lesssim \|(1 + (|u_1| + |u_2|)^\beta) w\|_{L_T^1 L^2} \\ &\leq T \|w\|_{L_T^\infty L^2} + \|(u_1, u_2)\|_{L^{6\beta} L^{6\beta}} \|w\|_{L^3 L^3} \end{aligned}$$

Choosing now  $\max(1/2, \alpha) < \beta < 2$  so that  $3 \leq 6\beta < 12$  and using the embedding  $W^{1/2,3} \hookrightarrow L^{12}$  we finally obtain

$$\|w\|_{Y_T} \lesssim T \|w\|_{Y_T} + T^{(2-\beta)/12} \|(u_1, u_2)\|_{X_T}^\beta \|w\|_{Y_T},$$

this implies  $w|_{[0,T]} = 0$  for  $T$  small enough, and then  $w \equiv 0$  on the interval of existence by connectedness.  $\square$

### Some further questions

- We did not derive smoothing estimates at other levels of regularity than  $H^{1/2}$ , though it would be particularly interesting to extend our results for the energy space  $H^1$ . By differentiating the equations and interpolation arguments, one should obtain local smoothing at any level  $H^s$ ,  $s \geq 1/2$ , leading to well-posedness results similar to Theorem 16.
- Similar results at the  $H^1$  regularity level would open naturally the question of existence of global solutions. Indeed contrarily to the homogeneous IBVP there is no conserved energy, but for defocusing non-linearities  $F = \partial V / \partial \bar{z}$ ,  $V \geq 0$  we still have the formal identity

$$\frac{d}{dt} \int_{\Omega} |\nabla u|^2 dx + V(u) dx = \operatorname{Re} \int_{\partial\Omega} \partial_n u \partial_t \bar{u} dS,$$

which may lead - with appropriate control of  $\partial_n u$ - to global existence results. The expected trace estimate

$$\|\partial_n u\|_{H^{s-1/2,2}} \lesssim \|u_0\|_{H^s} + \|g\|_{H^{s+1/2,2}},$$

is however not a clear consequence of local smoothing for  $s > 1/2$ . In the absence of such an estimate, one may not expect to obtain better than local existence results, independently of whether the nonlinearity is focusing or defocusing.

- Since for the homogeneous boundary value problem well-posedness was established up to  $\alpha = 4/(d-1)$  in [12] for  $d = 3$ , there is still a gap (at least for low regularity boundary data) for  $2/(d-1) \leq \alpha < 4/(d-1)$  when  $\Omega$  is the complement of a convex set.
- More generally, it would be interesting to obtain dispersive estimates for the IBVP when the obstacle is not convex by direct (non duality based) methods. The case where  $\Omega$  is the complement of a star shaped obstacle should be the most natural further step.

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