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# Arnold diffusion in arbitrary degrees of freedom and normally hyperbolic invariant cylinders

P. Bernard\*, V. Kaloshin†, K. Zhang‡

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## Abstract

We prove a form of Arnold diffusion in the a priori stable case. Let

$$H_0(p) + \varepsilon H_1(\theta, p, t), \quad \theta \in \mathbb{T}^n, \quad p \in B^n, \quad t \in \mathbb{T} = \mathbb{R}/\mathbb{T}$$

be a nearly integrable system of arbitrary degrees of freedom  $n \geq 2$  with a strictly convex  $H_0$ . We show that for a “generic”  $\varepsilon H_1$ , there exists an orbit  $(\theta, p)(t)$  satisfying

$$\|p(t) - p(0)\| > l(H_1) > 0,$$

where  $l(H_1)$  is independent of  $\varepsilon$ . The diffusion orbit travels along a co-dimension one resonance, and the only obstruction to our construction is a finite set of additional resonances.

For the proof we use a combination geometric and variational methods, and manage to adapt tools which have recently been developed in the a priori unstable case.

## 1 Introduction

On the phase space  $\mathbb{T}^n \times B^n$ , we consider the Hamiltonian system generated by the  $C^r$  time-periodic Hamiltonian

$$H_\varepsilon(\theta, p, t) = H_0(p) + \varepsilon H_1(\theta, p, t), \quad (\theta, p, t) \in \mathbb{T}^n \times B^n \times \mathbb{T},$$

where  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ ,  $B^n$  is the unit ball in  $\mathbb{R}^n$  around the origin, and  $\varepsilon \geq 0$  is a small parameter. The equations

$$\dot{\theta} = \partial_p H_0 + \varepsilon \partial_p H_1, \quad \dot{p} = -\varepsilon \partial_\theta H$$

imply that the momenta  $p$  are constant in the case  $\varepsilon = 0$ . A question of general interest in Hamiltonian dynamics is to understand the evolution of these momenta when  $\varepsilon > 0$  is small (see e.g. [Ar1, Ar2, AKN]). In the present paper, we assume that  $H_0$  is convex, and, more precisely,

$$(1/D) I \leq \partial_p^2 H_0 \leq D I, \tag{1}$$

and prove that a certain form of Arnold’s diffusion occur for many perturbations. We assume that  $r \geq 4$  and denote by  $\mathcal{S}^r$  the unit sphere in  $C^r(\mathbb{T}^n \times B^n \times \mathbb{T})$ .

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**Theorem 1.** *There exist two continuous functions  $\ell$  and  $\varepsilon_0$  on  $\mathcal{S}^r$ , which are positive on an open and dense set  $\mathcal{U} \subset \mathcal{S}^r$ , and an open and dense subset  $\mathcal{V}_1$  of*

$$\mathcal{V} := \{H_0 + \varepsilon H_1 : H_1 \in \mathcal{U}, 0 < \varepsilon < \varepsilon_0(H_1)\}$$

*such that the following property holds for each Hamiltonian  $H \in \mathcal{V}_1$ :*

*There exists an orbit  $(\theta(t), p(t))$  of  $H_\varepsilon$  and a time  $T \in \mathbb{N}$  such that*

$$\|p(T) - p(0)\| > \ell(H_1).$$

The key point in this statement is that  $\ell(H_1)$  does not depend on  $\varepsilon \in ]0, \varepsilon_0(H_1)[$ . In section 1.1, we give a more detailed description of the diffusion path. Moreover, an improved version of the main theorem provides an explicit lower bound on  $l(H_1)$  (see Theorem 2.1 and Remark 2.1).

The present work is in large part inspired by the work of Mather [Ma3, Ma4, Ma5]. In [Ma3], Mather announced a much stronger version of Arnold diffusion for  $n = 2$ . Our set  $\mathcal{V}$  is what Mather called a *cuspidal residual set*. As in Mather's work the instability phenomenon thus holds in an open dense subset of a cuspidal residual set. Our result is, however, quite different. We obtain a much more restricted form of instability, which holds for any  $n \geq 2$ . The restricted character of the diffusion comes from the fact that we do not really solve the problem of double resonance (but only finitely many, independent from  $\varepsilon$ , double resonances are really problematic). The proof of Mather's result is partially written (see [Ma4]), and he has given lectures about some parts of the proof [Ma5].<sup>1</sup>

The study of Arnold diffusion was initiated by the seminal paper of Arnold, [Ar1], where he describes a diffusion phenomenon on a specific example involving two independent perturbations. A lot of work has then been devoted to describe more general situations where similar constructions could be achieved. A unifying aspect of all these situations is the presence of a normally hyperbolic cylinder, as was understood in [Mo] and [DLS], see also [DGLS, DH, T1, T2, CY1, CY2, Be1]. These general classes of situations have been referred to as *a priori unstable*.

The Hamiltonian  $H_\varepsilon$  studied here is, on the contrary, called *a priori stable*, because no hyperbolic structure is present in the unperturbed system  $H_0$ . Our method will, however, rely on the existence of a normally hyperbolic invariant cylinder. The novelty here thus consists in proving that a priori unstable methods do apply in the a priori stable case. Application of normal forms to construct normally 3-dimensional hyperbolic invariant cylinders in a priori stable situation in 3 degrees of freedom had already been discussed in [KZZ] and in [Mar]. The existence of normally hyperbolic cylinders with a length independent from  $\varepsilon$  in the a priori stable case, in arbitrary dimension, have been proved in [Be3], see also [Be5]. In the present paper, we obtain an explicit lower bound on the length of such a cylinder. The quantity  $l(H_1)$  in the statement of Theorem 1 is closely related to this lower bound (see also Remark 2.1). Let us mention some additional works of interest around the problem of Arnold's diffusion [Be4, Be6, BB, BBB, Bs1, Bs2, Bo, BK, CL1, CL2, Cr, GR1, GR2, KS, KL1, KL2, KLS, LM, MS, Zha, Zhe, X] and many others.

## 1.1 Reduction to normal form

As is usual in the theory of instability, we build our unstable orbits around a resonance. A frequency  $\omega \in \mathbb{R}^n$  is said *resonant* if there exists  $k \in \mathbb{Z}^{n+1}$ ,  $k \neq 0$ , such that  $k \cdot (\omega, 1) = 0$ . The

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<sup>1</sup>After a preliminary version of this paper was completed for  $n = 2$  the problem of double resonance was solved and existence of a strong form of Arnold diffusion is given in [KZ2].

set of such integral vectors  $k$  forms a submodule  $\Lambda$  of  $\mathbb{Z}^{n+1}$ , and the dimension of this module (which is also the dimension of the vector subspace of  $\mathbb{R}^{n+1}$  it generates) is called the order, or the dimension of the resonant frequency  $\omega$ .

In order to apply our proof, we have to consider a resonance of order  $n - 1$  or, equivalently, of codimension 1. For definiteness and simplicity, we choose once and for all to work with the resonance

$$\omega^s = 0,$$

where

$$\omega = (\omega^s, \omega^f) \in \mathbb{R}^{n-1} \times \mathbb{R}.$$

Similarly, we use the notations

$$\theta = (\theta^s, \theta^f) \in \mathbb{T}^{n-1} \times \mathbb{T}, \quad p = (p^s, p^f) \in \mathbb{R}^{n-1} \times \mathbb{R},$$

which are the slow and fast variables associated to our resonance (see Section 2 for definitions). More precisely, we will be working around the manifold defined by the equation

$$\partial_{p^s} H_0(p) = 0$$

in the phase space. In view of (1), this equation defines a  $C^{r-1}$  curve  $\Gamma$  in  $\mathbb{R}^n$ , which can also be described parametrically as the graph of a  $C^{r-1}$  function  $p_*^s(p^f) : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ . We will also use the notation  $p_*(p^f) := (p_*^s(p^f), p^f)$ .

We define the averaged perturbation corresponding to the resonance  $\Gamma$ ,

$$Z(\theta^s, p) := \iint H_1(\theta^s, p^s, \theta^f, p^f, t) d\theta^f dt.$$

If the perturbation  $H_1(\theta, p, t)$  is expanded as

$$H_1(\theta, p, t) = H_1(\theta^s, \theta^f, p, t) = \sum_{k^s \in \mathbb{Z}^{n-1}, k^f \in \mathbb{Z}, l \in \mathbb{Z}} h_{[k^s, k^f, l]}(p) e^{2i\pi(k^s \cdot \theta^s + k^f \cdot \theta^f + l \cdot t)},$$

then

$$Z(\theta^s, p) = \sum_{k^s} h_{[k^s, 0, 0]}(p) e^{2i\pi(k^s \cdot \theta^s)}.$$

Our first generic assumption, which defines the set  $\mathcal{U} \subset \mathcal{S}^r$  in Theorem 1 is on the shape of  $Z$ . We assume that there exists a subarc  $\Gamma_1 \subset \Gamma$  such that :

**Hypothesis 1.** *There exists a real number  $\lambda \in ]0, 1/2[$  such that, for each  $p \in \Gamma_1$ , there exists  $\theta_*^s(p) \in \mathbb{T}^{n-1}$  such that the inequality*

$$Z(\theta^s, p) \leq Z(\theta_*^s(p), p) - \lambda d^2(\theta^s, \theta_*^s(p)) \tag{HZ\lambda}$$

holds for each  $\theta^s$ .

## 1.2 Single maximum

This condition implies that for each  $p \in \Gamma_1$  the averaged perturbation  $Z(\theta, p)$  has a unique non-degenerate maximum at  $\theta_*^s(p)$ . In Section 1.5 we relax this condition and allow bifurcations from one global maxima to a different one. The set of functions  $Z \in C^r(\mathbb{T}^{n-1} \times B^n)$  satisfying Hypothesis 1 on some arc  $\Gamma_1 \subset \Gamma$  is open and dense for each  $r \geq 2$ . As a consequence, the set  $\mathcal{U}$  of functions  $H_1 \in \mathcal{S}^r$  (the unit sphere in  $C^r(\mathbb{T}^n \times B^n \times \mathbb{T})$ ) whose average  $Z$  satisfies Hypothesis 1 on some arc  $\Gamma_1 \subset \Gamma$  is open and dense in  $\mathcal{S}^r$  if  $r \geq 2$ .

The general principle of averaging theory is that the dynamics of  $H_\varepsilon$  is approximated by the dynamics of the averaged Hamiltonian  $H_0 + \varepsilon Z$  in a neighborhood of  $\mathbb{T}^n \times \Gamma$ . The applicability of this principle is limited by the presence of additional resonances, that is points  $p \in \Gamma$  such that the remaining frequency  $\partial_{p^f} H_0$  is rational. Although additional resonances are dense in  $\Gamma$ , only finitely many of them, called punctures, are really problematic. More precisely, denoting by  $U_{\varepsilon^{1/3}}(\Gamma_1)$  the  $\varepsilon^{1/3}$ -neighborhood of  $\Gamma_1$  in  $B^n$  and by  $\mathcal{R}(\Gamma_1, \varepsilon, \delta) \subset C^r(\mathbb{T}^n \times B^n \times \mathbb{T})$  the set of functions  $R(\theta, p, t) : \mathbb{T}^n \times B^n \times \mathbb{T} \rightarrow \mathbb{R}$  such that

$$\|R\|_{C^2(\mathbb{T}^n \times U_{\varepsilon^{1/3}}(\Gamma_1) \times \mathbb{T})} \leq \delta.$$

We will prove in section 2 that :

**Proposition 1.1.** *For each  $\delta \in ]0, 1[$ , there exists a locally finite subset  $\mathcal{P}_\delta \subset \Gamma$  and  $\varepsilon_1 \in ]0, \delta[$ , such that :*

*For each compact arc  $\Gamma_1 \subset \Gamma$  disjoint from  $\mathcal{P}_\delta$ , each  $H_1 \in \mathcal{S}^r$ , and each  $\varepsilon \in ]0, \varepsilon_1[$ , there exists a  $C^r$  smooth canonical change of coordinates*

$$\Phi : \mathbb{T}^n \times B \times \mathbb{T} \rightarrow \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T}$$

*satisfying  $\|\Phi - id\|_{C^0} \leq \sqrt{\varepsilon}$  and such that, in the new coordinates, the Hamiltonian  $H_0 + \varepsilon H_1$  takes the form*

$$N_\varepsilon = H_0(p) + \varepsilon Z(\theta^s, p) + \varepsilon R(\theta, p, t), \quad (2)$$

*with  $R \in \mathcal{R}(\Gamma_1, \varepsilon, \delta)$ .*

The key aspects of this result is that the set  $\mathcal{P}_\delta$  is locally finite and independent from  $\varepsilon$ . Because it is essential to have these properties of  $\mathcal{P}_\delta$ , the conclusions on the smallness of  $R$  are not very strong. Yet they are sufficient to obtain:

**Theorem 1.2.** *Let us consider the  $C^r$  Hamiltonian*

$$N_\varepsilon(\theta, p, t) = H_0(p) + \varepsilon Z(\theta^s, p) + \varepsilon R(\theta, p, t), \quad (3)$$

*and assume that  $\|Z\|_{C^2} \leq 1$  and that  $(HZ\lambda)$  holds on some arc  $\Gamma_1 \subset \Gamma$  of the form*

$$\Gamma_1 := \{(p_*(p^f)), p^f \in [a_-, a_+]\}.$$

*Then there exist constants  $\delta > 0$  and  $\varepsilon_0$ , which depends only on  $n$ ,  $H_0$ , and  $\lambda$ , and such that, for each  $\varepsilon \in ]0, \varepsilon_0[$ , the following property holds for an open dense subset of functions  $R \in \mathcal{R}(\Gamma_1, \varepsilon, \delta)$  (for the  $C^r$  topology):*

*There exists an orbit  $(\theta(t), p(t))$  and an integer  $T \in \mathbb{N}$  such that  $\|p(0) - p_*(a_-)\| < \sqrt{\varepsilon}$  and  $\|p(T) - p_*(a_+)\| < \sqrt{\varepsilon}$ .*

### 1.3 Derivation of Theorem 1 using Proposition 1.1 and Theorem 1.2

Given  $l > 0$ , we denote by  $\mathcal{D}^r(l)$  the set of  $C^r$  Hamiltonians with the following property: There exists an orbit  $(\theta(t), p(t))$  and an integer  $T$  such that  $\|p(T) - p(0)\| > l$ . The set  $\mathcal{D}^r(l)$  is clearly open.

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We now prove the existence of a continuous function  $\varepsilon_0$  on  $\mathcal{S}^r$  which is positive on  $\mathcal{U}$  and such that each Hamiltonian  $H_\varepsilon = H_0 + \varepsilon H_1$  with  $H_1 \in \mathcal{U}$  and  $\varepsilon < \varepsilon_0(H_1)$  belongs to the closure of  $\mathcal{D}^r(\varepsilon_0(H_1))$ .

For each  $H_1 \in \mathcal{U}$ , there exists a compact arc  $\Gamma_1 \subset \Gamma$  and a number  $\lambda \in ]0, 1/4[$  such that the corresponding averaged perturbation  $Z$  satisfies Hypothesis 1 on  $\Gamma_1$  with constant  $2\lambda$ . We then consider the real  $\delta$  given by Theorem 1.2 (applied with the parameter  $\lambda$ ). By possibly reducing the arc  $\Gamma_1$ , we can assume in addition that this arc is disjoint from the set  $\mathcal{P}_\delta$  of punctures for this  $\delta$ . The following properties then hold:

- The averaged perturbation  $Z$  satisfies Hypothesis 1 on  $\Gamma_1$  with a constant  $\lambda' > \lambda$ .
- The parameter  $\delta$  is associated to  $\lambda$  by Theorem 1.2.
- The arc  $\Gamma_1$  is disjoint from the set  $\mathcal{P}_\delta$  of punctures.

We say that  $(\Gamma_1, \lambda, \delta)$  is a compatible set of data if they satisfy the second and third point above. Then, we denote by  $\mathcal{U}(\Gamma_1, \lambda, \delta)$  the set of  $H_1 \in \mathcal{S}^r$  which satisfy the first point. This is an open set, and we just proved that the union on all compatible sets of data of these open sets covers  $\mathcal{U}$ .

To each compatible set of data  $(\Gamma_1, \lambda, \delta)$  we associate the positive numbers  $\ell := \|p_- - p_+\|/2$ , where  $p_\pm$  are the extremities of  $\Gamma_1$ , and  $\varepsilon_2(\Gamma_1, \lambda, \delta) := \min(\varepsilon_1, \ell^2/5, \ell)$ , where  $\varepsilon_1$  is associated to  $\delta$  by Proposition 1.1.

Using a partition of the unity, we can build a continuous function  $\varepsilon_0$  on  $\mathcal{S}^r$  which is positive on  $\mathcal{U}$  and have the following property: For each  $H_1 \in \mathcal{U}$ , there exists a compatible set of data  $(\Gamma_1, \lambda, \delta)$  such that  $H_1 \in \mathcal{U}(\Gamma_1, \lambda, \delta)$  and  $\varepsilon_0(H_1) \leq \varepsilon_2(\Gamma_1, \lambda, \delta)$ .

For this function  $\varepsilon_0$ , we claim that each Hamiltonian  $H_\varepsilon = H_0 + \varepsilon H_1$  with  $H_1 \in \mathcal{U}$  and  $0 < \varepsilon < \varepsilon_0(H_1)$  belongs to the closure of  $\mathcal{D}^r(\varepsilon_0(H_1))$ .

Assuming the claim, we finish the proof of Theorem 1. For  $l > 0$ , let us denote by  $\mathcal{V}(l)$  the open set of Hamiltonians of the form  $H_0 + \varepsilon H_1$ , where  $H_1 \in \mathcal{U}$  satisfies  $\varepsilon_0(H_1) > l$  and  $\varepsilon \in ]0, \varepsilon_0(H_1)[$ . The claim implies that  $\mathcal{D}(l)$  is dense in  $\mathcal{V}(l)$  for each  $l > 0$ . The conclusion of the Theorem (with  $l(H_1) := \varepsilon_0(H_1)$ ) then holds with the open set  $\mathcal{V}_1 := \cup_{l>0} (\mathcal{V}(l) \cap \mathcal{D}(l))$ , which is open and dense in  $\mathcal{V} = \cup_{l>0} \mathcal{V}(l)$ .

To prove the claim, we consider a Hamiltonian  $H_\varepsilon = H_0 + \varepsilon H_1$ , with  $H_1 \in \mathcal{U}$  and  $\varepsilon \in ]0, \varepsilon_0(H_1)[$ . We take a compatible set of data  $(\Gamma_1, \lambda, \delta)$  such that  $H_1 \in \mathcal{U}(\Gamma_1, \lambda, \delta)$  and  $\varepsilon_0(H_1) \leq \varepsilon_2(\Gamma_1, \lambda, \delta)$ . We apply Proposition 1.1 to find a change of coordinates  $\Phi$  which transforms the Hamiltonian  $H_0 + \varepsilon H_1$  to a Hamiltonian in the normal form  $\Phi^* H_\varepsilon = N_\varepsilon$  with  $R \in \mathcal{R}(\Gamma_1, \varepsilon, \delta)$ . The change of coordinates  $\Phi$  is fixed for the sequel of this discussion, as well as  $\varepsilon$ . By Theorem 1.2, the Hamiltonian  $N_\varepsilon$  can be approximated in the  $C^r$  norm by Hamiltonians  $\tilde{N}_\varepsilon$  admitting an orbit  $(\theta(t), p(t))$  such that  $p(0) = p_-$  and  $p(T) = p_+$  for some  $T \in \mathbb{N}$ . Let us denote by  $\tilde{H}_\varepsilon := (\Phi^{-1})^* \tilde{N}_\varepsilon$  the expression in the original coordinates of  $\tilde{N}_\varepsilon$ . It can be made arbitrarily  $C^r$ -close to  $H_\varepsilon$  by taking  $\tilde{N}_\varepsilon$  sufficiently close to  $N_\varepsilon$ . Since  $\|\Phi - Id\|_{C^0} \leq \sqrt{\varepsilon}$ , the extended  $\tilde{H}_\varepsilon$ -orbit  $(x(t), y(t), t \bmod 1) := \Phi(\theta(t), p(t), t \bmod 1)$  satisfies  $\|p(0) - p_-\| \leq \sqrt{\varepsilon}$  and  $\|p(T) - p_+\| \leq \sqrt{\varepsilon}$ , hence

$$\|y(T) - y(0)\| \geq \|p_+ - p_-\| - 2\sqrt{\varepsilon} > \ell \geq \varepsilon_0(H_1).$$

In other words, we have  $\tilde{H}_\varepsilon \in \mathcal{D}(\varepsilon_0(H_1))$ . We have proved that  $H_\varepsilon$  belongs to the closure of  $\mathcal{D}(\varepsilon_0(H_1))$ . This ends the proof of Theorem 1.  $\square$

The Hamiltonian in normal form  $N_\varepsilon$  has the typical structure of what is called an a priori unstable system under Hypothesis 1. Actually, under the additional assumption that  $\|R\|_{C^2} \leq \delta$ , with  $\delta$  sufficiently small with respect to  $\varepsilon$ , the conclusion of Theorem 1.2 would follow from the various works on the a priori unstable case, see [Be1, CY1, CY2, DLS, GR2, T1, T2]. The difficulty here is the weak hypothesis made on the smallness of  $R$ , and, in particular, the fact that  $\varepsilon$  is allowed to be much smaller than  $\delta$ .

## 1.4 Proof of Theorem 1.2

We give a proof based on several intermediate results that will be established in the further sections of the paper. The first step is to establish the existence of a normally hyperbolic cylinder. It is detailed in Section 3. As a consequence of the difficulties of our situation, we get only a rough control on this cylinder, as was already the case in [Be3]. Some  $C^1$  norms might blow up when  $\varepsilon \rightarrow 0$  (see (4)).

The second step consists in building unstable orbits along this cylinder under additional generic assumptions. In the a priori unstable case, where a regular cylinder is present, several methods have been developed. Which of them can be extended to the present situation is unclear. Here we manage to extend the variational approach of [Be1, CY1, CY2] (which are based on Mather's work). We use the framework of [Be1], but also essentially appeal to ideas from [Mag] and [CY2] for the proof of one of the key genericity results. A self-contained proof of the required genericity with many new ingredients is presented in Section 5.

The second step consists of three main steps:

- Along a resonance  $\Gamma$  prove existence a normally hyperbolic cylinder  $\mathcal{C}$  and derive its properties (see Theorem 1.3).
- Show that this cylinder  $\mathcal{C}$  contains a family of Mañé sets  $\tilde{\mathcal{N}}(c)$ ,  $c \in \Gamma$ , each being of Aubry-Mather type, i.e. a Lipschitz graph over the circle (see Theorem 1.4).
- Using the notion of a forcing class [Be1] generically construct orbits diffusing along this cylinder  $\mathcal{C}$  (see Theorem 1.5).

### 1.4.1 Existence and properties of a normally hyperbolic cylinder $\mathcal{C}$

**Theorem 1.3.** *Let us consider the  $C^r$  Hamiltonian system (3) and assume that  $Z$  satisfies (HZ $\lambda$ ) on some arc  $\Gamma_1 \subset \Gamma$  of the form*

$$\Gamma_1 := \{(p_*(p^f)), p^f \in [a_-, a_+]\}.$$

*Then there exist constants  $C > 1 > \kappa > \delta > 0$ , which depend only on  $n$ ,  $H_0$ , and  $\lambda$ , and such that, for each  $\varepsilon$  in  $]0, \delta[$ , the following property holds for each function  $R \in \mathcal{R}(\Gamma_1, \varepsilon, \delta)$ :*

*There exists a  $C^2$  map*

$$(\Theta^s, P^s)(\theta^f, p^f, t) : \mathbb{T} \times [a_- - \kappa\varepsilon^{1/3}, a_+ + \kappa\varepsilon^{1/3}] \times \mathbb{T} \longrightarrow \mathbb{T}^{n-1} \times \mathbb{R}^{n-1}$$

*such that the cylinder*

$$\mathcal{C} = \{(\theta^s, p^s) = (\Theta^s, P^s)(\theta^f, p^f, t); \quad p^f \in [a_- - \kappa\varepsilon^{1/3}, a_+ + \kappa\varepsilon^{1/3}], \quad (\theta^f, t) \in \mathbb{T} \times \mathbb{T}\}$$

*is weakly invariant with respect to  $N_\varepsilon$  in the sense that the Hamiltonian vector field is tangent to  $\mathcal{C}$ . The cylinder  $\mathcal{C}$  is contained in the set*

$$W := \{(\theta, p, t); p^f \in [a_- - \kappa\varepsilon^{1/3}, a_+ + \kappa\varepsilon^{1/3}], \\ \|\theta^s - \theta_*^s(p^f)\| \leq \kappa, \quad \|p^s - p_*^s(p^f)\| \leq \kappa\sqrt{\varepsilon}\},$$

*and it contains all the full orbits of  $N_\varepsilon$  contained in  $W$ . We have the estimate*

$$\left\| \frac{\partial \Theta^s}{\partial p^f} \right\| \leq C \left( 1 + \sqrt{\frac{\delta}{\varepsilon}} \right), \quad \left\| \frac{\partial \Theta^s}{\partial (\theta^f, t)} \right\| \leq C(\sqrt{\varepsilon} + \sqrt{\delta}), \quad (4)$$

$$\left\| \frac{\partial P^s}{\partial p^f} \right\| \leq C, \quad \left\| \frac{\partial P^s}{\partial (\theta^f, t)} \right\| \leq C\sqrt{\varepsilon}, \quad (5)$$

$$\|\Theta^s(\theta^f, p^f, t) - \theta_*^s(p^f)\| \leq C\sqrt{\delta}.$$

A similar, weaker, result is proved in [Be3]. The present statement contains better quantitative estimates. It follows from Theorem 3.1 below, which makes these estimates even more explicit. The terms  $\kappa\varepsilon^{1/3}$  come from the fact that we only estimate  $R$  on the  $\varepsilon^{1/4}$ -neighborhood of  $\Gamma_1$ , see the definition of  $\mathcal{R}(\Gamma_1, \varepsilon, \delta)$ .

For convenience of notations we extend our system from  $\mathbb{T}^n \times B^n \times \mathbb{T}$  to  $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T}$ . It is more pleasant in many occasions to consider the time-one Hamiltonian flow  $\phi$  and the discrete system that it generates on  $\mathbb{T}^n \times \mathbb{R}^n$ . We will thus consider the cylinder

$$\mathcal{C}_0 = \{(q, p) \in \mathbb{T}^n \times \mathbb{R}^n : (q, p, 0) \in \mathcal{C}\}.$$

We will think of this cylinder as being  $\phi$ -invariant, although this is not precisely true, due to the possibility that orbits may escape through the boundaries. If  $r$  is large enough, it is possible to prove the existence of a really invariant cylinder closed by KAM invariant circles, but this is not useful here.

The presence of this normally hyperbolic invariant cylinder is another similarity with the a priori unstable case. The difference is that we only have rough control on the present cylinders, with some estimates blowing up when  $\varepsilon \rightarrow 0$ . As we will see, variational methods can still be used to build unstable orbits along the cylinder. We will use the variational mechanism of [Be1]. Variational methods for this problem were initiated by Mather, see [Ma2] in an abstract setting. In a quite different direction, they were also used by Bessi to study the Arnold's example of [Ar1], see [Bs1].

#### 1.4.2 Weak KAM and Mather theory

We will use standard notations of weak KAM and Mather theory, we recall here the most important ones for the convenience of the reader. We mostly use Fathi's presentation in terms of weak KAM solutions, see [Fa], and also [Be1] for the non-autonomous case. We consider the Lagrangian function  $L(\theta, v, t)$  associated to  $N_\varepsilon$  (see Section 4 for the definition) and, for each  $c \in \mathbb{R}^n$ , the function

$$G_c(\theta_0, \theta_1) := \min_{\gamma} \int_0^1 L(\gamma(t), \dot{\gamma}(t), t) - c \cdot \dot{\gamma}(t) dt,$$

where the minimum is taken on the set of  $C^1$  curves  $\gamma : [0, 1] \rightarrow \mathbb{T}^n$  such that  $\gamma(0) = \theta_0, \gamma(1) = \theta_1$ . It is a classical fact that this minimum exists, and that the minimizers is the projection of a Hamiltonian orbit. A (discrete) *weak KAM solution at cohomology c* is a function  $u \in C(\mathbb{T}^n, \mathbb{R})$  such that

$$u(\theta) = \min_{v \in \mathbb{R}^n} [u(\theta - v) + G_c(\theta - v, \theta) + \alpha(c)]$$

where  $\alpha(c)$  is the only real constant such that such a function  $u$  exists. For each curve  $\gamma(t) : \mathbb{R} \rightarrow \mathbb{T}^n$  and each  $S < T$  in  $\mathbb{Z}$  we thus have the inequalities

$$u(\gamma(T)) - u(\gamma(S)) \leq G_c(\gamma(S), \gamma(T)) + (T - S)\alpha(c) \leq \int_S^T L(\gamma(t), \dot{\gamma}(t), t) - c \cdot \dot{\gamma}(t) + \alpha(c) dt.$$

A curve  $\theta(t) : \mathbb{R} \rightarrow \mathbb{T}^n$  is said *calibrated* by  $u$  if

$$u(\theta(T)) - u(\theta(S)) = \int_S^T L(\theta(t), \dot{\theta}(t), t) - c \cdot \dot{\theta}(t) + \alpha(c) dt,$$

for each  $S < T$  in  $\mathbb{Z}$ . The curve  $\theta(t)$  is then the projection of a Hamiltonian orbit  $(\theta(t), p(t))$ , such an orbit is called a *calibrated orbit*. We denote by

$$\tilde{\mathcal{I}}(u, c) \subset \mathbb{T}^n \times \mathbb{R}^n$$



the union on all calibrated orbits  $(\theta, p)(t)$  of the sets  $(\theta, p)(\mathbb{Z})$ , or equivalently of the sets  $(\theta, p)(0)$ . In other words, these are the initial conditions the orbits of which are calibrated by  $u$ . By definition, the set  $\tilde{\mathcal{I}}(u, c)$  is invariant under the time one Hamiltonian flow  $\varphi$ , it is moreover compact and not empty. We also denote by

$$s\tilde{\mathcal{I}}(u, c) \subset \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T}$$

the suspension of  $\tilde{\mathcal{I}}(u, c)$ , or in other words the set of points of the form  $((\theta(t), p(t), t \bmod 1))$  for each  $t \in \mathbb{R}$  and each calibrated orbit  $(\theta, p)$ . The set  $s\tilde{\mathcal{I}}(u, c)$  is compact and invariant under the extended Hamiltonian flow. Note that  $s\tilde{\mathcal{I}}(u, c) \cap \{t = 0\} = \tilde{\mathcal{I}}(u, c) \times \{0\}$ . The projection

$$\mathcal{I}(u, c) \subset \mathbb{T}^n$$

of  $\tilde{\mathcal{I}}(u, c)$  on  $\mathbb{T}^n$  is the union of points  $\theta(0)$  where  $\theta$  is a calibrated curve. The projection

$$s\mathcal{I}(u, c) \subset \mathbb{T}^n \times \mathbb{T}$$

of  $s\tilde{\mathcal{I}}(u, c)$  on  $\mathbb{T}^n \times \mathbb{T}$  is the union of points  $(\theta(t), t \bmod 1)$  where  $t \in \mathbb{R}$  and  $\theta$  is a calibrated curve. It is an important result of Mather theory that  $s\tilde{\mathcal{I}}(u, c)$  is a Lipschitz graph above  $s\mathcal{I}(u, c)$  (hence  $\tilde{\mathcal{I}}(u, c)$  is a Lipschitz graph above  $\mathcal{I}(u, c)$ ). We finally define the Aubry and Mañé sets by

$$\tilde{\mathcal{A}}(c) = \cap_u \tilde{\mathcal{I}}(u, c), \quad s\tilde{\mathcal{A}}(c) = \cap_u s\tilde{\mathcal{I}}(u, c), \quad \tilde{\mathcal{N}}(c) = \cup_u \tilde{\mathcal{I}}(u, c), \quad s\tilde{\mathcal{N}}(c) = \cup_u s\tilde{\mathcal{I}}(u, c), \quad (6)$$

where the union and the intersection are taken on the set of all weak KAM solutions  $u$  at cohomology  $c$ . When a clear distinction is needed, we will call the sets  $s\tilde{\mathcal{A}}(c), s\tilde{\mathcal{N}}(c)$  the suspended Aubry (and Mañé) sets. We denote by  $s\mathcal{A}(c)$  and  $s\mathcal{N}(c)$  the projections on  $\mathbb{T}^n \times \mathbb{T}$ , of  $s\tilde{\mathcal{A}}(c)$  and  $s\tilde{\mathcal{N}}(c)$ . Similarly,  $\mathcal{A}(c)$  and  $\mathcal{N}(c)$  are the projections on  $\mathbb{T}^n$  of  $\tilde{\mathcal{A}}(c)$  and  $\tilde{\mathcal{N}}(c)$ . The Aubry set  $\tilde{\mathcal{A}}(c)$  is compact, non-empty and invariant under the time one flow. It is a Lipschitz graph above the projected Aubry set  $\mathcal{A}(c)$ . The Mañé set  $\tilde{\mathcal{N}}(c)$  is compact and invariant. Its orbits (under the time-one flow) either belong, or are bi-asymptotic, to  $\tilde{\mathcal{A}}(c)$ .

In [Be1], an equivalence relation is introduced on the cohomology  $H^1(\mathbb{T}^n, \mathbb{R}) = \mathbb{R}^n$ , called *forcing relation*. It will not be useful for the present exposition to recall the precise definition of this forcing relation. What is important is that, if  $c$  and  $c'$  belong to the same forcing class, then there exists an orbit  $(\theta(t), p(t))$  and an integer  $T \in \mathbb{N}$  such that  $p(0) = c$  and  $p(T) = c'$ . We will establish here that, in the presence of generic additional assumptions, the resonant arc  $\Gamma_1$  is contained in a forcing class, which implies the conclusion of Theorem 1.2, but also the existence of various types of orbits, see [Be1], Section 5, for more details. To prove that  $\Gamma_1$  is contained in a forcing class, it is enough to prove that each of its points is in the interior of its forcing class. This can be achieved using the mechanisms exposed in [Be1], called the Mather mechanism and the Arnold mechanism, under appropriate informations on the sets

$$\tilde{\mathcal{A}}(c) \subset \tilde{\mathcal{I}}(u, c) \subset \tilde{\mathcal{N}}(c), \quad c \in \Gamma_1.$$

### 1.4.3 Localization and a graph theorem

The first step is to relate these sets to the normally hyperbolic cylinder  $\mathcal{C}_0$  as follows:

**Theorem 1.4.** *In the context of Theorem 1.3, we can assume by possibly reducing the constant  $\delta > 0$  that the following additional property holds for each function  $R \in \mathcal{R}(\Gamma_1, \varepsilon, \delta)$  with  $\varepsilon \in ]0, \delta[$ :*

*For each  $c \in \Gamma_1$ , the Mañé set  $\tilde{\mathcal{N}}(c)$  is contained in the cylinder  $\mathcal{C}_0$ . Moreover, the restriction of the coordinate map  $\theta^f : \mathbb{T}^n \times \mathbb{R}^n \rightarrow \mathbb{T}$  to  $\tilde{\mathcal{I}}(u, c)$  is a Bi-Lipschitz homeomorphism for each Weak KAM solution  $u$  at cohomology  $c$ .*

*Proof.* The proof is based on estimates on Weak KAM solutions that will be established in Section 4. Let  $\kappa$  be as given by Theorem 1.3. Theorem 4.1 (which is stated and proved in Section 4) implies that the suspended Mañé set  $s\tilde{\mathcal{N}}(c)$  is contained in the set

$$\{\|\theta^s - \theta_*^s(c^f)\| \leq \kappa, \quad \|p^s - p_*^s(c^f)\| \leq \kappa\sqrt{\varepsilon}, \quad |p^f - c^f| \leq \kappa\sqrt{\varepsilon}\}$$

provided  $R \in \mathcal{R}(\Gamma_1, \varepsilon, \kappa^{16})$  and  $\varepsilon \in ]0, \varepsilon_0[$  (a constant depending on  $\kappa$ ). As a consequence, this inclusion holds for  $R \in \mathcal{R}(\Gamma_1, \varepsilon, \delta)$  and  $\varepsilon \in ]0, \delta[$ , with  $\delta = \min(\kappa^{16}, \varepsilon_0)$ . The suspended Mañé set  $s\tilde{\mathcal{N}}(c)$  is then contained in the domain called  $W$  in the statement of Theorem 1.3. It is thus contained in  $\mathcal{C}$ , hence  $\tilde{\mathcal{N}}(c) \subset \mathcal{C}_0$ .

Let us consider a Weak KAM solution  $u$  of  $N_\varepsilon$  at cohomology  $c$  and prove the projection part of the statement. Let  $(\theta_i, p_i), i = 1, 2$  be two points in  $\tilde{\mathcal{I}}(u, c)$ . By Theorem 4.2, we have

$$\|p_2 - p_1\| \leq 9\sqrt{D\varepsilon}\|\theta_2 - \theta_1\| \leq 9\sqrt{D\varepsilon}(\|\theta_2^f - \theta_1^f\| + \|\theta_2^s - \theta_1^s\|).$$

Since the points belong to  $\mathcal{C}_0$ , the last estimate in Theorem 1.3 implies that

$$\|\theta_2^s - \theta_1^s\| \leq C(1 + \sqrt{\delta/\varepsilon})(\|\theta_2^f - \theta_1^f\| + \|p_2 - p_1\|).$$

We get

$$\|p_2 - p_1\| \leq 9C\sqrt{D}(2\sqrt{\varepsilon} + \sqrt{\delta})\|\theta_2^f - \theta_1^f\| + 9C\sqrt{D}(\sqrt{\varepsilon} + \sqrt{\delta})\|p_2 - p_1\|.$$

If  $\delta$  is small enough and  $\varepsilon < \delta$ , then

$$9C\sqrt{D}(\sqrt{\varepsilon} + \sqrt{\delta}) \leq 9C\sqrt{D}(2\sqrt{\varepsilon} + \sqrt{\delta}) \leq \frac{1}{2}$$

hence

$$\|p_2 - p_1\| \leq 9C\sqrt{D}(2\sqrt{\varepsilon} + \sqrt{\delta})\|\theta_2^f - \theta_1^f\| + \frac{1}{2}\|p_2 - p_1\|,$$

thus

$$\|p_2 - p_1\| \leq 9C\sqrt{D}(4\sqrt{\varepsilon} + 2\sqrt{\delta})\|\theta_2^f - \theta_1^f\| \leq \|\theta_2^f - \theta_1^f\|.$$

□

#### 1.4.4 Structure of Aubry sets inside the cylinder and existence of diffusing orbits

This last result, in conjunction with the theory of circle homeomorphisms, has strong consequences:

All the orbits of  $\tilde{\mathcal{A}}_0(c)$  have the same rotation number  $\rho(c) = (\rho^f(c), 0)$ , with  $\rho^f(c) \in \mathbb{R}$ . Since the sub-differential  $\partial\alpha(c)$  of the convex function  $\alpha$  is the rotation set of  $\tilde{\mathcal{A}}(c)$ , we conclude that the function  $\alpha$  is differentiable at each point of  $\Gamma_1$ , with  $d\alpha(c) = (\rho^s(c), 0)$ .

When  $\rho^s(c)$  is rational, the Mather minimizing measures are supported on periodic orbits.

When  $\rho^s(c)$  is irrational, the invariant set  $\tilde{\mathcal{A}}(c)$  is uniquely ergodic. As a consequence, there exists one and only one weak KAM solution (up to the addition of an additive constant), hence  $\tilde{\mathcal{N}}(c) = \tilde{\mathcal{A}}(c)$ .

In the irrational case, we will have to consider homoclinic orbits. Such orbits can be dealt with by considering the two-fold covering

$$\begin{aligned} \xi : \mathbb{T}^n &\longrightarrow \mathbb{T}^n \\ \theta = (\theta^f, \theta_1^s, \theta_2^s, \dots, \theta_{n-1}^s) &\longmapsto \xi(\theta) = (\theta^f, 2\theta_1^s, \theta_2^s, \dots, \theta_{n-1}^s). \end{aligned}$$

The idea of using a covering to study homoclinic orbits comes from Fathi, see [Fa2]. This covering lifts to a symplectic covering

$$\begin{aligned} \Xi : \mathbb{T}^n \times \mathbb{R}^n &\longrightarrow \mathbb{T}^n \times \mathbb{R}^n \\ (\theta, p) = (\theta, p^f, p_1^s, p_2^s, \dots, p_{n-1}^s) &\longmapsto \Xi(\theta, p) = (\xi(\theta), p^f, p_1^s/2, p_2^s, \dots, p_{n-1}^s), \end{aligned}$$

and we define the lifted Hamiltonian  $\tilde{N} = N \circ \Xi$ . It is known, see [Fa2, CP, Be1] that

$$\tilde{\mathcal{A}}_{H \circ \Xi}(\xi^* c) = \Xi^{-1}(\tilde{\mathcal{A}}_H(c))$$

where  $\xi^* c = (c^f, c_1^s/2, c_2^s, \dots, c_{n-1}^s)$ . On the other hand, the inclusion

$$\tilde{\mathcal{N}}_{N \circ \Xi}(\xi^* c) \supset \Xi^{-1}(\tilde{\mathcal{N}}_N(c)) = \Xi^{-1}(\tilde{\mathcal{A}}_N(c))$$

is not an equality. More precisely, in the present situation, the set  $\tilde{\mathcal{A}}_{N \circ \Xi}(\tilde{c})$  is the union of two disjoint homeomorphic copies of the circle  $\tilde{\mathcal{A}}_N(\tilde{c})$ , and  $\tilde{\mathcal{N}}_{N \circ \Xi}(\tilde{c})$  contains heteroclinic connections between these copies (which are the liftings of orbits homoclinic to  $\tilde{\mathcal{A}}_N(c)$ ). More can be said if we are allowed to make a small perturbation to avoid degenerate situations. We recall that a metric space is called totally disconnected if its only connected subsets are its points. The hypothesis of total disconnectedness in the following statement can be seen as a weak form of transversality of the stable and unstable manifolds of the invariant circle  $\tilde{\mathcal{A}}_N(c)$ .

**Theorem 1.5.** *In the context of Theorems 1.3 and 1.4, the following property holds for a dense subset of functions  $R \in \mathcal{R}(\Gamma_1, \varepsilon, \delta_0)$  (for the  $C^r$  topology): Each  $c \in \Gamma_1$  is in one of the following cases:*

1.  $\theta^f(\mathcal{I}(u, c)) \subsetneq \mathbb{T}$  for each weak KAM solution  $u$  at cohomology  $c$ .
2.  $\rho(c)$  is irrational,  $\theta^f(\mathcal{N}_N(c)) = \mathbb{T}$  (hence,  $\tilde{\mathcal{N}}_N(c)$  is an invariant circle), and  $\tilde{\mathcal{N}}_{N \circ \Xi}(\xi^* c) - \Xi^{-1}(\tilde{\mathcal{N}}_N(c))$  is totally disconnected.

The arc  $\Gamma_1$  is then contained in a forcing class, hence the conclusion of Theorem 1.2 holds.

*Proof.* By general results on Hamiltonian dynamics, the set  $\mathcal{R}_1 \subset \mathcal{R}(\Gamma_1, \varepsilon, \delta_0)$  of functions  $R$  such that the flow map  $\phi$  does not admit any non-trivial invariant circle of rational rotation number is  $C^r$ -dense. This condition holds for example if  $N$  is Kupka Smale (in the Hamiltonian sense, see [RR] for example).

Since the coordinate map  $\theta^f$  is a homeomorphism in restriction to  $\tilde{\mathcal{I}}(u, c)$ , this set is an invariant circle if  $\theta^f(\mathcal{I}(u, c)) = \mathbb{T}$ . If  $R \in \mathcal{R}_1$ , this implies that the rotation number  $\rho^f(c)$  is irrational. In other words, for  $R \in \mathcal{R}_1$ , condition 1 can be violated only at points  $c$  when  $\rho^f(c)$  is irrational, and then  $\tilde{\mathcal{I}}(u, c) = \tilde{\mathcal{A}}(c) = \tilde{\mathcal{N}}(c)$  is an invariant circle.

When  $R \in \mathcal{R}_1$ , it is possible to perturb  $R$  away from  $\mathcal{C}_0$  in such a way that  $\tilde{\mathcal{N}}_{N \circ \Xi}(\xi^* c) - \Xi^{-1}(\tilde{\mathcal{N}}_N(c))$  is totally disconnected for each value of  $c$  such that  $\tilde{\mathcal{N}}(c)$  is an invariant circle. This second perturbation procedure is not easy because there are uncountably many such values of  $c$ . This is the result of Theorem 5.1. A result of this kind was obtained in [CY2], here we give a self-contained proof with many new ingredients, see Section 5.

We now explain, under the additional condition (1 or 2), how the variational mechanisms of [Be1] can be applied to prove that  $\Gamma_1$  is contained in a forcing class. It is enough to prove that each point  $c \in \Gamma_1$  is in the interior of its forcing class. We treat separately the two cases.

In the first case, we can apply the Mather mechanism, see (0.11) in [Be1]. In that paper, the subspace  $Y(u, c) \subset \mathbb{R}^n$ , defined as the set of cohomology classes of closed one-forms whose support is disjoint from  $\mathcal{I}(u, c)$ , is associated to each weak KAM solution  $u$  at cohomology  $c$  (in [Be1], the notation  $R(\mathcal{G})$  is used). In the present case, we know that the map  $\theta^f$  restricted

to  $\tilde{\mathcal{I}}(u, f)$  is a bi-Lipschitz homeomorphism which is not onto. We conclude that  $Y(u, c) = \mathbb{R}^n$ . Since this holds for each weak KAM solution  $u$ , we conclude that

$$Y(c) := \bigcap_u Y(u, c) = \mathbb{R}^n.$$

The result called Mather mechanism in [Be1] states that there is a small ball  $B \subset Y(c)$  centered at 0 in  $Y$  such that the forcing class of  $c$  contains  $c + B$ . In the present situation, we conclude that  $c$  is in the interior of its forcing class.

In the second case, we can apply the Arnold's Mechanism, see Section 9 in [Be1]. We work with the Hamiltonian  $N \circ \Xi$  lifted to the two-fold cover. By Proposition (7.3) in [Be1], it is enough to prove that  $\xi^*c$  is in the interior of its forcing class for the lifted Hamiltonian  $N \circ \Xi$ ; this implies that  $c$  is in the interior of its forcing class for  $N$ .

The preimage  $\Xi^{-1}(\tilde{\mathcal{N}}_N(c))$  is the union of two closed curves  $\tilde{\mathcal{S}}_1$  and  $\tilde{\mathcal{S}}_2$ . The set  $\tilde{\mathcal{N}}_{N \circ \Xi}(\xi^*c)$  contains these two curves, as well as a set  $\tilde{\mathcal{H}}_{12}$  of heteroclinic connections from  $\tilde{\mathcal{S}}_1$  to  $\tilde{\mathcal{S}}_2$ , and a set  $\tilde{\mathcal{H}}_{21}$  of heteroclinic connections from  $\tilde{\mathcal{S}}_2$  to  $\tilde{\mathcal{S}}_1$ . Theorem (9.2) in [Be1] states that  $\xi^*c$  is in the interior of its forcing class provided  $\tilde{\mathcal{H}}_{12}$  and  $\tilde{\mathcal{H}}_{21}$  are totally disconnected. Actually, the hypothesis is stated in [Be1] in a slightly different way, we explain in Appendix B that total disconnectedness actually implies the hypothesis of [Be1]. We conclude that each  $c \in \Gamma_1$  is in the interior of its forcing class. Since  $\Gamma_1$  is connected, it is contained in a single forcing class. It is then a simple consequence of the definition of the forcing relation, see [Be1], Section 5, that the conclusion of Theorem 1.2 holds. This ends the proof of Theorem 1.2, using the results proved in the rest of the paper.  $\square$

## 1.5 Bifurcation points and a longer diffusion path

This section discusses some improvements on Theorems 1 and 1.2. There are two limitations to the size of the resonant arc  $\Gamma_1 \subset \Gamma$  to which the above construction can be applied.

The first limitation comes from the assumption that hypothesis  $(HZ\lambda)$  should hold on  $\Gamma_1$ . Given a resonant arc  $\Gamma_2 \subset \Gamma$ , it is generic to satisfy this condition on a certain subarc  $\Gamma_1 \subset \Gamma_2$ , but it is not generic to satisfy  $(HZ\lambda)$  on the whole of  $\Gamma_2$ . The presence of values of  $c \in \Gamma_2$  such that  $Z(\cdot, c)$  has two nondegenerate maxima can't be excluded. In this section, we explain how a modification on the proof of Theorem 1.2 allows to get rid of this limitation.

The second limitation comes from the normal form theorem, and from the impossibility to incorporate a finite set of additional resonances (punctures) in the domain of our normal forms. This limitation is serious, and bypassing it would require a specific work around additional resonances which will not be discussed here. Some preprints on this issue appeared after the first version of the present works, see [C, KZ1, KZ2] (the latter ones being sequels to the present work, and the first one is independent). Here, the best we can achieve is to prove existence of diffusion orbits between two consecutive punctures. The number of punctures is independant from  $\varepsilon$ , it depends on the parameter  $\delta$  in Theorem 1.2, which can be computed using the non-degeneracy parameter  $\lambda$ , see Remark 2.1.

In order to get rid of the first limitation, we consider a second hypothesis on  $Z$ :

**Hypothesis 2.** *There exists a real number  $\lambda > 0$  and two points  $\vartheta_1^s, \vartheta_2^s$  in  $\mathbb{T}^{n-1}$  such that the balls  $B(\vartheta_1^s, 3\lambda)$  and  $B(\vartheta_2^s, 3\lambda)$  are disjoint and such that, for each  $p \in \Gamma_1$ , there exists two local maxima  $\theta_1^s(p) \in B(\vartheta_1^s, \lambda)$  and  $\theta_2^s(p) \in B(\vartheta_2^s, \lambda)$  of the function  $Z(\cdot, p)$  in  $\mathbb{T}^{n-1}$  satisfying*

$$\begin{aligned} \partial_{\theta^s}^2 Z(\theta_1^s(p), p) &\leq \lambda I \quad , \quad \partial_{\theta^s}^2 Z(\theta_2^s(p), p) \leq \lambda I, \\ Z(\theta^s, p) &\leq \max\{Z(\theta_1^f(p), p), Z(\theta_2^f(p), p)\} - \lambda(\min\{d(\theta^s - \theta_1^s), d(\theta^s - \theta_2^s)\})^2 \end{aligned}$$

for each  $p \in \Gamma_1$  and each  $\theta^s \in \mathbb{T}^{n-1}$ .

Given an arc  $\Gamma_2 \in \mathbb{R}^n$ , the following property is generic in  $C^r(\mathbb{T}^{n-1} \times \mathbb{R}^n, \mathbb{R})$ :

The arc  $\Gamma_2$  is a finite union of subarcs such that either Hypothesis 1 or Hypothesis 2 holds on each of these subarcs, with a common constant  $\lambda > 0$ .

We have the following improvement on Theorem 1.2:

**Proposition 1.6.** *For the system (3), assume that there exists  $\lambda > 0$  such that for each  $c \in \Gamma_1$ , either Hypothesis 1 or 2 hold for each  $c \in \Gamma_1$ . Then there exists  $\delta > 0$ , which depend only on  $n$ ,  $H_0$ , and  $\lambda$ , and such that, for each  $\varepsilon \in ]0, \delta[$ , the following property holds for a dense subset of functions  $R \in \mathcal{R}(\Gamma_1, \varepsilon, \delta)$  (for the  $C^r$  topology):*

*There exists an orbit  $(\theta(t), p(t))$  and an integer  $T \in \mathbb{N}$  such that  $p(0) = p_*(a_-)$  and  $p(T) = p_*(a_+)$ .*

*Proof of Proposition 1.6.* We use the same framework as in the proof of Theorem 1.2, so it is enough to prove that each element of  $\Gamma_1$  is in the interior of its forcing class.

Observe first that Theorem 3.1 can be applied to prove the existence of two invariant cylinders  $\mathcal{C}^1$  and  $\mathcal{C}^2$  in the extended phase space  $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T}$ . Moreover, we can chose the parameter  $\kappa$  smaller than  $\lambda$ , in such a way that

$$\theta^s(\mathcal{C}_1) \subset B(\vartheta_1^s, 2\lambda) \quad , \quad \theta^s(\mathcal{C}_2) \subset B(\vartheta_2^s, 2\lambda).$$

As earlier, we denote by  $\mathcal{C}_0^1$  and  $\mathcal{C}_0^2$  the intersections with the section  $\{t = 0\}$ . By Theorem 4.4, we have

$$\tilde{\mathcal{A}}(c) \subset \mathcal{C}_0^1 \cup \mathcal{C}_0^2$$

for each  $c \in \Gamma_1$ . Let us now introduce two smooth functions  $F_i(\theta^s) : \mathbb{T}^{n-1} \rightarrow [0, 1]$ ,  $i \in \{1, 2\}$ , with the property that  $F_1 = 1$  in  $B(\vartheta_2^s, 2\lambda)$ ,  $F_1 = 0$  outside of  $B(\vartheta_2^s, 3\lambda)$ ,  $F_2 = 1$  in  $B(\vartheta_1^s, 2\lambda)$  and  $F_2 = 0$  outside of  $B(\vartheta_1^s, 3\lambda)$

Considering the modified Hamiltonians  $N - F_i$  will help the description of the Mather sets of  $N$ . One can check by inspection in the proofs (using that  $F_i$  does not depend on  $p$ ) that Theorem 4.1 applies to  $N - F_i$ , and allows to conclude that the Mañé set  $\tilde{\mathcal{N}}_i(c)$  of  $N - F_i$  is contained in  $\mathcal{C}_0^i$ . Let us denote by  $\alpha_i(c)$  the  $\alpha$  function of  $N - F_i$ . These objects are closely related to Mather's local Aubry sets.

**Lemma 1.1.** *For each  $c \in \Gamma_1$ ,  $\alpha_i(c)$  are differentiable at  $c$ , and  $\alpha(c) = \max\{\alpha_1(c), \alpha_2(c)\}$ . Moreover,*

- *If  $\alpha(c) = \alpha_1(c) > \alpha_2(c)$ , then  $\tilde{\mathcal{N}}(c) = \tilde{\mathcal{N}}_1(c)$ ,*
- *If  $\alpha(c) = \alpha_2(c) > \alpha_1(c)$ , then  $\tilde{\mathcal{N}}(c) = \tilde{\mathcal{N}}_2(c)$ ,*
- *If  $\alpha(c) = \alpha_1(c) = \alpha_2(c)$ , then  $\tilde{\mathcal{N}}_1(c) \cup \tilde{\mathcal{N}}_2(c) \subsetneq \tilde{\mathcal{N}}(c)$ .*

*Proof.* The functions  $\alpha_i(c)$  are  $C^1$  for the same reason as  $\alpha(c)$  is  $C^1$  in the one peak case.

Since  $N - N_i \leq N$ , we have  $\alpha_i(c) \leq \alpha(c)$ . On the other hand, we know that

$$\alpha(c) = \max_{\mu} \left( c \cdot \rho(\mu) - \int p \partial_p N - N d\mu \right),$$

where the minimum is taken on the set of invariant measures  $\mu$ . Since we know that  $\tilde{\mathcal{A}}(c) \subset \mathcal{C}_0^1 \cup \mathcal{C}_0^2$ , and since the maximizing measures are supported on the Aubry set, we conclude that each ergodic maximizing measure is supported either on  $\mathcal{C}^1$  or on  $\mathcal{C}^2$ . If the measure is supported in  $\mathcal{C}^i$ , then we have

$$\alpha_i(c) \geq c \cdot \rho(\mu) - \int p \partial_p N - N + F_i d\mu = c \cdot \rho(\mu) - \int p \partial_p N - N d\mu = \alpha(c).$$

This proves the equality  $\alpha(c) = \max\{\alpha_1(c), \alpha_2(c)\}$ .

As is explained in the proof of Theorem 4.4, there are two possibilities for the Mañé set  $\tilde{\mathcal{N}}(c)$ : either it is contained in one of the  $\mathcal{C}_0^i$ , or it intersects both of them, and then also contains connections (because it is necessarily chain transitive).

If the Mañé set  $\tilde{\mathcal{N}}(c)$  intersects  $\mathcal{C}_0^i$ , then the intersection is a compact invariant set, which thus support an invariant measure. This measure must be maximizing the functional  $c \cdot \rho(\mu) - \int p \partial_p N - N d\mu$ , and thus also the functional  $c \cdot \rho(\mu) - \int p \partial_p N - N + F_i d\mu$ . As a consequence, we must have  $\alpha(c) = \alpha_i(c)$ .  $\square$

We can prove by the variational mechanisms of [Be1] that a point  $c$  is in the interior of its forcing class in the following three cases:

First case, the Mañé  $\tilde{\mathcal{N}}(c)$  set is contained in one of the cylinders  $\mathcal{C}_0^i$ , and it does not contain any invariant circle. Then the Mather mechanism applies as in the single peak case, and  $c$  is contained in the interior of its forcing class.

Second case, the Mañé set is an invariant circle (then necessarily contained in one of the cylinders  $\mathcal{C}_0^i$ ), it is uniquely ergodic, and  $\tilde{\mathcal{N}}_{N \circ \Xi}(c) - \Xi^{-1}(\tilde{\mathcal{N}}(c))$  is totally disconnected. Then the Arnold's mechanism applies as in the single peak case, and  $c$  is contained in the interior of its forcing class.

Third case, the sets  $\tilde{\mathcal{N}}_i(c)$  are both non-empty and uniquely ergodic, and  $\tilde{\mathcal{N}}(c) - (\tilde{\mathcal{N}}_1(c) \cup \tilde{\mathcal{N}}_2(c))$  is totally disconnected. Then the Arnold's mechanism applies directly (without taking a cover), and  $c$  is contained in the interior of its forcing class.

Each  $c \in \Gamma_1$  is in one of these three cases provided the following set of additional conditions holds:

- The sets  $\tilde{\mathcal{N}}_i(c)$  are uniquely ergodic.
- The equality  $\alpha_1(c) = \alpha_2(c)$  has finitely many solutions on  $\Gamma_1$ .
- The set  $\tilde{\mathcal{N}}(c) - (\tilde{\mathcal{N}}_1(c) \cup \tilde{\mathcal{N}}_2(c))$  is totally disconnected (and not empty) when  $\alpha_1(c) = \alpha_2(c)$ .
- The set  $\tilde{\mathcal{N}}_{N \circ \Xi}(c) - \Xi^{-1}(\tilde{\mathcal{N}}(c))$  is totally disconnected whenever  $\tilde{\mathcal{N}}(c)$  is an invariant circle.

Let us now explain how these conditions can be imposed by a  $C^r$  perturbation of  $R$ .

We first consider a perturbation  $R_1$  of  $R$  such that, for each rational number  $\rho \in \mathbb{Q} \times \{0\}$ , there exists a unique Mather minimizing measure of rotation number  $\rho$ . Such a condition is known to be generic (because it concerns only countably many rotation numbers) see [Mn, CP, BC, Be7].

We then consider a perturbation  $R_2$  of the form  $R_1 - sF_1$ , with a small  $s > 0$ . It is easy to see that the functions  $\alpha_i^2(c), c \in \Gamma_1$  associated to the Hamiltonian  $H_0 + \varepsilon Z + \varepsilon R_2$  are

$$\alpha_1^2(c) = \alpha_1^1(c) \quad , \quad \alpha_2^2(c) = \alpha_2^1(c) + s$$

where  $\alpha_i^1(c)$  are the functions associated to  $H_0 + \varepsilon Z + \varepsilon R_1$ . By Sard's theorem, there exist arbitrarily small regular values  $s$  of the difference  $\alpha_1^1 - \alpha_2^1$ . If  $s$  is such a value, then 0 is a regular value of the difference  $\alpha_1^2 - \alpha_2^2$ , hence the equation  $\alpha_1^2(c) = \alpha_2^2(c)$  has only finitely many solutions on  $\Gamma$ . Note that the perturbation is locally constant around the cylinders  $\mathcal{C}^i$ , hence this second perturbation does not destroy the first property.

We then perform new perturbations supported away from  $\mathcal{C}^i$ , which preserve the first two properties. The third property is not hard to obtain since it now concerns only finitely many values of  $c$ . The last property is obtained using arguments of Section 5.

We have proved that the Hamiltonian  $R$  can be perturbed in such a way that each point of  $\Gamma_1$  is in the interior of its forcing class.  $\square$

## 2 Normal forms

The goal of the present section is to prove Proposition 1.1 which allows to reduce Theorem 1 to Theorem 1.2. This reduction to the normal form does not use the convexity assumption. We put the initial Hamiltonian  $H_\varepsilon$  in normal form around a compact subarc  $\Gamma_2$  of the resonance

$$\Gamma = \{p^s = p_*(p^f)\} = \{(p \in \mathbb{R}^n, \partial_{p^s} H_0 = 0)\}.$$

This global normal form is obtained by using mollifiers to glue local normal forms that depends on the arithmetic properties of the frequencies. This allows a simpler proof for instability, as we avoid the need to justify transitions between different local coordinates.

Recall that study a resonance of order  $n-1$  or, equivalently, of codimension 1. The resonance of order  $n-1$  is given by a lattice  $\Lambda$  span by  $n-1$  linearly independent vectors  $k_1, \dots, k_{n-1} \in (\mathbb{Z}^n \setminus 0) \times \mathbb{Z}$ . Denote by  $\theta_j^s = k_j \cdot \theta$ ,  $\omega_j^s = k_j \cdot \nabla H_0(p)$ ,  $j = 1, \dots, n-1$ , and  $\theta^s = (\theta_1^s, \dots, \theta_{n-1}^s)$  the slow angles and by  $\omega^s = (\omega_1^s, \dots, \omega_{n-1}^s)$  the slow actions resp. Choose a complement angle  $\theta^f$  so that  $(\theta^s, \theta^f) \in \mathbb{T}^{n-1} \times \mathbb{T}$  form a basis.

For  $p \in \Gamma$  we have  $\omega(p) = (0, \partial_{p^f} H_0(p))$ . We say that  $p$  has an additional resonance if the remaining frequency  $\partial_{p^f} H_0(p)$  is rational. In order to reduce the system to an appropriate normal form, we must remove some additional resonances. More precisely, we denote by  $\mathcal{D}(K, s) \subset B$  the set of momenta  $p$  such that

- $\|\partial_{p^s} H_0(p)\| \leq s$ , and
- $|k^f \partial_{p^f} H_0(p) + k^t| \geq 3Ks$  for each  $(k^f, k^t) \in \mathbb{Z}^2$  satisfying  $\max(|k^f|, |k^t|) \in ]0, K]$ .

The following result, which does not use the convexity of  $H_0$ , is a refinement of Proposition 1.1:

**Theorem 2.1.** *[Normal Form] Let  $H_0(p)$  be a  $C^4$  Hamiltonian. For each  $\delta \in ]0, 1[$ , there exist positive parameters  $K_0, \varepsilon_0, \beta$  such that, for each  $C^4$  Hamiltonian  $H_1$  with  $\|H_1\|_{C^4} \leq 1$  and each  $K \geq K_0, \varepsilon \leq \varepsilon_0$ , there exists a smooth change of coordinates*

$$\Phi : \mathbb{T}^n \times B \times \mathbb{T} \longrightarrow \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T}$$

satisfying  $\|\Phi - id\|_{C^0} \leq \sqrt{\varepsilon}$  and  $\|\Phi - id\|_{C^2} \leq \delta$  and such that, in the new coordinates, the Hamiltonian  $H_0 + \varepsilon H_1$  takes the form

$$N_\varepsilon = H_0(p) + \varepsilon Z(\theta^s, p) + \varepsilon R(\theta, p, t),$$

with  $\|R\|_{C^2} \leq \delta$  on  $\mathbb{T}^n \times \mathcal{D}(K, \beta\varepsilon^{1/4}) \times \mathbb{T}$ . We can take  $K_0 = c\delta^{-2}$ ,  $\beta = c\delta^{-1-n}$ ,  $\varepsilon_0 = \delta^{6n+5}/c$ , where  $c > 0$  is some constant depending only on  $n$  and  $\|H_0\|_{C^4}$ .

The proof actually builds a symplectic diffeomorphism  $\tilde{\Phi}$  of  $\mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$  of the form

$$\tilde{\Phi}(\theta, p, t, e) = (\Phi(\theta, p, t), e + f(\theta, p, t))$$

and such that

$$N_\varepsilon + e = (H_\varepsilon + e) \circ \tilde{\Phi}.$$

We have the estimates  $\|\tilde{\Phi} - id\|_{C^0} \leq \sqrt{\varepsilon}$  and  $\|\tilde{\Phi} - id\|_{C^2} \leq \delta$ .

**Remark 2.1.** *[Distance between punctures] On the interval, the distance between 2 adjacent rationals with denominator at most  $K$  is  $1/K^2$ . Choose  $K = K_0$  as in Theorem 2.1, the distance between adjacent punctures is at least  $D^{-1}/K^2 \geq D^{-1}c^{-1}\delta^4$ .*

*The length of  $\Gamma_1$  is determined by the choice of  $\delta$ , which can be chosen optimally in Theorem 1.3 and Theorem 4.1. Upon inspection of the proof, it is not difficult to determine that  $\delta$  can be chosen to a power of  $\lambda$ , which shows the distance between punctures is polynomial in  $\lambda$ .*



To prove Theorem 2.1 we proceed in 3 steps. We first obtain a global normal form  $N_\varepsilon$  adapted to all resonances. We then show that this normal form takes the desired form on the domain  $\mathcal{D}(K, s)$ . However, the averaging procedure lowers smoothness, in particular, the technique requires the smoothness  $r \geq n + 5$ . To obtain a result that does not require this relation between  $r$  and  $n$ , we use a smooth approximation trick that goes back to Moser.

## 2.1 A global normal form adapted to all resonances.

We first state a result for autonomous systems. The time periodic version will come as a corollary. Consider the Hamiltonian  $H_\varepsilon(\phi, J) = H_0(J) + \varepsilon H_1(\phi, J)$ , where  $(\phi, J) \in \mathbb{T}^m \times \mathbb{R}^m$  (later, we will take  $m = n + 1$ ). Let  $B = \{|J| \leq 1\}$  be the unit ball in  $\mathbb{R}^m$ . Given any integer vector  $k \in \mathbb{Z}^m \setminus \{0\}$ , let  $[k] = \max\{|k_i|\}$ . To avoid zero denominators in some calculations, we make the unusual convention that  $[(0, \dots, 0)] = 1$ . We fix once and for all a bump function  $\rho : \mathbb{R} \rightarrow \mathbb{R}$  be a  $C^\infty$  such that

$$\rho(x) = \begin{cases} 1, & |x| \leq 1 \\ 0, & |x| \geq 2 \end{cases}$$

and  $0 < \rho(x) < 1$  in between. For each  $\beta > 0$  and  $k \in \mathbb{Z}^m$ , we define the function  $\rho_k(J) = \rho(\frac{k \cdot \partial_J H_0}{\beta \varepsilon^{1/4} [k]})$ , where  $\beta > 0$  is a parameter.

**Theorem 2.2.** *There exists a constant  $c_m > 0$ , which depends only on  $m$ , such that the following holds. Given:*

- A  $C^4$  Hamiltonian  $H_0(J)$ ,
- A  $C^r$  Hamiltonian  $H_1(\varphi, J)$  with  $\|H_1\|_{C^r} = 1$ ,
- Parameters  $r \geq m + 4$ ,  $\delta \in ]0, 1[$ ,  $\varepsilon \in ]0, 1[$ ,  $\beta > 0$ ,  $K > 0$ ,

satisfying

- $K \geq c_m \delta^{\frac{-1}{r-m-3}}$ ,
- $\beta \geq c_m (1 + \|H_0\|_{C^4}) \delta^{-1/2}$ ,
- $\beta \varepsilon^{1/4} \leq \|H_0\|_{C^4}$ ,

there exists a  $C^2$  symplectic diffeomorphism  $\Phi : \mathbb{T}^m \times B \rightarrow \mathbb{T}^m \times \mathbb{R}^m$  such that, in the new coordinates, the Hamiltonian  $H_\varepsilon = H_0 + \varepsilon H_1$  takes the form

$$H_\varepsilon \circ \Phi = H_0 + \varepsilon R_1 + \varepsilon R_2$$

with

- $R_1 = \sum_{k \in \mathbb{Z}^m, |k| \leq K} \rho_k(J) h_k(J) e^{2\pi i(k \cdot \phi)}$ , here  $h_k(J)$  is the  $k^{\text{th}}$  coefficient for the Fourier expansion of  $H_1$ ,
- $\|R_2\|_{C^2} \leq \delta$ ,
- $\|\Phi - id\|_{C^0} \leq \delta \sqrt{\varepsilon}$  and  $\|\Phi - id\|_{C^2} \leq \delta$ .

If both  $H_0$  and  $H_1$  are smooth, then so is  $\Phi$ .

We now prove Theorem 2.2. To avoid cumbersome notations, we will denote by  $c_m$  various different constants depending only on the dimension  $m$ . We have the following basic estimates about the Fourier series of a function  $g(\phi, J)$ . Given a multi-index  $\alpha = (\alpha_1, \dots, \alpha_m)$ , we denote  $|\alpha| = \alpha_1 + \dots + \alpha_m$ . Denote also  $\kappa_m = \sum_{\mathbb{Z}^m} [k]^{-m-1}$ .



**Lemma 2.1.** For  $g(\phi, J) \in C^r(\mathbb{T}^m \times B)$ , we have

1. If  $l \leq r$ , we have  $\|g_k(J)e^{2\pi i(k \cdot \varphi)}\|_{C^l} \leq [k]^{l-r} \|g\|_{C^r}$ .
2. Let  $g_k(J)$  be a series of functions such that the inequality  $\|\partial_{J^\alpha} g_k\|_{C^0} \leq M[k]^{-|\alpha|-m-1}$  holds for each multi-index  $\alpha$  with  $|\alpha| \leq l$ , for some  $M > 0$ . Then, we have  $\|\sum_{k \in \mathbb{Z}^m} g_k(J)e^{2\pi i(k \cdot \varphi)}\|_{C^l} \leq c\kappa_m M$ .
3. Let  $\Pi_K^+ g = \sum_{|k| > K} g_k(J)e^{2\pi i(k \cdot \varphi)}$ . Then for  $l \leq r - m - 1$ , we have  $\|\Pi_K^+ g\|_{C^l} \leq \kappa_m K^{m-r+l+1} \|g\|_{C^r}$ .

*Proof.* 1. Let us assume that  $k \neq 0$  and take  $j$  such that  $k_j = [k]$ . Let  $\alpha$  and  $\eta$  be two multi-indices such that  $|\alpha + \eta| \leq l$ . Finally, let  $b = r - l$ , and let  $\beta$  be the multi-index  $\beta = (0, \dots, 0, b, 0, \dots, 0)$ , where  $\beta_j = b$ . We have

$$g_k(J)e^{2\pi i(k \cdot \varphi)} = \int_{\mathbb{T}^m} g(\theta, J)e^{2i\pi(k \cdot \varphi - \theta)} d\theta = \int_{\mathbb{T}^m} g(\theta + \varphi, J)e^{-2i\pi(k \cdot \theta)} d\theta,$$

hence

$$\begin{aligned} \partial_{\varphi^\alpha J^\eta} (g_k(J)e^{2i\pi(k \cdot \varphi)}) &= \int_{\mathbb{T}^m} \partial_{\varphi^\alpha J^\eta} g(\theta + \varphi, J)e^{-2i\pi(k \cdot \theta)} d\theta, \\ &= \int_{\mathbb{T}^m} \frac{\partial_{\varphi^{\alpha+\beta} J^\eta} g(\theta + \varphi, J)}{(2i\pi k_j)^b} e^{-2i\pi(k \cdot \theta)} d\theta. \end{aligned}$$

Since  $|\alpha + \beta + \eta| \leq r$ , we conclude that

$$\|g_k(J)e^{2i\pi(k \cdot \varphi)}\|_{C^l} \leq \|g\|_{C^r} / (2\pi[k])^b \leq \|g\|_{C^r} [k]^{l-r}.$$

2. We have  $\|g_k(J)e^{2i\pi(k \cdot \varphi)}\|_{C^l} \leq$

$$\left\| \sum_{k \in \mathbb{Z}^m} h_k(J)e^{2\pi i(k \cdot \varphi)} \right\|_{C^l} \leq \sum_{k \in \mathbb{Z}^m} c_l |k|^{-r+l} M \leq c_l \kappa_m M,$$

recall that  $\kappa_m = \sum_{k \in \mathbb{Z}^m} |k|^{-m-1}$ .

3. Using 1., we get

$$\begin{aligned} \|\Pi_K^+ g\|_{C^l} &\leq \sum_{|k| > K} [k]^{l-r} \|g\|_{C^r} \leq \|g\|_{C^r} K^{m-r+l+1} \sum_{|k| > K} [k]^{-m-1} \\ &\leq \|g\|_{C^r} K^{m-r+l+1} \sum_{k \in \mathbb{Z}^m} [k]^{-m-1}. \end{aligned}$$

□

*Proof of Theorem 2.2.* Let  $\tilde{G}(\phi, J)$  be the function that solves the cohomological equation

$$\{H_0, \tilde{G}\} + H_1 = R_1 + R_+,$$

where  $R_+ = \Pi_K^+ H_1$ . Observing that  $\rho_k(J) = 1$  when  $k \cdot \partial_J H_0 = 0$ , we have the following explicit formula for  $G$ :

$$\tilde{G}(\varphi, J) = (2\pi i)^{-1} \sum_{|k| \leq K} \frac{(1 - \rho_k(J))h_k(J)}{k \cdot \partial_J H_0(J)} e^{2\pi i(k \cdot \varphi)}$$

where each of the functions  $(1 - \rho_k(J))h_k(J)/(k \cdot \partial_J H_0)$  is extended by continuity at the points where the denominator vanishes. This function hence takes the value zero at these points.  $G$  is well defined thanks to the smoothing terms  $1 - \rho_k$  we introduced, as whenever  $k \cdot \partial_J H_0 = 0$  we also have  $1 - \rho_k = 0$  and that term is considered non-present. Since  $\tilde{G}$  as defined above is only  $C^3$ , we will consider a smooth approximation

$$G(\varphi, J) = \sum_{|k| \leq K} g_k(J) e^{2\pi i(k \cdot \phi)}$$

where  $g_k(J)$  are smooth functions which are sufficiently close to  $\frac{(1 - \rho_k(J))h_k(J)}{(2\pi i)^k \cdot \partial_J H_0(J)}$  in the  $C^3$  norm.

Let  $\Phi^t$  be the Hamiltonian flow generated by  $\varepsilon G$ . Setting  $F_t = R_1 + R_+ + t(H_1 - R_1 - R_+)$ , we have the standard computation

$$\begin{aligned} \partial_t((H_0 + \varepsilon F_t) \circ \Phi^t) &= \varepsilon \partial_t F_t \circ \Phi^t + \varepsilon \{H_0 + \varepsilon F_t, G\} \circ \Phi^t \\ &= \varepsilon (\partial_t F_t + \{H_0, G\}) \circ \Phi^t + \varepsilon^2 \{F_t, G\} \circ \Phi^t \\ &= \varepsilon^2 \{F_t, G\} \circ \Phi^t, \end{aligned}$$

from which follows that

$$H_\varepsilon \circ \Phi^1 = H_0 + \varepsilon R_1 + \varepsilon R_+ + \varepsilon^2 \int_0^1 \{F_t, G\} \circ \Phi^t dt.$$

Let us estimate the  $C^2$  norm of the function  $R_2 := R_+ + \varepsilon \int_0^1 \{F_t, G\} \circ \Phi^t dt$ . It follows from Lemma 2.1 that

$$\|R_+\|_{C^2} \leq \kappa_m K^{-r+m+2} \|H_1\|_{C^r} \leq \frac{1}{2} \delta.$$

We now focus on the term  $\int_0^1 \{F_t, G\} \circ \Phi^t dt$ . To estimate the norm of  $F_t$ , it is convenient to write  $F_t = \tilde{F}_t + (1-t)R_1$ , where  $\tilde{F}_t = (1-t)R_+ + tH_1$ . Notice that the coefficients of the Fourier expansion of  $\tilde{F}_t$  is simply a constant times that of  $H_1$ , Lemma 2.1 then implies that

$$\|\tilde{F}_t\|_{C^3} \leq \sum_{k \in \mathbb{Z}^m} [k]^{3-r} \|H_1\|_{C^r} = \kappa_m \|H_1\|_{C^r}$$

provided that  $r \geq m + 4$ , where we set  $\kappa_m = \sum_{\mathbb{Z}^m} [k]^{-m-1}$ .

We now have to estimate the norm of  $R_1$  and  $G$ . This requires additional estimates of the smoothing terms  $\rho_k$  as well as the small denominators  $k \cdot \partial_J H_0$ . We always assume that  $l \in \{0, 1, 2, 3\}$  in the following estimates:

- $\rho_k(J) \neq 1 \Rightarrow |(k \cdot \partial_J H_0)^{-1}| \leq \beta^{-1} \varepsilon^{-1/4} |k|^{-1}$ .
- $\|(k \cdot \partial_J H_0)^{-1}\|_{C^l} \leq c_m \beta^{-l-1} \varepsilon^{-(l+1)/4} \|H_0\|_{C^4}^{l+1}$  on  $\{\rho_k \neq 1\}$ .
- $\|\rho_k(J)\|_{C^l} \leq c_m \beta^{-l} \varepsilon^{-l/4} \|H_0\|_{C^4}^l$  and  $\|1 - \rho_k(J)\|_{C^l} \leq c_m \beta^{-l} \varepsilon^{-l/4} \|H_0\|_{C^4}^l$ .

We have been using the following estimates on the derivative of composition of functions: For  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  and  $g : \mathbb{R}^m \rightarrow \mathbb{R}^m$  we have  $\|f \circ g\|_{C^l} \leq c_{m,l} \|f\|_{C^l} (1 + \|g\|_{C^l}^l)$ .

- For each multi-index  $|\alpha| \leq 3$ , we have that

$$\begin{aligned}
& \|\partial_{J^\alpha} ((1 - \rho_k(J))h_k(J)(k \cdot \partial_J H_0)^{-1})\|_{C^0} \\
& \leq \sum_{\alpha_1 + \alpha_2 + \alpha_3 = \alpha} \|1 - \rho_k(J)\|_{C^{|\alpha_1|}} \|h_k\|_{C^{|\alpha_2|}} \|(k \cdot \partial_J H_0)^{-1}\|_{C^{|\alpha_3|}} (\{\rho_k \neq 1\}) \\
& \leq c_m \sum_{\alpha_1 + \alpha_2 + \alpha_3 = \alpha} \left( \beta^{-|\alpha_1|} \varepsilon^{-|\alpha_1|/4} \|H_0\|_{C^4}^{|\alpha_1|} \cdot [k]^{-r + |\alpha_2|} \|H_1\|_{C^r} \right. \\
& \quad \left. \cdot \beta^{-|\alpha_3| - 1} \varepsilon^{-(|\alpha_3| + 1)/4} \|H_0\|_{C^4}^{|\alpha_3| + 1} \right) \\
& \leq c_m \beta^{-|\alpha| - 1} \varepsilon^{-(|\alpha| + 1)/4} [k]^{|\alpha| - r} \|H_0\|_{C^4}^{|\alpha| + 1} \|H_1\|_{C^r}.
\end{aligned}$$

In these computations, we have used the hypothesis  $\beta \varepsilon^{1/4} \leq \|H_0\|_{C^4}$ . Since  $G(\varphi, J) = \sum_{k \in \mathbb{Z}^m} (1 - \rho_k(J))h_k(J)(k \cdot \partial_J H_0)^{-1} e^{2\pi i(k \cdot \varphi)}$ , Lemma 2.1 implies (since  $r \geq m + 1$ ):

$$- \|G\|_{C^l} \leq c_m \beta^{-l-1} \varepsilon^{-(l+1)/4} \|H_0\|_{C^4}^{l+1} \|H_1\|_{C^r} \leq \varepsilon^{-1}.$$

We now turn our attention to  $R_1 = \sum_{|k| \leq K} \rho_k(J)h_k(J)e^{2i\pi(k \cdot \phi)}$ :

$$\begin{aligned}
- \|h_k\|_{C^l} & \leq [k]^{l-r} \|H_1\|_{C^r}. \\
- \|\rho_k h_k\|_{C^l} & \leq c_m \beta^{-l} \varepsilon^{-l/4} [k]^{-r+l} \|H_0\|_{C^4}^l \|H_1\|_{C^r}. \\
- \|R_1\|_{C^l} & \leq c_m \beta^{-l} \varepsilon^{-l/4} \|H_0\|_{C^4}^l \|H_1\|_{C^r}, \text{ provided } r \geq m + 4.
\end{aligned}$$

We obtain

$$\|F_t\|_{C^l} \leq \|R_1\|_{C^l} + \|\tilde{F}_t\|_{C^l} \leq c_m \beta^{-l} \varepsilon^{-l/4} \|H_0\|_{C^4}^l \|H_1\|_{C^r},$$

and

$$\|\{F_t, G\}\|_{C^2} \leq \sum_{|\alpha_1 + \alpha_2| \leq 3} \|F_t\|_{C^{|\alpha_1|}} \|G\|_{C^{|\alpha_2|}} \leq c_m \beta^{-4} \varepsilon^{-1} \|H_0\|_{C^4}^4 \|H_1\|_{C^r}^2.$$

Concerning the flow  $\Phi^t$ , we observe that  $\|\varepsilon G\|_{C^3} \leq 1$ , and get the following estimate (see *e. g.* [DH], Lemma 3.15):

$$\begin{aligned}
- \|\Phi^t - id\|_{C^2} & \leq c_m \varepsilon \|G\|_{C^3} \leq c_m \beta^{-4} \|H_0\|_{C^4}^4 \|H_1\|_{C^r} \leq \delta, \\
- \|\Phi^t - id\|_{C^0} & \leq c_m \varepsilon \|G\|_{C^1} \leq c_m \beta^{-2} \sqrt{\varepsilon} \|H_0\|_{C^4}^2 \|H_1\|_{C^2} \leq \delta \sqrt{\varepsilon}.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
\varepsilon \|\{F_t, G\} \circ \Phi^t\|_{C^2} & \leq c_m \varepsilon \|\{F_t, G\}\|_{C^2} \|\Phi^t\|_{C^2}^2 \\
& \leq c_m \beta^{-4} \|H_0\|_{C^4}^4 \|H_1\|_{C^r}^2 \leq \delta/2.
\end{aligned}$$

□

## 2.2 Normal form away from additional resonances

We now return to our non-autonomous system and apply Theorem 2.2 around the resonance under study. To the non-autonomous Hamiltonian

$$H_\varepsilon(\theta, p, t) = H_0(p) + \varepsilon H_1(\theta, p, t) : \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T} \longrightarrow \mathbb{R}$$

we associate the autonomous Hamiltonian

$$\tilde{H}_e(\varphi, J) = H_0(I) + e + \varepsilon H_1(\theta, I, t) : \mathbb{T}^{n+1} \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R},$$

where  $\varphi = (\theta, t)$  and  $J = (I, e)$ . We denote the frequencies  $\omega \in \mathbb{R}^{n+1}$  by  $\omega = (\omega^f, \omega^s, \omega^t) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$ , and define the set

$$\Omega(K, s) := \{\omega \in \mathbb{R}^{n+1} : \|\omega^s\| > s, |k^f \omega^f + k^t \omega^t| \geq 3sK \quad \forall (k^s, k^t) \in \mathbb{Z}_K^2\},$$

where we have denoted by  $\mathbb{Z}_K^2$  the set of pairs  $(k^f, k^t)$  of integers such that  $0 < \max(k^f, k^t) \leq K$ . Note that

$$\mathcal{D}(K, s) = \{p \in \mathbb{R}^n : (\partial_p H_0(p), 1) \in \Omega(K, s)\}.$$

**Corollary 2.2.** *There exists a constant  $c_n > 0$ , which depends only on  $n$ , such that the following holds. Given :*

- A  $C^4$  Hamiltonian  $H_0(p)$ ,
- A  $C^r$  Hamiltonian  $H_1(\theta, p, t)$  with  $\|H_1\|_{C^r} = 1$ ,
- Parameters  $r \geq n + 5$ ,  $\delta \in ]0, 1[$ ,  $\varepsilon \in ]0, 1[$ ,  $\beta > 0$ ,  $K > 0$ ,

satisfying

- $K \geq c_n \delta^{\frac{-1}{r-n-4}}$ ,
- $\beta \geq c_n (1 + \|H_0\|_{C^4}) \delta^{-1/2}$ ,
- $\beta \varepsilon^{1/4} \leq \|H_0\|_{C^4}$ ,

there exists a  $C^2$  symplectic diffeomorphism  $\tilde{\Phi}$  of  $\mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$  such that, in the new coordinates, the Hamiltonian  $H_\varepsilon = H_0 + \varepsilon H_1$  takes the form

$$N_\varepsilon = H_0 + \varepsilon Z + \varepsilon R,$$

with

- $\|R\|_{C^2} \leq \delta$  on  $\mathbb{T}^n \times \mathcal{D}(K, \beta \varepsilon^{1/4}) \times \mathbb{T}$ ,
- $\|\tilde{\Phi} - id\|_{C^0} \leq \delta \sqrt{\varepsilon}$  and  $\|\tilde{\Phi} - id\|_{C^2} \leq \delta$ .

The symplectic diffeomorphism  $\tilde{\Phi}$  is of the form

$$\tilde{\Phi}(\theta, p, t, e) = (\Phi(\theta, p, t), e + f(\theta, p, t))$$

where  $\Phi$  is a diffeomorphism of  $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T}$  fixing the last variable  $t$ . The maps  $\tilde{\Phi}$  and  $\Phi$  are smooth if  $H_0$  and  $H_1$  are.

*Proof.* We apply Theorem 2.2 with  $\tilde{H}_\varepsilon$ ,  $m = n + 1$  and  $\tilde{\delta} = \delta/2$ . We get a diffeomorphism  $\tilde{\Phi}$  of  $\mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$  as time-one flow of the Hamiltonian  $G$ . By inspection in the proof of Theorem 2.2, we observe that  $G$  does not depend on  $e$ , which implies that  $\tilde{\Phi}$  has the desired form. We have

$$\tilde{H}_\varepsilon \circ \tilde{\Phi} = \tilde{H}_0(J) + \varepsilon \tilde{R}_1 + \varepsilon \tilde{R}_2$$

where  $\|\tilde{R}_2\|_{C^2} \leq \delta/2$  and

$$\tilde{R}_1(\theta, p, t) = \sum_{[k] \leq K} \rho \left( \frac{k^f \cdot \partial_{p^f} H_0 + k^s \partial_{p^s} H_0 + k^t}{\beta \varepsilon^{1/4} [k]} \right) g_k(p) e^{2i\pi k \cdot (\theta, t)}.$$

Let us compute this sum under the assumption that  $p \in \mathcal{D}(K, \beta\varepsilon^{1/4})$  (or equivalently, that  $(\partial_p H_0, 1) \in \Omega(K, \beta\varepsilon^{1/4})$ ). We have

$$\left| \frac{k^f \cdot \partial_{p^f} H_0}{\beta\varepsilon^{1/4}[k]} \right| \leq 1$$

hence

$$\rho \left( \frac{k^f \cdot \partial_{p^f} H_0 + k^s \partial_{p^s} H_0 + k^t}{\beta\varepsilon^{1/4}[k]} \right) = 1$$

for  $k$  such that  $k^s = 0 = k^t$ . For the other terms, we have, by definition of  $\Omega(K, s)$ ,

$$\left| \frac{k^s \partial_{p^s} H_0 + k^t}{\beta\varepsilon^{1/4}[k]} \right| \geq \left| \frac{k^s \partial_{p^s} H_0 + k^t}{\beta\varepsilon^{1/4}K} \right| \geq 3,$$

hence

$$\left| \frac{k^f \cdot \partial_{p^f} H_0 + k^s \partial_{p^s} H_0 + k^t}{\beta\varepsilon^{1/4}[k]} \right| \geq 2$$

and these terms vanish in the expansion of  $\tilde{R}_1$ . We conclude that

$$\tilde{R}_1(\theta, p, t) = \sum_{k^f \in \mathbb{Z}^{n-1}, [k^f] \leq K} g_{(k^f, 0, 0)}(p) e^{2i\pi k^f \cdot \theta^f}$$

hence  $\tilde{R}_1 = Z - \Pi_K^+(Z)$ , with the notation of Lemma 2.1. Finally  $\tilde{H}_\varepsilon \circ \tilde{\Phi} = \tilde{H}_0 + \varepsilon Z + \varepsilon R_2$  with  $R_2 = \tilde{R}_2 - \Pi_K^+ Z$ . From Lemma 2.1, we see that

$$\|\Pi_K^+ Z\|_{C^2} \leq c_n K^{m+3-r} \|Z\|_{C^r} \leq c_n K^{m+3-r} \|H_1\|_{C^r} \leq c_n K^{m+3-r} \leq \delta/2.$$

On the other hand,  $\|\tilde{R}_2\|_{C^2} \leq \delta/2$ , hence  $\|R_2\|_{C^2} \leq \delta$ .  $\square$

### 2.3 Smooth approximation

We finally remove the restriction on  $r$  and obtain a smooth change of coordinates. If  $r < n+5$ , we use Lemma 2.3 below to approximate  $H_1$  by an analytic function  $H_1^* := S_\tau H_1$  (with a parameter  $\tau$  that will be specified later).

**Lemma 2.3.** [SZ] *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a  $C^r$  function, with  $r \geq 4$ . Then for each  $\tau > 0$  there exists an analytic function  $S_\tau f$  such that*

$$\|S_\tau f - f\|_{C^3} \leq c(n, r) \|f\|_{C^3} \tau^{r-3},$$

$$\|S_\tau f\|_{C^s} \leq c(n, r) \|f\|_{C^s} \tau^{-(s-r)},$$

for each  $s > r$ , where  $c(n, r)$  is a constant which depends only on  $n$  and  $r$ .

In order to obtain a smooth change of variables, it is also convenient to approximate  $H_0(p)$  in  $C^4(B)$  by a smooth  $H_0^*(p)$  (using a standard mollification). We then apply Corollary 2.2 to the Hamiltonian

$$H_\varepsilon^* := H_0^* + \varepsilon H_1^* = H_0^* + \varepsilon_2 H_2$$

with  $H_2 = H_1^* / \|H_1^*\|_{C^{r_2}}$ , with  $\varepsilon_2 = \varepsilon \|H_1^*\|_{C^{r_2}}$ , and with some parameters  $r_2 \geq r$  and  $\delta_2 \leq \delta$  to be specified later. We find a smooth change of coordinates  $\tilde{\Phi}$  such that

$$\tilde{H}_\varepsilon^* \circ \tilde{\Phi} = \tilde{H}_0^* + \varepsilon_2 Z_2 + \varepsilon_2 R_2 = \tilde{H}_0^* + \varepsilon Z^* + \varepsilon \|H_1^*\|_{C^{r_2}} R_2$$

and  $\|R_2\|_{C^2} \leq \delta_2$ , where  $Z_2(\theta^s, p) = \int H_2 d\theta^f dt$  and  $Z^*(\theta^s, p) = \int H_1^* d\theta^f dt$ . As usual, we have denoted by  $\tilde{H}_\varepsilon^*$  and  $\tilde{H}_0^*$  the automomized Hamiltonians  $\tilde{H}_\varepsilon^* = H_\varepsilon^* + e$  and  $\tilde{H}_0^* = H_0^* + e$ . With the same map  $\tilde{\Phi}$ , we obtain

$$\tilde{H}_\varepsilon \circ \tilde{\Phi} = \tilde{H}_0 + \varepsilon Z + \varepsilon R$$

with

$$R = \|H_1^*\|_{C^{r_2}} R_2 + (Z - Z^*) + (H_1^* - H_1) \circ \Phi + ((\tilde{H}_0^* - \tilde{H}_0) + (\tilde{H}_0 - \tilde{H}_0^*) \circ \tilde{\Phi})/\varepsilon$$

In the expression above, the map  $\Phi$  is the trace on the  $(\theta, p, t)$  variables of the map  $\tilde{\Phi}$ . Choosing  $\tau = \delta_2^{1/(r_2-3)}$ , and assuming that  $\|H_0^* - H_0\|_{C^4} \leq \varepsilon\delta/c(n, 4)$  we get

- $\|H_1^* - H_1\|_{C^3} \leq c(n, r_2)\delta_2^{\frac{r-3}{r_2-3}}$
- $\|H_1^*\|_{C^{r_2}} \leq c(n, r_2)\delta_2^{-\frac{r_2-r}{r_2-3}}$
- $\|Z^* - Z\|_{C^2} \leq \|H_1^* - H_1\|_{C^2} \leq c(n, r_2)\delta_2^{\frac{r-3}{r_2-3}}$
- $\|\tilde{\Phi} - id\|_{C^2} \leq \delta_2 \leq \delta \leq 1$ ,
- $\|(H_1^* - H_1) \circ \Phi\|_{C^2} \leq c(n, r_2)\|H_1^* - H_1\|_{C^2}(\|\Phi\|_{C^2} + \|\Phi\|_{C^2}^2) \leq c(n, 5)\|H_1^* - H_1\|_{C^2}$ .
- $\|(\tilde{H}_0 - \tilde{H}_0^*) \circ \tilde{\Phi}\|_{C^2} \leq \delta/c(n, r_2)$ .

and finally

$$\|R\|_{C^2} \leq c(n, r_2)\delta_2^{\frac{r-3}{r_2-3}} + \delta/c(n, r_2).$$

We now set

$$\delta_2 = \delta^{\frac{r_2-3}{r-3}}/c(n, r_2) \leq \delta/2$$

and get  $\|R\|_{C^2} \leq \delta$ . To apply Corollary 2.2 as we just did, we need the following conditions to hold on the parameters:

- $K \geq c(n, r_2)\delta^{\frac{r_2-3}{(r-3)(r_2-n-4)}}$ , which implies  $K \geq c_n\delta_2^{\frac{-1}{r-n-4}}$ ,
- $\beta \geq c(n, r_2)(2 + \|H_0\|_{C^4})\delta^{-\frac{r_2-3}{2(r-3)}}$  which implies  $\beta \geq c_n(1 + \|H_0^*\|_{C^4})\delta_2^{-1/2}$ ,
- $\beta\varepsilon^{1/4} \leq (1 + \|H_0\|_{C^4})\delta^{\frac{r_2-r}{4(r-3)}}$  which implies  $\beta\varepsilon_2^{1/4} \leq \|H_0^*\|_{C^4}$ .

We apply the above discussion with  $r_2 = 2n + 5$  and get Theorem 2.1. Note the estimate

$$\|id - \tilde{\Phi}\|_{C^0} \leq \delta_2\sqrt{\varepsilon_2} \leq \delta_2^{1-\frac{r_2-r}{2(r_2-3)}}\sqrt{\varepsilon} \leq \sqrt{\varepsilon}.$$

□

### 3 Normally hyperbolic cylinders

In this section, we study the  $C^2$  Hamiltonian

$$N_\varepsilon(\theta, p, t) = H_0(p) + \varepsilon Z(\theta^s, p) + \varepsilon R(\theta, p, t).$$

In the above notations we denote by  $p_*^s(p^f) \in \mathbb{R}^{n-1}$  the solution of the equation  $\partial_{p^s} H_0(p_*^s(p^f), p^f) = 0$ . We recall also the notation  $p_*(p^f) := (p_*^s(p^f), p^f)$  from the introduction. We assume that

$\|Z\|_{C^3} \leq 1$ , and that  $D^{-1}I \leq \partial_{pp}^2 H_0 \leq DI$  for some  $D \geq 1$ . To simplify notations, we will be using the  $O(\cdot)$  notation, where  $f = O(g)$  means  $|f| \leq Cg$  for a constant  $C$  independent of  $\varepsilon, \lambda, \delta, r, a^-, a^+$ . We will not be keeping track of the parameter  $D$ , which is considered fixed throughout the paper.

Given parameters

$$\lambda \in ]0, 1], \quad a^- < a^+,$$

we assume that for each  $p^f \in [a^-, a^+]$  there exists a local maximum  $\theta_*^s(p^f)$  of the map  $\theta^s \mapsto Z(\theta^s, p_*(p^f))$ , and that  $\theta_*^s$  is a  $C^2$  function of  $p^f$ . We assume in addition that

$$-I \leq \partial_{\theta^s \theta^s}^2 Z(\theta_*^s(p^f), p_*(p^f)) \leq -\lambda I \quad (7)$$

for each  $p^f \in [a^-, a^+]$ , where as before  $I$  is the identity matrix. We shall at some occasions lift the map  $\theta_*^s$  to a  $C^2$  map taking values in  $\mathbb{R}^{n-1}$  without changing its name.

**Theorem 3.1.** *The following conclusion holds if  $b \in ]0, 1[$  is a sufficiently small constant (how small does not depend on the parameters  $\varepsilon, \lambda, \delta, a^-, a^+$ ): If the parameters  $\lambda \in ]0, 1], a^- < a^+, \varepsilon, \delta$  satisfy*

$$0 < \varepsilon < b\lambda^{9/2} \quad , \quad 0 \leq \delta < b\lambda^{5/2},$$

if  $\|R\|_{C^2} \leq \delta$ , on the open set

$$\{(\theta, p, t) : p^f \in ]a^-, a^+[ , \quad \|p^s - p_*^s(p^f)\| < \varepsilon^{1/2}\}, \quad (8)$$

and if (7) holds for each  $p^f \in [a^-, a^+]$ , then there exists a  $C^2$  map

$$(\Theta^s, P^s)(\theta^f, p^f, t) : \mathbb{T} \times [a^- + \sqrt{\delta\varepsilon}, a^+ - \sqrt{\delta\varepsilon}] \times \mathbb{T} \longrightarrow \mathbb{T}^{n-1} \times \mathbb{R}^{n-1}$$

such that the cylinder

$$\mathcal{C} = \{(\theta^s, p^s) = (\Theta^s, P^s)(\theta^f, p^f, t); \quad p^f \in [a^- + \sqrt{\delta\varepsilon}, a^+ - \sqrt{\delta\varepsilon}], \quad (\theta^f, t) \in \mathbb{T} \times \mathbb{T}\}$$

is weakly invariant with respect to  $N_\varepsilon$  in the sense that the Hamiltonian vector field is tangent to  $\mathcal{C}$ . The cylinder  $\mathcal{C}$  is contained in the set

$$V := \{(\theta, p, t); p^f \in [a^- + \sqrt{\delta\varepsilon}, a^+ - \sqrt{\delta\varepsilon}]; \\ \|(\theta^s - \theta_*^s(p^f))\| \leq b^{1/5}\lambda^{3/2}, \quad \|p^s - p_*^s(p^f)\| \leq b^{1/5}\lambda^{3/2}\varepsilon^{1/2}\},$$

and it contains all the full orbits of  $N_\varepsilon$  contained in  $V$ . We have the estimates

$$\|\Theta^s(\theta^f, p^f, t) - \theta_*^s(p^f)\| \leq O(\lambda^{-1}\delta + \lambda^{-3/4}\sqrt{\varepsilon}),$$

$$\|P^s(\theta^f, p^f, t) - p_*^s(p^f)\| \leq \sqrt{\varepsilon}O(\lambda^{-3/4}\delta + \lambda^{-1/2}\sqrt{\varepsilon}),$$

$$\left\| \frac{\partial \Theta^s}{\partial p^f} \right\| = O\left( \frac{\lambda^{-2}\sqrt{\varepsilon} + \lambda^{-5/4}\sqrt{\delta}}{\sqrt{\varepsilon}} \right), \quad \left\| \frac{\partial \Theta^s}{\partial(\theta^f, t)} \right\| = O\left( \lambda^{-2}\sqrt{\varepsilon} + \lambda^{-5/4}\sqrt{\delta} \right), \\ \left\| \frac{\partial P^s}{\partial p^f} \right\| = O(1), \quad \left\| \frac{\partial P^s}{\partial(\theta^f, t)} \right\| = O(\sqrt{\varepsilon}).$$

Notice that the domain  $V$  is contained in the domain (8) where the assumption on  $R$  is made.

*Proof of Theorem 1.3.* We derive Theorem 1.3 from Theorem 3.1 as follows. We assume that Hypothesis  $(HZ\lambda)$  holds on

$$\Gamma_1 := \{(p_*(p^f)), p^f \in [a_-, a_+]\}.$$

Then the inequality

$$-I \leq \partial_{\theta^s \theta^s}^2 Z(\theta_*^s(p^f), p_*(p^f)) \leq -2\lambda I$$

holds for  $p^f \in [a_-, a_+]$ . Since  $\|Z\|_{C^3} \leq 1$ , the inequality

$$-I \leq \partial_{\theta^s \theta^s}^2 Z(\theta^s, p) \leq -\lambda I$$

holds for each  $(\theta^s, p)$  in the  $\lambda$ -neighborhood of  $(\theta_*^s(a_-), p_*(a_-))$ . The inequality

$$Z(\theta^s, p_*(a_-)) \leq Z(\theta_*^s(a_-), p_*(a_-)) - \lambda d^2(\theta^s, \theta_*^s(a_-))$$

implies that the function  $Z(\cdot, p_*(p^f))$  has a global maximum  $\theta_*^s(p^f)$ , which is contained in the ball  $B(\theta_*^s(a_-), \lambda)$ , provided  $|p^f - a_-| \leq b\lambda^3$  and  $b$  is small enough. By a similar reasoning at  $a_+$ , we extend the map  $p^f \mapsto \theta_*^s(p^f)$  to the interval  $[a_- - b\lambda^3, a_+ + b\lambda^3]$  in such a way that, for each  $p^f$  in this interval, the point  $\theta_*^s(p^f)$  is a local (and even global) maximum of the function  $Z(\cdot, p_*(p^f))$  which satisfies the inequalities

$$-I \leq \partial_{\theta^s \theta^s}^2 Z(\theta_*^s(p^f), p_*(p^f)) \leq -\lambda I.$$

Taking a small  $b > 0$ , we set  $\kappa = b^{1/5}\lambda^{3/2}$  and  $\delta = b^3\lambda^9$ . Assuming as in the statement of Theorem 1.3 that the estimate  $\|R\|_{C^2} < \delta$  holds on  $\mathbb{T}^n \times U_{\varepsilon^{1/3}} \times \mathbb{T}$ , hence on

$$\{(\theta, p, t) : p^f \in ]a_- - \varepsilon^{1/3}/2, a_+ + \varepsilon^{1/3}/2[, \|p^s - p_*^s(p^f)\| < \varepsilon^{1/3}/2\}.$$

and that  $\varepsilon \in ]0, \delta[$ , we apply Theorem 3.1 on the interval

$$[a^-, a^+] := [a_- - \varepsilon^{1/3}/2, a_+ + \varepsilon^{1/3}/2] \subset [a_- - b\lambda^3, a_+ + b\lambda^3].$$

If  $b$  (hence  $\kappa$ ) is small enough, then we have the inclusion

$$[a^- + \sqrt{\varepsilon\delta}, a^+ - \sqrt{\varepsilon\delta}] \supset [a_- - \kappa\varepsilon^{1/3}, a_+ + \kappa\varepsilon^{1/3}].$$

□

The proof of Theorem 3.1 occupies the rest of the section.

The Hamiltonian flow admits the following equation of motion :

$$\begin{cases} \dot{\theta}^s = \partial_{p^s} H_0 + \varepsilon \partial_{p^s} Z + \varepsilon \partial_{p^s} R \\ \dot{p}^s = -\varepsilon \partial_{\theta^s} Z - \varepsilon \partial_{\theta^s} R \\ \dot{\theta}^f = \partial_{p^f} H_0 + \varepsilon \partial_{p^f} Z + \varepsilon \partial_{p^f} R \\ \dot{p}^f = -\varepsilon \partial_{\theta^f} R \\ \dot{t} = 1 \end{cases} \quad (9)$$

The Hamiltonian structure of the flow is not used in the following proof.

It is convenient in the sequel to lift the angular variables to real variables and to consider the above system as defined on  $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ . We will see this system as a perturbation of the model system

$$\dot{\theta}^s = \partial_{p^s} H_0 \quad , \quad \dot{p}^s = -\varepsilon \partial_{\theta^s} Z \quad , \quad \dot{\theta}^f = \partial_{p^f} H_0 \quad , \quad \dot{p}^f = 0 \quad , \quad \dot{t} = 1. \quad (10)$$



The graph of the map

$$(\theta^f, p^f, t) \longmapsto (\theta_*^s(p^f), p_*^s(p^f))$$

on  $\mathbb{R} \times ]a^-, a^+[ \times \mathbb{R}$  is obviously invariant for the model flow. For each fixed  $p^f$ , the point  $(\theta_*^s(p^f), p_*^s(p^f))$  is a hyperbolic fixed point of the partial system

$$\dot{\theta}^s = \partial_{p^s} H_0(p^s, p^f) \quad , \quad \dot{p}^s = -\varepsilon \partial_{\theta^s} Z(\theta^s, p^s, p^f)$$

where  $p^f$  is seen as a parameter. This hyperbolicity is the key property we will use, through the theory of normally hyperbolic invariant manifolds. It is not obvious to apply this theory here because the model system itself depends on  $\varepsilon$ , and because we have to deal with the problem of non-invariant boundaries. We will however manage to apply the quantitative version exposed in Appendix A.

We perform some changes of coordinates in order to put the system in the framework of Appendix A. These coordinates appear naturally from the study of the model system as follows. We set

$$B(p^f) := \partial_{p^s p^s}^2 H_0(p_*^s(p^f)) \quad , \quad A(p^f) := -\partial_{\theta^s \theta^s}^2 Z(\theta_*^s(p^f), p_*^s(p^f)).$$

If we fix the variable  $p^f$  and consider the model system in  $(\theta^s, p^s)$ , we observed that this system has a hyperbolic fixed point at  $(\theta_*^s(p^f), p_*^s(p^f))$ . The linearized system at this point is

$$\dot{\theta}^s = B(p^f) p^s \quad , \quad \dot{p}^s = \varepsilon A(p^f) \theta^s.$$

To put this system under a simpler form, it is useful to consider the matrix

$$T(p^f) := (B^{1/2}(p^f)(B^{1/2}(p^f)A(p^f)B^{1/2}(p^f))^{-1/2}B^{1/2}(p^f))^{1/2}$$

which is symmetric, positive definite, and satisfies  $T^2(p^f)A(p^f)T^2(p^f) = B(p^f)$ , as can be checked by a direct computation. We finally introduce the symmetric positive definite matrix

$$\Lambda(p^f) := T(p^f)A(p^f)T(p^f) = T^{-1}(p^f)B(p^f)T^{-1}(p^f).$$

In the new variables

$$\xi = T^{-1}(p^f)\theta^s + \varepsilon^{-1/2}T(p^f)p^s \quad , \quad \eta = T^{-1}(p^f)\theta^s - \varepsilon^{-1/2}T(p^f)p^s,$$

the linearized system is reduced to the following block-diagonal form:

$$\dot{\xi} = \varepsilon^{1/2}\Lambda(p^f)\xi \quad , \quad \dot{\eta} = -\varepsilon^{1/2}\Lambda(p^f)\eta,$$

see [Be3] for more details. This leads us to introduce the following set of new coordinates for the full system:

$$\begin{aligned} x &= T^{-1}(p^f)(\theta^s - \theta_*^s(p^f)) + \varepsilon^{-1/2}T(p^f)(p^s - p_*^s(p^f)) \\ y &= T^{-1}(p^f)(\theta^s - \theta_*^s(p^f)) - \varepsilon^{-1/2}T(p^f)(p^s - p_*^s(p^f)), \\ I &= \varepsilon^{-1/2}p^f \quad , \quad \Theta = \gamma\theta^f, \end{aligned}$$

where  $\gamma$  is a parameter which will be taken later equal to  $\delta^{1/2}$ . Note that

$$\theta^s = \theta_*^s(\varepsilon^{1/2}I) + \frac{1}{2}T(\varepsilon^{1/2}I)(x + y), \quad p^s = p_*^s(\varepsilon^{1/2}I) + \frac{\varepsilon^{1/2}}{2}T^{-1}(\varepsilon^{1/2}I)(x - y).$$

**Lemma 3.1.** *We have  $\Lambda(p^f) \geq \sqrt{\lambda/D} I$  for each  $p^f \in [a^-, a^+]$ .*

*Proof.* The matrix  $\Lambda$  is symmetric, hence it satisfies  $\Lambda \geq \lambda_* I$ , where  $\lambda_* > 0$  is its smallest eigenvalue. The real number  $\lambda_*$  is then an eigenvalue of the matrix  $\begin{bmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{bmatrix}$  which is similar to  $\begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}$ . Since both  $A$  and  $B$  are square matrices of equal size, we conclude that  $\lambda_*^{-2}$  is an eigenvalue of  $A^{-1}B^{-1}$ . Since  $\|A^{-1}\| \leq \lambda^{-1}$  and  $\|B^{-1}\| \leq D$ , we have  $\lambda_*^{-2} \leq \|A^{-1}B^{-1}\| \leq D\lambda^{-1}$ . We conclude that  $\lambda_* \geq \sqrt{\lambda/D}$ .  $\square$

The links between the various parameters  $\varepsilon$ ,  $\delta$ ,  $\gamma$ ,  $\lambda$ ,  $\rho$  which appear in the computations below will be specified later. We will however assume from the beginning that

$$\delta \leq \rho \leq \lambda < 1 \quad , \quad \sqrt{\varepsilon} \leq \rho < 1 \quad , \quad 0 < \gamma \leq \lambda < 1.$$

Let us first collect some estimates that will be useful to see that the system (9) is indeed a perturbation of the model system.

**Lemma 3.2.** *We have the estimates*

$$\begin{aligned} \|T\| &= O(\lambda^{-1/4}), \quad \|T^{-1}\| = O(1), \quad \|\partial_{p_f} T\| \leq O(\lambda^{-5/4}), \quad \|\partial_{p_f} T^{-1}\| \leq O(\lambda^{-3/4}), \\ \|\partial_{p_f} \theta_*^s\| &\leq O(\lambda^{-1}), \quad \|p_*^s\|_{C^2} = O(1), \quad \|\theta^s - \theta_*^s\| \leq O(\lambda^{-1/4} \rho), \quad \|p^s - p_*^s\| \leq O(\varepsilon^{1/2} \rho), \end{aligned}$$

where  $\rho = \max(\|x\|, \|y\|)$ .

*Proof.* We recall that  $T = (B^{1/2}(B^{1/2}AB^{1/2})^{-1/2}B^{1/2})^{1/2}$  and  $T^{-1} = (B^{-1/2}(B^{1/2}AB^{1/2})^{1/2}B^{-1/2})^{1/2}$ . Since  $D^{-1}I \leq B \leq DI$  and  $\lambda I \leq A \leq I$ , we obtain that  $\|T\| \leq O(\lambda^{-1/4})$  and that  $\|T^{-1}\| \leq O(1)$ . To estimate the derivative of  $T$ , we consider the map  $F : M \mapsto M^{1/2}$  defined on positive symmetric matrices. It is known that

$$dF_M \cdot N = \int_0^\infty e^{-tM^{1/2}} N e^{-tM^{1/2}} dt.$$

To verify this one can diagonalize  $M$ , perform integration, and match terms in  $(M^{1/2} + \varepsilon dF_M \cdot N)(M^{1/2} + \varepsilon dF_M \cdot N) = M + \varepsilon N + O(\varepsilon^2)$ . This implies that

$$\|dF_M\| \leq \|M^{1/2}\|^{-1/2} \leq \|M^{-1/2}\|/2.$$

As a consequence, if  $M(p_f)$  is a positive symmetric matrix depending on  $p_f$ , we have

$$\|\partial_{p_f} M\| \leq \|M^{-1/2}\| \|\partial_{p_f} M\|/2.$$

We apply this bound several times to estimate  $\partial_{p_f} T$  and  $\partial_{p_f} T^{-1}$ . In our situation, we have  $\partial_{p_f} A = O(1)$ ,  $\partial_{p_f} B = O(1)$ . Using  $M = A$  and  $B$ , we get  $\partial_{p_f}(A^{1/2}) = O(\lambda^{-1/2})$  and  $\partial_{p_f}(B^{1/2}) = O(1)$  resp. Using  $M = B^{1/2}AB^{1/2}$  we get  $\partial_{p_f}[(B^{1/2}AB^{1/2})^{1/2}] = O(\lambda^{-1/2})$ , and then

$$\begin{aligned} \|\partial_{p_f}[T^{-1}]\| &\leq \|(B^{-1/2}(B^{1/2}AB^{1/2})^{1/2}B^{-1/2})^{-1/2}\| \|\partial_{p_f}[B^{-1/2}(B^{1/2}AB^{1/2})^{1/2}B^{-1/2}]\| \\ &= O(\lambda^{-1/4})O(\lambda^{-1/2}) = O(\lambda^{-3/4}). \end{aligned}$$

Recalling that

$$\|\partial_{p_f}(M^{-1})\| \leq \|M^{-1}\|^2 \|\partial_{p_f} M\|,$$

we obtain (with  $M = T^{-1}$ )

$$\|\partial_{p_f} T\| \leq \|T\|^2 \|\partial_{p_f}[T^{-1}]\| \leq \partial_{p_f}(M^{1/2}) = O(\lambda^{-5/4}).$$

The other estimates are straightforward.  $\square$

**Corollary 3.3.** *Let  $\tilde{V}$  be the image in the  $(x, y, I, \Theta, t)$  coordinates of the domain called  $V$  in the statement. We have*

$$\begin{aligned}\tilde{V} &\subset \{x : \|x\| \leq b^{1/6} \lambda^{5/4}\} \times \{y : \|y\| \leq b^{1/6} \lambda^{5/4}\} \times \mathbb{R} \times \left[ \frac{a^-}{\sqrt{\varepsilon}} + \sqrt{\delta}, \frac{a^+}{\sqrt{\varepsilon}} - \sqrt{\delta} \right] \times \mathbb{R}, \\ \tilde{V} &\supset \{x : \|x\| \leq 2b^{1/4} \lambda^{7/4}\} \times \{y : \|y\| \leq 2b^{1/4} \lambda^{7/4}\} \times \mathbb{R} \times \left[ \frac{a^-}{\sqrt{\varepsilon}} + \sqrt{\delta}, \frac{a^+}{\sqrt{\varepsilon}} - \sqrt{\delta} \right] \times \mathbb{R}\end{aligned}$$

provided  $b$  is small enough.

From now on, we work on the region

$$p^f \in [a^-, a^+], \quad \|x\| \leq \rho, \quad \|y\| \leq \rho.$$

In view of Lemma 3.2, this region is contained in the (image in the new coordinates of the) domain where the inequality  $\|R\|_{C^2} \leq \delta$  was assumed.

**Lemma 3.4.** *The equations of motion in the new coordinates take the form*

$$\begin{aligned}\dot{x} &= -\sqrt{\varepsilon} \Lambda(\sqrt{\varepsilon} I) x + \varepsilon^{1/2} O(\lambda^{-1/4} \delta + \lambda^{-3/4} \rho^2) + O(\varepsilon) \\ \dot{y} &= \sqrt{\varepsilon} \Lambda(\sqrt{\varepsilon} I) y + \varepsilon^{1/2} O(\lambda^{-1/4} \delta + \lambda^{-3/4} \rho^2) + O(\varepsilon) \\ \dot{I} &= O(\sqrt{\varepsilon} \delta),\end{aligned}$$

where  $\rho = \max(\|x\|, \|y\|)$  is assumed to satisfy  $\rho \leq \lambda$ . The expression for  $\dot{\Theta}$  is not useful here.

*Proof.* The last part of the statement is obvious. We prove the part concerning  $\dot{x}$ , the calculations for  $\dot{y}$  are exactly the same. In the original coordinates the vector field (9) can be written

$$\begin{aligned}\dot{\theta}^s &= B(p^f)(p^s - p_*^s(p^f)) + O(\|p^s - p_*^s(p^f)\|^2) + O(\varepsilon), \\ \dot{p}^s &= \varepsilon A(p^f)(\theta^s - \theta_*^s(p^f)) + O(\varepsilon \|\theta^s - \theta_*^s(p^f)\|^2) + O(\varepsilon \delta).\end{aligned}$$

As a consequence, we have

$$\begin{aligned}\dot{x} &= T^{-1} B(p^s - p_*^s) + \varepsilon^{1/2} T A(\theta^s - \theta_*^s) \\ &\quad + T^{-1} \cdot O(\|p^s - p_*^s\|^2 + \varepsilon) + \varepsilon^{1/2} T \cdot O(\|\theta^s - \theta_*^s\|^2 + \delta) \\ &\quad + (\partial_{p^f} T^{-1}) \dot{p}^f (\theta^s - \theta_*^s) + \varepsilon^{-1/2} (\partial_{p^f} T) \dot{p}^f (p^s - p_*^s) \\ &\quad - T^{-1} (\partial_{p^f} \theta_*^s) \dot{p}^f - \varepsilon^{-1/2} T (\partial_{p^f} p_*^s) \dot{p}^f.\end{aligned}$$

We use the estimates of Lemma 3.2 to simplify (recall also that  $\dot{p}^f = O(\varepsilon \delta)$ ):

$$\begin{aligned}\dot{x} &= T^{-1} B(p^s - p_*^s) + \varepsilon^{1/2} T A(\theta^s - \theta_*^s) \\ &\quad + O(\varepsilon \rho^2 + \varepsilon) + O(\varepsilon^{1/2} \lambda^{-3/4} \rho^2 + \varepsilon^{1/2} \lambda^{-1/4} \delta) \\ &\quad + O(\lambda^{-1} \varepsilon \delta \rho) + O(\lambda^{-5/4} \varepsilon \delta \rho) + O(\lambda^{-1} \varepsilon \delta + \lambda^{-1/4} \varepsilon^{1/2} \delta).\end{aligned}$$

□

**Lemma 3.5.** *In the new coordinate system  $(x, y, \Theta, I, t)$ , the linearized system is given by the matrix*

$$L = \begin{bmatrix} \sqrt{\varepsilon} \Lambda & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{\varepsilon} \Lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + O(\sqrt{\varepsilon} \delta \lambda^{-1/4} \gamma^{-1} + \sqrt{\varepsilon} \lambda^{-3/4} \rho + \varepsilon \lambda^{-5/4} + \sqrt{\varepsilon} \gamma),$$

where  $\rho = \max(\|x\|, \|y\|)$ .

*Proof.* Most of the estimates below are based on Lemma 3.2. In the original coordinates, the matrix of the linearized system is:

$$\tilde{L} = \begin{bmatrix} O(\varepsilon) & \partial_{p^s p^s}^2 H_0 + O(\varepsilon) & 0 & \partial_{p^f p^s}^2 H_0 + O(\varepsilon) & 0 \\ -\varepsilon \partial_{\theta^s \theta^s}^2 Z & O(\varepsilon) & 0 & O(\varepsilon) & 0 \\ O(\varepsilon) & O(1) & 0 & O(1) & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + O(\delta\varepsilon),$$

In our notations we have

$$\tilde{L} = \begin{bmatrix} O(\varepsilon) & B + O(\varepsilon + \sqrt{\varepsilon}\rho) & 0 & \partial_{p^f p^s}^2 H_0 + O(\varepsilon) & 0 \\ \varepsilon A + O(\varepsilon\lambda^{-1/4}\rho) & O(\varepsilon) & 0 & O(\varepsilon) & 0 \\ O(\varepsilon) & O(1) & 0 & O(1) & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + O(\delta\varepsilon),$$

In the new coordinates, the matrix is the product

$$L = \left[ \frac{\partial(x, y, \Theta, I, t)}{\partial(\theta^s, p^s, \theta^f, p^f, t)} \right] \cdot \tilde{L} \cdot \left[ \frac{\partial(\theta^s, p^s, \theta^f, p^f, t)}{\partial(x, y, \Theta, I, t)} \right].$$

We have

$$\left[ \frac{\partial(\theta^s, p^s, \theta^f, p^f, t)}{\partial(x, y, \Theta, I, t)} \right] = \begin{bmatrix} T/2 & T/2 & 0 & O(\sqrt{\varepsilon}\lambda^{-1}) & 0 \\ \sqrt{\varepsilon}T^{-1}/2 & -\sqrt{\varepsilon}T^{-1}/2 & 0 & \sqrt{\varepsilon}\partial_{p^f p^s}^2 H_0 + O(\varepsilon\lambda^{-3/4}\rho) & 0 \\ 0 & 0 & \gamma^{-1} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\varepsilon} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

hence

$$\tilde{L} \left[ \frac{\partial(\theta^s, p^s, \theta^f, p^f, t)}{\partial(x, y, \Theta, I, t)} \right] = O(\gamma^{-1}\delta\varepsilon) + \begin{bmatrix} \sqrt{\varepsilon}BT^{-1}/2 + O(\varepsilon\lambda^{-1/4}) & -\sqrt{\varepsilon}BT^{-1}/2 + O(\varepsilon\lambda^{-1/4}) & 0 & O(\varepsilon\lambda^{-3/4}\rho + \varepsilon^{3/2}\lambda^{-1}) & 0 \\ \varepsilon AT/2 + O(\varepsilon\lambda^{-1/2}\rho) & \varepsilon AT/2 + O(\varepsilon\lambda^{-1/2}\rho) & 0 & \varepsilon^{3/2}O(\lambda^{-5/4}\rho + \lambda^{-1}) & 0 \\ O(\sqrt{\varepsilon}) & O(\sqrt{\varepsilon}) & 0 & O(\sqrt{\varepsilon}) & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This expression is the result of a tedious, but straightforward, computation. Let us just detail the computation of the coefficient on the first line, fourth row, which contains an important cancellation:

$$\begin{aligned} & \sqrt{\varepsilon}\partial_{p^s p^s}^2 H_0 \partial_{p^f p^s}^2 H_0 + \sqrt{\varepsilon}\partial_{p^f p^s}^2 H_0 + O(\varepsilon\lambda^{-3/4}\rho + \varepsilon^{3/2}\lambda^{-1}) \\ &= \sqrt{\varepsilon}\partial_{p^f} (\partial_{p^s} H_0(p_*(p^f))) + O(\varepsilon\lambda^{-3/4}\rho + \varepsilon^{3/2}\lambda^{-1}) = O(\varepsilon\lambda^{-3/4}\rho + \varepsilon^{3/2}\lambda^{-1}). \end{aligned}$$

We now write

$$\left[ \frac{\partial(x, y, \Theta, I, t)}{\partial(\theta^s, p^s, \theta^f, p^f, t)} \right] = \begin{bmatrix} T^{-1} & \varepsilon^{-1/2}T & 0 & O(\varepsilon^{-1/2}\lambda^{-1/4}) & 0 \\ T^{-1} & -\varepsilon^{-1/2}T & 0 & O(\varepsilon^{-1/2}\lambda^{-1/4}) & 0 \\ 0 & 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & \varepsilon^{-1/2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and compute that

$$L = \begin{bmatrix} \sqrt{\varepsilon}\Lambda + O(\sqrt{\varepsilon}\lambda^{-3/4}\rho) & O(\sqrt{\varepsilon}\lambda^{-3/4}\rho) & 0 & O(\varepsilon\lambda^{-5/4}) & 0 \\ O(\sqrt{\varepsilon}\lambda^{-3/4}\rho) & -\sqrt{\varepsilon}\Lambda + O(\sqrt{\varepsilon}\lambda^{-3/4}\rho) & 0 & O(\varepsilon\lambda^{-5/4}) & 0 \\ O(\sqrt{\varepsilon}\gamma) & O(\sqrt{\varepsilon}\gamma) & 0 & O(\sqrt{\varepsilon}\gamma) & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ + O(\sqrt{\varepsilon}\delta\lambda^{-1/4}\gamma^{-1}).$$

□

In order to prove the existence of a normally hyperbolic invariant strip (for the lifted system), we apply Proposition A.4 to the system in coordinates  $(x, y, \Theta, I, t)$ . More precisely, with the notations of appendix A, we set:  $u = x, s = y, c_1 = (\Theta, t), c_2 = I$ , and consider the domain

$$\Omega = \mathbb{R}^2 \times \Omega^{c_2} = \mathbb{R}^2 \times \left[ \frac{a^-}{\sqrt{\varepsilon}} + \sqrt{\delta}, \frac{a^+}{\sqrt{\varepsilon}} - \sqrt{\delta} \right].$$

We fix

$$\gamma = \sqrt{\delta}, \quad \alpha = \sqrt{\varepsilon\lambda/4D}, \quad \sigma = \sqrt{\delta}, \quad (11)$$

observe that  $\sqrt{\varepsilon}\Lambda \geq 2\alpha I$ , by Lemma 3.1. We assume, as in the statement of the Theorem, that  $0 < \varepsilon < b\lambda^{9/2}$  and that  $0 \leq \delta < b\lambda^{5/2}$ . We apply Proposition A.4 with  $B^u = \{u : \|u\| \leq \rho\}$  and  $B^s = \{s : \|s\| \leq \rho\}$  under the constraint

$$b^{-1/4}(\lambda^{-3/4}\delta + \lambda^{-1/2}\sqrt{\varepsilon}) \leq \rho \leq b^{1/6}\lambda^{5/4}, \quad (12)$$

provided  $b \in ]0, 1[$  is small enough. Observe that, if  $b$  is small enough, the inequalities

$$b^{-1/4}(\lambda^{-3/4}\delta + \lambda^{-1/2}\sqrt{\varepsilon}) \leq 2b^{1/4}\lambda^{7/4} \leq b^{1/6}\lambda^{5/4}$$

holds under our assumptions on the parameters, hence values of  $\rho$  satisfying (12) do exist. It is easy to check under our assumptions on the parameters that such values of  $\rho$  exist. Let us check the isolating block condition under the condition (12). By Lemma 3.4, we have

$$\dot{x} \cdot x \geq 2\alpha\|x\|^2 - \|x\| \quad O(\varepsilon^{1/2}\lambda^{-1/4}\delta + \varepsilon^{1/2}\lambda^{-3/4}\rho^2 + \varepsilon)$$

if  $x \in B^u, y \in B^s$ . If in addition  $\|x\| = \rho$ , then

$$\lambda^{-3/4}\delta \leq b^{1/4}\|x\| \quad , \quad \lambda^{-5/4}\rho^2 \leq b^{1/6}\|x\| \quad , \quad \sqrt{\varepsilon/\lambda} \leq b^{1/4}\|x\|,$$

hence

$$\dot{x} \cdot x \geq 2\alpha\|x\|^2 - \|x\|^2 b^{1/6} O(\sqrt{\varepsilon\lambda}) \geq \alpha\|x\|^2$$

provided  $b$  is small enough. Similarly,  $\dot{y} \cdot y \leq -\alpha\|y\|^2$  on  $B^u \times \partial B^s$  provided  $b$  is small enough. Concerning the linearized system, we have

$$L_{uu} = \sqrt{\varepsilon}\Lambda + O(\sqrt{\varepsilon}\delta\lambda^{-1/4}\gamma^{-1} + \sqrt{\varepsilon}\lambda^{-3/4}\rho + \varepsilon\lambda^{-5/4} + \sqrt{\varepsilon}\gamma) \\ = \sqrt{\varepsilon}\Lambda + O(b^{1/6}\sqrt{\varepsilon\lambda}) \geq \alpha I, \\ L_{ss} = -\sqrt{\varepsilon}\Lambda + O(b^{1/6}\sqrt{\varepsilon\lambda}) \leq -\alpha I$$

on  $B^u \times B^s \times \Omega_r$ . These inequalities holds when  $b$  is small enough because  $\sqrt{\varepsilon}\Lambda \geq 2\alpha I$  and  $\sqrt{\varepsilon\lambda} \leq O(\alpha)$ . Finally, still with the notations of Proposition A.4, we can take

$$m = O(\sqrt{\varepsilon}\delta\lambda^{-1/4}\gamma^{-1} + \sqrt{\varepsilon}\lambda^{-3/4}\rho + \varepsilon\lambda^{-5/4} + \sqrt{\varepsilon}\gamma + \sqrt{\varepsilon}\delta/\sigma) \\ = \sqrt{\varepsilon\lambda} O(\sqrt{\delta}\lambda^{-3/4} + \rho\lambda^{-5/4} + \sqrt{\varepsilon}\lambda^{-7/4}) = \sqrt{\varepsilon\lambda} O(b^{1/6}). \quad (13)$$

If  $b$  is small enough, we have  $16m < \alpha$  hence

$$K \leq 2m/\alpha < 1/8,$$

and Proposition A.4 can be applied. The invariant strip obtained from the proof of Proposition A.4 does not depend on the choice of  $\rho$ , as long as (12) holds. It contains all the full orbits contained in

$$\{x : \|x\| \leq b^{1/6}\lambda^{5/4}\} \times \{y : \|y\| \leq b^{1/6}\lambda^{5/4}\} \times \mathbb{R} \times \left[ \frac{a^-}{\sqrt{\varepsilon}} + \sqrt{\delta}, \frac{a^+}{\sqrt{\varepsilon}} - \sqrt{\delta} \right] \times \mathbb{R} \supset \tilde{V},$$

where  $\tilde{V}$  is the image in the new coordinates of the domain  $V$  defined in the statement of Theorem 3.1 and where the last inclusion holds provided  $b$  is small enough, as follows from Corollary 3.3. So our invariant strip contains all the full orbits contained in  $\tilde{V}$ . On the other hand, we can take  $\rho = 2b^{1/4}\lambda^{7/4}$ , and since

$$\{x : \|x\| \leq 2b^{1/4}\lambda^{7/4}\} \times \{y : \|y\| \leq 2b^{1/4}\lambda^{7/4}\} \times \mathbb{R} \times \left[ \frac{a^-}{\sqrt{\varepsilon}} + \sqrt{\delta}, \frac{a^+}{\sqrt{\varepsilon}} - \sqrt{\delta} \right] \times \mathbb{R} \subset \tilde{V}$$

(still for  $b$  small enough, by Corollary 3.3), our invariant strip is contained in  $\tilde{V}$ .

The possibility of taking  $\rho = b_1^{-1/4}(\lambda^{-3/4}\delta + \lambda^{-1/2}\sqrt{\varepsilon})$  now implies that the cylinder is actually contained in the domain where

$$\|x\|, \|y\| \leq b_1^{-1/4}(\lambda^{-3/4}\delta + \lambda^{-1/2}\sqrt{\varepsilon}).$$

Moreover, with this choice of  $\rho$  and using that  $K = O(m/\sqrt{\varepsilon\lambda})$ , we can obtain an improved estimate of the Lipschitz constant  $K$  (notation from the appendix):

$$\begin{aligned} K &= O(\sqrt{\delta}\lambda^{-3/4} + \rho\lambda^{-5/4} + \sqrt{\varepsilon}\lambda^{-7/4}) \\ &= O(\sqrt{\delta}\lambda^{-3/4} + b_1^{-1/4}\delta\lambda^{-2} + b_1^{-1/4}\sqrt{\varepsilon}\lambda^{-7/4} + \sqrt{\varepsilon}\lambda^{-7/4}) \\ &= O(\sqrt{\delta}\lambda^{-3/4} + \sqrt{\delta}\lambda^{-1} + b_1^{-1/4}\sqrt{\varepsilon}\lambda^{-7/4}) \\ &= O(\sqrt{\delta}\lambda^{-1} + b_1^{-1/4}\sqrt{\varepsilon}\lambda^{-7/4}). \end{aligned}$$

Observe finally that, since the system is  $1/\gamma$ -periodic in  $\Theta$  and 1-periodic in  $t$ , so is the invariant strip given by Proposition A.4. We have obtained the existence of a  $C^1$  map

$$w^c = (w_u^c, w_s^c) : (\Theta, I, t) \in \mathbb{R} \times \left[ \frac{a^-}{\sqrt{\varepsilon}} + \sqrt{\delta}, \frac{a^+}{\sqrt{\varepsilon}} - \sqrt{\delta} \right] \times \mathbb{R} \longrightarrow \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$$

which is  $2K$ -Lipschitz,  $1/\gamma$ -periodic in  $\Theta$  and 1-periodic in  $t$ , and the graph of which is tangent to the vector field. Our last task is to return to the original coordinates by setting

$$\begin{aligned} \Theta^s(\theta^f, p^f, t) &= \theta_*^s(p^f) + \frac{1}{2}T(p^f) \cdot (w_u^c + w_s^c)(\gamma\theta^f, \varepsilon^{-1/2}p^f, t) \\ P^s(\theta^f, p^f, t) &= p_*^s(p^f) + \frac{\sqrt{\varepsilon}}{2}T^{-1}(p^f) \cdot (w_u^c - w_s^c)(\gamma\theta^f, \varepsilon^{-1/2}p^f, t). \end{aligned} \tag{14}$$

All the estimates stated in Theorem 3.1 follow directly from these expressions, and from the fact that  $\|dw^c\| \leq 2K$ . This concludes the proof of Theorem 3.1.  $\square$

## 4 Localization and Mather's projected graph theorem

We study the system in normal form  $N_\varepsilon = H_0 + \varepsilon Z + \varepsilon R$  of Theorem 1.2 from the point of view of Mather theory at a fixed cohomology  $c \in \mathbb{R}^n$  such that  $\partial_{p^s} H_0(c) = 0$  (or in other words such that  $c \in \Gamma$ ). We assume that  $\|Z\|_{C^2} \leq 1$ , and that  $\|R\|_{C^2} \leq \delta$  on  $\{\|p - c\| < \varepsilon^{1/3}\}$ . We continue to assume (1), and, for simplicity, we assume that  $D$  is large enough and  $\varepsilon$  small enough for the following inequality to also hold:

$$(1/D)I \leq \partial_p^2 N_\varepsilon \leq DI.$$

Most of our statement depend on the shape of the function  $Z_c : \theta^s \mapsto Z(\theta, c)$ . We will most of the time assume that  $(HZ\lambda)$  holds at  $c$ : There exists  $\theta_*^s$  such that  $Z(\theta^s, c) \leq Z(\theta_*^s, c) - \lambda d^2(\theta^s, \theta_*^s)$ . We will rewrite this inequality as

$$\hat{Z}_c(\theta^s) \leq -\lambda d^2(\theta^s, \theta_*^s)$$

with the notation  $\hat{Z}_c = Z_c - \max Z_c$ . Later in section 4.4, we also consider the double peak case, which is not necessary for the proof of Theorem 1.2, but is very natural. Our first statement localizes the Mañé set.

**Theorem 4.1.** *In the single peak case (when  $(HZ\lambda)$  holds at  $c$ ), if  $\delta > 0$  is small enough with respect to  $n, D, \lambda$  and  $\varepsilon$  is small enough with respect to  $n, D, \lambda, \delta$ , then the Mañé set at cohomology  $c$  of the Hamiltonian  $N_\varepsilon$  satisfies*

$$s\tilde{\mathcal{N}}(c) \subset B(\theta_*^s, \delta^{1/5}) \times \mathbb{T} \times B(c, \sqrt{\varepsilon} \delta^{1/16}) \times \mathbb{T} \subset \mathbb{T}^{n-1} \times \mathbb{T} \times \mathbb{R}^n \times \mathbb{T}.$$

This statement is proved in Section 4.2. Our second statement is a quantitative version of the celebrated Mather Lipschitz graph Theorem, it does not rely on any particular assumption on  $Z$ , besides  $\|Z\|_{C^2} \leq 1$ :

**Theorem 4.2.** *For each Weak KAM solution  $u$  of  $N_\varepsilon$  at cohomology  $c$ , the set  $\tilde{\mathcal{I}}(u, c) \subset \mathbb{T}^n \times \mathbb{R}^n$  is contained in a  $9\sqrt{D\varepsilon}$ -Lipshitz graph above  $\mathbb{T}^n$ .*

This theorem is proved in Section 4.3. We will always assume in this section that  $\delta$  is sufficiently small with respect to  $n, H_0$  and  $\lambda$ , and that  $\varepsilon$  is sufficiently small with respect to  $n, H_0, \lambda$  and  $\delta$ .

### 4.1 Some inequalities

We will denote by  $N$  the Hamiltonian  $N_\varepsilon$  and by  $L$  the associated Lagrangian function, which is defined by

$$L(\theta, v, t) = \max_{p \in \mathbb{R}^n} (p \cdot v - N(\theta, p, t)).$$

The function  $L$  is then  $C^2$ , and the maps

$$(\theta, p, t) \mapsto \partial_p N(\theta, p, t), \quad (\theta, v, t) \mapsto \partial_v L(\theta, v, t),$$

are diffeomorphisms of  $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T}$ , which are inverse of each other. The maximum in the definition of  $L$  is reached at  $p = \partial_v L(\theta, v, t)$ . Since  $I/D \leq \partial_{pp} N \leq DI$ , we have

$$I/D \leq \partial_{vv} L \leq DI.$$

We will also denote by  $L_0(v)$  the Lagrangian associated to  $H_0$ , or more explicitly  $L_0(v) := \sup_p (p \cdot v - H_0(p))$ . It satisfies

$$I/D \leq \partial_{vv} L_0 \leq DI.$$

**Lemma 4.1.** For each  $\rho \in [4D\varepsilon, \varepsilon^{1/4}]$ , the image of the open set  $\mathbb{T}^n \times B(c, \rho) \times \mathbb{T}$  under the diffeomorphism  $\partial_p N$  contains the set

$$\mathbb{T}^n \times B(\partial_p H_0(c), \rho/2D - 2\varepsilon) \times \mathbb{T}.$$

In particular, if  $\varepsilon$  is small enough, the image of  $\mathbb{T}^n \times B(c, \varepsilon^{1/4}) \times \mathbb{T}$  contains  $\mathbb{T}^n \times B(c, \varepsilon^{1/4}/4D) \times \mathbb{T}$ .

*Proof.* In view of the estimate  $\partial_p^2 H \geq I/D$ , each of the applications  $p \mapsto \partial_p N(\theta, p, t)$  sends the ball  $B(c, r)$  to a set which contains the ball  $B(\partial_p N(\theta, c, t), r/2D)$ . Since  $|\partial_p N(\theta, c, t) - \partial_p H_0(c)| \leq \varepsilon + \varepsilon\delta \leq 2\varepsilon$ , we conclude that the image contains  $B(\partial_p H_0(c), \rho/2D - 2\varepsilon)$ .  $\square$

**Lemma 4.2.** The estimates

$$\|\partial_{\theta v} L\|_{C^0} \leq 2D\varepsilon, \quad \|\partial_{\theta\theta} L\|_{C^0} \leq 3\varepsilon.$$

hold on  $\mathbb{T}^n \times B(c, \varepsilon^{1/3}/4D) \times \mathbb{T}$ .

*Proof.* Note first that the estimates

$$\|\partial_{\theta p} H\| \leq 2\varepsilon, \quad \|\partial_{\theta\theta} H\| \leq 2\varepsilon$$

hold on the domain  $\mathbb{T}^n \times B(c, \varepsilon^{1/3}) \times \mathbb{T}$ , which contains the image of  $\mathbb{T}^n \times B(c, \varepsilon^{1/3}/4D) \times \mathbb{T}$  under  $\partial_v L$ . Observing that  $\partial_{\theta} L = -\partial_{\theta} N(\theta, \partial_v L_\varepsilon(\theta, v))$ , which implies

$$\partial_{\theta v} L(\theta, v, t) = -\partial_{p\theta} N_\varepsilon(\theta, \partial_v L(\theta, v, t), t) \partial_{vv} L(\theta, v, t)$$

we deduce that  $\|\partial_{\theta v} L\| \leq 2D\varepsilon$  on  $\mathbb{T}^n \times B(c, \varepsilon^{1/3}/4D) \times \mathbb{T}$ . The equality

$$\partial_{\theta\theta} L(\theta, v, t) = -\partial_{\theta\theta} N(\theta, \partial_v L(\theta, v, t), t) - \partial_{p\theta} N(\theta, \partial_v L(\theta, v, t), t) \partial_{\theta v} L(\theta, v, t),$$

implies that  $\|\partial_{\theta\theta} L\| \leq 2\varepsilon + (2\varepsilon)(2D\varepsilon)$  on  $\mathbb{T}^n \times B(c, \varepsilon^{1/3}/4D) \times \mathbb{T}$ .  $\square$

**Lemma 4.3.** We have the estimate

$$|L(\theta, v, t) - (L_0(v) - \varepsilon Z(\theta^s, c))| \leq 2\varepsilon\delta$$

if  $|v - \partial_p H_0(c)| < \varepsilon^{1/3}/4D$ .

*Proof.* On the domain  $\{|p - c| < \varepsilon^{1/3}\}$ , we have

$$|N(\theta, p, t) - (H_0(p) + \varepsilon Z(\theta^s, c))| \leq \varepsilon^{5/4} + \varepsilon\delta \leq 2\varepsilon\delta.$$

If  $|v - \partial_p H_0(c)| < \varepsilon^{1/3}/4D$ , then by Lemma 4.1,

$$L(\theta, v, p) = \sup_{|p-c| < \varepsilon^{1/3}} [p \cdot v - N(\theta, p, t)]$$

and, by Lemma 4.1 applied with  $R \equiv 0$  and  $Z(\theta^s, p) \equiv Z(\theta^s, c)$

$$L_0(v) - \varepsilon Z(\theta^s, c) = \sup_p [p \cdot v - H_0(p) - \varepsilon Z(\theta^s, c)] = \sup_{|p-c| < \varepsilon^{1/3}} [p \cdot v - H_0(p) - \varepsilon Z(\theta^s, c)].$$

$\square$

Let us now estimate the value  $\alpha(c)$  of the Mather function of  $N$ . We use the notation  $Z_c(\theta^s) := Z(\theta^s, c)$ .



**Lemma 4.4.** *The value  $\alpha(c)$  of the Mather function of  $N$  satisfies*

$$|\alpha(c) - (H_0(c) + \varepsilon \max Z_c)| \leq 2\varepsilon\delta.$$

The reason behind this inequality is that the value  $\alpha(c)$  of the Hamiltonian  $H_0 + \varepsilon Z_c$  is  $H_0(c) + \varepsilon \max Z_c$ .

*Proof.* On one hand, we have

$$\alpha(c) \leq \max_{(t,\theta)} N_\varepsilon(t, \theta, c) \leq H_0(c) + \varepsilon \max Z_c + \varepsilon \max_{(t,\theta) \in \mathbb{T}^{n+1}} R(\theta, c, t) \leq H_0(c) + \varepsilon \max Z_c + \varepsilon\delta.$$

For the other inequality, we use that  $\partial_{p^s} H_0 = 0$ . We consider the Haar measure  $\mu$  of the torus  $\mathbb{T} \times \{\theta_*^s(c)\} \times \{\partial H_0(c)\} \times \mathbb{T}$ , where  $\theta_*^s(c)$  is any point maximizing  $Z_c$ . This measure is not necessarily invariant under the Lagrangian flow of  $L$ , but it is invariant under the Lagrangian flow of  $L_0 - Z_c$  (because  $\partial_{p^s} H_0 = 0$ ) hence it is closed, which means that  $\int \partial_t f + \partial_\theta f \cdot v d\mu(\theta, v, t) = 0$  for each smooth function  $f(t, \theta)$ . See [Ba, FS] (both inspired from [Mn]) for the notion of closed measures. Each closed measure  $\mu$  has a rotation vector  $\rho(\mu) := \int v d\mu(\theta, v, t) \in \mathbb{R}^n$ , and its action is not less than  $c \cdot \rho(\mu) - \alpha(c)$ . Here,  $\rho(\mu) = \partial_p H_0(c)$  hence

$$\begin{aligned} \alpha(c) &\geq c \cdot \partial_p H_0(c) - \int L d\mu = c \cdot \partial_p H_0(c) - L_0(\omega) + \varepsilon Z_c(\theta_*^s(c)) - 2\varepsilon\delta \\ &= H_0(c) + \varepsilon \max Z_c - 2\varepsilon\delta. \end{aligned}$$

□

**Lemma 4.5.** *If  $\varepsilon$  is small enough (with respect to  $D$  and  $\delta$ ), we have the estimates*

$$L(\theta, v, t) - c \cdot v + \alpha(c) \geq \|v - \partial H_0(c)\|^2 / (4D) - \varepsilon \hat{Z}_c(\theta^s) - 4\varepsilon\delta \quad (15)$$

$$L(\theta, v, t) - c \cdot v + \alpha(c) \leq D\|v - \partial H_0(c)\|^2 - \varepsilon \hat{Z}_c(\theta^s) + 4\varepsilon\delta \quad (16)$$

for each  $(\theta, v, t) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}$ , where  $\hat{Z}_c(\theta^s) := Z(\theta^s, c) - \max_{\theta^s} Z(\theta^s, c)$ .

*Proof.* It is a direct computation :

$$\begin{aligned} L(\theta, v, t) &\geq c \cdot v - N(\theta, c, t) + \|v - \partial_p N(\theta, c, t)\|^2 / 2D \\ &\geq c \cdot v - H_0(c) - \varepsilon Z_c(\theta^s) - \varepsilon\delta + (\|v - \partial_p H_0(c)\| - 2\varepsilon)^2 / 2D \\ &\geq c \cdot v - \alpha(c) + \varepsilon(\max Z_c - Z_c(\theta^s)) - 3\varepsilon\delta + \|v - \partial_p H_0(c)\|^2 / 4D - 16\varepsilon^2, \end{aligned}$$

$$\begin{aligned} L(\theta, v, t) &\leq c \cdot v - N(\theta, c, t) + D\|v - \partial_p N(\theta, c, t)\|^2 / 2 \\ &\leq c \cdot v - H_0(c) - \varepsilon Z_c(\theta^s) + \varepsilon\delta + D(\|v - \partial_p H_0(c)\| + 2\varepsilon)^2 / 2 \\ &\leq c \cdot v - \alpha(c) + \varepsilon(\max Z_c - Z_c(\theta^s)) + 3\varepsilon\delta + D\|v - \partial_p H_0(c)\|^2 + 8D\varepsilon^2. \end{aligned}$$

□

It is useful to consider suspended weak KAM solutions. Recall that we defined Weak KAM solutions associated to a Lagrangian  $L$  at cohomology  $c$  as functions  $u$  on  $\mathbb{T}^n$  such that, for each  $t \in \mathbb{N}$ ,

$$u(\theta) = \inf_\gamma \left( u(\gamma(0)) + \int_0^t L(\gamma(s), \dot{\gamma}(s), s) - c \cdot \dot{\gamma}(s) + \alpha(c) ds \right),$$

where the infimum is taken on the set of  $C^1$  curves  $\gamma : \mathbb{R} \rightarrow \mathbb{T}^n$  such that  $\gamma(t) = \theta$ . We can similarly define suspended weak KAM solutions as functions  $u : \mathbb{T}^n \times \mathbb{T} \rightarrow \mathbb{R}$  such that

$$u(\theta, T \bmod 1) = \inf_{\gamma} \left( u(\gamma(S), S \bmod 1) + \int_S^T L(\gamma(t), \dot{\gamma}(t), t) + c \cdot \dot{\gamma}(t) dt \right),$$

for each real times  $S \leq T$ , where the infimum is taken on the space of  $C^1$  curves  $\gamma : [S, T] \rightarrow \mathbb{T}^n$  such that  $\gamma(T) = \theta$ . There is a bijection between suspended weak KAM solution  $u(\theta, t)$  and genuine weak KAM solutions: Each suspended weak KAM solution  $u(\theta, t)$  restricts to a genuine weak KAM solution  $u(\theta) = u(\theta, 0)$ , and each genuine weak KAM solution  $u(\theta)$  is the restriction of a unique suspended weak KAM solution  $u(\theta, t)$  which can be defined by

$$u(\theta, t \bmod 1) = \inf_{\gamma} \left( u(\gamma(0)) + \int_0^t L(\gamma(s), \dot{\gamma}(s), s) + c \cdot \dot{\gamma}(s) + \alpha(c) ds \right),$$

for each  $t > 0$ , where the infimum is taken on  $C^1$  curves  $\gamma : \mathbb{R} \rightarrow \mathbb{T}^n$  such that  $\gamma(t) = \theta$ . We shall use the same notation for a weak KAM solution  $u$  and the associated suspended weak KAM solution. Curves  $\gamma$  calibrated by the weak KAM solutions  $u(\theta)$  are also calibrated by the corresponding suspended weak KAM solution in the sense that

$$u(\gamma(t_2), t_2 \bmod 1) - u(\gamma(t_1), t_1 \bmod 1) = \int_{t_1}^{t_2} L(\gamma(s), \dot{\gamma}(s), s) + c \cdot \dot{\gamma}(s) + \alpha(c) ds$$

for each time interval  $[t_1, t_2]$ . Let us now estimate the oscillation  $\text{osc } u := \max u - \min u$  of suspended weak KAM solutions. We consider a convex subset  $\Omega \in \mathbb{T}^{n-1}$ , meaning that it is the projection of a convex subset  $\tilde{\Omega}$  of  $\mathbb{R}^{n-1}$ , of diameter less than  $2\sqrt{n}$ .

**Lemma 4.6.** *Let  $u(\theta, t)$  be a suspended weak KAM solution of  $N$  at cohomology  $c$ . Given two points  $(\theta_1, t_1), (\theta_2, t_2) \in \mathbb{T} \times \Omega \times \mathbb{T}$ , we have*

$$u(\theta_2, t_2) - u(\theta_1, t_1) \leq 10\sqrt{nD\varepsilon(m + 4\delta)},$$

where  $m := -\inf_{\Omega} \hat{Z}_c$ . We can take in particular  $\Omega = \mathbb{T}^{n-1}$ , then  $m \leq 1$  and we conclude that  $\text{osc } u \leq 10\sqrt{2nD\varepsilon}$ .

*Proof.* We have  $0 \geq \hat{Z}_c \geq -m$  on  $\Omega$ . We take two points  $(\theta_i, t_i)$ ,  $i = 1$  or  $2$  in the domain  $\mathbb{T} \times \Omega \times \mathbb{T}$ , and consider the curve

$$\theta(t) = \theta_1 + (t - \tilde{t}_1) \frac{\tilde{\theta}_2 - \tilde{\theta}_1 + [(T + \tilde{t}_2 - \tilde{t}_1)\partial H_0(c)]}{T + \tilde{t}_2 - \tilde{t}_1}$$

where  $T \in \mathbb{N}$  is a parameter to be fixed later, where  $\tilde{t}_i \in [0, 1[$  and  $\tilde{\theta}_i \in [0, 1[\times\tilde{\Omega}$  are representatives of the angular variables  $t_i, \theta_i$ , and where  $[\omega] \in \mathbb{Z}^n$  is the component-wise integral part of  $\omega$ . Note that  $\theta(\tilde{t}_1) = \theta_1$  and  $\theta(\tilde{t}_2 + T) = \theta_2$ , hence

$$\begin{aligned} u(\theta_2, t_2) - u(\theta_1, t_1) &\leq \int_{\tilde{t}_1}^{\tilde{t}_2+T} L(\theta(t), \dot{\theta}(t), t) - c \cdot \dot{\theta}(t) + \alpha(c) dt \\ &\leq \int_{\tilde{t}_1}^{\tilde{t}_2+T} D\|\dot{\theta} - \partial H_0(c)\|^2 - \varepsilon \hat{Z}_c(\theta^s(t)) + 4\varepsilon\delta dt \\ &\leq \int_{\tilde{t}_1}^{\tilde{t}_2+T} \frac{9Dn}{(T + \tilde{t}_2 - \tilde{t}_1)^2} + \varepsilon m + 4\varepsilon\delta dt \\ &\leq \frac{9Dn}{(T + \tilde{t}_2 - \tilde{t}_1)} + (T + \tilde{t}_2 - \tilde{t}_1)\varepsilon(m + 4\delta). \end{aligned}$$

This inequality holds for all  $T \in \mathbb{N}$ , in particular, we can choose  $T \in \mathbb{N}$  so that

$$2\sqrt{\frac{nD}{\varepsilon(m+4\delta)}} \leq T + \tilde{t}_2 - \tilde{t}_1 \leq 3\sqrt{\frac{nD}{\varepsilon(m+4\delta)}}$$

and obtain  $u(\theta_2, t_2) - u(\theta_1, t_1) \leq 10\sqrt{nD\varepsilon(m+4\delta)}$ .  $\square$

## 4.2 Localization of the invariant sets

We prove Theorem 4.1. It is enough to prove that the inclusion

$$s\tilde{\mathcal{I}}(u, c) \subset B(\theta_*^s(c), \delta^{1/5}) \times \mathbb{T} \times B(c, \sqrt{\varepsilon}\delta^{1/16}) \times \mathbb{T}$$

holds for each (suspended) weak KAM solution  $u$ . We fix such a solution  $u(\theta, t)$  and prove the inclusion. The following preliminary localization, which does not use any assumption on the shape of  $Z$ , implies that the set  $s\tilde{\mathcal{I}}(u, c)$  is contained (when  $\varepsilon$  is small enough) in the domain  $\{\|p - c\| < \varepsilon^{1/3}\}$  where the assumption  $\|R\|_{C^2} \leq \delta$  is made.

**Lemma 4.7.** *Let  $(\theta(t), p(t)) : [t_1, t_2] \rightarrow \mathbb{T}^n \times \mathbb{R}^n$  be an orbit calibrated by  $u$ . If  $t_2 - t_1 \geq \varepsilon^{-1/2}$ , then*

$$\|p(t) - c\| \leq C\sqrt{\varepsilon}$$

for each  $t \in [t_1, t_2]$ , where  $C$  is a constant which depends on  $n$  and  $D$ . In particular,

$$s\tilde{\mathcal{I}}(u, c) \subset \mathbb{T}^n \times B(c, C\sqrt{\varepsilon}) \times \mathbb{T} \subset \mathbb{T}^n \times B(c, \varepsilon^{1/3}) \times \mathbb{T}.$$

*Proof.* We denote by  $C_i$  various positive constants which depend on  $n$  and  $D$ . Since  $\hat{Z}_c \leq 0$ , we have  $L(\theta, v, t) \geq \|v - \partial_p H_0(c)\|^2/4D - 4\varepsilon\delta$ . As a consequence,  $L(\theta, v, t) \geq 20\varepsilon\sqrt{nD}$  if  $\|v - \partial_p H_0(c)\| \geq C_1\sqrt{\varepsilon}$ . In view of Lemma 4.1, we thus have

$$L(\theta(t), \dot{\theta}(t), t) \geq 20\sqrt{nD}\varepsilon$$

for each  $t$  such that  $\|p(t) - c\| \geq C_2\sqrt{\varepsilon}$ . Since  $\theta$  is a calibrated curve, we have

$$\int_{t'_1}^{t'_2} L(\theta(t), \dot{\theta}(t), t) dt \leq \text{osc } u$$

for each  $[t'_1, t'_2] \subset [t_1, t_2]$ . In particular, by Lemma 4.6 we have  $20\varepsilon\sqrt{nD}(t_2 - t_1) > \text{osc } u$ . Therefore, there exists a time  $t_0 \in [t_1, t_2]$  such that  $\|p(t_0) - c\| = C_2\sqrt{\varepsilon}$ . Let  $t_3 \in [t_1, t_2]$  be the time maximizing  $\|p(t) - c\|$ . We assume for definiteness that  $t_3 \geq t_0$ , and that  $\|p(t) - c\| \geq C_2\sqrt{\varepsilon}$  for each  $t \in [t_0, t_3]$  (otherwise we reduce the interval). The equations of motion imply that  $\|\dot{p}\| \leq 2\varepsilon$  on  $[t_0, t_3]$ , hence  $t_3 \geq t_0 + (\|p(t_3) - c\| - C_2\sqrt{\varepsilon})/2\varepsilon$ , and using the above lower bound on  $L(\theta(t), \dot{\theta}(t), t)$

$$20\sqrt{nD}\varepsilon \geq \text{osc } u \geq \int_{t_0}^{t_3} L(\theta(t), \dot{\theta}(t), t) dt \geq 10\sqrt{nD}(\|p(t_3) - c\| - C_2\sqrt{\varepsilon})$$

which implies that  $\|p(t_3) - c\| \leq (2 + C_2)\sqrt{\varepsilon}$ .  $\square$

We now assume that  $Z(\theta^s, c) \leq Z(\theta_*^s, c) - \lambda d^2(\theta^s, \theta_*^s)$ , or, equivalently, that  $\hat{Z}_c(\theta^s) \leq -\lambda d^2(\theta^s, \theta_*^s)$ , and prove the horizontal part of Theorem 4.1, or more precisely that

$$s\mathcal{I}(u, c) \subset \mathbb{T} \times B(\theta_*^s(c), \delta^{1/5}) \times \mathbb{T}. \quad (17)$$

We consider the domain  $\Omega = B(\theta_*^s, 4\sqrt{\delta/\lambda})$ . On this domain, we have  $-8\delta/\lambda \leq \hat{Z}_c$ , hence, by Lemma 4.6, the oscillation of  $u$  on  $\mathbb{T} \times \Omega \times \mathbb{T}$  satisfies

$$\text{osc}_{\mathbb{T} \times \Omega \times \mathbb{T}} u \leq 40\sqrt{nD\varepsilon\delta/\lambda}.$$

For  $\theta^s \notin \Omega$ , we have

$$L(\theta, v, t) - c \cdot v - \alpha(c) \geq \|v - \partial_p H_0(c)\|^2/4D + \lambda\varepsilon d^2(\theta^s, \theta_*^s)/2 \geq \|v^s\|^2/4D + \lambda\varepsilon d^2(\theta^s, \theta_*^s)/2,$$

by Lemma 4.5. Let  $\theta(t) : \mathbb{R} \rightarrow \mathbb{T}^n$  be a curve calibrated by  $u$ , and let  $[t_1, t_2]$  be an excursion of  $\theta^s$  outside of  $\Omega$ , meaning that  $d(\theta^s(t), \theta_*^s) > 4\sqrt{\delta/\lambda}$  for each  $t \in ]t_1, t_2[$ , and that  $d(\theta^s(t_1), \theta_*^s) = 4\sqrt{\delta/\lambda} = d(\theta^s(t_2), \theta_*^s)$ . We have the inequalities

$$40\sqrt{nD\varepsilon\delta/\lambda} \geq \int_{t_1}^{t_2} L(\theta(t), \dot{\theta}(t), t) - c \cdot \dot{\theta}(t) + \alpha(c) dt \geq \int_{t_1}^{t_2} \frac{\|\dot{\theta}^s(t)\|^2}{4D} + \lambda\varepsilon \frac{d^2(\theta^s(t), \theta_*^s(c))}{2} dt.$$

If the curve  $\theta^s(t)$  is not contained in  $B(\theta_*^s, \delta^{1/5})$  on  $[t_1, t_2]$ , then there exists a time interval  $[t_3, t_4] \subset [t_1, t_2]$  such that  $d(\theta(t), \theta_*^s) > \delta^{1/5}/2$  on  $[t_3, t_4]$ ,  $d(\theta(t_3), \theta_*^s) = \delta^{1/5}/2 = d(\theta(t_4), \theta_*^s)$ , and  $\max_{t \in [t_3, t_4]} d(\theta(t), \theta_*^s) > \delta^{1/5}$ . We then have  $\int_{t_3}^{t_4} \|\dot{\theta}^s(t)\| dt \geq \delta^{1/5}$  hence

$$\begin{aligned} 40\sqrt{nD\varepsilon\delta/\lambda} &\geq \int_{t_1}^{t_2} \frac{\|\dot{\theta}^s(t)\|^2}{4D} + \lambda\varepsilon \frac{d^2(\theta^s(t), \theta_*^s(c))}{2} dt \\ &\geq \int_{t_3}^{t_4} \|\dot{\theta}^s(t)\|^2/(4D) + \lambda\varepsilon d^2(\theta^s(t), \theta_*^s(c))/2 dt \geq \frac{1}{4D(t_4 - t_3)} \left( \int_{t_3}^{t_4} \|\dot{\theta}^s(t)\| dt \right)^2 + \\ &\quad \lambda\varepsilon(t_4 - t_3)\delta^{2/5}/8 \geq \frac{1}{4D(t_4 - t_3)}\delta^{2/5} + \lambda\varepsilon\delta^{2/5}(t_4 - t_3)/8 \\ &\geq \frac{\sqrt{\lambda\varepsilon}}{8\sqrt{D}}\delta^{2/5} \end{aligned}$$

which is a contradiction when  $\delta$  is small enough with respect to  $n, D$  and  $\lambda$ . We have proved (17).  $\square$

We can now prove a better vertical localization of the set  $s\tilde{\mathcal{I}}(u, c)$  than was obtained in Lemma 4.7. On the domain  $\mathbb{T} \times B(\theta_*^s, \delta^{1/5}) \times \mathbb{T}$ , we have  $\hat{Z}_c \geq -\delta^{2/5}/2$ . We deduce from Lemma 4.6 that

$$10\delta^{1/5}\sqrt{nD\varepsilon} \geq u(\theta(t_2), t_2) - u(\theta(t_1), t_1) = \int_{t_1}^{t_2} L(\theta(t), \dot{\theta}(t), t) - c \cdot \dot{\theta}(t) + \alpha(c) dt$$

for each curve  $\theta : \mathbb{R} \rightarrow \mathbb{T}^n$  calibrated by  $u$  and each time interval  $[t_1, t_2]$ . We can chose the time interval  $[t_1, t_2]$  as a maximal excursion outside of  $\{\|p - c\| < \sqrt{\varepsilon}\delta^{1/16}/2\}$ . On  $[t_1, t_2]$ , we have  $\|\dot{\theta} - \partial_p H_0(c)\| \geq \sqrt{\varepsilon}\delta^{1/16}/5D$  (by Lemma 4.1) hence

$$L(\theta, \dot{\theta}, t) - c \cdot \dot{\theta} + \alpha(c) \geq \varepsilon\delta^{1/8}/100D^2 - 4\varepsilon\delta \geq \varepsilon\delta^{1/8}/200D.$$

We thus have

$$(t_2 - t_1)\varepsilon\delta^{1/8}/200D \leq 10\delta^{1/5}\sqrt{nD\varepsilon}$$

hence  $2\varepsilon(t_2 - t_1) \leq \sqrt{\varepsilon}\delta^{1/16}/2$  (if  $\delta$  is small enough). Since  $\|p\| \leq 2\varepsilon$  and  $\|p(t_1) - c\| = \sqrt{\varepsilon}\delta^{1/16}/2$ , we conclude that  $\|p(t) - c\| \leq \sqrt{\varepsilon}\delta^{1/16}$  on  $[t_1, t_2]$ . This ends the proof of Theorem 4.1.  $\square$

### 4.3 The Lipschitz constant

We prove Theorem 4.2. We will work here with weak KAM solutions rather than suspended weak KAM solutions. We recall the concept of semi-concave function on  $\mathbb{T}^n$ . A function  $u : \mathbb{T}^n \rightarrow \mathbb{R}$  is called  $K$ -semi-concave if the function

$$x \mapsto u(x) - K\|x\|^2/2$$

is concave on  $\mathbb{R}^n$ , where  $u$  is seen as a periodic function on  $\mathbb{R}^n$ . It is equivalent to require that, for each  $\theta \in \mathbb{T}^n$ , there exists a linear form  $l$  on  $\mathbb{R}^n$  such that the inequality

$$u(\theta + y) \leq u(\theta) + l \cdot y + K\|y\|^2/2$$

holds for each  $y \in \mathbb{R}^n$ . It is sufficient to check that, for each  $\theta$ , there exists  $l$  such that this inequality holds for  $\|y\| \leq 1$ . We will need the following regularity result of Fathi, see [Fa]:

**Lemma 4.8.** *Let  $u_1$  and  $u_2$  be  $K$ -semiconcave functions, and let  $\mathcal{I} \subset \mathbb{T}^n$  be the set of points where the sum  $u_1 + u_2$  is minimal. Then the functions  $u_1$  and  $u_2$  are differentiable at each point of  $\mathcal{I}$ , and the differential  $x \mapsto du_1(x)$  is  $6K$ -Lipshitz on  $\mathcal{I}$ .*

The Weak KAM solutions of cohomology  $c$  are the functions  $u : \mathbb{T}^n \rightarrow \mathbb{R}$  such that

$$u(\theta) := \min_{\gamma} \left( u(\gamma(0)) + \int_0^T L(\gamma(t), \dot{\gamma}(t), t) - c \cdot \dot{\gamma}(t) + \alpha(c) dt \right),$$

for each  $T \in \mathbb{N}$ , where the minimum is taken on the set of  $C^1$  curves  $\gamma : [0, T] \rightarrow \mathbb{T}^n$  satisfying the final condition  $\gamma(T) = \theta$ .

**Proposition 4.3.** *For each  $c \in \mathbb{R}^n$ , each Weak KAM solution  $u$  at cohomology  $c$  is  $3\sqrt{D\varepsilon}/2$ -semi-concave.*

*Proof.* Given  $T \in \mathbb{N}$  and  $\theta \in \mathbb{T}^n$ , there exists a curve  $\Theta : [0, T] \rightarrow \mathbb{T}^n$  such that  $\Theta(T) = \theta$  and which is calibrated by  $u$ , which means that

$$u(\theta) = u(\Theta(0)) + \int_0^T L(t, \Theta(t), \dot{\Theta}(t)) - c \cdot \dot{\Theta}(t) + \alpha(c) dt.$$

We assume that  $T \geq \varepsilon^{-1/2}$ , which implies by Lemma 4.7 that  $\|p(t) - c\| \leq C\sqrt{\varepsilon}$ , for a constant  $C$  independent of  $\varepsilon$  and  $\delta$ . We deduce that  $\|\dot{\Theta} - \partial_p H_0(c)\| \leq C\sqrt{\varepsilon}$  (with a higher constant  $C$ ) for each  $t \in [0, T]$ . We lift  $\Theta$  (and the point  $\theta = \Theta(T)$ ) to a curve in  $\mathbb{R}^n$  without changing its name, and consider, for each  $x \in \mathbb{R}^n$ , the curve

$$\Theta_x(t) := \Theta(t) + tx/T,$$

so that  $\Theta_x(T) = \theta + x$ . Each of the curves  $\Theta_x$ ,  $\|x\| \leq 1$ , satisfy  $\|\dot{\Theta}_x - \partial_p H_0(c)\| \leq C\sqrt{\varepsilon} \leq \varepsilon^{1/3}$  (provided  $\varepsilon$  is small enough). We have the inequality

$$u(\theta + x) - u(\theta) \leq \int_0^T L(\Theta_x(t), \dot{\Theta}_x(t), t) - L(\Theta(t), \dot{\Theta}(t), t) - c \cdot x/T dt.$$

Use Lemma 4.2, we get

$$\begin{aligned} L(\Theta_x(t), \dot{\Theta}_x(t), t) &\leq L(\Theta(t), \dot{\Theta}(t), t) \\ &\quad + \partial_{\theta} L(\Theta(t), \dot{\Theta}(t), t) \cdot tx/T + \partial_v L(\Theta(t), \dot{\Theta}(t), t) \cdot x/T \\ &\quad + 3\varepsilon|tx/T|^2/2 + 2D\varepsilon t|x/T|^2 + D|x/T|^2/2. \end{aligned} \tag{18}$$

Using the Euler-Lagrange equation and integrating by parts, we conclude that

$$u(\theta + x) - u(\theta) \leq (c + \partial_v L(T, \Theta(T), \dot{\Theta}(T))) \cdot x + (\varepsilon T/2 + D\varepsilon + D/2T)|x|^2$$

for each  $T \in \mathbb{N}$ ,  $T \geq \varepsilon^{-1/2}$ . Taking  $T \in [\sqrt{D/\varepsilon}, \sqrt{2D/\varepsilon}]$ , we obtain

$$u(\theta + x) - u(\theta) \leq (c + \partial_v L(T, \Theta(T), \dot{\Theta}(T))) \cdot x + 3\sqrt{D\varepsilon}|x|^2/2$$

for each  $x \in \mathbb{R}^n$ ,  $\|x\| \leq 1$ . This ends the proof of the semi-concavity.  $\square$

*Proof of Theorem 4.2.* Let  $u$  be a weak KAM solution, and let  $\tilde{u}$  be the conjugated dual weak KAM solution. Then the set  $\tilde{\mathcal{I}}(u, c)$  can be characterized as follows: Its projection  $\mathcal{I}(u, c)$  on  $\mathbb{T}^n$  is the set where  $u = \tilde{u}$ , and

$$\tilde{\mathcal{I}}(u, c) = \{(x, c + du(x)), x \in \mathcal{I}(u, c)\}.$$

Since  $-\tilde{u}$  is semi-concave, it is a consequence of Lemma 4.8 that the differential  $du(x)$  exists for  $x \in \mathcal{I}(u, c)$ . Moreover, we can prove exactly as in Proposition 4.3 that  $-\tilde{u}$  is  $3D\varepsilon/2$ -semi-concave. Lemma 4.8 then implies that the map  $x \mapsto du(x)$  is  $9\sqrt{D\varepsilon}$ -Lipschitz on  $\mathcal{I}(u, c)$ .  $\square$

#### 4.4 Double peak case

We now localize the Aubry and Mañé sets in the more general case where  $(HZ\lambda)$  is replaced by:

$$\hat{Z}_c(\theta^s) \leq -\lambda(\min\{d(\theta^s - \theta_1^s), d(\theta^s - \theta_2^s)\})^2.$$

It is natural to relax  $(HZ\lambda)$  in this way because, for a generic family of functions  $\hat{Z}_c, c \in \Gamma$ , there exist values of  $c$  for which  $\hat{Z}_c$  has two degenerate maxima. Note that Theorem 4.2 is still valid in this case, its proof does not use  $(HZ\lambda)$ . On the other hand, Theorem 4.1 is replaced by:

**Theorem 4.4.** *If  $\delta > 0$  is small enough with respect to  $n, D, \lambda$  and if  $\varepsilon$  is small enough with respect to  $n, D, \lambda, \delta$ , then the Aubry set at cohomology  $c$  of the Hamiltonian  $N_\varepsilon$  satisfies*

$$s\tilde{\mathcal{A}}(c) \subset (B(\theta_1^s, \delta^{1/5}) \cup B(\theta_2^s, \delta^{1/5})) \times \mathbb{T} \times B(c, \sqrt{\varepsilon}\delta^{1/16}) \times \mathbb{T} \subset \mathbb{T}^{n-1} \times \mathbb{T} \times \mathbb{R}^n \times \mathbb{T}.$$

*If, moreover, the projection  $\theta^s(s\mathcal{A}(c)) \subset \mathbb{T}^{n-1}$  is contained in one of the (disjoint) balls  $B(\theta_i^s, \delta^{1/5})$ , then the projection  $\theta^s(s\mathcal{N}(c)) \subset \mathbb{T}^{n-1}$  of the Mañé set is contained in the same ball  $B(\theta_i^s, \delta^{1/5})$ .*

*Proof.* We assume that  $\theta_1^s \neq \theta_2^s$ , and that  $\delta$  is small enough for the balls  $B(\theta_i^s, 2\delta^{1/5})$  to be disjoint. We first show that

$$\theta^s(s\mathcal{A}(c)) \subset B(\theta_1^s, \delta^{1/5}) \cup B(\theta_2^s, \delta^{1/5}).$$

As in the single peak case, we set  $r_1 = 4\sqrt{\delta/\lambda}$ , and observe that

$$L(\theta, v, t) - c \cdot v - \alpha(c) \geq \|v^s\|^2/4D + \lambda\varepsilon(\min\{d(\theta^s - \theta_1^s), d(\theta^s - \theta_2^s)\})^2/2$$

for  $\theta^s \notin B(\theta_1^s, r_1) \cup B(\theta_2^s, r_1)$ . The  $\theta^s$  component of each orbit of the Aubry set spends a finite amount of time outside of  $B(\theta_1^s, r_1) \cup B(\theta_2^s, r_1)$ . There are four type of excursions that the orbits of  $\mathcal{A}(c)$  can perform outside of this union : From  $B(\theta_i^s, r_1)$  to  $B(\theta_j^s, r_1)$  for  $i \in \{1, 2\}$  and  $j \in \{1, 2\}$ . Exactly as in the single pick case, the orbits segments connecting  $B(\theta_i^s, r_1)$  to

itself are contained in  $B(\theta_i^s, \delta^{1/5})$ . So the claim holds, provided there exists no orbit segment in  $s\mathcal{A}(c)$  connecting  $B(\theta_i^s, r_1)$  to  $B(\theta_j^s, r_1)$  with  $i \neq j$ .

Assume for example that there exists an orbit segment  $\theta(t) : [t_1, t_2] \rightarrow \mathbb{T}^n$  connecting  $B(\theta_1^s, r_1)$  to  $B(\theta_2^s, r_1)$ . Then, given any suspended weak KAM solution  $u$ , the same action estimates as in the single peak case imply that

$$u(\theta(t_2), t_2) - u(\theta(t_1), t_1) \geq \frac{\sqrt{\lambda\varepsilon}}{8\sqrt{D}}\delta^{2/5}.$$

Since the Aubry set is chain recurrent, there must exist an orbit segment  $\check{\theta}(t) : [\check{t}_1, \check{t}_2] \rightarrow \mathbb{T}^n$  connecting  $B(\theta_2^s, r_1)$  to  $B(\theta_1^s, r_1)$ , and we have

$$u(\check{\theta}(\check{t}_2), \check{t}_2) - u(\check{\theta}(\check{t}_1), \check{t}_1) \geq \frac{\sqrt{\lambda\varepsilon}}{8\sqrt{D}}\delta^{2/5}.$$

By using Lemma 4.6 with  $\Omega = B(\theta_1^s, r_1)$  and  $\Omega = B(\theta_2^s, r_1)$ , we get that

$$u(\check{\theta}(\check{t}_2), \check{t}_2) - u(\theta(t_1), t_1) \leq 40\sqrt{nD\varepsilon\delta/\lambda} \quad \text{and} \quad u(\theta(t_2), t_2) - u(\check{\theta}(\check{t}_1), \check{t}_1) \leq 40\sqrt{nD\varepsilon\delta/\lambda}.$$

All these inequalities together imply that

$$40\sqrt{nD\varepsilon\delta/\lambda} \geq \frac{\sqrt{\lambda\varepsilon}}{8\sqrt{D}}\delta^{2/5},$$

which does not hold if  $\delta$  is small enough. This contradiction proves that no excursion connecting  $B(\theta_1^s, r_1)$  to  $B(\theta_2^s, r_1)$  can exist in the Aubry set. Note that we have used the chain recurrence of the Aubry set, and that the conclusion does not in general apply to the Mañé set. We have proved that

$$s\mathcal{A}(c) \subset \mathbb{T} \times (B(\theta_1^s, \delta^{1/5}) \cup B(\theta_2^s, \delta^{1/5})) \times \mathbb{T}.$$

The vertical part of the localisation follows exactly as in the single peak case.

In general, such a localization does not hold for the Mañé set, which may contain connections from one of the regions  $\mathbb{T} \times B(\theta_i^s, \delta^{1/5}) \times \mathbb{T}$  to the other (but, in view of the calculations above, not in both direction). If such a connection exists, then its  $\alpha$ -limit is contained in one of the domains  $\mathbb{T} \times B(\theta_i^s, \delta^{1/5}) \times \mathbb{T}$ , say  $\mathbb{T} \times B(\theta_1^s, \delta^{1/5}) \times \mathbb{T}$ , and its  $\omega$ -limit is contained in the other domain  $\mathbb{T} \times B(\theta_2^s, \delta^{1/5}) \times \mathbb{T}$ . Recalling that the  $\alpha$  and  $\omega$  limits of the Mañé set are contained in the Aubry set, we conclude that each of the intersections

$$s\mathcal{A}(c) \cap (\mathbb{T} \times B(\theta_i^s, \delta^{1/5}) \times \mathbb{T})$$

is non empty. This proves the last part of the statement □

## 5 Nondegeneracy of the barrier functions

In this section we prove:

**Theorem 5.1.** *In the context of Theorem 1.5, by possibly taking a smaller  $\delta_0$ , for a residue set of  $R \in \mathcal{R} = \mathcal{R}(r, \varepsilon, \delta_0)$  the following hold: for any  $c \in \Gamma_1$  such that  $\rho(c)$  is irrational and  $\theta^f(\mathcal{N}_N(c)) = \mathbb{T}$ , the set  $\tilde{\mathcal{N}}_{N \circ \Xi}(\xi^*c) - \Xi^{-1}(\tilde{\mathcal{N}}_N(c))$  is totally disconnected.*

This is a delicate perturbation problem, and a version of it for *a priori unstable* systems appeared in [CY2] and was discussed in [Mag]. In this section we give a self-contained proof with many new ingredients.

## 5.1 Outline of the proof

In this section we prove Theorem 5.1 assuming some statements to be proven in later subsections. Let  $L$  denote the Lagrangian associated to  $N$ .

- We define  $\mathcal{R}_1 \subset \mathcal{R}(r, \varepsilon, \delta)$  to be the set of  $R$  such that  $\theta^f(\mathcal{N}_N(c)) \neq \mathbb{T}$  whenever  $\rho^f(c)$  is rational. The set  $\mathcal{R}_1$  is a residue subset of  $\mathcal{R}$ . We also abuse notations and denote by  $\mathcal{R}_1$  the set of Hamiltonians of the form  $N = H_0 + \varepsilon Z + \varepsilon R$ ,  $R \in \mathcal{R}_1$ .

- We define

$$\Gamma_*(N) = \{c \in \Gamma_1 : \theta^f(\mathcal{N}_N(c)) = \mathbb{T}\},$$

according to the previous item, for  $N \in \mathcal{R}_1$  and  $c \in \Gamma_*(N)$ , we necessarily have  $\rho^f(c)$  irrational. In particular,  $\mathcal{A}_N(c) = \mathcal{N}_N(c)$  contains a unique static class. In view of the upper semi-continuity of the Mañé set,  $\Gamma_*(N)$  is a compact subset of  $\Gamma_1$ .

- If  $N \in \mathcal{R}_1$  and  $c \in \Gamma_*(N)$ , then the Aubry set  $\tilde{\mathcal{A}}_{N \circ \Xi}(\xi^*c) = \Xi^{-1}\tilde{\mathcal{A}}_N(c)$  contains exactly two static classes denoted  $\tilde{\mathcal{S}}_1, \tilde{\mathcal{S}}_2$  (with projections  $\mathcal{S}_1, \mathcal{S}_2$ ). Then the Mañé set is the disjoint union

$$\tilde{\mathcal{N}}_{N \circ \Xi}(\xi^*c) = \tilde{\mathcal{S}}_1 \cup \tilde{\mathcal{S}}_2 \cup \tilde{\mathcal{H}}_{12} \cup \tilde{\mathcal{H}}_{21}, \quad (19)$$

where  $\tilde{\mathcal{H}}_{12}$  (and  $\tilde{\mathcal{H}}_{21}$ ) is the set of heteroclinic orbits from  $\tilde{\mathcal{S}}_1$  to  $\tilde{\mathcal{S}}_2$  (and vice versa). Projections are denoted  $\mathcal{H}_{12}, \mathcal{H}_{21}$ . Note that  $\tilde{\mathcal{N}}_{N \circ \Xi}(\xi^*c) - \Xi^{-1}\tilde{\mathcal{N}}_N(c) = \tilde{\mathcal{H}}_{12} \cup \tilde{\mathcal{H}}_{21}$ . We will also use the notations  $\tilde{\mathcal{S}}_i(N, c)$  and  $\tilde{\mathcal{H}}_{ij}(N, c)$  when discussing the dependence on  $N, c$ .

- For  $N \in \mathcal{R}_1$  and  $c \in \Gamma_*(N)$ , the static classes  $\tilde{\mathcal{S}}_1, \tilde{\mathcal{S}}_2$  determine two elementary forward and two backward weak KAM solutions

$$h(\zeta_1, \cdot), h(\zeta_2, \cdot), \quad h(\cdot, \zeta_1), h(\cdot, \zeta_2), \quad \zeta_i \in \mathcal{S}_i, i = 1, 2,$$

where the barrier functions are evaluated for  $N \circ \Xi$  and  $\xi^*c$ . The associated pseudographs are denoted  $\mathcal{E}_i(N, c)$  and  $\tilde{\mathcal{E}}_i(N, c)$ ,  $i = 1, 2$  respectively, they do not depend of the choices of points  $\zeta_1 \in \mathcal{S}_1, \zeta_2 \in \mathcal{S}_2$ . Define

$$b_{N,c}^-(\theta) = h(\zeta_1, \theta) + h(\theta, \zeta_2) - h(\zeta_1, \zeta_2)$$

and  $b_{N,c}^+$  similarly defined with  $\zeta_1, \zeta_2$  switched. The functions  $b_{N,c}^\pm$  do not depend on the choice of points  $\zeta_1 \in \mathcal{S}_1, \zeta_2 \in \mathcal{S}_2$ , they are non-negative, and vanish, respectively, on  $\mathcal{H}_{12} \cup \mathcal{S}_1 \cup \mathcal{S}_2$  and  $\mathcal{H}_{21} \cup \mathcal{S}_1 \cup \mathcal{S}_2$ .

Given  $\underline{c} \in \Gamma_1$ , we consider the compact subset  $\underline{\mathcal{K}} \subset \mathbb{T}^n$  formed by points  $\theta$  such that  $d(\theta^s(\underline{c}), \theta^s) \geq 1/10$ . There exists  $\sigma > 0$  such that the Mañé set  $\mathcal{N}(N, c)$  is disjoint from  $\underline{\mathcal{K}}$  for each  $c \in \Gamma_1 \cap \mathcal{B}_\sigma(\underline{c})$  and  $N \in \mathcal{R}(r, \varepsilon, \delta_0)$ . The compact set  $\mathcal{K} = \xi^{-1}(\underline{\mathcal{K}})$  ( $\xi$  is the double covering) is then disjoint from  $\mathcal{A}_{N \circ \Xi}(\xi^*c)$ . Moreover, for these  $N$  and  $c$ , the set  $\pi^{-1}(\mathcal{K})$  intersects each orbit of  $\tilde{\mathcal{N}}_{N \circ \Xi}(\xi^*c) - \tilde{\mathcal{A}}_{N \circ \Xi}(\xi^*c)$ .

Since the compact interval  $\Gamma_1$  is the union of finitely compact segments, each contained in a ball of the form  $B_\sigma(\underline{c})$ , it suffices to prove Theorem 5.1 for each segment. Therefore, we can assume without loss of generality that  $\Gamma_1$  is actually contained in one of these balls. Then, there exists a compact set  $\mathcal{K}$  such that

- For each  $c \in \Gamma_1$  and  $N \in \mathcal{R}(r, \varepsilon, \delta_0)$ ,  $\mathcal{K}$  is disjoint from  $\mathcal{A}_{N \circ \Xi}(\xi^*c)$  and  $\pi^{-1}(\mathcal{K})$  intersects each orbit of  $\tilde{\mathcal{N}}_{N \circ \Xi}(\xi^*c) - \tilde{\mathcal{A}}_{N \circ \Xi}(\xi^*c)$ .

We make this additional assumption for the sequel of the section.



**Lemma 5.1.** *For each  $(N, c) \in \mathcal{R}_1 \times \Gamma_1$ , the set  $\tilde{\mathcal{N}}_{N \circ \Xi}(\xi^* c) - \Xi^{-1} \tilde{\mathcal{N}}_N(c)$  is totally disconnected if and only if the set*

$$\mathcal{N}_{N \circ \Xi}(\xi^* c) \cap \mathcal{K} = (\mathcal{H}_{12}(N, c) \cup \mathcal{H}_{21}(N, c)) \cap \mathcal{K}$$

*is totally disconnected.*

*Proof.* The set  $\mathcal{N}_{N \circ \Xi}(\xi^* c) \cap \mathcal{K}$  is a compact metric space, so it is totally disconnected if and only if it has topological dimension zero, see [HW]. Assuming that this property holds, The set  $\tilde{\mathcal{N}}_{N \circ \Xi}(\xi^* c) \cap \pi^{-1}(K)$  is the disjoint union of two homeomorphic copies of  $\mathcal{N}_{N \circ \Xi}(\xi^* c) \cap \mathcal{K}$ , hence it is compact and of zero topological dimension. As a consequence, each of the sets  $\phi^k(\mathcal{N}_{N \circ \Xi}(\xi^* c) \cap \mathcal{K})$ ,  $k \in \mathbb{Z}$  is compact and of zero topological dimension, where  $\phi^k$  is the time  $k$  Hamiltonian flow of  $N$ . The countable union

$$\tilde{\mathcal{N}}_{N \circ \Xi}(\xi^* c) - \tilde{\mathcal{A}}_{N \circ \Xi}(\xi^* c) = \bigcup_{k \in \mathbb{Z}} \phi^k(\mathcal{N}_{N \circ \Xi}(\xi^* c) \cap \mathcal{K})$$

is then also of zero dimension. As a consequence the projection  $\mathcal{N}_{N \circ \Xi}(\xi^* c) - \mathcal{A}_{N \circ \Xi}(\xi^* c)$  is of zero topological dimension, hence it is totally disconnected.  $\square$

We want to prove that a dense  $G_\delta$  of Hamiltonians  $N \in \mathcal{R}_1$  have the property that  $\mathcal{N}_{N \circ \Xi}(\xi^* c) \cap \mathcal{K}$  is totally disconnected for each  $c \in \Gamma_*(N)$ . The  $G_\delta$  part follows from the next Lemma.

**Lemma 5.2.** *Let  $J \subset \Gamma_1$  and  $K \subset \mathbb{T}^n$  be compact subsets, then the set of  $R \in \mathcal{R}$  such that all  $c \in (J \cap \Gamma_*(N))$  satisfies*

$$Q(N, c, K) := \mathcal{N}_{N \circ \Xi}(\xi^* c) \cap K \tag{20}$$

*is totally disconnected is a  $G_\delta$  set.*

*Proof.* Consider  $N$  satisfying the conditions of the lemma, then for each  $c \in (J \cap \Gamma_*(N))$ ,  $Q(N, c, K)$  is compact and totally disconnected, and hence has zero topological dimension.

Let's call a compact subset  $1/k$  disconnected if it admits a finite disjoint covering by compact subsets of diameter at most  $1/k$ . If  $N$  satisfies the conditions of the Lemma, then  $\mathcal{N}_{N \circ \Xi}(\xi^* c) \cap K$  is  $1/k$  disconnected for each  $k \in \mathbb{N}$  and each  $c \in (J \cap \Gamma_*(N))$ . Since the Mañé set is upper semi-continuous in the Hamiltonian (in the  $C^2$  topology), so is  $\mathcal{N}_{N \circ \Xi}(\xi^* c) \cap K$  and we have, for each fixed  $k$ :

There exists an open set  $\Gamma'$  containing  $\Gamma_*(N) \cap J$  and a neighborhood  $\mathcal{U}$  of  $N$  in  $C^2$  such that the set  $\mathcal{N}_{N' \circ \Xi}(\xi^* c') \cap K$  is  $1/k$  disconnected for all  $c' \in \Gamma'$  and  $N' \in \mathcal{U}$ .

We now use the observation that  $\Gamma_*(N)$  is upper semi-continuous in  $N$ , hence so is  $J \cap \Gamma_*(N)$  since  $J$  is compact. We deduce the existence of a smaller neighborhood  $\mathcal{U}' \subset \mathcal{U}$  of  $N$ , such that  $J \cap \Gamma_*(N') \subset \Gamma'$  for each  $N' \in \mathcal{U}'$ . We have proved: the property that  $\mathcal{N}_{N \circ \Xi}(\xi^* c) \cap K$  is  $1/k$  disconnected for each  $c \in \Gamma_*(N) \cap J$  is  $C^2$  open (and hence  $C^r$  open). The Lemma follows by taking the intersection on  $k$ .  $\square$

We now adress the density part. Let us consider the product space  $C^r(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T}) \times \mathbb{R}^n$  with the standard norms on both spaces. Define the following subset

$$\mathcal{Q} = \{(N, c) : N \in \mathcal{R}_1, c \in \Gamma_*(N)\} \subset \mathcal{R} \times \Gamma_1 \subset C^r(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T}) \times \mathbb{R}^n.$$

The following proposition allows us to perturb the function  $b_{N,c}^\pm$  locally simultaneously for an open set of  $c$ . The proof is given in section 5.2.

**Proposition 5.2.** *Let  $(N_0, c_0) \in \mathcal{Q}$  and  $K \subset \mathbb{T}^n$  be a compact set disjoint from  $\mathcal{A}_{N_0 \circ \Xi}(\xi^* c_0)$ . Then there exists  $\sigma > 0$  such that for all  $N \in \mathcal{R}_1 \cap B_\sigma(N_0)$ ,  $\theta_0 \in K \cap \mathcal{H}_{12}(N_0, c_0)$ , and  $\varphi \in C_c^r(B_\sigma(\theta_0))$  with  $\|\varphi\|_{C^r} < \sigma$ , there exists a Hamiltonian  $N_\varphi$  such that:*

1. *For all  $c \in B_\sigma(c_0)$ , the Aubry set  $\tilde{\mathcal{A}}_{N_\varphi \circ \Xi}(\xi^* c)$  coincides with  $\tilde{\mathcal{A}}_{N \circ \Xi}(\xi^* c)$ , with the same static classes. In particular,  $B_\sigma(c_0) \cap \Gamma_*(N) = B_\sigma(c_0) \cap \Gamma_*(N_\varphi)$ .*
2. *For all  $c \in B_\sigma(c_0) \cap \Gamma_*(N)$ , there exists a constant  $e \in \mathbb{R}$  such that*

$$b_{N_\varphi, c}^+(\theta) = b_{N, c}^+(\theta) + \varphi(\theta) + e, \quad \theta \in B_\sigma(\theta_0). \quad (21)$$

*The same holds for  $\theta_0 \in K \cap \mathcal{H}_{21}(N_0, c_0)$ , with  $b^+$  replaced with  $b^-$  in (21). Moreover, for each  $N \in \mathcal{R}_1 \cap B_\sigma(N_0)$ ,  $\|N_\varphi - N\|_{C^r} \rightarrow 0$  when  $\|\varphi\|_{C^r} \rightarrow 0$ .*

We will use Proposition 5.2 to perturb all barrier functions near a given  $c_0$  simultaneously. Because we are perturbing an uncountable family of functions, we need an additional information on how the functions  $b_{N, c}^\pm$  depends on  $c$ . The proof is given in Section 5.3.

**Proposition 5.3.** *For each  $N \in \mathcal{R}_1$ , the maps  $c \mapsto b_{N, c}^+, b_{N, c}^-$  are 1/2-Hölder from  $\Gamma_*(N)$  to  $C^0(\mathbb{T}^n, \mathbb{R})$ .*

This regularity implies that the set  $\{b_{N, c}^\pm, c \in \Gamma_*(N)\}$  is compact and has Hausdorff dimension at most 2 in  $C^0(\mathbb{T}^n, \mathbb{R})$ . The following Lemma will allow to take advantage of this fact:

**Lemma 5.3.** *Let  $\mathcal{F} \subset C^0([-1, 1]^n, \mathbb{R})$  be a compact set of finite Hausdorff dimension. The following property is satisfied on a residue set of functions  $\varphi \in C^r(\mathbb{R}^n, \mathbb{R})$  (with the uniform  $C^r$  norm):*

*For each  $f \in \mathcal{F}$ , the set of minima of the function  $f + \varphi$  on  $[-1, 1]^n$  is totally disconnected.*

*As a consequence, for each open neighborhood  $\Omega$  of  $[-1, 1]^n$  in  $\mathbb{R}^n$ , there exists arbitrarily  $C^r$ -small compactly supported functions  $\varphi : \Omega \rightarrow \mathbb{R}$  satisfying this property.*

*Proof.* We first consider the case  $n = 1$ . The set  $\tilde{\mathcal{F}} = \{c - f, f \in \mathcal{F}, c \in \mathbb{R}\}$  is compact and of finite Hausdorff dimension (one more than the dimension of  $\mathcal{F}$ ). For each compact subinterval  $J \subset [-1, 1]$ , the set  $\tilde{\mathcal{F}}_J \subset C(J, \mathbb{R})$  is also compact and finite dimensional, since the restriction map is Lipschitz. If  $J$  is non trivial, the complement

$$\Phi(J) := C^r(\mathbb{R}, \mathbb{R}) - (\tilde{\mathcal{F}}_J \cap C^r(\mathbb{R}, \mathbb{R}))$$

is open and dense in  $C^r(\mathbb{R}, \mathbb{R})$ . To prove density, we consider a subspace  $H \subset C^r(\mathbb{R}, \mathbb{R})$  of finite dimension larger than the Hausdorff dimension of  $\tilde{\mathcal{F}}$ . We moreover assume that all functions of  $H$  are compactly supported inside the interior of  $J$ . Given  $\varphi \in C^r(\mathbb{R}, \mathbb{R})$ , we consider the affine space  $\varphi + H$ . Considering the  $C^0([-1, 1], \mathbb{R})$  distance, the Hausdorff dimension of  $\tilde{\mathcal{F}}_J \cap (\varphi + H)$  is not greater than the Hausdorff dimension of  $\tilde{\mathcal{F}}$ , hence it is less than the dimension of  $H$ . This implies that the complement  $(\varphi + H) - \tilde{\mathcal{F}}$  is dense in  $\varphi + H$  endowed with the  $C^0$  distance. Since the  $C^0$  and  $C^r$  norms are equivalent on the finite dimensional space  $\varphi + H$ , we conclude that  $\varphi$  belongs to the closure of  $\Phi(J)$  in  $C^r(\mathbb{R}, \mathbb{R})$ .

Let  $J_k$  be a sequence of compact subintervals of  $[-1, 1]$  such that each open interval contains one of the  $J_k$ . Then if  $\varphi \in \cap_k \Phi(J_k)$  (this intersection is a dense  $G_\delta$ ), each of the functions  $f + \varphi, f \in \mathcal{F}$  has the property that it is not constant on any open interval, hence its set of minima in  $[-1, 1]$  is totally disconnected.

Let us now turn to the general case. We denote by  $\pi_i : [-1, 1]^n \rightarrow [-1, 1]$  the projections on the factors. We associate to each function  $f \in C^0([-1, 1]^n, \mathbb{R})$  the functions

$$f_i : [-1, 1] \ni x_i \mapsto f_i(x_i) = \min_{\pi_i(x)=x_i} f(x).$$

For each  $k$  and  $i$ , the following property holds on an open and dense subset of functions  $\varphi \in C^r(\mathbb{R}^n, \mathbb{R})$ : None of the functions  $(f + \varphi)_i, f \in \mathcal{F}$  is constant on  $J_k$ .

To prove density, we consider a function  $\varphi \in C^r(\mathbb{R}^n, \mathbb{R})$ . The map  $f \mapsto f_i$  is Lipschitz hence the set  $\mathcal{F}_i(\varphi) = \{(f + \varphi)_i, f \in \mathcal{F}\} \subset C^0([-1, 1], \mathbb{R})$  is compact and has finite Hausdorff dimension. We can apply the result for  $n = 1$  to this family and obtain that for generic  $\varphi_1 \in C^r(\mathbb{R}, \mathbb{R})$ , none of the functions

$$(f + \varphi)_i + \varphi_i = (f + \varphi + \varphi_i)_i$$

for  $f \in \mathcal{F}$  is constant on the interval  $J_k$ .

By taking the intersection on  $n$  and  $k$ , we obtain that, for generic  $\varphi \in C^r(\mathbb{R}^n, \mathbb{R})$ , each of the functions  $(f + \varphi)_i$  has a totally disconnected set of minima in  $[-1, 1]$ .

Since  $\pi_i(\operatorname{argmin}(f + \varphi)) \subset \operatorname{argmin}(f + \varphi)_i$ , this implies that  $\operatorname{argmin}(f + \varphi)$  is totally disconnected.  $\square$

*Proof of Theorem 5.1.* Let  $\mathcal{R}_2 \subset \mathcal{R}_1$  be the set of Hamiltonians  $N$  which have the property that  $\mathcal{N}_{N \circ \Xi}(\xi^* c) \cap \mathcal{K}$  is totally disconnected for each  $c \in \Gamma_*(N)$ .

By Lemma 5.1, it is enough to prove that  $\mathcal{R}_2$  is a dense  $G_\delta$ . By Lemma 5.2,  $\mathcal{R}_2$  is a  $G_\delta$ , we have to prove density.

Let us fix  $N_0 \in \mathcal{R}_1$ . For each  $\theta_0 \in \mathcal{N}_{N_0 \circ \Xi}(\xi^* c) \cap \mathcal{K}$ , we consider  $\sigma > 0$  small enough so that Proposition 5.2 applies. We define the cube

$$D_\sigma(\theta_0) = \{\theta : \max_i |\theta^i - \theta_0^i| \leq \sigma / (2\sqrt{n})\} \subset B_\sigma(\theta_0).$$

In view of Proposition 5.3, we can apply Lemma 5.3 to the family of functions  $b_{N,c}^\pm, c \in \Gamma_1 \cap \Gamma_*(N)$  on the cube  $D_\sigma(\theta_0)$  for each  $N \in \mathcal{R}_1$ . We find arbitrarily small functions  $\varphi$  compactly supported in  $B_\sigma(\theta_0)$  and such that each of the functions  $b_{N,c}^\pm + \varphi, c \in \Gamma_*(N) \cap \Gamma_1$  have a totally disconnected set of minima in  $D_\sigma(\theta_0)$ . If  $N \in \mathcal{R}_1 \cap B_\sigma(N_0)$ , we can apply Proposition 5.2 to get Hamiltonians  $N_\varphi$  approximating  $N$ . We obtain:

- The set of Hamiltonians  $N$  such that  $\mathcal{N}_{N \circ \Xi}(\xi^* c) \cap D_\sigma(\theta_0)$  is totally disconnected for each  $c \in \Gamma_*(N)$  is dense in  $\mathcal{R}_1 \cap B_\sigma(N_0)$ . By Lemma 5.2, it is a  $G_\delta$ .

Since  $\mathcal{K}$  is compact, there is a finite cover  $\mathcal{K} \subset \bigcup_{i=1}^k D_{\sigma_i}(\theta_i)$ , such that the above can be applied on each  $D_{\sigma_i}(\theta_i)$  some constant  $\sigma_i > 0$ . For  $\sigma_0 = \min \sigma_i > 0$ , we obtain:

- For a residue set of  $N \in B_{\sigma_0}(N_0)$ , the set  $\mathcal{N}_{N \circ \Xi}(\xi^* c) \cap D_{\sigma_i}(\theta_i)$  is totally disconnected for all  $i = 1, \dots, k$  and  $c \in \Gamma_*(N)$ .

Taking the intersection over  $i$ , we obtain :

- For a residue set of  $N \in B_{\sigma_0}(N_0)$ , the set  $\mathcal{N}_{N \circ \Xi}(\xi^* c) \cap \mathcal{K}$  is totally disconnected for all  $c \in \Gamma_*(N)$ .

In particular,  $N_0$  is in the closure of  $\mathcal{R}_2$ .  $\square$

## 5.2 Perturbing the Peierls' barrier functions

Let  $L$  be the Lagrangian for  $N = H_0 + \varepsilon Z + \varepsilon R$ . We define the generating function  $\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  by

$$G_N(x, x') = \min \int_0^1 L(\gamma, \dot{\gamma}, t) dt, \quad \gamma(0) = x, \quad \gamma(1) = x'.$$

Note that  $G_N(x+k, x'+k) = G_L(x, x')$  for all  $x, x' \in \mathbb{R}^n$  and  $k \in \mathbb{Z}^n$ . If  $\varepsilon$  is sufficiently small, there is a one-to-one correspondence between the time-1 map of the Euler-Lagrange flow of  $L$ , and the generating function  $G$ . We will also consider the generating function of the Hamiltonian  $N \circ \Xi$  (pull back of the double covering), which satisfies

$$G_{N \circ \Xi}(x, x') = G_N(\xi x, \xi x'), \quad (22)$$

where we have lifted  $\xi$  to a map  $\mathbb{R}^n \rightarrow \mathbb{R}^n$ . It is important to keep in mind that  $G_{N \circ \Xi}$  has an additional symmetry  $G_{N \circ \Xi}(x, x') = G_{N \circ \Xi}(x + \frac{1}{2}e_1, x' + \frac{1}{2}e_1)$  where  $e_1 = (1, 0, \dots, 0)$ , corresponding to the deck transformation of  $\xi$ . We also denote

$$A_{N,c}^M(\theta_1, \theta_2) = \min \int_0^M L(\gamma, \dot{\gamma}, t) - c \cdot \dot{\gamma} + \alpha_N(c) dt, \quad \gamma(0) = \theta_1, \quad \gamma(M) = \theta_2 \in \mathbb{T}^n,$$

and note that  $A_{N,c}^M$  and therefore  $h_{N,c}$  is completely determined by  $G_N$ . We will perturb the barrier functions by perturbing  $G_N$ .

Let  $U, V \subset \mathbb{R}^n$  be open sets which projects injectively to  $\mathbb{T}^n$ , namely  $U \cap (U+k) = \emptyset$  for all  $k \in \mathbb{Z}^d$ . We define a *perturbation block* to be the set

$$\mathcal{B}_N(U, V) := \phi_N(U \times \mathbb{R}^n) \cap (V \times \mathbb{R}^n) \subset \mathbb{R}^n \times \mathbb{R}^n,$$

in other words, the set of  $(\theta, p)$  such that  $\theta \in V$  and  $\pi_\theta \Phi_N^{-1}(\theta, p) \in U$ , where  $\phi_N$  is the time-1-map of the Hamiltonian  $N$ . We can also consider  $\mathcal{B}_N$  as a subset of  $\mathbb{T}^n \times \mathbb{R}^n$  since  $V$  projects injectively to  $\mathbb{T}^n$ .

Given  $U_1 \subset U_2 \subset \mathbb{R}^n$  and  $V \subset \mathbb{R}^n$  as before, for  $\varphi \in C_c^r(V)$ , we define a perturbation of the generating function (depending on  $\varphi, U_1, U_2, V$ ) as follows:

$$G_\varphi(x, x') = G_N(x, x') + \rho(x)\varphi(x'), \quad (23)$$

and extends it by periodicity  $G_\varphi(x+k, x'+k) = G_\varphi(x, x')$  for all  $k \in \mathbb{Z}^n$ . Here  $\rho : \mathbb{R}^n \rightarrow \mathbb{R}^+ \cup \{0\}$  is a standard mollifier function such that

$$\rho|_{U_1} = 1, \quad \rho|_{(U_2)^c} = 0.$$

**Lemma 5.4.** *When  $\|\varphi\|_{C^r}$  is small enough, there exists a Tonelli Hamiltonian  $N_\varphi$  whose generating function is equal to  $G_\varphi$ . Moreover,  $\|N_\varphi - N\|_{C^r} \rightarrow 0$  as  $\|\varphi\|_{C^r} \rightarrow 0$ .*

*Proof.* Let  $g(x, x') = \rho(x)\varphi(x')$ , extended by periodicity, then  $\|g\|_{C^r} \leq C\|\varphi\|_{C^r}$  for some  $C > 0$  depending on  $\rho$ . Let  $G_t(x, x')$  be the generating function of the time- $t$  map of the Hamiltonian  $N$ , we consider the following functions

$$G'_t(x, x') = G_t(x, x') + s(t)g(x, x'),$$

where  $s : [0, 1] \rightarrow [0, 1]$  is a  $C^\infty$  mollifier function with  $s(t) = 0$  on  $[0, \frac{1}{3}]$  and  $s(t) = 1$  on  $[\frac{2}{3}, 1]$ . When  $\|g\|_{C^2}$  is small enough, the functions  $G'_t$  uniquely determines exact symplectic maps  $\psi_t : \mathbb{T}^n \times \mathbb{R}^n \rightarrow \mathbb{T}^n \times \mathbb{R}^n$ .

It's easy to see that there exists an exact symplectic isotopy between  $\psi_t$  and  $\phi_t$ , then there is an exact symplectic isotopy between  $(\phi_t)^{-1}\psi_t$  and  $id$ . In view of Proposition 9.19 and Corollary 9.20 of [MDS], we get  $\{\psi_t\}_{0 \leq t \leq 1}$  is a Hamiltonian isotopy. Moreover, since  $\frac{d}{dt}\psi_t$  is periodic in  $t$ , it must be generated by a time periodic Hamiltonian  $N'(\theta, p, t)$ . The maps are  $C^{r-1}$  in  $(\theta, p)$  and  $C^\infty$  in  $t$ , the vector fields are  $C^{r-1}$  and the Hamiltonians are  $C^r$ .

Moreover, it's easy to see that  $\psi_t(\phi_t)^{-1}$  converges in  $C^{r-1}$  to identity uniformly over  $t$  as  $\|g\|_{C^r} \rightarrow 0$ . Since  $\psi_t(\phi_t)^{-1}$  has the Hamiltonian function  $-N_t \circ \phi_t + N'_t \circ \phi_t$  (see [MDS] Proposition 10.2) we conclude that  $\|N_t - N'_t\|_{C^r} \rightarrow 0$  as  $\|g\|_{C^r} \rightarrow 0$ .  $\square$

The following lemma prepares us for the perturbation. For an orbit contained in the pseudograph  $\mathcal{E}_1(N, c)$ , there exists a perturbation block that the orbit of  $(\theta, p)$  never returns to in backward time. Moreover, the orbit also does not return to the ‘‘copy’’ of the perturbation block under the deck transformation of  $\Xi$ . This is important because we would like to perturb the generating function  $G_{N \circ \Xi}$  by perturbing only  $N$ .

**Lemma 5.5.** *Consider  $(N_0, c_0) \in \mathcal{Q}$ , and  $(\theta_0, p_0) \in \tilde{\mathcal{H}}_{12}(N_0, c_0)$ . Then there exists  $\sigma > 0$ , and open sets  $V \ni \theta_0$  and  $U_1 \subset U_2 \subset \mathbb{R}^n$ , such that*

- The covering map  $\xi : \mathbb{T}^n \rightarrow \mathbb{T}^n$  is injective on  $\overline{U_2}, \overline{V}$ .
- $\overline{U_2} \cup (\overline{U_2} + \frac{1}{2}e_1), \overline{V} \cup (\overline{V} + \frac{1}{2}e_1)$  are disjoint from  $\mathcal{A}_{N_0 \circ \Xi}(\xi^*c)$ .

The following hold for each  $(N, c) \in \mathcal{Q} \cap B_\sigma(N_0, c_0)$ .

1. For  $\theta \in V$ , let  $(\theta, p)$  be contained in the closure of the pseudograph  $\overline{\mathcal{E}_1(N, c)}$ .
  - (a)  $(\theta, p) \in \mathcal{B}_{N \circ \Xi}(U_1, V)$ .
  - (b) The backward orbit  $\phi_{N \circ \Xi}^{-k}(\theta, p)$  is asymptotic to  $\tilde{\mathcal{S}}_1(N, c)$ .
  - (c) For  $k \geq 1$ ,  $\phi_{N \circ \Xi}^{-k}(\theta, p)$  is not contained in  $\mathcal{B}_{N \circ \Xi}(U_2, V)$  or  $\mathcal{B}_{N \circ \Xi}(U_2 + \frac{1}{2}e_1, V + \frac{1}{2}e_1)$ .
2. For  $\theta \in V$ , let  $(\theta, p)$  be contained in the closure of the pseudograph  $\overline{\mathcal{E}_2(N, c)}$ .
  - (a) The forward orbit  $\phi_N^k(\theta, p)$  is asymptotic to  $\tilde{\mathcal{S}}_2$ .
  - (b) For  $k \geq 1$ ,  $\phi_N^k(\Xi(\theta, p))$  is not contained in  $\mathcal{B}_{N \circ \Xi}(U_2, V)$  or  $\mathcal{B}_{N \circ \Xi}(U_2 + \frac{1}{2}e_1, V + \frac{1}{2}e_1)$ .

Moreover, an analogous statement holds for  $\mathcal{H}_{21}$ , where the roles of  $\mathcal{E}_1, \tilde{\mathcal{E}}_2$  are replaced by  $\mathcal{E}_2$  and  $\tilde{\mathcal{E}}_1$ .

*Proof.* First we claim: for any  $\iota > 0$ , there is  $\sigma > 0$  such that: if  $\|\theta - \theta_0\| < \sigma$ ,  $(N, c) \in B_\sigma(N_0, c_0) \cap \mathcal{Q}$ , then  $(\theta, p) \in \tilde{\mathcal{E}}_1(N, c)$  implies:

- (c1)  $\|p - p_0\| < \iota$ .
- (c2) The backward orbit  $\phi_{N \circ \Xi}^{-k}(\theta, p)$  is asymptotic to  $\tilde{\mathcal{S}}_1(N, c)$ .
- (c3) There exists  $M > 0$  such that  $k > M$  implies  $\text{dist}(\phi_{N \circ \Xi}^{-k}(\theta, p), \tilde{\mathcal{S}}_1(N, c)) < \iota$ .

We note that  $\theta_0 \in \mathcal{H}_{12}(N_0, c_0)$  implies the weak KAM solution  $h(\zeta_1, \cdot)$  is differentiable at  $\theta_0$ , and therefore  $p_0$  is the unique super-differential. Item (c1) then follows from semi-continuity of super-differentials, see Proposition C.1.

Since  $\theta_0 \in \mathcal{H}_{12}(N_0, c_0)$ , we have for  $h = h_{N_0 \circ \Xi, \xi^*c}$

$$h(\zeta_1, \theta_0) + h(\theta_0, \zeta_2) = \min_{\theta} (h(\zeta_1, \cdot) + h(\cdot, \zeta_2)) = h(\zeta_1, \zeta_2). \quad (24)$$

Assume by contradiction that for  $(N_k, c_k) \rightarrow (N_0, c_0)$  in  $\mathcal{Q}$ , and  $(\theta_k, p_k) \in \mathcal{E}_1(N_k, c_k)$  with  $\theta_k \rightarrow \theta_0$ , the backward orbit of  $(\theta_k, p_k)$  accumulates to  $\mathcal{S}_2(N_k, c_k)$ . This implies

$$h_{N_k \circ \Xi, \xi^* c_k}(\zeta_1^k, \theta_k) = h_{N_k \circ \Xi, \xi^* c_k}(\zeta_1^k, \zeta_2^k) + h_{N_k \circ \Xi, \xi^* c_k}(\zeta_2^k, \theta_k), \quad \zeta_1^k \in \mathcal{S}_1, \zeta_2^k \in \mathcal{S}_2.$$

Taking limit as  $k \rightarrow \infty$  (by Proposition C.1), we obtain

$$h_{N_0 \circ \Xi, \xi^* c_0}(\zeta_1, \theta_0) = h_{N_0 \circ \Xi, \xi^* c_0}(\zeta_1, \zeta_2) + h_{N_0 \circ \Xi, \xi^* c_0}(\zeta_2, \theta_0), \quad \zeta_1 \in \mathcal{S}_1, \zeta_2 \in \mathcal{S}_2.$$

Combine with (24) we get (omitting the subscript of  $h$ )

$$h(\zeta_1, \zeta_2) = h(\zeta_1, \theta_0) + h(\theta_0, \zeta_2) = h(\zeta_1, \zeta_2) + h(\zeta_2, \theta_0) + h(\theta_0, \zeta_2),$$

or  $h(\zeta_2, \theta_0) + h(\theta_0, \zeta_2) = 0$  this is a contradiction with  $\theta_0 \notin \mathcal{S}_2$ .

To prove (c3) we again argue by contradiction. Let  $N_k, c_k, \theta_k, p_k$  be as before, we assume that there exists  $M_k \rightarrow \infty$  such that  $\text{dist}(\phi_{N_k \circ \Xi}^{-M_k}(\theta_k, p_k), \mathcal{S}_1(N, c)) \geq \varepsilon$ . Denote  $m_k = \pi \phi_{N_k \circ \Xi}^{-M_k}(\theta_k, p_k)$ , using the fact that backward orbit of  $(\theta_k, p_k)$  is calibrated, we have

$$h_{N_k \circ \Xi, \xi^* c_k}(\zeta_1, \theta_k) = h_{N_k \circ \Xi, \xi^* c_k}(\zeta_1, m_k) + A_{N_k \circ \Xi, \xi^* c_k}^{M_k}(m_k, \theta_k).$$

Up to taking a subsequence, assume  $m_k \rightarrow m_0$ , take limit as  $k \rightarrow \infty$ , we obtain

$$h(\zeta_1, \theta_0) \geq h(\zeta_1, m_0) + h(m_0, \theta_0) = h(\zeta_1, m_0) + \min_{i=1,2} (h(m_0, \zeta_i) + h(\zeta_i, \theta_0)),$$

where  $h$  are evaluated at  $N_0 \circ \Xi, \xi^* c_0$ . Since  $h(\zeta_1, m_0) + h(m_0, \zeta_1) > 0$ , the above minimum is not reached at  $\zeta_1$ . Therefor  $h(\zeta_1, \theta_0) \geq h(\zeta_1, m_0) + h(m_0, \zeta_2) + h(\zeta_2, \theta_0) \geq h(\zeta_1, \zeta_2) + h(\zeta_2, \theta_0)$ , but we showed (in the proof of (c2)) this is also impossible.

We now define the sets  $U, V$ . Since  $\phi_{N_0 \circ \Xi}^{-k}(\theta_0, p_0)$  is asymptotic to  $\mathcal{S}_1(N_0, c_0)$ , project via  $\Xi$  implies  $\phi_{N_0}^{-k}(\Xi(\theta_0, p_0))$  is asymptotic to  $\Xi(\mathcal{S}_1) = \mathcal{A}_{N_0}(c_0)$ . There exists  $\iota_1 > 0$  such that

$$\phi_N^{-k}(\Xi(\theta_0, p_0)) \cap \Xi(B_\varepsilon(\theta_0, p_0)) = \emptyset,$$

and  $\xi(B_{\iota_1}(\theta_0)) \cap \mathcal{A}_N = \emptyset$  for all  $N \in B_{\iota_1}(N_0) \cap \mathcal{R}_1$ .

Apply claim (c1)-(c3) to  $\iota = \iota_1/2$ , and obtain the parameters  $\sigma, M$ . Since the orbit of  $(\theta_0, p_0)$  is wandering, there exists  $0 < \sigma_1 < \sigma$  such that  $(\theta, p) \in B_{\sigma_1}(\theta_0, p_0)$ ,  $N \in B_{\sigma_1}(N_0)$  implies

$$\phi_N^{-k}(\Xi(B_{\sigma_1}(\theta_0, p_0))) \cap \Xi(B_{\sigma_1}(\theta_0, p_0)) = \emptyset, \quad 1 \leq k \leq M.$$

apply the relation  $\Xi \circ \phi_{N \circ \Xi} = \phi_N \circ \Xi$  we get

$$\phi_{N \circ \Xi}^{-k}(\Xi(B_{\sigma_1}(\theta_0, p_0))) \cap \Xi^{-1} \Xi(B_{\sigma_1}(\theta_0, p_0)) = \emptyset, \quad 1 \leq k \leq M. \quad (25)$$

For a later determined  $\sigma_2 < \sigma_1$ , choose  $\sigma_3 < \sigma_2$  using claim (c1) again to ensure any  $(\theta, p) \in \mathcal{E}_1(N, c)$  with  $\|\theta - \theta_0\| < \sigma_3$  implies  $\|p - p_0\| < \sigma_2$ . Define  $V = B_{\sigma_3}(\theta_0)$ ,

$$U_1 = \bigcup_{N \in B_{\sigma_3}(N_0)} \pi \phi_{N \circ \Xi}^{-1}(B_{\sigma_3}(\theta_0) \times B_{\sigma_2}(p_0)), \quad (26)$$

$U_2 = B_{\sigma_2}(U_1)$ . Since  $U_1 \rightarrow \pi \phi_{N_0 \circ \Xi}^{-1}(\theta_0, p_0)$ , as  $\sigma_2, \sigma_3 \rightarrow 0$ , we can choose  $\sigma_2, \sigma_3$  small enough such that

$$\mathcal{B}_{N \circ \Xi}(\overline{U_2}, \overline{V}) \subset B_{\sigma_1}(\theta_0, p_0), \quad \forall N \in B_{\sigma_3}(N_0).$$

We now verify that for  $\theta \in V$  and  $(\theta, p) \in \mathcal{E}_1(N, c)$ ,  $\phi_{N \circ \Xi}^{-1}(\theta, p) \in U_1$  due to (26). Moreover, since

$$\mathcal{B}_{N \circ \Xi}(\overline{U_2}, \overline{V}) \cup \mathcal{B}_{N \circ \Xi}(\overline{U_2} + \frac{1}{2}e_1, \overline{V} + \frac{1}{2}e_1) \subset \Xi^{-1} \Xi B_{\sigma_1}(\theta_0, p_0),$$

(25) implies 1(c) for  $1 \leq k \leq M$ . On the other hand, (c3) ensures the same for  $k > M$  as well.

The proof of 2(a)(b) and the moreover part is analogous and we omit it.  $\square$

*Proof of Proposition 5.2.* Given  $\theta_0 \in K \cap \mathcal{H}_{12}(N_0, c_0)$ , let  $(\theta_0, p_0)$  be the corresponding point in  $\tilde{\mathcal{H}}_{12}(N_0, c_0)$ . Choose  $\sigma > 0$ ,  $U_1, U_2, V$  as in Lemma 5.5. For  $\varphi \in C_c^r(\xi V)$ , consider perturbation  $N_\varphi$  via (23) using the neighborhoods  $\xi U_1, \xi U_2, \xi V$ . Note that for  $W = U_i, V$ , we have  $\xi^{-1}\xi W = W \cup (W + \frac{1}{2}e_1)$  and we will use this notation throughout the proof. First, notice that according to Lemma 5.4,  $\|N_\varphi - N\|_{C^r} \rightarrow 0$  as  $\|\varphi\|_{C^r} \rightarrow 0$ .

*Item 1.* We first show that the perturbation  $N_\varphi$  does not affect Aubry set and static classes. Lemma 5.5 asserts  $\xi^{-1}\xi\overline{U_2}, \xi^{-1}\xi\overline{V}$  are disjoint from  $\mathcal{A}_{N_0 \circ \Xi}(\xi^* c_0)$ . For  $(N, c) \in B_\sigma(N_0, c_0)$  and  $\sigma$  small enough, using semi-continuity,  $\xi^{-1}\xi\overline{U_2}, \xi^{-1}\xi\overline{V}$  are disjoint from  $\mathcal{A}_{N \circ \Xi}(\xi^* c)$  and  $\mathcal{A}_{N_\varphi \circ \Xi}(\xi^* c)$ . Then (23) and (22) implies the  $L_{N \circ \Xi}$  action and  $L_{N_\varphi \circ \Xi}$  action coincide on orbits of  $\tilde{\mathcal{A}}_{N \circ \Xi}(\xi^* c)$  and  $\tilde{\mathcal{A}}_{N_\varphi \circ \Xi}(\xi^* c)$ . As a result  $\tilde{\mathcal{A}}_{N \circ \Xi}(\xi^* c)$  and  $\tilde{\mathcal{A}}_{N_\varphi \circ \Xi}(\xi^* c)$  must coincide with the same static classes.

*Item 2.* We proceed to prove (21). Let  $(\theta, p) \in \overline{\mathcal{E}_1(N, c)}$ , then  $\gamma(t) := \pi_\theta \circ \phi^t(\theta, p)$  is a calibrated orbit (on  $(-\infty, 0]$ ) for the weak KAM solution  $h_{N_\varphi \circ \Xi, \xi^* c}(\zeta_1, \cdot)$ , with  $\zeta_1 \in \mathcal{S}_1$ . Write  $\gamma_t = \gamma(t)$ . Since  $\gamma(t)$  is backward asymptotic to  $\mathcal{S}_1$ , there is  $i_k \rightarrow \infty$  such that

$$\begin{aligned} h_{N_\varphi \circ \Xi, \xi^* c}(\zeta_1, \theta) &= \lim_{k \rightarrow \infty} A_{N_\varphi \circ \Xi, \xi^* c}^{i_k}(\gamma_{-i_k}, \gamma_0) \\ &= \lim_{k \rightarrow \infty} \sum_{j=-i_k}^{-1} (G_{N_\varphi \circ \Xi}(\gamma_j, \gamma_{j+1}) - \xi^* c \cdot (\gamma_{j+1} - \gamma_j) + \alpha_{N_\varphi \circ \Xi}(\xi^* c)), \end{aligned} \quad (27)$$

where in the last line  $\gamma$  is lifted to  $\mathbb{R}^n$ . In view of 1(c) and (23), for any  $j \leq -2$ , we have

$$G_{N_\varphi \circ \Xi}(\gamma_j, \gamma_{j+1}) = G_{N_\varphi}(\xi\gamma_j, \xi\gamma_{j+1}) = G_N(\xi\gamma_j, \xi\gamma_{j+1}) = G_{N \circ \Xi}(\gamma_j, \gamma_{j+1}).$$

By the same reasoning, we have

$$G_{N_\varphi \circ \Xi}(\gamma_{-1}, \gamma_0) = G_{N \circ \Xi}(\gamma_{-1}, \gamma_0) + \rho(\gamma_{-1})\varphi(\gamma_0) = G_{N \circ \Xi}(\gamma_{-1}, \gamma_0).$$

Using (27), we get

$$h_{N_\varphi \circ \Xi, \xi^* c}(\zeta_1, \theta) = \lim_{k \rightarrow \infty} A_{N \circ \Xi, \xi^* c}^{i_k}(\gamma_{-i_k}, \gamma_0) \leq h_{N \circ \Xi, \xi^* c}(\zeta_1, \theta).$$

Observe that the previous arguments holds when  $N_\varphi$  and  $N$  are switched, the last displayed formula becomes an equality. By the same reasoning, using Lemma 5.5, 2(a),(b), we obtain

$$h_{N_\varphi \circ \Xi, \xi^* c}(\theta, \zeta_2) = h_{N \circ \Xi, \xi^* c}(\theta, \zeta_2), \quad \zeta_2 \in \mathcal{S}_2.$$

These (21) follows. The proof for  $b^-$  is identical with two static classes switched.  $\square$

### 5.3 Hölder continuity of the barrier functions

We prove Proposition 5.3 by relating the barriers to the stable and unstable manifolds of the Aubry sets.

Recall that the system  $N$  admit a weakly invariant cylinder  $\mathcal{C}$  which contains the Aubry set  $\tilde{\mathcal{A}}_N(c)$  for  $c \in \Gamma_1$ . Using the covering map  $\Xi$ , we obtain  $\Xi^{-1}\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$  and denote  $\tilde{\mathcal{S}}_i(N, c) = \mathcal{C}_i \cap \Xi^{-1}(\tilde{\mathcal{A}}(c))$ ,  $i = 1, 2$  for all  $c \in \Gamma_*(N)$ .

Recall that  $\Gamma_*(N)$  is the set of  $c \in \Gamma_1$  such that  $\mathcal{A}_N(c)$  is an invariant curve contained in  $\mathcal{C}$ . Let  $c^\pm$  be the  $c \in \Gamma_*(N)$  with the smallest and largest  $p^f$  component. Then the component of  $\mathcal{C}$  bounded by  $\mathcal{A}_N(c^\pm)$  is an invariant set for  $\phi_N$ , we denote it  $\Lambda_*$ . Let  $\Lambda_1, \Lambda_2$  be the lifts under  $\Xi$ , then  $\Lambda_i \subset \mathcal{C}_i$  are normally hyperbolic invariant manifolds for  $\phi_{N \circ \Xi}$ .

They admit  $C^2$  center stable and center unstable manifolds  $W^{cs/cu}$ , which are locally graphs above  $(\theta, p^f)$ . These manifolds are foliated by the strong stable and unstable manifolds  $W^{s,u}(z)$



of the points of  $\Lambda_i$ , see Appendix A. The leaves  $W^{s,u}(z)$  of this foliation are  $C^2$ , they are locally graphs above  $\theta^s$ . The foliation itself is  $C^1$ .

Consider  $c \in \Gamma_*(N)$ , then for  $i = 1, 2$ ,  $\tilde{\mathcal{S}}_i(N, c)$  is a Lipschitz invariant curve. Define the sets

$$W_i^{u/s}(N, c) = \bigcup_{z \in \tilde{\mathcal{S}}_i(N, c)} W^{u/s}(z).$$

Since  $\tilde{\mathcal{S}}_i(N, c)$  are Lipschitz graphs over  $\theta^f$ , and since  $W^{u,s}$  are a  $C^1$  foliation whose leaves are graphs over  $\theta^s$ ,  $W_i^{u/s}(N, c)$  are Lipschitz graphs over  $\theta$  in a neighborhood of  $\tilde{\mathcal{S}}_i$ . We will show that they coincides with the pseudographs  $\mathcal{E}_i(N, c)$  in a neighborhood of  $\mathcal{S}_i(N, c)$ .

**Lemma 5.6.** *For  $i, j = 1, 2$ , if  $(\theta, p) \in \overline{\mathcal{E}_i(N, c)}$  is backward asymptotic to  $\mathcal{S}_j(N, c)$ , then there exists  $M > 0$  such that  $\phi_{N \circ \Xi}^{-k} \in W_j^u(N, c)$  for each  $k > M$ .*

Suppose an orbit is backward asymptotic to  $\mathcal{S}_1(N, c)$ , then it is asymptotic to the normally hyperbolic set  $\Lambda_1$ . This orbit is contained in the strong manifold of a point  $z' \in \Lambda_1$  which is asymptotic to  $\mathcal{S}_1(N, c)$ , but which in principle may not belong to  $\mathcal{S}_1(N, c)$ . To prove that  $z' \in \mathcal{S}_1(N, c)$ , we need an argument similar to Theorem 1.4.

We need the following version of Proposition 4.3.

**Proposition 5.4.** *Suppose  $k \geq 1/\sqrt{\varepsilon}$ , then for each semi-concave function  $u_0$ , the function  $u_k = T_c^k u_0$  is  $6D\sqrt{\varepsilon}$ -semi-concave and  $6D\sqrt{n\varepsilon}$ -Lipschitz. Similar statement holds for  $\tilde{T}_c^k u$ . As a result, for any weak KAM solution  $u$  and  $k \geq 1/\sqrt{\varepsilon}$ , the set*

$$\phi_N^{-k}(\overline{\mathcal{G}_{c,u}})$$

is a  $6D\sqrt{\varepsilon}$ -Lipschitz graph over the  $\theta$  component.

*Proof.* We observe that the proof of Proposition 4.3 applies as long as we replace  $u(\theta)$  by  $u_k$  and  $u(\Theta(0))$  by  $u_0(\Theta(0))$ . The assumption  $k \geq \frac{1}{\sqrt{\varepsilon}}$  ensures we can choose  $T \in [1/2\sqrt{\varepsilon}, 1/\sqrt{\varepsilon}]$  in that proof.

For the second part, observe that

$$\phi_N^{-k}(\overline{\mathcal{G}_{c,u}}) \subset \mathcal{G}_{c,u} \tilde{\wedge} \tilde{\mathcal{G}}_{c, \tilde{T}^k u}$$

and the proof is similar to Theorem 4.1.  $\square$

For the rest of this section,  $\phi$  denotes  $\phi_{N \circ \Xi}$ .

*Proof of Lemma 5.6.* We only prove for the case  $i = j = 1$  as the others are similar. Since  $z := (\theta, p)$  is backward asymptotic to  $\mathcal{S}_1(N, c) \subset \Lambda_1$ , then there exists  $z_1 \in \Lambda_1$  such that  $(\theta, p) \in W^u(z_1)$ . Necessarily  $\phi^{-k}(z_1)$  converges to  $\mathcal{S}_1(N, c)$ . We will show  $z_1 \in \mathcal{S}_1(N, c)$ .

Arguing by contradiction, suppose  $z_1 \notin \mathcal{S}_1(N, c)$ , then using the fact that  $TC_1$  is the central direction,  $\text{dist}(\phi^{-k}z_1, \mathcal{S}_1(N, c))$  converges at a maximal rate of  $\rho^n$ .

Denote  $z_1^k = \phi^{-k}(z_1)$ ,  $\mathcal{S}_1(N, c)$  projects onto  $\theta^f$  component, for any  $k \in \mathbb{N}$ , there is  $z_2^k \in \mathcal{S}_1(N, c)$  such that  $\theta^f(z_1^k) = \theta^f(z_2^k)$ . According to Theorem 3.1, there exists  $D_1 > 1$  such that  $\mathcal{C}$  is an  $D_1/\sqrt{\varepsilon}$  graph over  $(\theta^f, p^f)$ , which implies

$$\|p^f(z_1^k) - p^f(z_2^k)\| \geq \sqrt{\varepsilon}/D_1 \|z_1^k - z_2^k\| \geq D_2^{-1} \sqrt{\varepsilon} \rho^k \quad (28)$$

for some  $D_2 > 1$ . Let  $z^k = \phi^{-k}(z)$ , we have  $\|z^k - z_1^k\| < C\lambda^k$ . Suppose  $k$  is large enough such that  $C\lambda^k < \frac{1}{2}D_2^{-1}\sqrt{\varepsilon}\rho^k$ , then

$$\|p^f(z^k) - p^f(z_2^k)\| \geq \|p^f(z_1^k) - p^f(z_2^k)\| - \|p^f(z^k) - p^f(z_1^k)\| \geq \frac{1}{2}\|p^f(z_1^k) - p^f(z_2^k)\|. \quad (29)$$



Assume  $k \geq 1/\sqrt{\varepsilon}$ . We now use Proposition 5.4 to get for some  $D_3 > 1$ ,

$$\begin{aligned} \|p(z^k) - p(z_2^k)\| &\leq D_3\sqrt{\varepsilon} (\|\theta^s(z^k) - \theta^s(z_3^k)\| + \|\theta^f(z^k) - \theta^f(z_3^k)\|) \\ &\leq D_3\sqrt{\varepsilon} (\|\theta^s(z^k) - \theta^s(z_3^k)\| + \|\theta^f(z^k) - \theta^f(z_1^k)\|) \\ &\leq D_2\sqrt{\varepsilon} (\|\theta^s(z^k) - \theta^s(z_2^k)\|) + D_2D_3\sqrt{\varepsilon}\lambda^k, \end{aligned} \quad (30)$$

keep in mind that  $\theta^f(z_1^k) = \theta^f(z_2^k)$ . Since  $z_1^k, z_2^k \in \mathcal{C}_1$ , using Theorem 1.4, we get for small  $\varepsilon$ ,

$$\begin{aligned} \|\theta^s(z^k) - \theta^s(z_2^k)\| &\leq \|\theta^s(z_1^k) - \theta^s(z_2^k)\| + C\lambda^k \\ &\leq \frac{1 + \sqrt{\delta/\varepsilon}}{\kappa} (\|\theta^f(z_1^k) - \theta^f(z_2^k)\| + \|p^f(z_1^k) - p^f(z_2^k)\|) + C\lambda^k \\ &\leq 4\kappa^{-1}\delta^{\frac{1}{2}}\varepsilon^{-\frac{1}{2}}\|p^f(z^k) - p^f(z_2^k)\| + C\lambda^k \end{aligned}$$

Combine with (30), we get

$$\|p(z^k) - p(z_2^k)\| \leq 4C\kappa^{-1}\delta^{\frac{1}{2}}\|p(z^k) - p(z_2^k)\| + 2D_2D_3\sqrt{\varepsilon}\lambda^k.$$

When  $\kappa^{-1}\delta^{\frac{1}{2}} < \frac{1}{2}$  we get  $\|p(z^k) - p(z_2^k)\| \leq 4D_2D_3\sqrt{\varepsilon}\lambda^k$ , but this contradicts with (28) and (29).  $\square$

**Lemma 5.7.** *For  $(N, c_0) \in \mathcal{Q}$ , there is  $\sigma_1, \sigma_2, M > 0$  such that for all  $c \in B_{\sigma_1}(c_0) \cap \Gamma_*(N)$ , we have for  $i = 1, 2$ ,*

1.

$$\overline{\mathcal{E}_i(N, c)} \cap \pi^{-1}(B_{\sigma_2}(\mathcal{S}_i(N, c_0))) \subset W_i^u(N, c).$$

*This also implies  $\mathcal{E}_i(N, c) = \overline{\mathcal{E}_i(N, c)}$  and is  $C^1$  over  $B_{\sigma_2}(\mathcal{S}_i(N, c_0))$ .*

2. *For each  $(\theta, p) \in \overline{\mathcal{E}_i(N, c)}$ , there exists  $k \leq M$  such that*

$$\phi^{-k}(\theta, p) \in B_{\sigma_2}(\mathcal{S}_1(N, c) \cup \mathcal{S}_2(N, c)).$$

*Proof.* We prove item 1. for  $i = 1$ , the proof for  $i = 2$  is identical. We first prove the statement for  $c = c_0$  then extend to a neighborhood by continuity. First of all, we refer to [Be1] Lemma 4.4, to get the existence of  $\sigma_3 > 0$  such that every  $(\theta, p) \in \mathcal{E}_1(N, c)$  with  $\theta \in B_{\sigma_3}(\mathcal{S}_1(N, c_0))$  is backward asymptotic to  $\mathcal{S}_1$ . By Lemma 5.6, there exists  $k$  such that  $\phi^{-k}(\theta, p) \in W_1^u(N, c)$ . We now show that  $k$  can be chosen uniformly for all  $\theta \in \overline{B_{\sigma_3/2}(\mathcal{S}_1(N, c_0))}$ . Arguing by contradiction, if there is  $k_i \rightarrow \infty$  and  $\phi^{-j}(\theta_i, p_i) \notin W_1^u(N, c)$  for all  $0 \leq j \leq k_i$ , after taking a convergent subsequence, we get  $(\theta_i, p_i) \rightarrow (\theta_*, p_*) \in \mathcal{E}_1(N, c)$  whose backward orbit does not intersect  $W_1^u(N, c)$ . This is a contradiction. Using a similar compactness argument over  $c$ , we obtain:

There exists  $\sigma_4, \sigma_5 > 0$  and  $M > 0$ , such that for all  $c \in B_{\sigma_4}(c_0) \cap \Gamma_*(N)$  and  $(\theta, p) \in B_{\sigma_5}(\mathcal{S}(N, c_0))$ , we have  $\phi^{-k}(\theta, p) \in W_1^u(N, c)$  for all  $k \geq M$ .

Finally, we choose  $\sigma_6$  small enough so that  $B_{\sigma_6}(\mathcal{S}_1(N, c_0)) \subset \phi^{-M}(B_{\sigma_5}(\mathcal{S}_1(N, c_0)))$ . Since  $\mathcal{S}_1(N, c)$  is semi-continuous in  $c$ , this property extends to a small neighborhood of  $c \in \Gamma_*(N)$ .

We now prove item 2, for  $i = 1$ . Assume there exists  $\sigma_7 > 0$ ,  $k_i \rightarrow \infty$ ,  $(\theta_i, p_i) \in \mathcal{E}_1(N, c_i)$  with  $c_i \rightarrow c_0$ , such that  $\phi^{-j}(\theta_i, p_i) \notin B_{\sigma_7}(\mathcal{S}_1 \cup \mathcal{S}_2)$  for all  $0 \leq j \leq k_i$ . Taking limit up to a subsequence, we obtain an orbit  $(\theta_*, p_*) \in \mathcal{E}_1(N, c_0)$  not backward asymptotic to  $\mathcal{S}_1 \cup \mathcal{S}_2$ , a contradiction.  $\square$

For each  $c \in \Gamma_*(N)$ , the set  $\tilde{\mathcal{S}}_1(N, c)$  is a graph over  $\theta^f$ , hence there exists a map  $\eta_c : \mathbb{T} \rightarrow \mathbb{T}^n \times \mathbb{R}^n$  such that  $\mathcal{S}_1(N, C)$  is the image of  $\eta_c$  and  $\pi_{\theta^f} \circ \eta_c(s) = s$ .

**Lemma 5.8.** *There exists  $C_1 > 0$  such that*

$$\sup_s \|\eta_c(s) - \eta_{c'}(s)\| \leq C_1 \|c - c'\|^{\frac{1}{2}}$$

for each  $c$  and  $c'$  in  $\Gamma_*(N)$ .

*Proof.* We denote by  $D_i$  different positive constants that may depend on  $\varepsilon$  and  $\delta$ . Since  $\mathcal{C}_1$  is a Lipschitz graph over  $(\theta^f, p^f)$ ,

$$\sup_s \|\eta_c(s) - \eta_{c'}(s)\| \leq D_1 \sup_s \|\pi_{p^f} \eta_c(s) - \pi_{p^f} \eta_{c'}(s)\|. \quad (31)$$

Each Weak KAM solution  $u_c$  is differentiable on  $\mathcal{S}_1(N, c)$ , and we have  $\pi_p \circ \eta_c = c + du_c(\pi_\theta \circ \eta_c)$ . We have

$$\int_\eta pd\theta = \int_\eta cd\theta + \int_\eta du_c(\pi_\theta \circ \eta_c)d\theta = \pi_{p^f}(c),$$

hence the symplectic area  $A(\eta_c, \eta_{c'})$  of the domain of  $\mathcal{C}_1$  delimited by the curves  $\eta_c$  and  $\eta_{c'}$  is

$$A(\eta_c, \eta_{c'}) = \left( \int_\eta - \int_{\eta_{c'}} \right) pd\theta = \pi_{p^f}(c) - \pi_{p^f}(c').$$

Recall that the cylinder  $\mathcal{C}_1$  is given by a graph  $(\theta^s, p^s) = (\Theta^s, P^s)(\theta^f, p^f)$ . The estimates (4) imply that, if  $v, v'$  are two vectors tangent to  $\mathcal{C}_1$ , then  $|(d\Theta^s \wedge dP^s)(v, v')| \leq C\sqrt{\delta}|d\theta^f \wedge dp^f(v, v')|$ , hence, if  $\delta$  is small enough,

$$|(d\Theta \wedge dP)(v, v')| \geq \frac{1}{2}|(d\theta^f \wedge dp^f)(v, v')|.$$

Note that given two  $C$  Lipschitz functions  $\gamma_1, \gamma_2 : \mathbb{T} \rightarrow \mathbb{R}$  with  $\gamma_1(s) > \gamma_2(s)$ ,

$$\int (\gamma_1 - \gamma_2) ds \geq \frac{1}{4C} \sup \|\gamma_1(s) - \gamma_2(s)\|^2.$$

Let  $\Omega$  denote the region on  $\mathcal{C}_1$  between  $\eta_c$  and  $\eta_{c'}$ . For  $c, c' \in \Gamma_*$ , there is  $D_3, D_4 > 1$  such that

$$\begin{aligned} D_3 \|c - c'\| &\geq \|\pi_{p^f}(c) - \pi_{p^f}(c')\| = |A(\eta_c, \eta_{c'})| \geq \frac{1}{2} \left| \int_\Omega d\theta^f \wedge dp^f \right| \\ &= \frac{1}{2} \left| \int (\pi_{p^f} \circ \eta_c(s) - \pi_{p^f} \circ \eta_{c'}(s)) dt \right| \geq \frac{1}{D_4} \sup \|\pi_{p^f} \circ \eta_c(s) - \pi_{p^f} \circ \eta_{c'}(s)\|^2. \end{aligned} \quad (32)$$

Combine with (31) we get our conclusion.  $\square$

**Lemma 5.9.** *In the context of Lemm 5.7, consider for  $c, c' \in B_{\sigma_1}(c_0) \cap \Gamma_*(N)$ , and  $\zeta_1 \in \mathcal{S}_1(N, c)$  and  $\zeta'_1 \in \mathcal{S}_1(N, c')$ , denote*

$$u_c(\cdot) = h_{\xi^*c}(\zeta_1, \cdot) = h_{N \circ \Xi, \xi^*c}(\zeta_1, \cdot), \quad u_{c'}(\cdot) = h_{\xi^*c'}(\zeta_2, \cdot) = h_{N \circ \Xi, \xi^*c'}(\zeta'_1, \cdot).$$

Then for  $\theta \in B_{\sigma_2}(\mathcal{S}_1(N, c_0))$  :

1.  $|\nabla u_c(\theta) - \nabla u_{c'}(\theta)| \leq C_2 \|c - c'\|^{\frac{1}{2}}$ ;
2.  $|u_c(\theta) - u_{c'}(\theta) - C_3| \leq C_2 \|c - c'\|^{\frac{1}{2}}$ .

Moreover, the same holds with  $\mathcal{S}_1$  replaced with  $\mathcal{S}_2$ .

*Proof.* For  $\theta \in B_{\sigma_2}(\mathcal{S}_1(N, c_0))$ , let  $y = (\theta, \nabla u_c(\theta))$ , and let  $z \in \mathcal{S}_1(N, c)$  be such that  $y \in W^s(z)$ . We then define  $z' \in \mathcal{S}_1(N, c')$  be the unique such point with  $\theta^f(z') = \theta^f(z)$ . Finally, define  $y' \in W^u(z')$  such that  $\theta^s(y') = \theta^s(y)$ , which is possible since  $W^u(z')$  is locally a graph over  $\theta^s$ .

We note that within the center unstable manifold  $W^u(\Lambda)$ , the NHIC  $\Lambda$  on one hand, and  $\theta^s = \theta^s(y)$  on the other hand serves as two transversals to the strong unstable foliation  $\{W^u(\cdot)\}$ . Since the foliation is  $C^1$ , there exists  $D_1 > 0$  such that

$$\|y - y'\| \leq D_1 \|z - z'\| \leq C_1 D_1 \|c - c'\|^{\frac{1}{2}},$$

where  $C_1$  is from Lemma 5.8. Denote  $w = (\theta, \nabla u_{c'}(\theta))$ , and noting  $y' \in W_1^u(N, c') = \{(x, \nabla u_{c'}(x))\}$  which is locally a  $C^1$  graph, we get for  $D_2 > 0$

$$\|w - y'\| \leq D_2 \|\pi_\theta(w) - \pi_\theta(y')\| = D_2 \|\pi_\theta(y) - \pi_\theta(y')\| \leq D_2 \|y - y'\|,$$

therefore

$$\|\nabla u_c(\theta) - \nabla u_{c'}(\theta)\| \leq \|w - y\| \leq \|w - y'\| + \|y - y'\| \leq D_3 \|y - y'\| \leq D_4 \|c - c'\|^{\frac{1}{2}}.$$

Item 1 follows. For item 2, we consider  $\theta, \theta_0 \in B_{\sigma_2}(\mathcal{S}_1(N, c_0))$ , then integrating item 1 leads to

$$|u_c(\theta) - u_{c'}(\theta) - (u_c(\theta_0) - u_{c'}(\theta_0))| \leq D_5 \|c - c'\|^{\frac{1}{2}}. \quad (33)$$

Item 2 follows by taking  $C_3 = u_c(\theta_0) - u_{c'}(\theta_0)$ .  $\square$

*Proof of Proposition 5.3.* Fix  $(N, c_0) \in \mathcal{Q}$ , we consider  $c \in B_{\sigma_2}(c_0) \cap \Gamma_*(N)$  in the context of Lemma 5.7. From item 2 of that lemma, for every  $\theta \in \mathbb{T}^n$ , there exists a calibrated orbit  $\gamma : (-\infty, 0] \rightarrow \mathbb{T}^n$  with  $\gamma(0) = \theta$ , such that  $\gamma(t) \in B_{\sigma_2}(\mathcal{S}_1(N, c) \cup \mathcal{S}_2(N, c))$  whenever  $t < -M$ . Then (omitting the subscript  $N \circ \Xi$ )

$$h_{\xi^*c}(\zeta_1, \theta) = \min_{i=1,2} \min_{k \leq M} \min_{\theta' \in B_{\sigma_2}} \{h_{\xi^*c}(\zeta_1, \zeta_i) + h_{\xi^*c}(\zeta_i, \theta') + A_{\xi^*c}^k(\theta', \theta)\}.$$

Since  $h_{\xi^*c}(\zeta_i, \theta')$  are uniformly  $\frac{1}{2}$  Holder in  $c$  for  $\theta' \in B_{\sigma_2}(\mathcal{S}_i(N, c))$  and  $c \in B_{\sigma_2}(c_0) \cap \Gamma_*(N)$ , each  $A_{\xi^*c}^k$  are uniformly Lipschitz in  $c$ , the family  $h_{\xi^*c}(\zeta_1, \theta)$  is  $\frac{1}{2}$  Holder in  $c$ .  $\square$

## A Normally hyperbolic manifold

Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a  $C^1$  vector field. We give sufficient conditions for the existence of a Normally hyperbolic invariant graph of  $F$ . We split the space  $\mathbb{R}^n$  as  $\mathbb{R}^{n_u} \times \mathbb{R}^{n_s} \times \mathbb{R}^{n_c}$ , and denote by  $x = (u, s, c)$  the points of  $\mathbb{R}^n$ . We denote by  $(F_u, F_s, F_c)$  the components of  $F$ :

$$F(x) = (F_u(x), F_s(x), F_c(x)).$$

We study the flow of  $F$  in the domain

$$\Omega = B^u \times B^s \times \Omega^c$$

where  $B^u$  and  $B^s$  are the open Euclidean balls of radius  $r_u$  and  $r_s$  in  $\mathbb{R}^{n_u}$  and  $\mathbb{R}^{n_s}$ , and  $\Omega^c$  is a convex open subset of  $\mathbb{R}^{n_c}$ . We denote by

$$L(x) = dF(x) = \begin{bmatrix} L_{uu}(x) & L_{us}(x) & L_{uc}(x) \\ L_{su}(x) & L_{ss}(x) & L_{sc}(x) \\ L_{cu}(x) & L_{cs}(x) & L_{cc}(x) \end{bmatrix}$$

the linearized vector field at point  $x$ . We assume that  $\|L(x)\|$  is bounded on  $\Omega$ , which implies that each trajectory of  $F$  is defined until it leaves  $\Omega$ . We denote by  $W^c$  the union of full orbits contained in  $\Omega$ . In other words, this is the set of initial conditions  $\underline{x} \in \Omega$  such that there exists a solution  $x(t) : \mathbb{R} \rightarrow \Omega$  of the equation  $\dot{x} = F(x)$  satisfying  $x(0) = \underline{x}$ . We denote by  $W^{sc}$  the set of points whose positive orbit remains inside  $\Omega$ . In other words, this is the set of initial conditions  $\underline{x} \in \Omega$  such that there exists a solution  $x(t) : [0, \infty) \rightarrow \Omega$  of the equation  $\dot{x} = F(x)$  satisfying  $x(0) = \underline{x}$ . Finally, we denote by  $W^{uc}$  the set of points whose negative orbit remains inside  $\Omega$ . In other words, this is the set of initial conditions  $\underline{x} \in \Omega$  such that there exists a solution  $x(t) : (\infty, 0] \rightarrow \Omega$  of the equation  $\dot{x} = F(x)$  satisfying  $x(0) = \underline{x}$ . These sets have specific features under the following assumptions:

**Hypothesis 3** (Isolating block). *We have:*

- $F_c = 0$  on  $B^u \times B^s \times \partial\Omega^c$ .
- $F_u(u, s, c) \cdot u > 0$  on  $\partial B^u \times \bar{B}^s \times \bar{\Omega}^c$ .
- $F_s(u, s, c) \cdot s < 0$  on  $\bar{B}^u \times \partial B^s \times \bar{\Omega}^c$ .

**Hypothesis 4.** *There exist positive constants  $\alpha$  and  $m$  such that:*

- $L_{uu}(x) \geq \alpha I$ ,  $L_{ss}(x) \leq -\alpha I$  for each  $x \in \Omega$  in the sense of quadratic forms.
- $\|L_{us}(x)\| + \|L_{uc}(x)\| + \|L_{su}(x)\| + \|L_{sc}(x)\| + \|L_{cu}(x)\| + \|L_{cs}(x)\| + \|L_{cc}(x)\| \leq m$  for each  $x \in \Omega$ .

**Theorem A.1.** *Assume that Hypotheses 3 and 4 hold, and that*

$$0 \leq K := \frac{m}{\alpha - 2m} \leq \frac{1}{\sqrt{2}}.$$

*Then the set  $W^{sc}$  is the graph of a  $C^1$  function*

$$w^{sc} : B^s \times \Omega^c \rightarrow B^u,$$

*the set  $W^{uc}$  is the graph of a  $C^1$  function*

$$w^{uc} : B^u \times \Omega^c \rightarrow B^s,$$

*and the set  $W^c$  is the graph of a  $C^1$  function*

$$w^c = (w_u^c, w_s^c) : \Omega^c \rightarrow B^u \times B^s.$$

*Moreover, we have the estimates*

$$\|dw^{sc}\| \leq K, \quad \|dw^{uc}\| \leq K, \quad \|dw^c\| \leq 2K.$$

*Proof.* This results could be reduced to several already existing ones, see [Fe, HPS, McG, Ch] or proved directly by well-known methods. We shall use Theorem 1.1 in [Ya] which is the closest to our needs because it is expressed in terms of vector fields. We first derive some conclusions from the isolating block conditions. We denote by  $\pi^{sc}$  the projection  $(u, s, c) \mapsto (s, c)$ , and so on.

**Lemma A.1.** *If Hypothesis 3 holds, then*

$$\pi^{sc}(W^{sc}) = B^s \times \Omega^c \quad \text{and} \quad \pi^{uc}(W^{uc}) = B^u \times \Omega^c$$

*Moreover, the closures of  $W^{sc}$  and  $W^{uc}$  satisfy*

$$\bar{W}^{sc} \subset B^u \times \bar{B}^s \times \bar{\Omega}^c, \quad \bar{W}^{uc} \subset \bar{B}^u \times B^s \times \bar{\Omega}^c.$$

*Proof.* Let us define  $T^+(x) \in [0, \infty]$  as the first positive time where the orbit of  $x$  hits the boundary  $\partial\Omega$ . Let us denote by  $\varphi(t, x)$  the flow of  $F$ . If  $T^+(x) < \infty$  (which is equivalent to  $x \notin W^{sc}$ ), we have  $\varphi(T^+(x), x) \in \partial B^u \times B^s \times \Omega^c$ , as follows from Hypothesis 3. Then, it is easy to check that the function  $T^+$  is continuous, and even  $C^1$ , at  $x$ .

We prove the first equality of the Lemma by contradiction, and assume that there exists a point  $(s, c) \in B^s \times \Omega^c$  such that  $W^{sc}$  does not intersect the disc  $B^u \times \{s\} \times \{c\}$ . Then, the first exit map

$$B^u \ni u \mapsto \pi^u \circ \varphi(T^+(u, s, c), (u, s, c)) \in \partial B^u,$$

extends by continuity to a continuous retraction from  $\bar{B}^u$  to its boundary  $\partial B^u$ . Such a retraction does not exist. The proof of the other equality is similar.

Finally, we have

$$\bar{W}^{sc} \subset \bar{B}^u \times \bar{B}^s \times \bar{\Omega}^c = (B^u \times \bar{B}^s \times \bar{\Omega}^c) \cup (\partial B^u \times \bar{B}^s \times \bar{\Omega}^c).$$

Hypothesis 3 implies that each point of  $\partial B^u \times \bar{B}^s \times \bar{\Omega}^c$  has a neighborhood formed of points which leave  $\Omega$  after a small time. As a consequence, the set  $\partial B^u \times \bar{B}^s \times \bar{\Omega}^c$  can't intersect  $\bar{W}^{uc}$ , and we have proved that  $\bar{W}^{sc} \subset B^u \times \bar{B}^s \times \bar{\Omega}^c$ . The other inclusion can be proved in a similar way.  $\square$

In order to prove the statement of the Theorem concerning  $W^{sc}$ , we apply Theorem 1.1 of [Ya]. More precisely, using the notation of that paper, we set

$$a = u/K, \quad z = (s, c), \quad f(a, z) = F_u(Ka, z)/K, \quad g(a, z) = (F_s(Ka, z), F_c(Ka, z)).$$

We have the estimates

$$\partial_a f = L_{uu} \geq \alpha, \quad \partial_z g = \begin{bmatrix} L_{ss} & L_{sc} \\ L_{cs} & L_{cc} \end{bmatrix} \leq m$$

in the sense of quadratic forms. Moreover, we have the estimates

$$\|\partial_z f\| \leq \frac{m}{K}, \quad \|\partial_a g\| \leq Km.$$

Since

$$m + m/K + Km < 2m + m/K = \alpha$$

we conclude that Hypothesis 2 of [Ya] is satisfied. Hypothesis 1 of [Ya] is verified by the domain  $\Omega$ , and Hypothesis 3 is precisely the conclusion of Lemma A.1. As a consequence, we can apply Theorem 1.1 of [Ya], and conclude that the set  $W^{sc}$  is the graph of a  $C^1$  and 1-Lipschitz map above  $B^s \times \Omega^c$  in  $(a, z)$  coordinates, and therefore the graph of a  $K$ -Lipschitz  $C^1$  map  $w^{sc} : B^s \times \Omega^c \rightarrow B^u$  in  $(u, s, c)$  coordinates.

In order to prove the statement concerning  $W^{uc}$ , we apply Theorem 1.1 of [Ya] with

$$a = s/K, \quad z = (u, c),$$

$$f(a, z) = -F_s(Ka, z)/K, \quad g(a, z) = -(F_u(Ka, z), F_c(Ka, z)).$$

It is easy to check as above that all hypotheses are satisfied.

Let us now study the set  $W^c = W^{sc} \cap W^{uc}$ . First, let us prove that  $W^c$  is a  $C^1$  graph above  $\Omega^c$ . We know that  $W^{sc}$  is the graph of a  $K$ -Lipshitz  $C^1$  function  $w^{sc}(s, c)$  and that  $W^{uc}$  is the graph of a  $K$ -Lipshitz  $C^1$  function  $w^{uc}(u, c)$ . The point  $(u, s, c)$  belongs to  $W^c$  if and only if

$$u = w^{sc}(s, c) \quad \text{and} \quad s = w^{uc}(u, c),$$

or in other words if and only if  $(u, s)$  is a fixed point of the  $K$ -Lipschitz  $C^1$  map

$$(u, s) \mapsto (w^{sc}(s, c), w^{uc}(u, c)).$$

For each  $c$ , this contracting map has a unique fixed point in  $\bar{B}^u \times \bar{B}^s$ , which corresponds to a point of  $\bar{W}^{sc} \cap \bar{W}^{uc}$ . It follows from Lemma A.1 that this point is contained in  $B^u \times B^s$ . Then, it depends in a  $C^1$  way of the parameter  $c$ . We have proved that  $W^c$  is the graph of a  $C^1$  function  $w^c$ . In order to estimate the Lipschitz constant of this graph, we consider two points  $(u_i, s_i, c_i), i = 0, 1$  in  $W^c$ . We have

$$\|u_1 - u_0\|^2 \leq K^2(\|s_1 - s_0\|^2 + \|c_1 - c_0\|^2)$$

and

$$\|s_1 - s_0\|^2 \leq K^2(\|u_1 - u_0\|^2 + \|c_1 - c_0\|^2).$$

Taking the sum gives

$$(1 - K^2)(\|u_1 - u_0\|^2 + \|s_1 - s_0\|^2) \leq 2K^2\|c_1 - c_0\|^2$$

and

$$\|(u_1, s_1) - (u_0, s_0)\| \leq \sqrt{\frac{2K^2}{1 - K^2}}\|c_1 - c_0\| \leq 2K\|c_1 - c_0\|,$$

since  $K \leq 1/\sqrt{2}$ . We conclude that  $w^c$  is  $2K$ -Lipschitz.  $\square$

It is useful to go a bit further in the study of the invariant manifold  $W^c = \{(w_u^c(c), w_s^c(c), c)\}$ . This manifold is a partially hyperbolic invariant set, hence by the usual theory, to each point  $x \in W^c$  is attached a strong stable manifold  $W^s(x)$  and a strong unstable manifold  $W^u(x)$ , which are  $C^1$  (and even  $C^r$  if  $F$  is  $C^r$ ). The manifolds  $W^u(x), x \in W^c$  partition  $W^{uc}$ , although this partition is not usually a  $C^1$  foliation. For each  $x \in W^{uc}$ , we denote by  $E^u(x)$  the strong unstable space, which is the tangent space at  $x$  of the only unstable manifold  $W^u(x_0)$  which contains  $x_0$ . We define the exponents

$$\begin{aligned} e_u &:= - \sup_{x \in W^c, v \in E^u(x)} \limsup_{t \rightarrow \infty} \log(\|v(-t)\|)/t, \\ &= - \sup_{x \in W^{uc}, v \in E^u(x)} \limsup_{t \rightarrow \infty} \log(\|v(-t)\|)/t, \\ e_c^+ &:= \sup_{x \in W_c, v \in T_x W_c} \limsup_{t \rightarrow \infty} \log(\|v(t)\|)/t, \\ e_c^- &:= \inf_{x \in W_c, v \in T_x W_c} \liminf_{t \rightarrow \infty} \log(\|v(t)\|)/t, \end{aligned}$$

where  $v(t)$  is the solutions of the linearized equation  $\dot{v}(t) = dF_{x(t)} \cdot v(t)$  with initial condition  $v(0) = v$ , and  $x(t)$  is the solution of  $\dot{x}(t) = F \circ x(t)$  starting from  $x(0) = x$ .

**Lemma A.2.**

$$-m - 2mK \leq e_c^- \leq e_c^+ \leq m + 2mK.$$

*Proof.* We consider an orbit  $x(t) \in W^c$ , and a variational orbit  $v(t) = (u'(t), s'(t), c'(t))$  tangent to  $W^c$ . Observe that  $\|(u', s')\| \leq 2K\|c'\|$  for each  $t$ , which implies:

$$\left| \frac{d}{dt} \|c'\|^2 \right| = 2|\langle c', L_{cu}u' + L_{cs}s' + L_{cc}c' \rangle| \leq 2(m + 2Km)\|c'\|^2.$$

$\square$

The next Lemma implies that the manifolds  $W^{s,c}(x)$  are the graphs of  $C^1$  and  $K$ -Lipschitz maps  $w_x^s : B^s \rightarrow B^u \times \Omega^c, w_x^u : B^u \rightarrow B^s \times \Omega^c$ .

**Lemma A.3.** *If  $x(t) : ]T^-, T^+[ \rightarrow \Omega$  is an orbit of  $F$ , then the linearized equation  $\dot{v}(t) = dF_{x(t)} \cdot v(t)$  preserves the cone  $C^u = \{\|(s', c')\| \leq K\|u'\|\}$  in forward time, and the cone  $C^s = \{\|(u', c')\| \leq K\|s'\|\}$  in backward time.*

*We have  $E^u(x) \subset C^u$  for each  $x \in W^{uc}$ ,  $E^s(x) \subset C^s$  for each  $x \in W^{sc}$ .*

*Finally we have the estimate*

$$e_u \geq \alpha - 2mK > \alpha/2.$$

*Proof.* Let  $v(t) = (u'(t), s'(t), c'(t))$  be a solution of the linearized equation along  $x(t)$ . Then

$$\frac{d}{dt}\|u'\|^2 = \langle u', L_{uu}u' + L_{us}s' + L_{uc}c' \rangle \geq \alpha\|u'\|^2 - m\|(s', c')\|\|u'\| \geq (\alpha - mK)\|u'\|^2$$

(this estimate will also provide the desired growth rate in the unstable direction) and

$$\begin{aligned} \frac{d}{dt}\|(s', c')\|^2 &= \langle s', L_{su}u' + L_{ss}s' + L_{sc}c' \rangle + \langle c', L_{cu}u' + L_{cs}s' + L_{cc}c' \rangle \\ &\leq m\|(s', c')\|(\|u'\| + \|(s', c')\|) \leq mK(1 + K)\|u'\|^2. \end{aligned}$$

This implies that

$$\frac{d}{dt}(K^2\|u'\|^2 - \|(s', c')\|^2) \geq K^2(\alpha - mK - m - m/K)\|u'\|^2 \geq 0,$$

recalling that  $m + m/K + mK < \alpha$ . The estimates concerning  $C^s$  are similar.  $\square$

In general, the maps  $w_x^s$  and  $w_x^u$  are not better than (Hölder)-continuous in  $x$ , but we can obtain a better regularity under stronger hypotheses:

**Theorem A.2.** *In the context of Theorem A.1, let us assume the additional assumptions that  $F$  is  $C^2$  and  $K < 1/8$  (or equivalently,  $m < \alpha/6$ ). Then each of the manifolds  $W^c, W^{uc}, W^{sc}$  is  $C^2$ , and the manifolds  $W^u(x), x \in W^c$  form a  $C^1$  foliation of  $W^{uc}$  (similarly for  $W^s$  in  $W^{us}$ ). The foliations are  $C^1$  in the strongest possible sense, namely the map  $x \mapsto E^u(x)$  is  $C^1$  on  $E^{cu}$ , which imply that the foliation admits  $C^1$  charts, and that the local holonomies are  $C^1$ .*

*Proof.* An easy computation shows that  $m + 2mK < \alpha/4$ , hence we obtain

$$e_u > \alpha/2, \quad e_c^+ < \alpha/4, \quad e_c^- > -\alpha/4.$$

This implies that  $e_u > 2e_c^+$ , hence  $W^c$  is 2-normally hyperbolic, hence it is  $C^2$ , as well as  $W^{uc}$  and  $W^{sc}$ , see [Fe, HPS].

Moreover, we have the bunching condition  $e_u > e_c^+ - e_c^-$ , which implies the  $C^1$  regularity of the unstable foliation, see [Fe2, PSW, DLS].  $\square$

We need the following easy addendum:

**Proposition A.3.** *Assume in addition that there exists a translation  $g$  of  $\mathbb{R}^{n_c}$  such that*

$$g(\Omega^c) = \Omega^c \quad \text{and} \quad F \circ (id \otimes id \otimes g) = F.$$

*Then we have*

$$w^{sc} \circ (id \otimes g) = w^{sc}, \quad w^{uc} \circ (id \otimes g) = w^{uc}, \quad w^c \circ g = w^c.$$

*Proof.* It follows immediately from the definition of the sets  $W^{sc}$ ,  $W^{uc}$  and  $W^c$  that  $g(W^{sc}) = W^{sc}$ ,  $g(W^{uc}) = W^{uc}$  and  $g(W^c) = W^c$ .  $\square$

In applications the first condition of Hypothesis 3 is usually not satisfied, except in the case where  $\Omega^c = \mathbb{R}^{n_c}$ . In view of the applications we have in mind, it is useful to split the central variables into two groups and consider

$$\Omega^c = \mathbb{R}^{n_c^1} \times \Omega^{c_2},$$

where  $\Omega^{c_2}$  is a convex open set in  $\mathbb{R}^{n_c^2}$ ,  $n_c^1 + n_c^2 = n_c$ . Given a positive parameter  $\sigma$ , let  $\Omega_\sigma^{c_2}$  be the set of points  $c_2 \in \mathbb{R}^{n_c^2}$  such that  $d(c, \Omega^{c_2}) < \sigma$ . This is a convex open subset of  $\mathbb{R}^{n_c^2}$  containing  $\Omega^{c_2}$ . We denote by  $\Omega_\sigma^c$  the product  $\mathbb{R}^{n_c^1} \times \Omega_\sigma^{c_2}$  and by  $\Omega_\sigma$  the product  $B^u \times B^s \times \Omega_\sigma^c$ . With the notation  $F_c = (F_{c_1}, F_{c_2})$ , and denoting by  $W^{sc}(F, \Omega)$ ,  $W^{uc}(F, \Omega)$ ,  $W^c(F, \Omega)$  the set of positive half orbits (resp. negative half orbits, full orbits) of  $F$  contained in  $\Omega$ , we have:

**Proposition A.4.** *Let  $F : \mathbb{R}^{n_u} \times \mathbb{R}^{n_s} \times \Omega_\sigma^c \rightarrow \mathbb{R}^{n_u} \times \mathbb{R}^{n_s} \times \mathbb{R}^{n_c}$  be a  $C^2$  vector field. Assume that there exists  $\lambda, m, \sigma > 0$  such that*

- $F_u(u, s, c) \cdot u > 0$  on  $\partial B^u \times \bar{B}^s \times \bar{\Omega}_\sigma^c$ .
- $F_s(u, s, c) \cdot s < 0$  on  $\bar{B}^u \times \partial B^s \times \bar{\Omega}_\sigma^c$ .
- $L_{uu}(x) \geq \alpha I$ ,  $L_{ss}(x) \leq -\alpha I$  for each  $x \in \Omega_\sigma$  in the sense of quadratic forms.
- $\|L_{us}(x)\| + \|L_{uc}(x)\| + \|L_{ss}(x)\| + \|L_{sc}(x)\| + \|L_{cu}(x)\| + \|L_{cs}(x)\| + \|L_{cc}(x)\| \leq m$  for each  $x \in \Omega_\sigma$ .
- $\|L_{us}(x)\| + \|L_{uc}(x)\| + \|L_{ss}(x)\| + \|L_{sc}(x)\| + \|L_{cu}(x)\| + \|L_{cs}(x)\| + \|L_{cc}(x)\| + 2\|F_{c_2}(x)\|/\sigma \leq m$  for each  $x \in \Omega_\sigma - \Omega$ .

Assume furthermore that

$$K := \frac{m}{\alpha - 2m} \leq \frac{1}{8},$$

then there exist  $C^2$  maps

$$w^{sc} : B^s \times \Omega_\sigma^c \rightarrow B^u, \quad w^{uc} : B^u \times \Omega_\sigma^c \rightarrow B^s, \quad w^c : \Omega_\sigma^c \rightarrow B^u \times B^s$$

satisfying the estimates

$$\|dw^{sc}\| \leq K, \quad \|dw^{uc}\| \leq K, \quad \|dw^c\| \leq 2K,$$

the graphs of which respectively contain  $W^{sc}(F, \Omega)$ ,  $W^{uc}(F, \Omega)$ ,  $W^c(F, \Omega)$ . Moreover, the graphs of the restrictions of  $w^{sc}$ ,  $w^{uc}$  and  $w^c$  to, respectively,  $B^s \times \Omega^c$ ,  $B^u \times \Omega^c$  and  $\Omega^c$ , are tangent to the flow.

There exists an invariant  $C^1$  foliation of the graph of  $w^{uc}$  whose leaves are graphs of  $K$ -Lipschitz maps above  $B^u$ . The set  $W^{uc}(F, \Omega)$  is a union of leaves : it has the structure of an invariant  $C^1$  lamination. Two points  $x, x'$  belong to the same leaf of this lamination if and only if  $d(x(t), x'(t))e^{t\alpha/4}$  is bounded on  $\mathbb{R}^-$ .

If in addition there exists a group  $G$  of translations of  $\mathbb{R}^{n_{c_1}}$  such that  $F \circ (id \otimes id \otimes g \otimes id) = F$  for each  $g \in G$ , then the maps  $w^*$  can be chosen such that

$$w^{sc} \circ (id \otimes g \otimes id) = w^{sc}, \quad w^{uc} \circ (id \otimes g \otimes id) = w^{uc}, \quad w^c \circ (g \otimes id) = w^c \quad (34)$$

for each  $g \in G$ . The lamination is also translation invariant.



In contrast to the earlier results of this section, the map  $w^{sc}$  is not uniquely defined, and neither is its restriction to  $B^s \times \Omega^c$ . Moreover, the intersection with  $\Omega$  of the graph of  $w^{sc}$  is not necessarily positively invariant. It can contain strictly the set  $W^{sc}(F, \Omega)$ . Similar remarks apply to  $w^{uc}$  and  $w^c$ .

*Proof.* We take a function  $\rho : \Omega_\sigma^{c_2} \rightarrow [0, 1]$  such that :

- $\rho = 0$  near the boundary of  $\Omega_\sigma^{c_2}$ ,
- $\rho = 1$  on  $\Omega^{c_2}$ ,
- $\|d\rho\| \leq 2/\sigma$  uniformly.

We claim that the vectorfield

$$\tilde{F}(u, s, c) := (F_u(u, s, c_1, c_2), F_s(u, s, c_1, c_2), F_{c_1}(u, s, c_1, c_2), \rho(c_2)F_{c_2}(u, s, c_1, c_2))$$

satisfies all the hypotheses of Theorem A.1 on  $\Omega_\sigma$ . Note also that  $\tilde{F} = F$  on  $\Omega$ . Denoting by  $\tilde{L}_{**}$  the variational matrix associated to  $\tilde{F}$ , we see that

$$\begin{aligned} \tilde{L}_{cu}(u, s, c) &= \rho(c_2)L_{cu}(u, s, c), & \tilde{L}_{cs}(u, s, c) &= \rho(c_2)L_{cs}(u, s, c), \\ \tilde{L}_{c_1c_1}(u, s, c) &= \rho(c_2)L_{c_1c_1}(u, s, c), & \tilde{L}_{c_1c_2}(u, s, c) &= \rho(c_2)L_{c_1c_2}(u, s, c), \end{aligned}$$

and

$$\tilde{L}_{c_2c_2}(u, s, c) = \rho(c_2)L_{c_2c_2}(u, s, c) + d\rho(c_2) \otimes F_{c_2}(u, s, c).$$

As a consequence, we have

$$\begin{aligned} & \|\tilde{L}_{us}(x)\| + \|\tilde{L}_{uc}(x)\| + \|\tilde{L}_{ss}(x)\| + \|\tilde{L}_{sc}(x)\| + \|\tilde{L}_{cu}(x)\| + \|\tilde{L}_{cs}(x)\| + \|\tilde{L}_{cc}(x)\| \\ &= \rho(c_2)(\|L_{us}(x)\| + \|L_{uc}(x)\| + \|L_{ss}(x)\| + \|L_{sc}(x)\| + \|L_{cu}(x)\| + \|L_{cs}(x)\| + \|L_{cc}(x)\|) \\ &+ \|F_{c_2}(x)\| \|d\rho(c_2)\| \leq m. \end{aligned}$$

The claim is proved. We define  $w^{sc}, w^{uc}, w^c$  as the maps given by Theorem A.1 applied to  $\tilde{F}$  on  $\Omega_\sigma$ . Since  $\tilde{F} = F$  on  $\Omega$ , we have  $W^*(F, \Omega) \subset W^*(\tilde{F}, \Omega_\sigma)$  for  $* = sc, uc$  or  $c$ . These maps may depend on the choice of the function  $\rho$  but, once the function  $\rho$  is chosen, they are uniquely defined. In the case where a group of translation  $G$  exists as in the statement, then we have  $\tilde{F} \circ (id \otimes id \otimes g \otimes id) = \tilde{F}$  for each  $g \in G$ . The uniqueness then implies (34). By definition,  $W^*(\tilde{F}, \Omega_\sigma)$  is the graph of  $w^*$ , the statement follows from this observation.  $\square$

## B Disconnectedness of Heteroclinics

We consider a Tonelli Hamiltonian  $H$ , a cohomology  $c$ , and the associated Aubry and Mañé sets  $\tilde{\mathcal{A}}$  and  $\tilde{\mathcal{N}}$ . We assume that the Aubry set is the union of two static classes  $\tilde{\mathcal{S}}_i, i = 1, 2$ . The Mañé set can then be written as the disjoint union

$$\tilde{\mathcal{N}} = \tilde{\mathcal{S}}_1 \cup \tilde{\mathcal{S}}_2 \cup \tilde{\mathcal{H}}_{12} \cup \tilde{\mathcal{H}}_{21},$$

where  $\tilde{\mathcal{H}}_{12}$  is a set of heteroclinic orbits from  $\tilde{\mathcal{S}}_1$  to  $\tilde{\mathcal{S}}_2$ , and  $\tilde{\mathcal{H}}_{21}$  is a set of heteroclinic orbits from  $\tilde{\mathcal{S}}_2$  to  $\tilde{\mathcal{S}}_1$ . Moreover, the sets

$$\tilde{\mathcal{I}}_{12} := \tilde{\mathcal{S}}_1 \cup \tilde{\mathcal{S}}_2 \cup \tilde{\mathcal{H}}_{12} \quad \text{and} \quad \tilde{\mathcal{I}}_{21} := \tilde{\mathcal{S}}_1 \cup \tilde{\mathcal{S}}_2 \cup \tilde{\mathcal{H}}_{21}$$

are invariant compact Lipschitz graphs. In the notations of [Be1], we have  $\tilde{\mathcal{I}}_{12} = \tilde{\mathcal{I}}(E_{\mathcal{S}_1}) = E_{\mathcal{S}_1} \wedge \check{E}_{\mathcal{S}_2}$ ,  $\tilde{\mathcal{I}}_{21} = \tilde{\mathcal{I}}(E_{\mathcal{S}_2}) = E_{\mathcal{S}_2} \wedge \check{E}_{\mathcal{S}_1}$ .

In [Be1], Section 9, it is proved that the cohomology  $c$  is in the interior of its forcing class provided each of the sets  $\tilde{\mathcal{H}}_{12}$  and  $\tilde{\mathcal{H}}_{21}$  is neat in the following sense:

The set  $\tilde{\mathcal{H}}_{12}$  is neat if there exists a compact subset  $\tilde{\mathcal{K}}_{12}$  which contains one and only one point in each orbits of  $\varphi|_{\tilde{\mathcal{H}}_{12}}$  and which is acyclic, which means that there exist an open neighborhood  $U$  of  $\mathcal{K}_{12}$  in  $TM$  such that the inclusion of  $U$  into  $TM$  generates the null map in homology.

In Section 1.4 of the present paper, we apply this result under the assumption that the sets  $\tilde{\mathcal{H}}_{12}$  and  $\tilde{\mathcal{H}}_{21}$  are totally disconnected. We can do so in view of the following:

**Proposition B.1.** *The set  $\tilde{\mathcal{H}}_{12}$  (or  $\tilde{\mathcal{H}}_{21}$ ) is neat if it is totally disconnected.*

*Proof.* We first recall that a compact metric space is totally disconnected if and only if it has dimension zero, which means that each of its points has a basis of neighborhood made of open and closed sets, see [HW], section II.4.

By removing small open neighborhoods of  $\tilde{\mathcal{S}}_1$  and  $\tilde{\mathcal{S}}_2$  in  $\tilde{\mathcal{I}}_{12}$ , we form a compact subset of  $\tilde{\mathcal{H}}_{12}$  which contains at least one point in each orbit. This compact subset is totally disconnected (it is a subset of  $\tilde{\mathcal{H}}_{12}$ ) hence each of its points is contained in an open and closed set which is disjoint from both  $\tilde{\mathcal{S}}_1$  and  $\tilde{\mathcal{S}}_2$ . We cover our compact by finitely many of these neighborhood. Their union is a compact and open subset  $\tilde{\mathcal{Q}}$  of  $\tilde{\mathcal{H}}_{12}$  which contains at least one point in each orbit. The set  $\tilde{\mathcal{K}}_{12} := \tilde{\mathcal{Q}} - \varphi(\tilde{\mathcal{Q}})$  is then compact and open, and it contains exactly one point of each  $\varphi$ -orbit. It is totally disconnected, and therefore acyclic, in view of the following Lemma.  $\square$

**Lemma B.1.** *Let  $M$  be a manifold and let  $K \subset M$  be a totally disconnected compact subset of  $M$ . Then  $K$  is acyclic.*

*Proof.* The subset  $K$  has dimension 0, see [HW]. As a consequence, each point of  $K$  is contained in an open, closed, and acyclic neighborhood (small open sets are contained in discs hence are acyclic). We cover  $K$  by finitely many of these subsets  $U_1, \dots, U_k$  and set  $V_1 = U_1$ ,  $V_2 = U_2 - V_1$ ,  $V_i = U_i - V_{i-1}$ . We obtain  $k$  open acyclic subsets  $V_i$  which are pairwise disjoint and cover  $K$ . This implies that  $K$  is acyclic.  $\square$

## C Continuity property of the Peierls' barrier function

We consider here a general Tonelli Lagrangian  $L$ . We recall, see [Be2], section 4, that the difference of two weak KAM solutions is constant on each static class.

**Proposition C.1.** *Let  $L_k \rightarrow L$  be a sequence of Tonelli Lagrangians  $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T}$  converging in the  $C^2$  compact open topology, and  $c_k \rightarrow c \in \mathbb{R}^n \simeq H^1(\mathbb{T}^n, \mathbb{R})$ . Assume that  $\mathcal{A}_L(c)$  has finitely many static classes. Let  $\zeta_k \in \mathcal{A}_{L_k}(c_k)$  be such that  $\zeta_k \rightarrow \zeta_0 \in \mathcal{A}_L(c)$ , then for any  $\theta \in \mathbb{T}^n$ ,*

$$\lim_{k \rightarrow \infty} h_{L_k, c_k}(\zeta_k, \theta) = h_c(\zeta_0, \theta).$$

*Proof.* First, since each  $A_{L, c}^M(\theta_1, \theta_2)$  is continuous in  $L$  and  $c$ , we obtain

$$\lim_{k \rightarrow \infty} \Phi_{L_k, c_k}(\theta_1, \theta_2) \leq \lim_{k \rightarrow \infty} (A_{L_k, c_k}^M(\theta_1, \theta_2)) = A_{L, c}^M(\theta_1, \theta_2)$$

taking infimum over  $N$ , we get  $\lim_{k \rightarrow \infty} \Phi_{L_k, c_k}(\theta_1, \theta_2) \leq \Phi_{L, c}(\theta_1, \theta_2)$ . Since  $h_{L, c}(\theta_1, \theta_2) = \Phi_{L, c}(\theta_1, \theta_2)$  if either  $\theta_1$  or  $\theta_2$  is in  $\mathcal{A}_L(c)$ , we obtain

$$\lim_{k \rightarrow \infty} h_{c_k}(\zeta_k, \theta) \leq h_c(\zeta_0, \theta).$$

Given  $\varepsilon_k \rightarrow 0$ , let  $\gamma_k : [-Q_k, 0] \rightarrow \infty$  be a sequence of extremal curves such that  $\gamma_k(-Q_k) = \zeta_k$ ,  $\gamma_k(0) = \theta$ , and

$$A_{L_k, c_k}^{Q_k}(\zeta_k, \theta) \leq h_{L_k, c_k}(\zeta_k, \theta) + \varepsilon_k.$$

We note that on each interval  $[i, j] \subset [-Q_k, 0]$ , we have

$$\begin{aligned} & A_{L_k, c_k}^{j-i}(\gamma_k(i), \gamma_k(j)) \\ &= A^{Q_k}(\gamma_k(-Q_k), \gamma_k(0)) - A^{i+Q_k}(\gamma_k(-Q_k), \gamma_k(i)) - A^{-j}(\gamma_k(j), \gamma_k(0)) \\ &\leq h(\zeta_k, \theta) - \varepsilon_k - (h(\zeta_k, \gamma_k(i)) - h(\zeta_k, \gamma_k(-Q_k))) - (h(\zeta_k, \gamma_k(0)) - h(\zeta_k, \gamma_k(j))) \\ &\leq h_{L_k, c_k}(\zeta_k, \gamma_k(j)) - h_{L_k, c_k}(\zeta_k, \gamma_k(i)) + \varepsilon_k, \end{aligned} \quad (35)$$

since  $h(\zeta_k, \gamma_k(-Q_k)) = h(\zeta_k, \zeta_k) = 0$  and  $\gamma_k(0) = \theta$ . Note we omit the subscript  $L_k, c_k$  in the intermediate calculations.

Let  $i_k, i'_k$  be two consecutive visit of  $\gamma_k(i)$  to  $U = B_\delta(\mathcal{A}_L(c))$ , we first show that  $i'_k - i_k$  must be bounded as  $k \rightarrow \infty$ . Assume otherwise, then the curves  $\gamma_k(t + i_k + 1)|_{[0, i'_k - i_k - 2]}$  converges in uniformly over compact sets to  $\gamma_* : [0, \infty) \rightarrow \mathbb{T}^n$ . Assume the weak KAM solutions  $h_{L_k, c_k}(\zeta_k, \cdot)$  converges uniformly to a weak KAM solution  $u$  of  $L, c$ , taking limit in (35) implies  $\gamma_*$  must be calibrated by  $u$ . Therefore  $\gamma_*$  must accumulates to  $\mathcal{A}_L(c)$  which is a contradiction.

Let  $\mathcal{S}_1, \dots, \mathcal{S}_r$  be the static classes of  $\mathcal{A}(L)$ . Denote  $U_q = B_\delta(\mathcal{S}_q)$  and assume  $\delta$  is small enough so that  $U_q$  are all disjoint. Let us note each  $\gamma_k$  determines sequences  $q_s \in \{1, \dots, r\}$ ,  $s = 1, \dots, r$ , and  $0 = i_0 \leq j_0 \leq \dots \leq i_r \leq j_r \leq Q_k$  as follows.

- Set  $i_0 = j_0 = 0$ .
- Let  $i_1$  be the first visit of  $\gamma(-i)$  to  $\bigcup_q U_q$  and  $U_{q_1}$  is the set that  $\gamma(-i_1)$  visits. Let  $j_1$  be the last visit to  $U_{q_1}$ , namely  $j_1 = \max\{i : \gamma(-i) \in U_{q_1}\}$ .
- The process stops if  $j_{s-1} = -Q_k$ , we set then set  $i_s = j_s = \dots = i_r = j_r = Q_k$ , and  $q_s = \dots = q_r = q_{s-1}$ .

Otherwise, let  $i_s$  be the first visits to  $\bigcup_q U_q$  for  $i > j_{s-1}$ , and  $U_{q_s}$  the set it visits. Define  $j_s$  be the last visit to  $U_{q_s}$  and continue.

Then

$$\begin{aligned} & h_{L_k, c_k}(\zeta_k, \theta) + \varepsilon_k \geq A_{L_k, c_k}^{Q_k}(\gamma_k(-Q_k), \gamma_k(0)) \\ &= \sum_{s=1}^r A^{i_s - j_{s-1}}(\gamma_k(-i_s), \gamma_k(-j_{s-1})) + \sum_{s=1}^r A^{j_s - i_s}(\gamma_k(-j_s), \gamma_k(-i_s)) \\ &\geq \sum_{s=1}^r A^{i_s - j_{s-1}}(\gamma_k(-i_s), \gamma_k(-j_{s-1})) + \sum_{s=1}^r (h(\zeta_k, \gamma_k(-i_s)) - h(\zeta_k, \gamma_k(-j_s))) - r\varepsilon_k, \end{aligned} \quad (36)$$

where the subscript  $L_k, c_k$  was omitted in the last two lines. By restricting to a subsequence, we may assume that for all  $\gamma_k$ , the ordering  $q_1, \dots, q_r$  are identical. Our previous observation implies for  $s = 1, \dots, r$ ,  $i_s - j_{s-1}$  are bounded as  $k \rightarrow \infty$ . By restricting to another subsequence, we may assume  $i_s - j_{s-1}$  is constant for all  $k$ , and  $\gamma_k(-i_s) \rightarrow \theta_s$ ,  $\gamma_k(-j_s) \rightarrow \theta'_s$  as  $k \rightarrow \infty$ . Note that for  $s = 1, \dots, r$ ,  $\theta_s, \theta'_s \in \overline{B_\delta(\mathcal{S}_{q_s})}$ , therefore, there exists  $\eta_s, \eta'_s \in \mathcal{S}_{q_s}$  such that  $\|\theta_s - \eta_s\|, \|\theta'_s - \eta'_s\| \leq \delta$ . Let us also note, by definition  $\theta_0 = \theta'_0 = \theta$ ,  $\theta_r = \theta'_r = \zeta_0$ . Define

$\eta_0 = \eta'_0 = \theta$  and  $\eta_r = \eta'_r = \zeta_0$ . Up to taking a subsequence, assume the weak KAM solutions  $h_{L_k, c_k}(\zeta_k, \cdot) \rightarrow u(\cdot)$  uniformly. Take limit as  $k \rightarrow \infty$  in (36), we obtain

$$\begin{aligned}
\lim_{k \rightarrow \infty} h_{L_k, c_k}(\zeta_k, \theta) &\geq \sum_{s=1}^r \left( A_{L, c}^{i_s - j_{s-1}}(\theta_s, \theta'_{s-1}) + u(\theta_s) - u(\theta'_s) \right) \\
&\geq \sum_{s=1}^r \left( A_{L, c}^{i_s - j_{s-1}}(\eta_s, \eta'_{s-1}) + u(\eta_s) - u(\eta'_s) - 4C\delta \right) \\
&= \sum_{s=1}^r \left( A_{L, c}^{i_s - j_{s-1}}(\eta_s, \eta'_{s-1}) + h_{L, c}(\zeta_0, \eta_s) - h_{L, c}(\zeta_0, \eta'_s) - 4C\delta \right) \\
&\geq \sum_{s=1}^r \left( h_{L, c}(\zeta_0, \eta'_{s-1}) - h_{L, c}(\zeta_0, \eta'_s) - 4C\delta \right) \\
&= h_{L, c}(\zeta_0, \eta'_0) - h_{L, c}(\zeta_0, \eta'_r) - 4rC\delta = h_{L, c}(\zeta_0, \theta) - 4rC\delta.
\end{aligned}$$

Since  $\delta$  is arbitrary, we obtain  $\lim_{k \rightarrow \infty} h_{L_k, c_k}(\zeta_k, \theta) \geq h_{L, c}(\zeta_0, \theta)$ .  $\square$

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