



HAL
open science

Constructing reductions for creative telescoping

Joris van der Hoeven

► **To cite this version:**

| Joris van der Hoeven. Constructing reductions for creative telescoping. 2017. hal-01435877v1

HAL Id: hal-01435877

<https://hal.science/hal-01435877v1>

Preprint submitted on 15 Jan 2017 (v1), last revised 28 Dec 2019 (v5)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Constructing reductions for creative telescoping

JORIS VAN DER HOEVEN

Laboratoire d'informatique, UMR 7161 CNRS
Campus de l'École polytechnique
1, rue Honoré d'Estienne d'Orves
Bâtiment Alan Turing, CS35003
91120 Palaiseau

Email: vdhoeven@lix.polytechnique.fr

January 15, 2017

The class of *reduction-based* algorithms was introduced recently as a new approach towards *creative telescoping*. Starting with Hermite reduction of rational functions, various reductions have been introduced for increasingly large classes of holonomic functions. In this paper we show how to construct reductions for general holonomic functions, in the purely differential setting.

KEYWORDS: creative telescoping, holonomic function, Hermite reduction, residues

A.C.M. SUBJECT CLASSIFICATION: I.1.2 Algebraic algorithms

A.M.S. SUBJECT CLASSIFICATION: 33F10, 68W30

1. INTRODUCTION

Let \mathbb{K} be an effective field of characteristic zero and let ϕ be a square-free polynomial. Consider the system of differential equations

$$\phi^t y' = Ay, \tag{1}$$

where $A \in \mathbb{K}[x]^{r \times r}$ is an $r \times r$ matrix with entries in $\mathbb{K}[x]$, where y is a vector of r unknown functions, and where $t \in \mathbb{N}$. Notice that any system of differential equations $y' = B y$ with $B \in \mathbb{K}(x)$ can be rewritten in this form by taking ϕ^t to be a multiple of all denominators.

Let y be a formal solution to (1) and consider the $\mathbb{K}[x, \phi^{-1}]$ -module M of linear combinations $\lambda^\top y = \lambda_1 y_1 + \dots + \lambda_r y_r$ with $\lambda \in \mathbb{K}[x, \phi^{-1}]^r$. Then M has the natural structure of a D-module for the derivation ∂ defined by $(\lambda^\top y)' = ((\lambda^\top)' + \phi^{-t} \lambda^\top A) y$. A \mathbb{K} -linear mapping $[\cdot]: M \rightarrow M$ is said to be a *reduction* if $[f] - f \in \text{Im } \partial$ for all $f \in M$. Such a reduction is said to be *confined* if its image is a finite dimensional subspace of M and *normal* if $[f'] = 0$ for all $f \in M$.

In this paper, we will show how to construct normal confined reductions. Such reductions are interesting for their application to creative telescoping [8, 5], as we briefly recall in section 2. The first reduction of this kind is Hermite reduction [1, 2], in which case $A = 0$. The existence of normal confined reductions has also been shown in increasingly general cases and most noticeably so for Fuchsian differential equations [4, 3]. We refer to [2, 6] for more details and the application to creative telescoping.

Our construction of confined reductions proceeds in two stages. In section 4, we first focus on the $\mathbb{K}[x]$ -submodule M^\sharp of M of linear combinations $\lambda^\top y$ with $\lambda \in \mathbb{K}[x]^r$. We will construct a \mathbb{K} -linear *head reduction* $[\cdot]: M^\sharp \rightarrow M^\sharp$ such that $[f] - f \in \text{Im } \partial$ and $\deg [f]$ is bounded from above for all $f \in M^\sharp$. Here we understand that $\deg(\lambda^\top y) := \deg \lambda := \max(\deg \lambda_1, \dots, \deg \lambda_r)$ for all $\lambda \in \mathbb{K}[x]^r$. The construction uses a variant of Gaussian elimination that will be described in section 3.

In section 5 we turn to the natural supplement M^b of M^\sharp that is defined as follows. Every element $a \in \mathbb{K}[x, \phi^{-1}]$ admits a natural finite ϕ -adic expansion $a = \sum_{i \in \mathbb{Z}} a_i \phi^i$ whose coefficients $a_i \in \mathbb{K}[x]$ are of degree $\deg a_i < \deg \phi$. Such expansions extend componentwise to linear combinations $f = \lambda^\top y \in M$ with $\lambda \in \mathbb{K}[x, \phi^{-1}]^r$. Now we define $M^b = \{f \in M : \forall i \geq 0, f_i = 0\}$. Notice that $M^\sharp = \{f \in M : \forall i < 0, f_i = 0\}$, so that $M = M^\sharp \oplus M^b$. We show that it is possible to define a \mathbb{K} -linear *tail reduction* $[\cdot]: M \rightarrow M$ such that $[f] - f \in \text{Im } \partial$ and $\text{val}_\phi [f]$ is bounded from below for all $f \in M$. Here val_ϕ stands for the ϕ -adic valuation on M .

The head reduction extends to $f \in M$ by setting $\lceil f \rceil = \lceil f^\sharp \rceil + f^b$, where $f^\sharp \in M^\sharp$ and $f^b \in M^b$ are unique such that $f = f^\sharp + f^b$. Combining the head and the tail reductions, we obtain a reduction $[f] := \lceil [f] \rceil$ with the property that $\deg_x [f]^\sharp$ is bounded from above and $\text{val}_\phi [f]$ is bounded from below. This means that we constructed a confined reduction. Using straightforward linear algebra, one can further turn this reduction into a normal one, as will be shown in detail in section 6.

2. CREATIVE TELESCOPING

Let \mathbb{k} be an effective subfield of \mathbb{C} and let $\partial_x = \partial / \partial x$ and $\partial_u = \partial / \partial u$ denote the partial derivations with respect to x and u . Consider a system of differential equations

$$\begin{cases} \phi^t \partial_x y = Ay \\ \phi^t \partial_u y = By, \end{cases} \quad (2)$$

where $\phi \in \mathbb{k}[x, u]$ is square-free, $t \in \mathbb{N}$, and $A, B \in \mathbb{k}[x, u]^{r \times r}$ are such that $AB = BA$. Setting $\mathbb{K} = \mathbb{k}(u)$, the first part of (2) then becomes of the form (1). Notice that any bivariate holonomic function is an entry of a solution to a system of the form (2).

Let y be a complex analytic solution of the above system of equations and let M be the $\mathbb{K}[x, \phi^{-1}]$ -module generated by the entries of y . Notice that M is stable under both ∂_x and ∂_u . For any $f = \lambda^\top y \in M$ with $\lambda \in \mathbb{K}[x, \phi^{-1}]^r$ and any non-singular contour \mathcal{C} in \mathbb{C} between two points $\alpha, \beta \in \mathbb{k}$, we may consider the integral

$$F(u) = \int_{\mathcal{C}} f(x, u) dx,$$

which defines a function in the single variable u . It is natural to ask under which conditions F is a holonomic function and how to compute a differential operator $L \in \mathbb{K}[\partial_u]$ with $LF = 0$.

The idea of *creative telescoping* is to compute a differential operator $K \in \mathbb{K}[\partial_u]$ and a function $\xi = \partial_x \chi$ with $\chi \in M$ such that

$$Kf(x, u) = \xi(x, u). \quad (3)$$

Integrating over \mathcal{C} , we then obtain

$$KF(u) = \int_{\mathcal{C}} \frac{\partial \chi}{\partial x}(x, u) dx = \chi(\beta, u) - \chi(\alpha, u).$$

If the contour \mathcal{C} has the property that $\chi(\beta) = \chi(\alpha)$ for all $\chi \in M$ (where the equality is allowed to hold at the limit if necessary), then $L = K$ yields the desired annihilator with $LF = 0$. In general, we need to multiply K on the left with an annihilator of $\chi(\beta, u) - \chi(\alpha, u)$.

Now assume that we have a computable confined reduction $[\cdot]: M \rightarrow M$. Then the functions in the sequence $[f], [\partial_u f], [\partial_u^2 f], \dots$ can all be computed and they belong to a finite dimensional \mathbb{K} -vector space V . Using linear algebra, that means that we can compute a relation

$$K_0 [f] + \dots + K_s [\partial_u^s f] = [K_0 f + \dots + K_s \partial_u^s f] = 0 \quad (4)$$

with $K_0, \dots, K_s \in \mathbb{K}$. Taking

$$\xi = (K_0 f + \dots + K_s \partial_u^s f) - [K_0 f + \dots + K_s \partial_u^s f] \in \partial_x M,$$

we thus obtain (3). If the relation (4) has minimal order s and the reduction $[\cdot]$ is normal, then it can be shown [6] that there exist no relations of the form (3) of order lower than s .

3. ROW SWEPT FORMS

Let $U \in \mathbb{K}^{r \times r}$ be a matrix and denote the i -th row of U by U_i . Assuming that $U_i \neq 0$, its *leading index* ℓ_i is the smallest index j with $U_{i,j} \neq 0$. We say that U is in *row swept form* if there exists a $k \in \{0, \dots, r\}$ such that $U_1 \neq 0, \dots, U_k \neq 0, U_{k+1} = \dots = U_r = 0$ and $U_{i',\ell_i} = 0$ for all $i < i' \leq k$. Notice that U has rank k in this case.

An invertible matrix $S \in \mathbb{K}^{r \times r}$ such that SU is in row swept form will be called a *row sweeper* for U . We may compute such a matrix S using the routine **RowSweeper** below, which is really a variant of Gaussian elimination. Whenever we apply this routine to a matrix U such that the truncated matrix \tilde{U} with rows $U_1, \dots, U_k, 0, \dots, 0$ is in row swept form, we notice that these first k rows are left invariant by the row sweeping process. In other words, the returned row sweeper S is of the form $S = \begin{pmatrix} \text{Id}_k & 0 \\ * & * \end{pmatrix}$.

Algorithm RowSweeper(U)

$S := \text{Id}_r, R := U$

for i **from** 1 **to** r **do**

if $R_{i',j} = 0$ for all $i' \geq i$ and j **then return** S

 Let $i' > i$ be minimal such that $R_{i',j} \neq 0$ for some j

 Swap the i -th and i' -th rows of S and R

$v := R_{i,\ell_i}^{-1}$

for i' **from** $i + 1$ **to** r **do**

$S_{i'} := S_{i'} - v R_{i',\ell_i} S_i, R_{i'} := R_{i'} - v R_{i',\ell_i} R_i$

return S

In what follows, we will also need to consider matrices $U \in \bar{\mathbb{L}}^{r \times r}$, where $\bar{\mathbb{L}} = \mathbb{K}[x]/(\psi)$ for some square-free, but not necessarily irreducible polynomial $\psi \in \mathbb{K}[x]$. In that case, we say that U is in *row swept form* if there exists a $k \in \{0, \dots, r\}$ such that $U_1 \neq 0, \dots, U_k \neq 0, U_{k+1} = \dots = U_r = 0$, the entries U_{i,ℓ_i} with $i \leq k$ are invertible, and $U_{i',\ell_i} = 0$ for all $i < i' \leq k$. An invertible matrix $S \in \bar{\mathbb{L}}^{r \times r}$ such that SU is in row swept form is again called a *row sweeper* for U . We may still use the above algorithm to compute row sweepers, but the algorithm potentially fails in the case when we hit a non zero pivot R_{i,ℓ_i} that is not invertible. Whenever such a failure occurs, the non invertible pivot $c = R_{i,\ell_i}$ induces a non trivial factorization $\psi = \psi_1 \psi_2$ of ψ by taking $\psi_1 = \text{gcd}(\psi, c)$ and $\psi_2 = \psi / \psi_1$.

4. HEAD REDUCTION

For any matrix $T \in \phi^t \mathbb{K}[x]^{r \times r}$ and

$$U = \phi^{-t} T A + T',$$

the equation (1) implies

$$(Ty)' = Uy.$$

For any constant matrix $C \in \mathbb{K}^{r \times r}$ and $i \in \mathbb{Z}$, we also obtain

$$(C x^i Ty)' = C x^{i-1} (Ux + iT) y.$$

We may regard $Ux + iT$ as a polynomial with matrix coefficients in $\mathbb{K}^{r \times r}$. Our next aim is to show that we may take T in such a way that the leading coefficient of this polynomial is an invertible matrix in $\mathbb{K}^{r \times r}$ for almost all i . We will then use that in order to construct the head reduction.

More precisely, given $T \in \phi^t \mathbb{K}[x]^{r \times r}$, let $d = \max(\deg U, \deg T - 1)$ and let σ be a formal indeterminate. We say that T is a *head chopper* for (1) if the $D = U_d + T_{d+1} \sigma$ is invertible as a matrix in $\mathbb{K}(\sigma)^{r \times r}$. In that case, the determinant of D is a non zero polynomial in $\mathbb{K}[\sigma]$ of degree at most r , so there are at most r integers i for which $U_d + T_{d+1} i$ is not invertible. We may compute a head chopper using the following algorithm:

Algorithm HeadChopper(ϕ, t, A)

$(T, U) := (\phi^t \text{Id}_r, A + t \phi' \phi^{t-1} \text{Id}_r)$

repeat

$d := \max(\deg U, \deg T - 1)$

if $U_d + T_{d+1} \sigma$ is invertible **then return** T

$S := \mathbf{RowSweeper}(U_d)$

$(T, U) := (ST, SU)$

 Let k be the number of non-zero rows of U_d and $\Delta := \begin{pmatrix} \text{Id}_k & 0 \\ 0 & \text{Id}_{r-k} x \end{pmatrix}$

$(T, U) := (\Delta T, \Delta U + \Delta' T)$

PROPOSITION 1. *The algorithm **HeadChopper** terminates and returns a head chopper.*

Proof. In the line where we compute the number k , let us decompose

$$T = \begin{pmatrix} T^- \\ T^+ \end{pmatrix} \quad U = \begin{pmatrix} U^- \\ U^+ \end{pmatrix},$$

where T^- and U^- are the first k rows of T and U , so that $U_d^+ = 0$. From the specification of **RowSweeper**, we also know that U_d^- has full rank k . Let us prove the following assertions:

1. The matrix $T_{\deg T}^+$ has full rank and $\deg T \leq d$.
2. The number k increases at every iteration and the degree d never changes.
3. The degree of T increases by one at every iteration.

The first time that reach the line where we compute k , we have $T = \phi^t S$ and S is an invertible scalar matrix. In particular, both $T_{\deg T}$ and $T_{\deg T}^+$ have full rank. We also cannot have $\deg T = d + 1$, since this would imply that $U_d + T_{d+1} \sigma$ is invertible. Consequently, $\deg T \leq d$.

Now assume that $T_{\deg T}^+$ has full rank and $\deg T \leq d$ when we reach the line where we compute k . Denoting $\tilde{T} := \Delta T$ and $\tilde{U} := \Delta U + \Delta' T$, we then have $\deg \tilde{T} = \max(\deg T^-, \deg T^+ + 1) = \deg T + 1 \leq d + 1$ and $\deg(\Delta' T) \leq d$. We also get that $d = \deg U$, whence $k \neq 0$. From the fact that $U_d^+ = 0$, it follows that $\deg(\Delta U) = d$ and the first k rows of U and ΔU coincide. Combined with the facts that $\deg \tilde{T} \leq d + 1$ and $\deg(\Delta' T) \leq d$, this shows that $\max(\deg \tilde{U}, \deg \tilde{T} - 1) = d$, whence d does not change at the next iteration.

Moreover, the first k rows of $\Delta' T$ vanish, whence the first k rows of U and \tilde{U} coincide. In particular, these first k lines still don't change when multiplying with the next row sweeper \tilde{S} , which shows that k can only increase at the next iteration. Finally, the first k lines of $\tilde{T}_{\deg \tilde{T}}$ coincide with $T_{\deg T+1}^- = 0$, whereas the last $r - k$ lines coincide with $(x T^+)_{\deg T+1} = T_{\deg T}^+$ which has full rank. If $\tilde{U}_d + \tilde{T}_{d+1} \sigma$ is not invertible, then this means that $\deg \tilde{T} \leq d$ and the last $r - k$ rows of $\tilde{S} \tilde{T}_{\deg \tilde{T}}$ are again linearly independent. This completes the proof of our assertions.

Since the degree of T increases by one at every iteration, there comes a point when $U_d + T_{d+1} \sigma$ is invertible or $\deg T$ exceeds $d + 1$, the latter being impossible. We conclude that the algorithm terminates and it clearly returns a head chopper. \square

Let T be a head chopper for (1), let $U = \phi^{-t} T A + T'$, and $d := \max(\deg U, \deg T - 1)$. Write \mathcal{I} for the set of indices $i \in \mathbb{N}$ for which $U_d + T_{d+1} i$ is not invertible. Now assume that we are given a vector $\lambda \in \mathbb{K}[x]^r$, considered as a polynomial $\lambda = \sum \lambda_i x^i$ with coefficients λ_i in \mathbb{K}^r . Then the following routine **HeadReduce** computes a vector $\tilde{\lambda} \in \mathbb{K}[x]^r$ with $(\tilde{\lambda} - \lambda)^\top y \in \partial M$ and $\tilde{\lambda}_{i+d} \neq 0 \Rightarrow i \in \mathcal{I}$ for all $i \in \mathbb{N}$. We will call $\lceil \lambda^\top y \rceil := \tilde{\lambda}^\top y$ the *head reduction* of $\lambda^\top y$. We will show that $\lceil \cdot \rceil: M^\# \rightarrow M^\#$ is \mathbb{K} -linear. By construction, the image $\lceil M^\# \rceil := \{\lceil f \rceil: f \in M^\#\}$ is finite dimensional and we have $\lceil \lambda^\top y \rceil - \lambda^\top y \in \partial M$ for all $\lambda \in \mathbb{K}[x]^r$.

Algorithm HeadReduce(λ)

repeat

if $\lambda_{i+d} = 0$ for all $i \in \mathbb{N} \setminus \mathcal{I}$ **then return** λ

 Let $i \in \mathbb{N} \setminus \mathcal{I}$ be maximal with $\lambda_{i+d} \neq 0$

$\delta := (U_d^\top + i T_{d+1}^\top)^{-1} \lambda_{i+d}$

$\lambda := \lambda - x^{i-1} (U^\top x + i T^\top) \delta$

PROPOSITION 2. *The routine **HeadReduce** terminates and is correct.*

Proof. Let λ^{orig} be the input value of λ . Each time that we set $\tilde{\lambda} := \lambda - x^{i-1} (U^\top x + i T^\top) \delta$, we have

$$(\tilde{\lambda} - \lambda)^\top y = -\delta^\top x^{i-1} (Ux + iT) y = (-\delta^\top x^i T y)' \in \partial M.$$

Consequently, the running value of λ satisfies $(\lambda - \lambda^{\text{orig}})^\top y \in \partial M$ throughout the algorithm. This shows that the algorithm is correct.

We also have $\deg(\tilde{\lambda} - \lambda) \leq i + d$ and $\tilde{\lambda}_{i+d} = \lambda_{i+d} - (U_d^\top + i T_{d+1}^\top) \delta_i = 0$. This means that the maximal $\tilde{i} \in \mathbb{N} \setminus \mathcal{I}$ with $\lambda_{\tilde{i}+d} \neq 0$ (if such an \tilde{i} exists at all) satisfies $\tilde{i} < i$. In other words, the value of i strictly decreases during successive iterations, whence the algorithm terminates.

Denoting by $\lceil \lambda \rceil$ the returned value of λ , let us show that the map $\lambda \mapsto \lceil \lambda \rceil$ is \mathbb{K} -linear. The projection $\pi: \mathbb{K}[x]^r \rightarrow \mathbb{K}[x]^r$ defined by $\pi(\lambda) = \sum_{i \in \mathbb{N} \setminus \mathcal{I}} \lambda_{i+d} x^{i+d}$ is certainly \mathbb{K} -linear. Let us prove that $\lceil \nu \rceil = \lceil \lambda \rceil + \lceil \mu \rceil$ for all $\lambda, \mu \in \mathbb{K}[x]^r$ and $\nu = \lambda + \mu$, by induction over $i = \max(\deg \pi(\lambda), \deg \pi(\mu)) - d$. If $\pi(\lambda) = \pi(\mu) = 0$, then we have nothing to prove. Otherwise, $i \in \mathbb{N} \setminus \mathcal{I}$, and we have $\lambda_{i+d} \neq 0$ or $\mu_{i+d} \neq 0$. Let $P = x^{i-1} (U^\top x + i T^\top)$, $Q = (U_d^\top + i T_{d+1}^\top)^{-1}$, $\tilde{\lambda} = \lambda - PQ \lambda_{i+d}$, $\tilde{\mu} = \mu - PQ \mu_{i+d}$ and $\tilde{\nu} = \nu - PQ \nu_{i+d} = \tilde{\lambda} + \tilde{\mu}$. If $\lambda_{i+d} \neq 0$, then we replace $\lambda := \tilde{\lambda}$ in the main loop, so $\lceil \lambda \rceil = \lceil \tilde{\lambda} \rceil$. If $\lambda_{i+d} = 0$, then we clearly have $\lceil \lambda \rceil = \lceil \tilde{\lambda} \rceil$. Similarly, $\lceil \mu \rceil = \lceil \tilde{\mu} \rceil$ and $\lceil \nu \rceil = \lceil \tilde{\nu} \rceil$. Now $\max(\deg \pi(\tilde{\lambda}), \deg \pi(\tilde{\mu})) < i + d$, whence $\lceil \nu \rceil = \lceil \tilde{\nu} \rceil = \lceil \tilde{\lambda} \rceil + \lceil \tilde{\mu} \rceil = \lceil \lambda \rceil + \lceil \mu \rceil$ by the induction hypothesis. Using a similar induction, one may show that $\lceil c \lambda \rceil = \lceil \lambda \rceil$ for all $c \in \mathbb{K}$. \square

Remark 3. The algorithm **HeadReduce** is not very efficient. The successive values of δ can be regarded as the coefficients of a polynomial $\Delta \in \mathbb{K}[x]^r$ such that the returned vector $\tilde{\lambda}$ satisfies $(\tilde{\lambda} - \lambda)^\top y = (-\Delta^\top T y)'$. Instead of computing the coefficients of Δ one by one, it is faster to evaluate $(-\Delta^\top T y)'$ in a relaxed manner [7], while ensuring that $\tilde{\lambda}_{i+d} = 0$ for all $i \in \mathbb{N} \setminus \mathcal{I}$.

Remark 4. It is straightforward to adapt **HeadReduce** so that it returns $\Delta \in \mathbb{K}[x]^r$ with $(\tilde{\lambda} - \lambda)^\top y = (-\Delta^\top T y)'$ instead of $\tilde{\lambda}$. This information is useful for the application to creative telescoping.

Remark 5. The algorithm **HeadReduce** also works for matrices $\lambda \in \mathbb{K}[x]^{r \times c}$ with c columns. This allows for the simultaneous reduction of several elements in M , something that might again be interesting for the application to creative telescoping.

5. TAIL REDUCTION

Head reduction essentially allows us to reduce the valuation in x^{-1} of elements in M via the subtraction of elements in ∂M . Tail reduction aims at reducing the valuation in ϕ in a similar way. More generally, we may wish to reduce the valuation in any divisor $\psi \in \mathbb{K}[x] \setminus \mathbb{K}$ of ϕ .

So consider a divisor $\psi \in \mathbb{K}[x] \setminus \mathbb{K}$ of ϕ and notice that ψ is necessarily square-free. We let $\mathbb{L} = \mathbb{K}[x]_{<\deg \psi} = \{a \in \mathbb{K}[x] : \deg a < \deg \psi\}$. We also introduce $\bar{\mathbb{L}} = \mathbb{K}[x]/(\psi)$, which comes with the projection $\mathbb{K}[x] \rightarrow \bar{\mathbb{L}}; a \mapsto \bar{a} = a \bmod \psi$. If ψ is irreducible, then we notice that $\bar{\mathbb{L}}$ is a field. In general, $\bar{\mathbb{L}}$ is a finite product of fields. We finally denote the ψ -adic completion of $\mathbb{K}(x)$ by $\mathbb{K}(x)_\psi$. So the elements a of $\mathbb{K}(x)_\psi$ are infinite series $a = \sum_{i \geq \nu} a_i \psi^i$ with $\nu \in \mathbb{Z}$ and $a_i \in \bar{\mathbb{L}}$. We denote by $\text{val}_\psi a$ the valuation of a in ψ and notice that $\text{val}_\psi \phi = 1$. Elements of $\mathbb{K}(x)_\psi^{r \times r}$ can both be regarded as matrices with entries in $\mathbb{K}(x)_\psi$ or as elements of the ψ -adic completion of $\mathbb{K}(x)^{r \times r}$. In particular we may regard such elements as series in ψ with coefficients in $\bar{\mathbb{L}}^{r \times r}$.

As before, we have $(Ty)' = Uy$ for any matrix $T \in \phi^t \mathbb{K}[x, \psi^{-1}]^{r \times r}$ and $U = \phi^{-t} T A + T'$. For any matrix $C \in \bar{\mathbb{L}}^{r \times r}$ and $i \in \mathbb{Z}$, we also have

$$(C\psi^i Ty)' = \psi^{i-1} (C(U\psi + i\psi' T) + C' T \psi) y.$$

Recall that $\bar{\psi}'$ is invertible in $\bar{\mathbb{L}}$ since ψ is square-free. Regarding $U\psi + i\psi' T$ as a series in ψ , let $v = \min(\text{val}_\psi U, \text{val}_\psi T - 1) \in \mathbb{Z}$ and let σ be a formal indeterminate. We say that T is a ψ -tail chopper for (1) if $\bar{U}_v + \bar{\psi}' \bar{T}_{v+1} \sigma$ is invertible as a matrix in $\bar{\mathbb{L}}(\sigma)^{r \times r}$. If ψ is irreducible, then we may compute such a tail chopper using the following adaptation of **HeadChopper**:

Algorithm SpecialTailChopper($\phi, t, A; \psi$)

$(T, U) := (\phi^t \text{Id}_r, A + t \phi' \phi^{t-1} \text{Id}_r)$

repeat

$v := \min(\text{val}_\psi U, \text{val}_\psi T - 1)$

if $\bar{U}_v + \bar{\psi}' \bar{T}_{v+1} \sigma$ is invertible **then return** T

Lift $\bar{S} := \mathbf{RowSweeper}(\bar{U}_v)$ into $S \in \bar{\mathbb{L}}^{r \times r}$

$(T, U) := (ST, SU + S'T)$

Let k be the number of non-zero rows of \bar{U}_v and $\Delta := \begin{pmatrix} \text{Id}_k & 0 \\ 0 & \text{Id}_{r-k} \psi^{-1} \end{pmatrix}$

$(T, U) := (\Delta T, \Delta U + \Delta' T)$

PROPOSITION 6. *If ψ is irreducible, then the routine **SpecialTailChopper** terminates and is correct.*

Proof. Analogous to the proof of Proposition 1. □

Remark 7. It can be shown that we may rather take $\Delta := \begin{pmatrix} \text{Id}_k \psi & 0 \\ 0 & \text{Id}_{r-k} \end{pmatrix}$ in the algorithm, thereby avoiding denominators; see also Lemma 8 below.

If ψ is not necessarily irreducible, then we may still use the routine **SpecialTailChopper** to compute a ψ -tail chopper, but function calls $\mathbf{RowSweeper}(\bar{U}_v)$ may potentially fail, as explained at the end of section 3. Nevertheless, whenever failure occurs, we may compute an explicit non trivial factorization $\psi = \psi_1 \psi_2$. This allows us to recursively compute a ψ_1 -tail chopper T_1 and a ψ_2 -tail chopper T_2 . In order to recombine T_1 and T_2 into a ψ -tail chopper, we need several lemmas.

LEMMA 8. *Let T be a ψ -tail chopper for (1). Then $\psi^\delta T$ is a ψ -tail chopper as well, for any $\delta \in \mathbb{Z}$.*

Proof. Let $U = \phi^{-t} T A + T'$ and $v = \min(\text{val}_\psi U, \text{val}_\psi T - 1)$. Also let $\psi^\dagger = \psi' / \psi$, $\tilde{T} = T \psi^\delta$, $\tilde{U} = \phi^{-t} \tilde{T} A + \tilde{T}' = (U + \delta \psi^\dagger T) \psi^\delta$ and $\tilde{v} = \min(\text{val}_\psi \tilde{U}, \text{val}_\psi \tilde{T} - 1)$. Setting $D(\sigma) = U_v + \psi' T_{v+1} \sigma$ and $\tilde{D}(\sigma) = \tilde{U}_{\tilde{v}+\delta} + \psi' \tilde{T}_{\tilde{v}+\delta+1} \sigma = D(\sigma + \delta)$, the fact that $\tilde{D}(\sigma)$ is invertible as a matrix in $\bar{\mathbb{L}}(\sigma)^{r \times r}$ implies that $\tilde{D}(\sigma) = \tilde{D}(\sigma + \delta)$ is invertible. Hence $\tilde{D}(\sigma) \neq 0$ and $\tilde{v} = v + \delta$. We finally have $\tilde{D}(\sigma) = \tilde{U}_{\tilde{v}} + \psi' \tilde{T}_{\tilde{v}+1} \sigma$ and $\tilde{D}(\sigma)$ is an invertible matrix in $\bar{\mathbb{L}}(\sigma)^{r \times r}$. \square

LEMMA 9. *Let $T \in \phi^t \mathbb{K}[x, \psi^{-1}]^{r \times r}$ be a tail chopper for (1) and ψ . Denote $U = \phi^{-t} T A + T'$ and $v = \min(\text{val}_\psi U, \text{val}_\psi T - 1)$. Let $\tilde{T} \in \phi^t \mathbb{K}[x, \psi^{-1}]^{r \times r}$ be such that*

$$\text{val}_\psi(\tilde{T} - T) > v + \max(1, t - \text{val}_\psi A).$$

Then \tilde{T} is a ψ -tail chopper for (1).

Proof. Denoting $\tilde{U} = \phi^{-t} \tilde{T} A + \tilde{T}'$ and $w = \min(\text{val}_\psi(\tilde{U} - U), \text{val}_\psi(\tilde{T} - T) - 1)$, we have $w \geq \text{val}_\psi(\tilde{T} - T) + \min(-1, \text{val}_\psi A - t) > v$. Hence $\tilde{v} := \min(\text{val}_\psi \tilde{U}, \text{val}_\psi \tilde{T} - 1) = v$ and $D(\sigma) := U_v + \psi' T_{v+1} \sigma = \tilde{U}_{\tilde{v}} + \psi' \tilde{T}_{\tilde{v}+1} \sigma =: \tilde{D}(\sigma)$. This shows in particular that $\tilde{D}(\sigma)$ is invertible as a matrix in $\bar{\mathbb{L}}(\sigma)^{r \times r}$. \square

LEMMA 10. *Let ψ_1 and ψ_2 be two non scalar divisors of ϕ with $\text{gcd}(\psi_1, \psi_2) = 1$ and $\psi = \psi_1 \psi_2$. For $i = 1, 2$, let $T_i \in \phi^t \mathbb{K}[x]^{r \times r}$ be such that T_i is a ψ_i -tail chopper for (1) and denote $U_i = \phi^{-t} T_i A + T_i'$. Assume that $v := \min(\text{val}_{\psi_1} U_1, \text{val}_{\psi_1} T_1 - 1) = \min(\text{val}_{\psi_2} U_2, \text{val}_{\psi_2} T_2 - 1)$ and let $p = v + 1 + \max(1, t - \text{val}_{\psi_1} A, t - \text{val}_{\psi_2} A)$. Now consider $T \in \phi^t \mathbb{K}[x]^{r \times r}$ with $T \equiv T_1 \pmod{\psi_1^p}$ and $T \equiv T_2 \pmod{\psi_2^p}$. Then T is a ψ -tail chopper for (1).*

Proof. For $i = 1, 2$, our assumptions imply that $\text{val}_{\psi_i}(T - T_i) \geq p > \max(1, t - \text{val}_{\psi_i} A)$, whence T is a ψ_i -tail chopper for (1) by Lemma 9. Let $H_i(\sigma) := [(U + \psi_i' T \sigma) \psi_i^{-v}] \text{rem } \psi_i$ be the remainder of the Euclidean division of $(U + \psi_i' T \sigma) \psi_i^{-v}$ by ψ_i . Then we notice that $H_i(\sigma)$ coincides with the coefficient of ψ_i^v in the ψ_i -adic expansion of $U + \psi_i' T \sigma$. Using that $T \equiv T_i \pmod{\psi_i^p}$, we also have $H_i(\sigma) = [(U_i + \psi_i' T_i \sigma) \psi_i^{-v}] \text{rem } \psi_i$. In particular, $H_i(\sigma) \text{ mod } \psi_i$ admits an inverse $V_i(\sigma)$ in $(\mathbb{K}[x] / (\psi_i))(\sigma)^{r \times r}$.

Now let $H(\sigma) := [(U + \psi' T \sigma) \psi^{-v}] \text{rem } \psi$. Computing modulo ψ_1 , we have

$$\begin{aligned} H_1(\sigma) &\equiv (U + \psi_1' T \sigma) \psi_1^{-v} \\ &\equiv \psi_2^v (U + \psi_1' \psi_2 T (\sigma \psi_2^{-1})) \psi^{-v} \\ &\equiv \psi_2^v (U + \psi' T (\sigma \psi_2^{-1})) \psi^{-v} \\ &\equiv \psi_2^v H(\sigma \psi_2^{-1}) \pmod{\psi_1}, \end{aligned}$$

whence

$$\begin{aligned} H(\sigma) &\equiv \psi_2^{-v} H_1(\sigma \psi_2) \pmod{\psi_1} \\ H(\sigma) &\equiv \psi_1^{-v} H_2(\sigma \psi_1) \pmod{\psi_2}. \end{aligned}$$

Now $\text{gcd}(\psi_1, \psi_2) = 1$, so the Chinese remainder theorem implies the existence of a matrix $V(\sigma) \in (\mathbb{K}[x] / (\psi))(\sigma)^{r \times r}$ with $V(\sigma) \equiv \psi_2^v H_1(\sigma \psi_2) \pmod{\psi_1}$ and $V(\sigma) \equiv \psi_1^v H_2(\sigma \psi_1) \pmod{\psi_2}$. It follows that $H(\sigma) V(\sigma) \equiv 1 \pmod{\psi_1}$ and $H(\sigma) V(\sigma) \equiv 1 \pmod{\psi_2}$, showing that $H(\sigma) V(\sigma) \equiv 1 \pmod{\psi}$. Since $H(\sigma)$ is the leading coefficient of the ψ -adic expansion of $U + \psi' T \sigma$, we conclude that T is a ψ -tail chopper. \square

We are now in a position to state an improved version of **SpecialTailChopper** that works for general square-free ψ . The computed ψ -tail chopper T always belongs to $\phi^t \mathbb{K}[x]^{r \times r}$.

Algorithm TailChopper($\varphi, t, A; \psi$)

 Try computing $T := \mathbf{SpecialTailChopper}(\varphi, t, A; \psi)$
if no error occurred **then return** $T\psi^{t-\text{val}_\psi T}$

 Compute a non trivial factorization $\psi = \psi_1 \psi_2$
 $T_1 := \mathbf{TailChopper}(\varphi, t, A; \psi_1)$, $U_1 := \phi^{-t} T_1 A + T_1'$
 $T_2 := \mathbf{TailChopper}(\varphi, t, A; \psi_2)$, $U_2 := \phi^{-t} T_2 A + T_2'$
 $v_1 := \min(\text{val}_{\psi_1} U_1, \text{val}_{\psi_1} T_1 - 1)$, $v_2 := \min(\text{val}_{\psi_2} U_2, \text{val}_{\psi_2} T_2 - 1)$
 $v := \max(v_1, v_2)$, $T_1 := T_1 \psi_1^{v-v_1}$, $T_2 := T_2 \psi_2^{v-v_2}$
 $p := v + 1 + \max(1, t - \text{val}_{\psi_1} A, t - \text{val}_{\psi_2} A)$

 Compute $T \in \phi^t \mathbb{K}[x]^{r \times r}$ with $T \equiv T_1 \pmod{\psi_1^p}$ and $T \equiv T_2 \pmod{\psi_2^p}$
return T

PROPOSITION 11. *The routine **TailChopper** is correct.*

Proof. If no error occurs, then $T \psi^{t-\text{val}_\psi T} \in \phi^t \mathbb{K}[x]^{r \times r}$ is a ψ -tail chopper by Lemma 8. Otherwise, the recursive calls yield a ψ_1 -tail chopper $T_1 \in \phi^t \mathbb{K}[x]^{r \times r}$ and a ψ_2 -tail chopper $T_2 \in \phi^t \mathbb{K}[x]^{r \times r}$. Applying Lemma 8 twice more, we get that the corrected $\tilde{T}_1 := T_1 \psi_1^{v-v_1} \in \phi^t \mathbb{K}[x]^{r \times r}$ and $\tilde{T}_2 := T_2 \psi_2^{v-v_2} \in \phi^t \mathbb{K}[x]^{r \times r}$ are still tail choppers. Moreover, setting $\tilde{U}_i := \phi^{-t} \tilde{T}_i A + \tilde{T}_i'$ and $\tilde{v}_i := \min(\text{val}_{\psi_i} \tilde{U}_i, \text{val}_{\psi_i} \tilde{T}_i - 1)$ for $i = 1, 2$, the arguments from the proof of Lemma 8 imply that $\tilde{v}_1 = v_1 + (v - v_1) = v = v_2 + (v - v_2) = \tilde{v}_2$. We conclude by Lemma 10. \square

Assume now that a ϕ -tail chopper T is known for (1). Let $U = \phi^{-t} T A + T'$ and $v := \max(\text{val}_\phi U, \text{val}_\phi T - 1)$. Write \mathcal{I} for the set of indices $i \in \mathbb{Z}$ for which $\bar{U}_v + \bar{\phi}' \bar{T}_{v+1} i$ is not invertible. Let $\lambda \in \mathbb{K}[x, \phi^{-1}]^r \subseteq \mathbb{K}[x]_\phi^r$, considered as a series in ϕ with a finite number of non-zero coefficients in \mathbb{L}^r . Then the following routine **TailReduce** computes a vector $\tilde{\lambda} \in \mathbb{K}[x, \phi^{-1}]^r$ with $(\tilde{\lambda} - \lambda)^\top y \in \partial M$ and $\tilde{\lambda}_{i+v} \neq 0 \Rightarrow (i \in \mathcal{I} \vee i + v \geq 0)$ for all $i \in \mathbb{Z}$. We call $[\lambda^\top y] := \tilde{\lambda}^\top y$ the *tail reduction* of $\lambda^\top y$. The map $[\cdot]: M \rightarrow M$ is \mathbb{K} -linear. By construction, $[\lambda^\top y] - \lambda^\top y \in \partial M$ for all $\lambda \in \mathbb{K}[x, \phi^{-1}]^r$ and there exists a lower bound B such that $\text{val}_\psi [\lambda^\top y] \geq B$ for all $\lambda \in \mathbb{K}[x, \phi^{-1}]^r$.

Algorithm TailReduce(λ)

repeat
if $\lambda_{i+v} = 0$ for all $i < -v$ with $i \notin \mathcal{I}$ **then return** λ

 Let $i < -v$ with $i \notin \mathcal{I}$ be minimal with $\lambda_{i+v} \neq 0$

 Lift $\bar{\delta} := (\bar{U}_v^\top + i \bar{\phi}' \bar{T}_{v+1}^\top)^{-1} \bar{\lambda}_{i+v}$ into $\delta^\top \in (\mathbb{L}^r)^\top$
 $\lambda := \lambda - \phi^{i-1} (U^\top \phi + i \phi' T^\top) \delta - \phi^i T^\top \delta'$

PROPOSITION 12. *The routine **TailReduce** terminates and is correct.*

Proof. Analogous to the proof of Proposition 2. \square

6. NORMALIZING THE REDUCTION

Given a system (1) of differential equations, Propositions 1 and 11 show how to compute head and ϕ -tail choppers for (1). The algorithms **HeadReduce** and **TailReduce** then allow us to compute the head reduction $[f] \in M^\sharp$ of any $f \in M^\sharp$ and the tail reduction $[f] \in M$ of any $f \in M$. The head reduction extends to $f \in M$ by setting $[f] = [f^\sharp] + f^\flat$ for any $f \in M$. We finally obtain a confined reduction $[\cdot]: M \rightarrow M$ by setting $[f] = [[f]]$ for all $f \in M$. It remains to be shown how we can turn this reduction into a normal confined reduction $[\![\cdot]\!]: M \rightarrow M$.

Consider the finite dimensional \mathbb{K} -vector space $[M] := \{[f]: f \in M\}$. Let $d = \max \{\deg_x \lambda^\sharp: \lambda^\top y \in [M]\} \in \mathbb{N}$ and $v = \min \{\min \{\text{val}_\phi \lambda: \lambda^\top y \in [M]\}, -1\} \in -\mathbb{N}$. Let Ω be the \mathbb{K} -subvector space of M of all $f = \lambda^\top y \in M$ with $\deg_x \lambda^\sharp \leq d + 1$ and $\text{val}_\phi \lambda \geq v + 1$. We notice that $f \in \Omega$ for any $f = \lambda^\top y \in M$ with $f' \in [M]$. Now let $V := \partial\Omega \cap [M]$ and let W be a supplement of V in $[M]$ so that $[M] = V \oplus W$. We may compute bases of V and W using straightforward linear algebra. The canonical \mathbb{K} -linear projections $\pi_V: [M] \rightarrow V$ and $\pi_W: [M] \rightarrow W$ with $\pi_V + \pi_W = \text{Id}$ are also computable. We claim that we may take $\llbracket f \rrbracket := \pi_W([f])$ for every $f \in M$.

PROPOSITION 13. *The mapping $\llbracket \cdot \rrbracket: M \rightarrow M; f \mapsto \pi_W([f])$ defines a computable normal confined reduction on M .*

Proof. The mapping $\llbracket \cdot \rrbracket$ is clearly a computable confined reduction on M . It remains to be shown that $\llbracket f' \rrbracket = 0$ for all $f \in M$. Now $[f'] - f' \in \partial M$, so $[f'] \in \partial M$ and there exists a $g \in M$ with $g' = [f']$. Since $g' \in [M]$, it follows that $g \in \Omega$ and $g' \in \partial\Omega \cap [M] = V$. In other words, $[f'] = g' \in V$ and $\llbracket f' \rrbracket = \pi_W([f']) = 0$. \square

BIBLIOGRAPHY

- [1] A. Bostan, F. Chen, S. Chyzak, and Z. Li. Complexity of creative telescoping for bivariate rational functions. In *Proc. ISSAC '12*, pages 203–210. New York, NY, USA, 2010. ACM.
- [2] S. Chen. *Some applications of differential-difference algebra to creative telescoping*. PhD thesis, École Polytechnique, 2011.
- [3] S. Chen, M. van Hoeij, M. Kauers, and C. Koutschan. Reduction-based creative telescoping for Fuchsian D-finite functions. Technical Report, ArXiv, 2016. <http://arxiv.org/abs/1611.07421>.
- [4] S. Chen, M. Kauers, and C. Koutschan. Reduction-based creative telescoping for algebraic functions. In *Proc. ISSAC '16*, pages 175–182. New York, NY, USA, 2016. ACM.
- [5] F. Chyzak. *The ABC of Creative Telescoping — Algorithms, Bounds, Complexity*. Habilitation, École polytechnique, 2014.
- [6] L. Dumont. *Efficient algorithms for the symbolic computation of some contour integrals depending on one parameter*. PhD thesis, École Polytechnique, 2016.
- [7] J. van der Hoeven. Relax, but don't be too lazy. *JSC*, 34:479–542, 2002.
- [8] D. Zeilberger. The method of creative telescoping. *JSC*, 11(3):195–204, 1991.