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Abstract

In this paper, we extend the optimal securitization model of Pagès [41] and Possamaï and Pagès [42] between an investor and a bank to a setting allowing both moral hazard and adverse selection. Following the recent approach to these problems of Cvitanić, Wan and Yang [12], we characterize explicitly and rigorously the so-called credible set of the continuation and temptation values of the bank, and obtain the value function of the investor as well as the optimal contracts through a recursive system of first-order variational inequalities with gradient constraints. We provide a detailed discussion of the properties of the optimal menu of contracts.

Key words: bank monitoring, securitization, moral hazard, adverse selection, principal-agent problem

AMS 2000 subject classification: 60H30, 91G40

JEL classifications: G21, G28, G32

1 Introduction

Principal-Agent problems with moral hazard have an extremely rich history, dating back to the early static models of the 70s, see among many others Zeckhauser [59], Spence and Zeckhauser [54], or Mirrlees [33, 34, 35, 36], as well as the seminal papers by Grossman and Hart [19], Jewitt, [25], Holmström [23] or Rogerson [48]. If moral hazard results from the inability of the Principal to monitor, or to contract upon, the actions of the Agent, there is a second fundamental feature of the Principal-Agent relationship which has been very frequently studied in the literature, namely that of adverse selection, corresponding to the inability to observe private information of the Agent, which is often referred to as his type. In this case, the Principal offers to the Agent a menu of contracts, each having been designed for a specific type. The so-called *revelation principle*, states then that it is always optimal for the Principal to propose menus for which it is optimal for the Agent to truthfully reveal his type. Pioneering research in the latter direction were due to Mirrlees [37], Mussa and Rosen [38], Roberts [46], Spence [53], Baron and Myerson [6], Maskin and Riley [29], Guesnerie and Laffont

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[20], and later by Salanié [50], Wilson [58], or Rochet and Choné [47]. However, despite the early realisation of the importance of considering models involving both these features at the same time, the literature on Principal-Agent problems involving both moral hazard and adverse selection has remained, in comparison, rather scarce. As far as we know, they were considered for the first time by Antle [1], in the context of auditor contracts, and then, under the name of generalised Principal-Agent problems, by Myerson [39][1]. These generalised agency problems were then studied in a wide variety of economic settings, notably by Dionne and Lasserre [14], Laffont and Tirole [27], McAfee and McMillan [30], Picard [44], Baron and Besanko [3, 4], Melumad and Reichelstein [31, 32], Guesnerie, Picard and Rey [21], Page [40], Zou[60], Caillaud, Guesnerie and Rey [9], Lewis and Sappington [28], or Bhattacharyya [7][2].

All the previous models are either in static or discrete-time settings. The first study of the continuous time problem with moral hazard and adverse selection was made by Sung [55], in which the author extends the seminal finite horizon and continuous-time model of Holmström and Milgrom [24]. A more recent work, to which our paper is mostly related has been treated by Civtanić, Wan and Yang [12], where the authors extend the famous infinite horizon model of Sannikov [51] to the adverse selection setting. If one of the main contributions of Sannikov [51] was to have identified that the *continuation value* of the Agent was a fundamental state variable for the problem of the Principal, [12] shows that in a context with both moral hazard and adverse selection, the Principal has also to keep track of the so-called *temptation value*, that is to say the continuation utility of the Agent who would not reveal his true type. Although close to the latter paper, our work is foremost an extension of the bank incentives model of Pagès and Possamaï [42], which studies the contracting problem between competitive investors and an impatient bank who monitors a pool of long-term loans subject to Markovian contagion (we also refer the reader to the companion paper by Pagès [41] for the economic intuitions and interpretations of the model). In the model of [42], moral hazard emerges because the bank has more "skin a game" than the investors, and has the opportunity, *ex ante* and *ex post*, to exercise a (costly) monitoring of the non-defaulted loans. This is a stylised way to sum up all the actions than the bank can enter into to ensure itself of the solvability of the borrowers. Since the investors cannot observe the monitoring effort of the bank, they offer CDS type contracts offering remuneration to the bank, and giving it incentives through postponement of payments and threat of stochastic liquidation of the contract (similarly to the seminal paper of Biais, Mariotti, Rochet and Villeneuve [8]). In the present paper, we assume furthermore that there are two types of banks, which we coin good and bad, co-existing in the market, differing by their efficiency in using their remuneration (or equivalently differing by their monitoring costs). Even if the investor is supposed to know the distribution of the type of banks, he cannot know whether the one is entering into a contract with is good or bad.

Mathematically speaking, we follow both the general dynamic programming approach of Cvitanić, Possamaï and Touzi [11], as well as the take on adverse selection problems initiated by [12]. Intuitively, these approaches require first, using martingale (or more precisely backward SDEs) arguments, to solve the (non-Markovian) optimal control problem faced by the two type of banks when choosing each contracts. This requires obviously, using the terminology introduced above, to keep track of both the *continuation value* and the *temptation value* of the banks, when they choose the contract designed for them or not. The problem of the Principal rewrites then as two standard stochastic control problems,

[1]There were earlier attempts in this direction, but providing a less systematic treatment of the problem; see the income tax model of Mirrlees [37], the Soviet incentive scheme study of Weitzman [57], or the papers by Baron and Holmström [5] and Baron [2].

[2]We refer the interested reader to the more recent works of Faynzilberg and Kumar [16], Theilen [56], Jullien, Salanié and Salanié [26], Gottlieb and Moreira [18].

one in which he hires the good bank, and one in which he hires the bad one. Each of these problems uses in turn the aforementioned two state variables (and these two only, because the horizon is infinite and the Principal is risk-neutral), with truth-telling constraint, asserting that the continuation value should always be greater than the temptation value. This leads to optimal control problems with state constraints, and thus to Hamilton-Jacobi-Bellman (HJB for short) equations (or more precisely variational inequalities with gradient constraints, since our problem is actually a singular stochastic control problem) in a domain, which, following [12], we call the credible set. This set is defined as the set containing the pair of value functions of the good and bad bank under every admissible contract offered by the investor. The determination of this set is the first fundamental step in our approach. Following the the orignal ideas of [12], we prove that the determination of the boundaries of this set can be achieved by solving two so-called double-sided moral hazard problems, in which one of the type of banks is actually hiring the other one. Fortunately for us, it turned out to be possible to obtain rigorously[3] explicit expressions for these boundaries by solving the associated system of HJB equations and using verification type arguments. We also would like to emphasise that unlike in [12], there is certain dynamic component in our model, since we have to keep track of the number of non-defaulted loans, through a time inhomogeneous Poisson process. This leads to a dynamic credible set, as well as, in the end, to a *recursive system* of HJB equations characterising the value function of the Principal.

After having determined the credible set itself, we pursue our study by concentrating on two specific forms of contracts: the shutdown contract in which the investor designs a contract which will be accepted only by the good bank, and the more classical screening contract, corresponding to a menu of contracts, one for each type of bank, which provides incentives to reveal her true type and choose the contract designed for her. These two contracts correspond simply to the offering, over the correct domain of expected utilities of the banks (so as to satisfy the proper truth-telling and participation constraints), of the best contracts that the investor can design independently for hiring the good and the bad bank.

Since we characterise, under classical verification type arguments, the value function of the investor through a system of HJB equations, we also have classically access to the optimal contracts through this value function and its derivatives. This allows us to provide an associated qualitative and quantitative analysis. It turns out that he optimal contracts designed for the good and the bad bank share the same attributes, and are close in spirit to the ones derived in the pure moral hazard case in [42]. On the boundaries of the credible set, the value function of the bad bank plays the role of a state process. The payments of the optimal contracts are postponed until the moment the state process reaches a sufficiently high level, depending on the current size of the project. Similarly, when one of the loans of the pool defaults, the project is liquidated with a probability that decreases with the value of the state process. If the value function of the bad bank at the default time is below some critical level, the project will be liquidated for sure under the optimal contracts. On the other side, if the value function of the bad bank is high enough at the default time, the project will be maintained. In the interior of the credible set, the continuation value and the temptation value of the banks are the state processes for the optimal contracts. It is possible to identify zones of *good performance* inside of the credible set, where the agents are remunerated and the project is maintained in case a default occurs. It is also possible to identify zones of *bad performance*, where the agents are not paid and the project is liquidated in case of default. In the rest of the credible set the optimal contracts provide intermediary situations.

[3]Notice that in this respect the study in [12] was more formal, and our paper provides, as far as we know, the first rigorous derivation of this credible set.

The rest of the paper is organised as follows. In Section 2, we present the model, we define the set of admissible contracts and we state the investor's problem. In Section 3, we recall the results obtained in [42] for the case of pure moral hazard, which will be useful later on for us. In Section 4, we formally study the credible set and obtain an explicit expression for it. In Section 5, we study both the optimal shutdown and screening contract, describing their characteristics and the behaviour of the banks when they accept these contracts. The Appendix contains all the technical proofs of the paper.

Notations: Let $\mathbb{N}$ denote the set of non–negative integers. For any $n \in \mathbb{N}\backslash\{0\}$, we identify $\mathbb{R}^n$ with the set of $n-$dimensional column vectors. The associated inner product between two elements $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ will be denoted by $x \cdot y$. For simplicity of notations, we will sometimes write column vectors in a row form, with the usual transposition operator $\top$, that is to say $(x_1, \ldots, x_n)^\top \in \mathbb{R}^n$ for some $x_i \in \mathbb{R}$, $1 \leq i \leq n$. Let $\mathbb{R}_+$ denote the set of non–negative real numbers, and $\mathcal{B}(\mathbb{R}_+)$ the associated Borel $\sigma-$algebra. For any fixed non–negative measure ν on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, the Lebesgue–Stieljes integral of a measurable map $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ will be denoted indifferently

$$\int_{[u,t]} f(s)d\nu_s \text{ or } \int_u^t f(s)d\nu_s,\ 0 \leq u \leq t.$$

2 The model

This section is dedicated to the description of the model we are going to study, presenting the contracts as well as the criterion of both the Principal and the Agent. As recalled in the Introduction, it is actually an adverse selection extension of the model introduced first by Pagès in [41] and studied in depth by Pagès and Possamaï [42].

2.1 Preliminaries

We consider a model in continuous time, indexed by $t \in [0, \infty)$. Without loss of generality and for simplicity, the risk–free interest rate is taken to be 0[4]. Our first player will be a bank (the Agent, referred to as "she"), who has access to a pool of I unit loans indexed by $j = 1, \ldots, I$ which are *ex ante* identical. Each loan is a perpetuity yielding cash flow μ per unit time until it defaults. Once a loan defaults, it gives no further payments. As is commonplace in the Principal-Agent literature, especially since the paper of Sannikov [51], the infinite maturity assumption is here for simplicity and tractability, since it makes the problem stationary, in the sense that the value function of the Principal will not be time–dependent. We assume that the banks in the market are different, and that two types of banks coexist, each one being characterised by a parameter taking values in the set $\mathfrak{R} := \{\rho_g, \rho_b\}$ with $\rho_g > \rho_b$. We call the bank good (respectively bad) if its type is ρ_g (respectively ρ_b). Furthermore, it is considered to be common knowledge that the proportion of the banks of type ρ_i, $i \in \{g, b\}$, is p_i.

Denote by

$$N_t := \sum_{j=1}^{I} \mathbf{1}_{\{\tau^j \leq t\}},$$

the sum of individual loan default indicators, where τ^j is the default time of loan j. The current size of the pool is, at some time $t \geq 0$, $I - N_t$. Since all loans are *a priori* identical, they can be reindexed in any order after defaults. The action of the banks consists in deciding at each time $t \geq 0$ whether

[4] As already pointed out in the seminal paper of Biais, Mariotti, Rochet and Villeneuve [8], see also [42], the only quantity of interest here is the difference between the discounting factors of the Principal and the Agent.

they monitor any of the loans which have not defaulted yet. These actions are summarised by the functions $e_t^{j,i}$, where for $1 \leq j \leq I - N_t,\ i \in \{g, b\}$, $e_t^{j,i} = 1$ if loan j is monitored at time t by the bank of type ρ_i, and $e_t^{j,i} = 0$ otherwise. Non-monitoring renders a private benefit $B > 0$ per loan and per unit time to the bank, regardless of its type. The opportunity cost of monitoring is thus proportional to the number of monitored loans. Once more, more general cost structures could be considered, but this choice has been made for the sake of simplicity.

The rate at which loan j defaults is controlled by the hazard rate α_t^j specifying its instantaneous probability of default conditional on history up to time t. Individual hazard rates are assumed to depend on the monitoring choice of the bank and on the size of the pool. In particular, this allows to incorporate a type of contagion effect in the model. Specifically, we choose to model the hazard rate of a non–defaulted loan j at time t, when it is monitored (or not) by a bank of type ρ_i as

$$\alpha_t^{j,i} := \alpha_{I-N_t} \left(1 + \left(1 - e_t^{j,i}\right) \varepsilon\right),\ t \geq 0,\ j = 1, \ldots, I - N_t,\ i \in \{b, g\}, \tag{2.1}$$

where the parameters $\{\alpha_j\}_{1 \leq j \leq I}$ represent individual "baseline" risk under monitoring when the number of loans is j and $\varepsilon > 0$ is the proportional impact of shirking on default risk. We assume that the impact of shirking is independent of the type of the bank. Actually, we found out that differentiating between the banks in this regard created degeneracy in the model. We refer the reader to Section H in the Appendix for a more detailed explanation.

For $i \in \{b, g\}$, we define the shirking process k^i as the number of loans that the bank of type ρ_i fails to monitor at time $t \geq 0$. Then, according to (2.1), the corresponding aggregate default intensity is given by

$$\lambda_t^{k^i} := \sum_{j=1}^{I-N_t} \alpha_t^{j,i} = \alpha_{I-N_t} \left(I - N_t + \varepsilon k_t^i\right). \tag{2.2}$$

The banks can fund the pool internally at a cost $r \geq 0$. They can also raise funds from a competitive investor (the Principal, referred to as "he") who values income streams at the prevailing risk–less interest rate of zero. We assume that both the banks and the investor observe the history of defaults and liquidations, as well as the parameters p_b and p_g, but the monitoring choices and the type of the bank are unobservable for the investor.

2.2 Description of the contracts

Before going on, let us now describe the stochastic basis on which we will be working. We will always place ourselves on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which N is a Poisson process with intensity λ_t^0 (which is defined by (2.2)). We denote by $\mathbb{F} := (\mathcal{F}_t^N)_{t \geq 0}$ the $\mathbb{P}-$completion of the natural filtration of N. We call τ the liquidation time of the whole pool and let $H_t := \mathbf{1}_{\{t \geq \tau\}}$ be the liquidation indicator of the pool. We denote by $\mathbb{G} := (\mathcal{G}_t)_{t \geq 0}$ the minimal filtration containing $\mathbb{F}$ and that makes τ a $\mathbb{G}-$stopping time. We note that this filtration satisfies the usual hypotheses of completeness and right–continuity.

Contracts are offered by the investor to the bank and agreed upon at time 0. As usual in contracting theory, the bank can accept or refuse the contract, but once accepted, both the bank and the investor are fully committed to the contract. More precisely, the investor offers a menu of contracts $\Psi_i := (k^i, \theta^i, D^i)$, $i \in \{g, b\}$ specifying on the one hand a desired level of monitoring k^i for the bank of type ρ_i, which is a $\mathbb{G}-$predictable process such that for any $t \geq 0$, k_t^i takes values in $\{0, \ldots, I - N_t\}$ (this set is denoted by $\mathfrak{K}$), as well as a flow of payment D^i. These payments belong to set $\mathcal{D}$ of processes

which are càdlàg, non–decreasing, non–negative, $\mathbb{G}-$predictable and such that

$$\mathbb{E}^{\mathbb{P}}[D^i_\tau] < +\infty.$$

We do not rule out the possibility of immediate lump–sum payments at the initialisation of the contract, and therefore the processes in $\mathcal{D}$ are assumed to satisfy $D_{0^-} = 0$. Hence, if $D_0 \neq 0$, it means that a lump–sum payment has indeed been made.

The contract also specifies when liquidation occurs. We assume that liquidations can only take the form of the stochastic liquidation of all loans following immediately default[5] Hence, the contract specifies the probability θ^i_t, which belongs to the set Θ of $[0,1]-$valued, $\mathbb{G}-$predictable processes, with which the pool is maintained given default ($dN_t = 1$), so that at each point in time, if the bank has indeed chosen the contract Ψ_i

$$dH_t = \begin{cases} 0 \text{ with probability } \theta^i_t, \\ dN_t \text{ with probability } 1 - \theta^i_t. \end{cases}$$

With our notations, given a contract Ψ_i, the hazard rates associated with the default and liquidation processes N_t and H_t are, if the bank does choose the contract Ψ_i, $\lambda^{k^i}_t$ and $\left(1-\theta^i_t\right)\lambda^{k^i}_t$, respectively.

The above properties translate into

$$\mathbb{P}\left(\tau \in \left\{\tau^1, ..., \tau^I\right\}\right) = 1, \text{ and } \mathbb{P}(\tau = \tau^j | \mathcal{F}_{\tau^j}, \tau > \tau^{j-1}) = 1 - \theta^i_{\tau^j}, \ j \in \{1, \ldots, I\}.$$

For ease of notations, a contract $\Psi := (k, \theta, D)$ will be said to be admissible if $(k, \theta, D) \in \mathfrak{K} \times \Theta \times \mathcal{D}$. As is commonplace in the Principal–Agent literature, we assume that the monitoring choices of the banks affect only the distribution of the size of the pool. To formalise this, recall that, by definition, any shirking process $k \in \mathfrak{K}$ is $\mathbb{G}-$predictable and bounded. Then, by Girsanov Theorem, we can define a probability measure $\mathbb{P}^k$ on $(\Omega, \mathcal{F})$, equivalent to $\mathbb{P}$, such that $N_t - \int_0^t \lambda^k_t ds$, is a $\mathbb{P}^k-$martingale. More precisely, we have on $\mathcal{G}_t$

$$\frac{d\mathbb{P}^k}{d\mathbb{P}} = Z^k_t,$$

where Z^k is the unique solution of the following SDE

$$Z^k_t = 1 + \int_0^t Z^k_{s^-}\left(\frac{\lambda^k_s}{\lambda^0_s} - 1\right)\left(dN_s - \lambda^0_s ds\right), \ 0 \leq t \leq T, \ \mathbb{P} - a.s.$$

Then, if the bank of type ρ_i chooses the contract Ψ_i, her utility at $t = 0$, if she follows the recommendation k^i, is given by

$$u^i_0(k^i, \theta^i, D^i) := \mathbb{E}^{\mathbb{P}^{k^i}}\left[\int_0^\tau e^{-rs}(\rho_i dD^i_s + B k^i_s \, ds)\right], \tag{2.3}$$

while that of the investor is

$$v_0\left((\Psi_i)_{i\in\{g,b\}}\right) := \sum_{i\in\{g,b\}} p_i \mathbb{E}^{\mathbb{P}^{k^i}}\left[\int_0^\tau (I - N_s)\,\mu ds - dD^i_s\right]. \tag{2.4}$$

The parameter ρ_i actually discriminates between the two types of banks through the way they derive utility from the cash–flows delivered by the investor. Hence, for a same level of salary, the good bank will get more utility than a bad bank. Such a form of adverse selection is also considered in the paper of Civtanić, Wan and Yang [12].

[5]Obviously, several other liquidations procedures could be considered. In the pure moral hazard case treated in [42] (see also the thesis [45, Chapter 8, Section 4]), which will be reviewed below in Section 3, some heuristic justifications are given, which lead to thinking that this should in general be, at least, not too far from optimality.

2.3 Formulation of the investor's problem

We assume for simplicity that the reservation utility for banks of both type is R_0. The investor's problem is to offer a menu of admissible contracts $(\Psi_i)_{i\in\{g,b\}} := (k^i, \theta^i, D^i)_{i\in\{g,b\}}$ which maximises his utility (2.4), subject to the three following constraints

$$u_0^i(k^i, \theta^i, D^i) \geq R_0, \ i \in \{g, b\}, \tag{2.5}$$

$$u_0^i(k^i, \theta^i, D^i) = \sup_{k\in\mathfrak{K}} u_0^i(k, \theta^i, D^i), \ i \in \{g, b\}, \tag{2.6}$$

$$u_0^i(k^i, \theta^i, D^i) \geq \sup_{k\in\mathfrak{K}} u_0^i(k, \theta^j, D^j), \ i \neq j, \ (i, j) \in \{g, b\}^2. \tag{2.7}$$

Condition (2.5) is the usual participation constraint for the banks. Condition (2.6) is the so–called incentive compatibility condition, stating that given (θ^i, D^i) the optimal monitoring choice of the bank of type ρ_i is the recommended effort k^i. Finally, Condition (2.7) means that if a bank adversely selects a contract, she cannot get more utility than if she had truthfully revealed her type at time 0. Following the literature, we call such a contract a *screening* contract.

In the sequel, we will start by deriving the optimal contract in the pure moral-hazard case, then we will look into the so–called optimal shutdown contract, for which the investor deliberately excludes the bad bank, before finally investigating the optimal screening contract.

3 The pure moral hazard case

In this section, we assume that the type of the bank is publicly known and is fixed to be some ρ_i, $i \in \{g, b\}$, which makes the problem exactly similar to the one considered in [42] (up to the modification of some constants). In particular, the investor only offers one contract. We will briefly explain how to solve the general maximisation problem for the bank and then recall the results obtained in [42]. Furthermore, the results we obtain here, in particular the dynamics of the continuation utilities of the banks, will be crucial to the study of the shutdown and screening contracts later on. Therefore, they will be used throughout without further references.

In this setting, the utility of the investor, when he offers a contract $(k^i, \theta^i, D^i) \in \mathfrak{K} \times \Theta \times \mathcal{D}$ is given by

$$v_0^{\mathrm{pm}}(k^i, \theta^i, D^i) := \mathbb{E}^{\mathbb{P}^{k^i}}\left[\int_0^\tau (I - N_s)\,\mu ds - dD_s^i\right], \tag{3.1}$$

for which we define the following dynamic version for any $t \geq 0$

$$v_t^{\mathrm{pm}}(k^i, \theta^i, D^i) := \mathbb{E}^{\mathbb{P}^{k^i}}\left[\int_{t\wedge\tau}^\tau (I - N_s)\,\mu ds - dD_s^i \middle| \mathcal{G}_t\right].$$

3.1 The bank's problem

3.1.1 Dynamics of the bank's value function

As usual, the so–called continuation value of the bank (that is to say her future expected payoff) when offered $(\theta^i, D^i) \in \Theta \times \mathcal{D}$ plays a central role in the analysis. It is defined, for any $(t, k) \in \mathbb{R}_+ \times \mathfrak{K}$ by

$$u_t^i(k, \theta^i, D^i) := \mathbb{E}^{\mathbb{P}^k}\left[\int_{t\wedge\tau}^\tau e^{-r(s-t)}\left(\rho_i dD_s^i + k_s B ds\right) \middle| \mathcal{G}_t\right].$$

We also define the value function of the bank for any $t \geq 0$

$$U_t^i(\theta^i, D^i) := \operatorname*{ess\,sup}_{k \in \mathfrak{K}} u_t^i(k, \theta^i, D^i).$$

Departing slightly from the usual approach in the literature, initiated notably by Sannikov [51, 52], we reinterpret the problem of the bank in terms of BSDEs, which, we believe, offers an alternative approach which may be easier to apprehend for the mathematical finance community. Of course, such an interpretation of optimal stochastic control problem with control on the drift is far from being original, and we refer the interested reader to the seminal papers of Hamadène and Lepeltier [22] and El Karoui and Quenez [15] for more information, as well as to the recent articles by Cvitanić, Possamaï and Touzi [10, 11] for more references and a systematic treatment of Principal–Agent type problems with this backward SDE approach. Before stating the related result, let us denote by (Y^i, Z^i) the unique (super–)solution (existence and uniqueness will be justified below) to the following BSDE

$$Y_t^i = 0 - \int_t^\tau g^i(s, Y_s^i, Z_s^i)ds + \int_t^\tau Z_s^i \cdot d\widetilde{M}_s^i + \int_t^\tau dK_s^i,\ 0 \leq t \leq \tau,\ \mathbb{P} - a.s., \tag{3.2}$$

where

$$M_t := (N_t, H_t)^\top,\ \widetilde{M}_t^i := M_t - \int_0^t \lambda_s^0 (1, 1 - \theta_s^i)^\top ds,$$

$$g^i(t, y, z) := \inf_{k \in \{0, \dots, I - N_t\}} f^i(t, k, y, z) = ry - (I - N_t)\left(\alpha_{I-N_t} \varepsilon z \cdot (1, 1 - \theta_t^i)^\top - B\right)^-.$$

We have the following proposition, which is basically a reformulation of [42, Proposition 3.2]. The proof is postponed to Appendix A

Proposition 3.1. *For any $(\theta^i, D^i) \in \Theta \times \mathcal{D}$, the value function of the bank has the dynamics, for $t \in [0, \tau]$, $\mathbb{P} - a.s.$*

$$dU_t^i(\theta^i, D^i) = \left(rU_t^i(\theta^i, D^i) - Bk_t^{\star,i} + \lambda_t^{k^{\star,i}} Z_t^i \cdot (1, 1 - \theta_t^i)^\top\right) dt - \rho_i dD_t^i - Z_t^i \cdot d\widetilde{M}_t^i,$$

where Z^i is the second component of the solution to the BSDE (3.2)*. In particular, the optimal monitoring choice of the bank is given by*

$$k_t^{\star,i} = (I - N_t)\mathbf{1}_{\{Z_t^i \cdot (1, 1 - \theta_t^i)^\top < b_t\}}.$$

Notice that the above result implies that the monitoring choices of the bank are necessarily of bang–bang type, in the sense that she either monitors all the remaining loans, or none at all, which in turn implies that the investor can never give the bank incentives to monitor only a fraction of the loans at a given time[6].

3.1.2 Introducing feasible sets

Following the terminology of Cvitanić, Wan and Yang [12], let us discuss the so–called feasible set for the banks.

Definition 3.1. *We call $\mathcal{V}_t^i$ the feasible set for the expected payoff of banks of type ρ_i, starting from some time $t \geq 0$, that is to say all the possible utilities that a bank of type ρ_i can get from all the admissible contracts offered by the investor from time t on.*

[6] We assume here, as is commonplace in the Principal–Agent literature, that in the case where the bank is indifferent with respect to her monitoring decision, that is when $Z_t^i \cdot (1, 1 - \theta_t^i)^\top = b_t$, she acts in the best interest of the investors, and thus monitors all the $I - N_t$ remaining loans.

Our first result gives an explicit form of the the feasible set $\mathcal{V}_t^i$, which turns out to be independent of the type of the bank. The proof is relegated to Appendix A

Lemma 3.1. *For $i \in \{g, b\}$ and for any $t \geq 0$, we have that $\mathcal{V}_t^i = \mathcal{V}_t$, with*

$$\mathcal{V}_t := \left[\frac{B(I - N_t)}{r + \lambda_t^{I-N_t}}, +\infty \right).$$

To describe the results of [42], we need to limit our subsequent analysis (for this section only), to contracts enforcing a constant monitoring from the banks, that is to say contracts incentive–compatible with $k = 0$. Obviously, for such contracts, the feasible set of the banks are not equal to $\mathcal{V}_t$, although we will see next that in this case again, it does not depend on the type of the bank.

Definition 3.2. *The set $\mathcal{V}_t^{0,i} \subset \mathcal{V}_t$ is called the feasible set for the expected payoff of the banks of type ρ_i, starting from some time $t \geq 0$, when the investors can only offer contracts enforcing $k = 0$.*

This sets can also be obtained explicitly, see Appendix A for the proof.

Lemma 3.2. *We have for $i \in \{g, b\}$ and for any $t \geq 0$ that $\mathcal{V}_t^{0,i} = \mathcal{V}_t^0$, with $\mathcal{V}_t^0 := [b_t, +\infty)$.*

3.2 The investor's problem and the optimal full–monitoring contract

As mentioned above, in this section only, we follow [42] and consider that the only acceptable behaviour for the bank, from the social point of view, is that she never shirks away from her monitoring responsibilities[7]. In other words, we only allow contracts with a recommendation of $k = 0$. Therefore, the value function of the investor becomes

$$V_t^{\mathrm{pm},0}(R_0) := \operatorname*{ess\,sup}_{(D^i,\theta^i)\in\mathcal{A}^{0,i}(t,R_0)} \mathbb{E}^{\mathbb{P}^0}\left[\int_{t\wedge\tau}^{\tau} (I - N_s)\mu ds - dD_s^i \middle| \mathcal{G}_t \right],$$

where the set of admissible contracts $\mathcal{A}^{0,i}(t, R_0)$ is defined for $R_0 \geq b_t$, by

$$\mathcal{A}^{0,i}(t, R_0) := \left\{ (\theta^i, D^i) \in \Theta \times \mathcal{D}, \text{ s.t. } (\theta^i, D^i) \text{ enforces } k = 0 \text{ and } U_t^i(\theta^i, D^i) \geq R_0 \right\}.$$

The main findings of [42] require the following assumptions. Define for any $t \geq 0$ and $j \in \{1, \ldots, I\}$,

$$\frac{j}{\overline{\alpha}_j} := \sum_{i=1}^{j} \frac{1}{\alpha_i},\ \widehat{\lambda}_j^0 := \alpha_j j,\ \widehat{b}_j := \frac{B}{\alpha_j \varepsilon}.$$

Assumption 3.1. *(i)* $\mu \geq \overline{\alpha}_I$.

(ii) We have for all $j \leq I$, $rB(1 + \varepsilon) \leq (\mu\varepsilon - B)\varepsilon\overline{\alpha}_j$.

(iii) Individual default risk is non–decreasing with past default, $\alpha_j \leq \alpha_{j-1}$, for all $j \leq I$.

Define next for $x > 0$

$$\phi(x) := \left(\frac{1 + x}{1 + 2x} \right)^{\frac{1}{x}-1},\ \psi(x) := \frac{\phi(x) - x}{(1 - x)\phi(x)}.$$

[7] We refer however to Example 3.1 below, where we show that this may not always be optimal for the investor, which is reason why we will forego this assumption later in the paper.

Let us then define some family of concave functions, unique solutions to the following system of ODEs

$$\begin{cases} \left(ru+\widehat{\lambda}^0_j\widehat{b}_j\right)(v^i_j)'(u)+j\mu-\widehat{\lambda}^0_j\left(v^i_j(u)-\dfrac{u-\widehat{b}_j}{\widehat{b}_{j-1}}v^i_{j-1}(\widehat{b}_{j-1})\right)=0,\ u\in\left(\widehat{b}_j,\widehat{b}_j+\widehat{b}_{j-1}\right], \\ \left(ru+\widehat{\lambda}^0_j\widehat{b}_j\right)(v^i_j)'(u)+j\mu-\widehat{\lambda}^0_j\left(v^i_j(u)-v^i_{j-1}(u-\widehat{b}_j)\right)=0,\ u\in\left(\widehat{b}_j+\widehat{b}_{j-1},\gamma^i_j\right], \\ \rho_i(v^i_j)'(u)+1=0,\ u>\gamma^i_j, \end{cases} \tag{3.3}$$

with initial values $\gamma^i_1:=\widehat{b}_1$ and

$$v^i_1(u):=\overline{v}^i_1-\frac{1}{\rho_i}(u-\widehat{b}_1), u\geq\widehat{b}_1,\ \overline{v}^i_1:=\frac{\mu}{\widehat{\lambda}^0_1}-\frac{\widehat{b}_1(r+\widehat{\lambda}^0_1)}{\rho_i\widehat{\lambda}^0_1},$$

and where for $j\geq 2$, γ^i_j is defined recursively by $r/\widehat{\lambda}^0_j-1\in\partial v^i_{j-1}(\gamma^i_j-\widehat{b}_j)$, where ∂v^i_{j-1} is the super–differential of the concave function v^i_{j-1}. The main result of [42] is

Theorem 3.1. *Assume that the $\left(\widehat{\lambda}^0_j\right)_{1\leq j\leq I}$ satisfy the following recursive conditions for $j\geq 2$*

$$\frac{r}{\widehat{\lambda}^0_j}-1\leq\frac{v^i_{j-1}\left(\widehat{b}_{j-1}\right)}{\widehat{b}_{j-1}}\ \text{and}\ \left((v^i_{j-1})'\left(\widehat{b}_{j-1}\right)\right)^+\frac{\widehat{b}_{j-1}}{v^i_{j-1}\left(\widehat{b}_{j-1}\right)}\leq\psi\left(\frac{r}{\widehat{\lambda}^0_j}\right).$$

Then, under Assumption 3.1, the system (3.3) *is well–posed and we have*

$$V^{\mathrm{pm},0}_t(R_0)=\sup_{u_t\geq R_0}v^i_{I-N_t}(u_t),$$

where $(u_s)_{s\geq t}$ is defined as the unique solution to the SDE on $[t,\tau)$

$$\begin{aligned} du_s=&\left(ru_s+\lambda^0_{I-N_s}\widehat{b}_{I-N_s}\right)ds-\rho_i dD^{\star,i}_s \\ &-\left(\mathbf{1}_{\{u_s\in[\widehat{b}_{I-N_s},\widehat{b}_{I-N_s-1}+\widehat{b}_{I-N_s})\}}(u_s-\widehat{b}_{I-N_s-1})+\widehat{b}_{I-N_s}\mathbf{1}_{\{u_s\in[\widehat{b}_{I-N_s}+\widehat{b}_{I-N_s-1},\gamma^i_{I-N_s})\}}\right)dN_s \\ &-\left(\mathbf{1}_{\{u_s\in[\widehat{b}_{I-N_s},\widehat{b}_{I-N_s-1}+\widehat{b}_{I-N_s})\}}\widehat{b}_{I-N_s-1}+(u_s-\widehat{b}_{I-N_s})\mathbf{1}_{\{u_s\in[\widehat{b}_{I-N_s}+\widehat{b}_{I-N_s-1},\gamma^i_{I-N_s})\}}\right)dH_s, \end{aligned}$$

with initial value u_t at t, and where we defined for $s\in[t,\tau)$ and $j=1,\dots,I$

$$\begin{aligned} &D^{\star,i}_s:=\mathbf{1}_{\{s=t\}}\frac{(u_t-\gamma^i_{I-N_t})^+}{\rho_i}+\int_t^s\delta^{I-N_r}_i(u_r)dr,\ \theta^\star_s:=\theta^{I-N_s}_i(u_s), \\ &\delta^j_i(u):=\mathbf{1}_{\{u=\gamma^i_j\}}\frac{\widehat{\lambda}^0_j\widehat{b}_j+r\gamma^i_j}{\rho_i},\ \theta^j_i(u):=\mathbf{1}_{\{u\in[\widehat{b}_j,\widehat{b}_{j-1}+\widehat{b}_j)\}}\frac{u-\widehat{b}_j}{\widehat{b}_{j-1}}+\mathbf{1}_{\{u\in[\widehat{b}_j+\widehat{b}_{j-1},\gamma^i_j)\}}. \end{aligned}$$

We finish this section with an example showing that forcing the bank to always monitor all the loans may not always be optimal for the Principal, which we explain why we forego this assumption in the rest of the paper.

Example 3.1. *Consider the case when there is one loan in the project, $I=1$. The value function of the investor is given by*

$$V^{\mathrm{pm},0}_t(R_0)=\sup_{u_t\geq R_0}v^i_1(u_t)=\begin{cases}\overline{v}^i_1-\frac{1}{\rho_i}(R_0-\widehat{b}_1), & R_0\geq\widehat{b}_1,\\ \overline{v}^i_1, & R_0<\widehat{b}_1.\end{cases}$$

It follows from Lemma 3.1 that the contract given by $\theta \equiv 0$, $D \equiv 0$ is the only one such that the banks get utility equal to $\frac{B}{r+\widehat{\lambda}_1^1}$ under it. Therefore, the value function of the investor at the point of minimum utility is equal to

$$V_t^{\mathrm{pm}}\left(\frac{B}{r+\widehat{\lambda}_1^1}\right) = \frac{\mu}{\widehat{\lambda}_1^1}.$$

If $R_0 \leq \frac{B}{r+\widehat{\lambda}_1^1}$ and $\mu < \frac{\widehat{\lambda}_1^0 \widehat{b}_1 (r+\widehat{\lambda}_1^0)}{\rho_i(\widehat{\lambda}_1^1 - \widehat{\lambda}_0^1)}$, then $V_t^{\mathrm{pm}}\left(\frac{B}{r+\widehat{\lambda}_1^1}\right) < V_t^{\mathrm{pm},0}\left(\frac{B}{r+\widehat{\lambda}_1^1}\right)$ and it is not optimal for the investor to offer contracts under which the banks never shirks.

4 Credible set

In this section we come back to the case in which there are two types of banks in the market, and study the so–called credible set, which is formed by the pairs of value functions of the banks under the admissible contracts.

As in [12], we do not expect all the points in the feasible set to correspond to a pair of reachable values of the banks under some admissible contract. We will therefore follow the approach initiated by [12] and we will characterize the credible set. We emphasise an important difference with [12] though, in the sense that in our context, the credible set becomes dynamic as it depends on the current size of the pool.

In this section we work with generic contracts $(\theta, D) \in \Theta \times \mathcal{D}$, not necessarily designed for a particular type of bank.

4.1 Definition of the credible set and its boundaries

We first define $\widehat{\mathcal{V}}_j := [Bj/(r+\widehat{\lambda}_j^{SH}), \infty)$. Observe that the feasible set

$$\mathcal{V}_t = \left[\frac{B(I-N_t)}{r+\lambda_t^{I-N_t}}, +\infty\right),$$

satisfies $\mathcal{V}_t = \widehat{\mathcal{V}}_{I-N_t}$ for every t, so the only dependence of the feasible set in time is due to the number of loans left. The formal definition of the credible set is the following.

Definition 4.1. *For any time $t \geq 0$, we define the credible set $\mathcal{C}_{I-N_t}$ as the set of $(u^b, u^g) \in \widehat{\mathcal{V}}_{I-N_t} \times \widehat{\mathcal{V}}_{I-N_t}$ such that there exists some admissible contract $(\theta, D) \in \Theta \times \mathcal{D}$ satisfying $U_t^b(\theta, D) = u^b$, $U_t^g(\theta, D) = u^g$ and $(U_s^b(\theta, D), U_s^g(\theta, D)) \in \widehat{\mathcal{V}}_{I-N_s} \times \widehat{\mathcal{V}}_{I-N_s}$ for every $s \in [t, \tau)$, $\mathbb{P}-a.s.$*

Given a starting time $t \geq 0$ and $u^b \in \widehat{\mathcal{V}}_{I-N_t}$, define the set of contracts under which the value function of the bad bank at time t is equal to u^b,

$$\mathcal{A}^b(t, u^b) = \left\{(\theta, D) \in \Theta \times \mathcal{D},\ U_t^b(\theta, D) = u^b\right\}.$$

We denote by $\mathfrak{U}_t(u^b)$ the largest value $U_t^g(\theta, D)$ that the good bank can obtain from all the contracts $(\theta, D) \in \mathcal{A}^b(t, u^b)$. Once again, this set only depends on t through the value of $I - N_t$, so that we will also use the notation $\widehat{\mathfrak{U}}_{I-N_t}(u^b) := \mathfrak{U}_t(u^b)$. We also denote the lowest one by $\mathfrak{L}_t(u^b)$ and $\widehat{\mathfrak{L}}_{I-N_t}(u^b)$ indifferently. Next, define

$$\widehat{\mathcal{C}}_j := \left\{(u^b, u^g) \in \widehat{\mathcal{V}}_j \times \widehat{\mathcal{V}}_j,\ \widehat{\mathfrak{L}}_j(u^b) \leq u^g \leq \widehat{\mathfrak{U}}_j(u^b)\right\}.$$

We will prove in Proposition 4.3 below that $\widehat{\mathcal{C}}_j = \mathcal{C}_j$ for every $j = 1, \dots I$. Therefore, we will call respectively the functions $\widehat{\mathfrak{L}}_j$ and $\widehat{\mathfrak{U}}_j$ the lower and upper boundary of the credible set when there are j loans left. The aim of the next sections is to obtain explicit formulas for these boundaries.

4.2 Utility of not monitoring

We introduce some notations, and denote by k^{SH} the strategy of a bank which does not monitor any loan at any time, *i.e.* $k_s^{SH} = I - N_s$ for every $s \geq 0$. We also denote by $\widehat{\lambda}_j^{SH}$ the default intensity under k^{SH} when there are j loans left, *i.e.* $\widehat{\lambda}_j^{SH} := \alpha_j j(1+\varepsilon)$. We observe that $\widehat{\lambda}_j^{SH} = \lambda_t^{k^{SH}} = \alpha_{I-N_t}(I-N_t)(1+\varepsilon)$, for every $t \geq 0$ such that $I - N_t = j$. Now consider any starting time t such that $I - N_t = j$ and any $\theta \in \Theta$. The continuation utility that the banks get from always shirking (without considering the payments) is

$$u_t^g(k^{SH}, \theta, 0) = u_t^b(k^{SH}, \theta, 0) = \mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_{t\wedge\tau}^{\tau} e^{-r(s-t)} B k_s^{SH} ds \middle| \mathcal{G}_t\right]. \tag{4.1}$$

This quantity is obviously increasing in θ, so that (4.1) attains its minimum value under any contract with $\theta \equiv 0$, which is equal to $c(j,1) := Bj/(r + \widehat{\lambda}_j^{SH})$. Moreover, if the pool is liquidated exactly after the next m defaults, with $m \in \{2, \dots, j\}$, (4.1) is equal to (see Appendix B)

$$c(j,m) := \frac{Bj}{r + \widehat{\lambda}_j^{SH}} + \sum_{i=j-m+1}^{j-1} \frac{Bi}{r + \widehat{\lambda}_i^{SH}} \prod_{\ell=i+1}^{j} \frac{\widehat{\lambda}_\ell^{SH}}{r + \widehat{\lambda}_\ell^{SH}}.$$

In particular, under any contract such that $\theta \equiv 1$, (4.1) attains its maximum value, which is equal to

$$C(j) := c(j,j) = \frac{Bj}{r + \widehat{\lambda}_j^{SH}} + \sum_{i=1}^{j-1} \frac{Bi}{r + \widehat{\lambda}_i^{SH}} \prod_{\ell=i+1}^{j} \frac{\widehat{\lambda}_\ell^{SH}}{r + \widehat{\lambda}_\ell^{SH}}. \tag{4.2}$$

4.3 Lower boundary of the credible set

The lower boundary of the credible set is the simplest of the two boundaries and it can be computed directly. We will see that it is a piecewise linear function corresponding to two lines with different slopes. The next proposition states the main inequalities that determine the lower boundary.

Lemma 4.1. *For any $t \in [0, \tau]$ and any admissible contract $(\theta, D) \in \Theta \times \mathcal{D}$, the value functions of the good and the bad banks satisfy, $\mathbb{P} - a.s.$*

$$U_t^g(\theta, D) \geq U_t^b(\theta, D), \tag{4.3}$$

$$U_t^g(\theta, D) \geq \frac{\rho_g}{\rho_b} U_t^b(\theta, D) - \frac{(\rho_g - \rho_b)}{\rho_b} C(I - N_t), \tag{4.4}$$

where the function $C(j)$ is defined in (4.2).

Using Lemma 4.1, we prove the following characterisation of the lower boundary of the credible set.

Proposition 4.1. *For any $j \in \{1, \dots, I\}$, the lower boundary when there are j loans left is given by*

$$\widehat{\mathfrak{L}}_j(u^b) = \begin{cases} u^b, \ c(j,1) \leq u^b \leq C(j), \\ \dfrac{\rho_g}{\rho_b} u^b - \dfrac{(\rho_g - \rho_b)}{\rho_b} C(j), \ C(j) \leq u^b < +\infty. \end{cases}$$

4.4 Upper boundary of the credible set

The upper boundary of the credible set is not as simple to obtain as the lower boundary and we have to solve a specific stochastic control problem to identify it. Notice that this approach is similar to the one used in [12].

Let us fix any contract $(\theta, D) \in \Theta \times \mathcal{D}$. We remind the reader that thanks to Proposition 3.1, we know that there exist $\mathbb{G}-$predictable and integrable processes $(h^{1,g}(\theta, D), h^{2,g}(\theta, D))$ such that

$$\begin{aligned} dU_s^g(\theta, D) = & \left(rU_s^g(\theta, D) - Bk_s^{\star,g}(\theta, D)\right) ds - \rho_g dD_s - h_s^{1,g}(\theta, D) \left(dN_s - \lambda_s^{k^{\star,g}(\theta,D)} ds\right) \\ & - h_s^{2,g}(\theta, D) \left(dH_s - (1-\theta_s)\lambda_s^{k^{\star,g}(\theta,D)} ds\right), \ s \in [0, \tau], \end{aligned} \tag{4.5}$$

where we recall that the optimal monitoring choice $k^{\star,g}(\theta, D)$ is given by

$$k_s^{\star,g}(\theta, D) = (I - N_s)\mathbf{1}_{\{h_s^{1,g}(\theta,D) + (1-\theta_s)h_s^{2,g}(\theta,D) < b_s\}}.$$

Similarly, there exist $\mathbb{G}-$predictable and integrable processes $(h^{1,b}(\theta, D), h^{2,b}(\theta, D))$ such that

$$\begin{aligned} dU_s^b(\theta, D) = & \left(rU_s^b(\theta, D) - Bk_s^{\star,b}(\theta, D)\right) ds - \rho_b dD_s - h_s^{1,b}(\theta, D) \left(dN_s - \lambda_s^{k^{\star,b}(\theta,D)} ds\right) \\ & - h_s^{2,b}(\theta, D) \left(dH_s - (1-\theta_s)\lambda_s^{k^{\star,b}(\theta,D)} ds\right), \ s \in [0, \tau], \end{aligned} \tag{4.6}$$

with $k_s^{\star,b}(\theta, D) = (I - N_s)\mathbf{1}_{\{h_s^{1,b}(\theta,D) + (1-\theta_s)h_s^{2,b}(\theta,D) < b_s\}}$. We will use the dynamics (4.5)–(4.6) to define a simple set of admissible contracts in which we will reinterpret both the value functions of the agents as controlled diffusion processes, where the controls are $(D, \theta, h^{1,g}, h^{2,g}, h^{1,b}, h^{2,b})$ and satisfying the instanteneous conditions (A.2). Obviously, doing so makes us look at a larger class of "contracts", in the sense that in the above representation of the value functions of the bank, the choice of the processes $(h^{1,g}, h^{2,g}, h^{1,b}, h^{2,b})$ is not free, since they are completely determined by the choice of (θ, D). Nonetheless, we will prove later a verification result that will ensure us that the solution of the stochastic control problem we consider provides us the upper boundary of the credible set.

Let us therefore denote by $\mathcal{H}$ the set of non–negative, $\mathbb{G}-$predictable and integrable processes. We abuse notations and define, for every $\Psi := (D, \theta, h^{1,g}, h^{2,g}, h^{1,b}, h^{2,b}) \in \mathcal{D} \times \Theta \times \mathcal{H}^4$, the processes $U^g(\Psi)$ and $U^b(\Psi)$ which satisfy the following (linear) SDEs (well–posedness is trivial)

$$dU_s^g(\Psi) = rU_s^g(\Psi) - Bk_s^{\star,g}(\Psi) - \rho_g dD_s - h_s^{1,g} \left(dN_s - \lambda_s^{k^{\star,g}(\Psi)} ds\right) - h_s^{2,g} \left(dH_s - (1-\theta_s)\lambda_s^{k^{\star,g}(\Psi)} ds\right), \tag{4.7}$$

$$dU_s^b(\Psi) = rU_s^b(\Psi) - Bk_s^{\star,b}(\Psi) - \rho_b dD_s - h_s^{1,b} \left(dN_s - \lambda_s^{k^{\star,b}(\Psi)} ds\right) - h_s^{2,b} \left(dH_s - (1-\theta_s)\lambda_s^{k^{\star,b}(\Psi)} ds\right), \tag{4.8}$$

where we defined

$$k_s^{\star,g}(\Psi) := (I - N_s)\mathbf{1}_{\{h_s^{1,g} + (1-\theta_s)h_s^{2,g} < b_s\}}, \ k_s^{\star,b}(\Psi) := (I - N_s)\mathbf{1}_{\{h_s^{1,b} + (1-\theta_s)h_s^{2,b} < b_s\}}.$$

Remark 4.1. *In the model, there is no need to consider $h^{1,g}$ and $h^{1,b}$ as positive processes and we do this just for technical reasons. Intuitively, the optimal contracts should satisfy this additional constraint because the investor does not benefit from earlier defaults and if a contract increases the banks' continuation utilities after one of the defaults, the banks should increase the default intensity as much as possible.*

For any starting time $t \in [0,\tau]$ and for every $u^b \geq B(I-N_t)/(r+\widehat{\lambda}^{SH}_{I-N_t})$ define

$$\overline{\mathcal{A}}^b(t,u^b) := \Bigg\{ \Psi = (D,\theta,h^{1,b},h^{2,b}) \in \mathcal{D}\times\Theta\times\mathcal{H}^2, \text{ such that } \forall s\in[t,\tau],$$
$$U^b_{s^-}(\Psi) = h^{1,b}_s + h^{2,b}_s,\ U^b_{s^-}(\Psi) - h^{1,b}_s \geq \frac{B(I-N_s)}{r+\lambda^{I-N_s}_s},\ U^b_t(\Psi) = u^b \Bigg\}.$$

We will abuse notations and also call elements of $\overline{\mathcal{A}}^b(t,u^b)$ contracts. The upper boundary $\mathfrak{U}_t$ solves the following control problem

$$\mathfrak{U}_t(u^b) = \operatorname*{ess\,sup}_{\Psi\in\overline{\mathcal{A}}^b(t,u^b)} \mathbb{E}^{\mathbb{P}^{k^g(\Psi)}}\left[\int_t^\tau e^{-r(s-t)}(\rho_g dD_s + Bk^g_s(\Psi)ds)\middle|\mathcal{G}_t\right],$$

subject to the dynamics

$$U^b_r(\Psi_g) = u^b + \int_t^r \left(ru^b_s - Bk^{\star,b}_s(\Psi) + h^{1,b}_s\lambda^{k^{\star,b}}_s + h^{2,b}_s(1-\theta_s)\lambda^{k^{\star,b}}_s\right)ds - \rho_b\int_t^r dD_s$$
$$- \int_t^r h^{1,b}_s dN_s - \int_t^r h^{2,b}_s dH_s,\ t\leq r\leq\tau,$$

with

$$k^{\star,b}_s(\Psi) = (I-N_s)\mathbf{1}_{\{h^{1,b}_s+(1-\theta_s)h^{2,b}_s<\widehat{b}_{I-N_s}\}},\ k^g(\Psi)\in\operatorname*{argmax}_{k\in\mathfrak{K}}\mathbb{E}^{\mathbb{P}^k}\left[\int_t^\tau e^{-r(s-t)}(\rho_g dD_s + Bk_s ds)\middle|\mathcal{G}_t\right].$$

Indeed, the above stochastic control problem corresponds to the highest value that the good bank can obtain from any admissible contract, while ensuring that when the bad bank takes it, she receives exactly u^b, which is exactly the definition of the upper boundary of the credible set. Also, notice that the dependence of $\mathfrak{U}$ on the time is only through the number of loans left at time t.

The next subsections are devoted to first obtaining the HJB equation associated with the above problem, its resolution and then finally to the proof of a verification theorem adapted to our framework. Notice that the above is actually a singular stochastic control problem, since the control D is a non–decreasing process, which is not necessarily absolutely continuous with respect to the Lebesgue measure. We refer the reader to the monograph by Fleming and Soner [17] for more details. In particular, this implies that the HJB equation associated to the problem will be a variational inequality with gradient constraints.

4.4.1 HJB equation for the upper boundary

Fix some $1\leq j\leq I$, and define for every $k=0,1,\cdots,j$, $\widehat{\lambda}^k_j := \alpha_j(j+k\varepsilon)$. The system of HJB equations associated to the previous control problem is given by $\widehat{\mathcal{U}}_0\equiv 0$, and for any $1\leq j\leq I$

$$\min\left\{ -\sup_{(\theta,h^1,h^2)\in C^j}\left\{\begin{array}{c}\widehat{\mathcal{U}}'_j(u^b)\left(ru^b - Bk^b + [h^1+(1-\theta)h^2]\widehat{\lambda}^{k^b}_j\right)\\ +\widehat{\lambda}^{k^g}_j\theta\widehat{\mathcal{U}}_{j-1}(u^b-h^1) - (\widehat{\lambda}^{k^g}_j + r)\widehat{\mathcal{U}}_j(u^b) + Bk^g\end{array}\right\},\ \widehat{\mathcal{U}}'_j(u^b) - \frac{\rho_g}{\rho_b}\right\} = 0, \quad (4.9)$$

for every $u^b \geq \frac{Bj}{r+\widehat{\lambda}^{SH}_j}$, with the boundary condition $\widehat{\mathcal{U}}_j(Bj/(r+\widehat{\lambda}^{SH}_j)) = Bj/(r+\widehat{\lambda}^{SH}_j)$, where $k^b := j\mathbf{1}_{\{h^1+(1-\theta)h^2<\widehat{b}_j\}}$, $k^g := j\mathbf{1}_{\{\widehat{\mathcal{U}}_j(u^b)-\theta\widehat{\mathcal{U}}_{j-1}(u^b-h^1)<\widehat{b}_j\}}$, and the set of constraints is defined by

$$C^j := \left\{(\theta,h^1,h^2)\in[0,1]\times\mathbb{R}^2_+,\ h^1+h^2 = u^{b,c},\ h^2\geq\frac{B(j-1)}{r+\widehat{\lambda}^{SH}_{j-1}}\right\}.$$

Remark 4.2. *Note that the incentive compatibility condition for the good bank is implicit in the HJB equation. Indeed, at every s we have*

$$\widehat{\mathcal{U}}_{I-N_s}(U^b_s(\Psi)) - \widehat{\mathcal{U}}_{I-N_{s^-}}(U^b_{s^-}(\Psi)) = \left(\widehat{\mathcal{U}}_{I-N_{s^-}-1}(U^b_{s^-}(\Psi) - h^{1,b}_s(\Psi)) - \widehat{\mathcal{U}}_{I-N_{s^-}}(U^b_{s^-}(\Psi))\right)\Delta N_s \\ - \widehat{\mathcal{U}}_{I-N_{s^-}-1}(U^b_{s^-}(\Psi) - h^{1,b}_s(\Psi))\Delta H_s,$$

which implies that on the upper boundary $h^{1,g}_s(\Psi) = \widehat{\mathcal{U}}_{I-N_{s^-}}(U^b_{s^-}(\Psi)) - \widehat{\mathcal{U}}_{I-N_{s^-}-1}(U^b_{s^-}(\Psi) - h^{1,b}_s(\Psi))$ and $h^{2,g}_s(\Psi) = \widehat{\mathcal{U}}_{I-N_{s^-}-1}(U^b_{s^-}(\Psi) - h^{1,b}_s(\Psi))$. Therefore

$$h^{1,g}_s(\Psi) + (1-\theta^g_s)h^{2,g}_s(\Psi) = \widehat{\mathcal{U}}_{I-N_{s^-}}(U^b_{s^-}(\Psi)) - \theta^g_s\widehat{\mathcal{U}}_{I-N_{s^-}-1}(U^b_{s^-}(\Psi) - h^{1,b}_s(\Psi)).$$

At the points where $\widehat{\mathcal{U}}'_j(u^b) > \rho_g/\rho_b$, the first term of the variational inequality (4.9) must be equal to zero, so the upper boundary must satisfy the following equation

$$r\widehat{\mathcal{U}}_j(u^b) = \sup_{(\theta,h^1,h^2)\in C^j} \left\{ \begin{array}{c} \widehat{\mathcal{U}}'_j(u^b)\left(ru^b - Bk^b + [h^1 + (1-\theta)h^2]\widehat{\lambda}^{k^b}_j\right) \\ +[\widehat{\mathcal{U}}_{j-1}(u^b - h^1)\theta - \widehat{\mathcal{U}}_j(u^b)]\widehat{\lambda}^{k^g}_j + Bk^g \end{array} \right\}. \tag{4.10}$$

We will refer to this equation as the diffusion equation.

- **Step 1: case of 1 loan, solving the diffusion equation**

Before dealing with the variational inequality (4.9), we will solve the diffusion equation (4.10). When $j = 1$, it reduces to

$$r\widehat{\mathcal{U}}_1(u^b) = \widehat{\mathcal{U}}'_1(u^b)\left(ru^b - Bk^b + u^b\widehat{\lambda}^{k^b}_1\right) - \widehat{\mathcal{U}}_1(u^b)\widehat{\lambda}^{k^g}_1 + Bk^g, \tag{4.11}$$

with $k^b = \mathbf{1}_{\{u^b < \widehat{b}_1\}}$, $k^g = \mathbf{1}_{\{\widehat{\mathcal{U}}(u^b) < \widehat{b}_1\}}$.

Remark 4.3. *Notice that the boundary condition $\widehat{\mathcal{U}}_1\left(\frac{B}{r+\widehat{\lambda}^1_1}\right) = \frac{B}{r+\widehat{\lambda}^1_1}$ is implicit in the equation.*

Our first result is the following, whose proof is deferred to Appendix F

Lemma 4.2. *There is a family of continuously differentiable solutions to the diffusion equation, indexed by some constant $C > 0$, which are given by*

$$\widehat{\mathcal{U}}^C_1(u^b) := \begin{cases} C^{\frac{r+\widehat{\lambda}^1_1}{r+\widehat{\lambda}^0_1}}\left(u^b - \dfrac{B}{r+\widehat{\lambda}^1_1}\right) + \dfrac{B}{r+\widehat{\lambda}^1_1},\ u^b \in \left[\dfrac{B}{r+\widehat{\lambda}^1_1}, x^\star_C\right), \\ C\widehat{b}_1^{\frac{\widehat{\lambda}^1_1 - \widehat{\lambda}^0_1}{r+\widehat{\lambda}^1_1}}\left(\dfrac{r+\widehat{\lambda}^1_1}{r+\widehat{\lambda}^0_1}\right)^{\frac{r+\widehat{\lambda}^0_1}{r+\widehat{\lambda}^1_1}}\left(u^b - \dfrac{B}{r+\widehat{\lambda}^1_1}\right)^{\frac{r+\widehat{\lambda}^0_1}{r+\widehat{\lambda}^1_1}},\ u^b \in [x^\star_C, \widehat{b}_1), \\ Cu^b,\ u^b \in [\widehat{b}_1, +\infty), \end{cases}$$

where $x^{C,\star}_1 := \left(\dfrac{1}{C}\right)^{\frac{r+\widehat{\lambda}^1_1}{r+\widehat{\lambda}^0_1}}\widehat{b}_1\dfrac{r+\widehat{\lambda}^0_1}{r+\widehat{\lambda}^1_1} + \dfrac{B}{r+\widehat{\lambda}^1_1}$.

- **Step 2: case of 1 loan, solving the HJB equation**

In this case the variational inequality (4.9) reduces to

$$\min\left\{r\widehat{\mathcal{U}}_1(u^b) - \widehat{\mathcal{U}}'(u^b)\left(ru^b - Bk^b + u^b\widehat{\lambda}^{k^b}_1\right) + \widehat{\mathcal{U}}_1(u^b)\widehat{\lambda}^{k^g}_1 - Bk^g,\ \widehat{\mathcal{U}}'_1(u^b) - \frac{\rho_g}{\rho_b}\right\} = 0. \tag{4.12}$$

We already found the solutions of the diffusion equation inside of this variational inequality and now we will take care of the whole HJB equation. We expect the upper boundary to saturate the second term in the variational inequality for big values of u^b, so we will search for a solution of (4.12) satisfying the following condition: there exists $x^\star \in [B/(r+\widehat{\lambda}_1^1), \infty)$ such that

$$\widehat{\mathcal{U}}_1'(x^\star) = \frac{\rho_g}{\rho_b} \text{ and } \widehat{\mathcal{U}}_1'(u^b) > \frac{\rho_g}{\rho_b}, \text{ for } u^b < x^\star. \tag{4.13}$$

At first sight it could seem that by doing this we face the risk of not finding the correct solution of the dynamic programming equation. Nevertheless, this is not the case and we will prove later a verification result which assures us that the solution that we find under this condition corresponds indeed to the upper boundary of the credible set. The proof of the following Lemma will be given in Appendix F.

Lemma 4.3. *The unique solution of the HJB equation which satisfies condition* (4.13) *is given by, defining* $x_1^\star := x_1^{\rho_g/\rho_b,\star}$

$$\widehat{\mathcal{U}}_1^\star(u^b) := \widehat{\mathcal{U}}_1^{\rho_g/\rho_b,\star}(u^b) = \begin{cases} \left(\frac{\rho_g}{\rho_b}\right)^{\frac{r+\widehat{\lambda}_1^1}{r+\widehat{\lambda}_1^0}} \left(u^b - \frac{B}{r+\widehat{\lambda}_1^1}\right) + \frac{B}{r+\widehat{\lambda}_1^1}, \ u^b \in \left[\frac{B}{r+\widehat{\lambda}_1^1}, x_1^\star\right), \\ \frac{\rho_g}{\rho_b} b_1^{\frac{\widehat{\lambda}_1^1-\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1}} \left(\frac{r+\widehat{\lambda}_1^1}{r+\widehat{\lambda}_1^0}\right)^{\frac{r+\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1}} \left(u^b - \frac{B}{r+\widehat{\lambda}_1^1}\right)^{\frac{r+\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1}}, \ u^b \in [x_1^\star, \widehat{b}_1), \\ \frac{\rho_g}{\rho_b} u^b, \ u^b \in [\widehat{b}_1, +\infty). \end{cases} \tag{4.14}$$

As an illustration, in Figure 1 we show the credible set which corresponds to the region delimited by its upper and lower boundaries. In this example, we considered $r = 0.02$, $B = 0.002$, $\varepsilon = 0.25$, $\alpha_1 = 0.055$, $\frac{\rho_g}{\rho_b} = 2$.

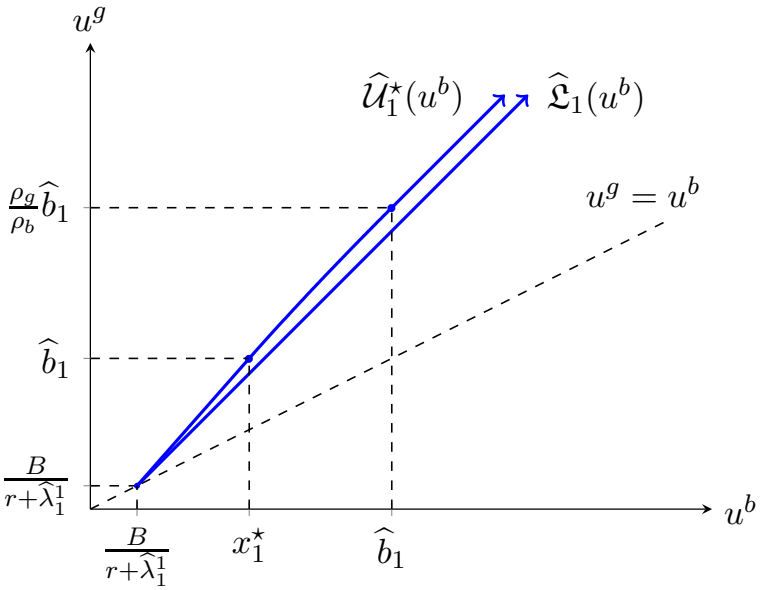

Figure 1: Credible set with one loan left.

- **Step 3: solving the HJB equation in the general case**

In the general case, when $j > 1$, we can reduce the number of variables and rewrite the diffusion

equation (4.10) in an equivalent form

$$r\widehat{\mathcal{U}}_j(u^b) = \sup_{(\theta,h^1)\in\widehat{C}^j} \left\{ \begin{array}{c} \widehat{\mathcal{U}}_j'(u^b)\left(ru^b - Bk^b + [u^b - \theta(u^b-h^1)]\widehat{\lambda}_j^{k^b}\right) \\ + \left(\widehat{\mathcal{U}}_{j-1}(u^b-h^1)\theta - \widehat{\mathcal{U}}_j(u^b)\right)\widehat{\lambda}_j^{k^g} + Bk^g \end{array} \right\}, \tag{4.15}$$

where we recall that $k^b = \mathbf{1}_{\{u^b-\theta(u^b-h^1)<\widehat{b}_j\}}$, $k^g = \mathbf{1}_{\{\widehat{\mathcal{U}}_j(u^b)-\theta\widehat{\mathcal{U}}_{j-1}(u^b-h^1)<\widehat{b}_j\}}$ and the set of constraints is now given by

$$\widehat{C}^j := \left\{ (\theta,h^1)\in[0,1]\times\mathbb{R}_+,\ u^b \geq h^1 + \frac{B(j-1)}{r+\widehat{\lambda}_{j-1}^{SH}} \right\}. \tag{4.16}$$

When we proved that the lower boundary of the credible set is reachable we used contracts of maximum duration, which maintain the pool until the last default. This gives us the intuition that the longer the contract lasts, the smaller the difference between the utilities of the banks will be. Therefore the upper boundary of the credible set, where the difference between both utilities is maximal, should be reachable with contracts of minimum duration, which terminate the contractual relationship immediately after the first default. In the model this means that θ is equal to zero and the resulting HJB equation for the upper boundary has the same form that the one in the case with one loan left. We expect then that the solution of the diffusion equation will be the of the same form as (4.14). The object of the next proposition is to prove our guess rigorously. We postpone the proof to Appendix F.

Proposition 4.2. *For any $j \geq 1$, the function $\widehat{\mathcal{U}}_j^\star$ defined by*

$$\widehat{\mathcal{U}}_j^\star(u^b) := \begin{cases} \left(\frac{\rho_g}{\rho_b}\right)^{\frac{r+\widehat{\lambda}_j^{SH}}{r+\widehat{\lambda}_j^0}} \left(u^b - \frac{Bj}{r+\widehat{\lambda}_j^{SH}}\right) + \frac{Bj}{r+\widehat{\lambda}_j^{SH}},\ u^b \in \left[\frac{Bj}{r+\widehat{\lambda}_j^{SH}}, x_j^\star\right), \\ \frac{\rho_g}{\rho_b} b_j^{\frac{\widehat{\lambda}_j^{SH}-\widehat{\lambda}_j^0}{r+\widehat{\lambda}_j^{SH}}} \left(\frac{r+\widehat{\lambda}_j^{SH}}{r+\widehat{\lambda}_j^0}\right)^{\frac{r+\widehat{\lambda}_j^0}{r+\widehat{\lambda}_j^{SH}}} \left(u^b - \frac{Bj}{r+\widehat{\lambda}_j^{SH}}\right)^{\frac{r+\widehat{\lambda}_j^0}{r+\widehat{\lambda}_j^{SH}}},\ u^b \in [x_j^\star, \widehat{b}_j), \\ \frac{\rho_g}{\rho_b} u^b,\ u^b \in [\widehat{b}_j, +\infty), \end{cases} \tag{4.17}$$

where $x_j^\star := \left(\frac{\rho_b}{\rho_g}\right)^{\frac{r+\widehat{\lambda}_j^{SH}}{r+\widehat{\lambda}_j^0}} \widehat{b}_j \frac{r+\widehat{\lambda}_j^0}{r+\widehat{\lambda}_j^{SH}} + \frac{Bj}{r+\lambda_j^{SH}}$, *is a solution of the HJB equation* (4.9).

4.4.2 Verification Theorem

According to the maximisers in equation (4.15) we define the following controls

$$\begin{cases} \delta^j(u^b) := \mathbf{1}_{\{u^b\geq\widehat{b}_j\}} \frac{u^b(r+\widehat{\lambda}_j^0)}{\rho_b}, \\ \theta^j(u^b) := 0, \\ h^{1,b,j}(u^b) := u^b,\ h^{2,b,j}(u^b) := 0, \\ k^{b,j}(u^b) := j\mathbf{1}_{\{u^b<\widehat{b}_j\}},\ k^{g,j}(u^b) := j\mathbf{1}_{\{\widehat{\mathcal{U}}_j^\star(u^b)<\widehat{b}_j\}}. \end{cases} \tag{4.18}$$

Before stating the verification result for the upper boundary, we make a comment about the domain of the functions $\widehat{\mathcal{U}}_j^\star$. Rigorously speaking, it is possible for the utilities of the banks to be zero but this happens only at time τ when all the pools are liquidated. The domain of $\widehat{\mathcal{U}}_j^\star$ is the set $\widehat{\mathcal{V}}_j$ but in the proof of the verification theorem it will be implicitly understood that $\widehat{\mathcal{U}}_j^\star(0) = 0$. In any case, we do not need the functions $\widehat{\mathcal{U}}_j^\star$ to be defined at zero because Itô's formula will be used on intervals which do not contain τ.

Theorem 4.1. *Consider any starting time $t \geq 0$. For any $u^b \geq \frac{B(I-N_t)}{r+\lambda^{SH}_{I-N_t}}$, let the process $(u^b_s)_{s\in[t,\tau]}$ be the unique solution of the following SDE*

$$u^b_v = u^b + \int_t^v \left[(r+\lambda^{k^{b,I-N_t}}_s)u^b_s - Bk^{b,I-N_t}(u^b_s) - \rho_b\delta^{I-N_t}(u^b_s)\right] ds - \int_t^v u^b_{s^-} dN_s,\ v \in [t,\tau]. \quad (4.19)$$

Then, under the contract $\Psi^\star := (D^{g,\star}, \theta^{g,\star}, h^{1,b,\star}, h^{2,b,\star}) \in \mathcal{D}\times\Theta\times\mathcal{H}^2$ defined for $s\in[t,\tau]$ by

$$dD^\star_s := \delta^{I-N_t}(u^b_s)ds,\ \theta^\star_s \equiv 0,\ h^{1,b,\star}_s := h^{1,b,I-N_t}(u^b_s),\ h^{2,b,\star}_s \equiv 0,$$

the value function of the bad bank is $U^b_t(\Psi^\star) = u^b$ and the one of good bank is $U^g_t(\Psi^\star) = \widehat{\mathcal{U}}^\star_{I-N_t}(u^b)$. Moreover, $\Psi^\star \in \overline{\mathcal{A}}^b(t,u^b)$ and for any other contract which belongs to $\overline{\mathcal{A}}^b(t,u^b)$, the value function of the good bank under such a contract is less or equal to $\widehat{\mathcal{U}}^\star_{I-N_t}(u^b)$. In particular, this implies that

$$\widehat{\mathcal{U}}^\star_{I-N_t}(u^b) = \widehat{\mathfrak{U}}_{I-N_t}(u^b).$$

To conclude the section, we state that $\overline{\mathcal{C}_j}$ is indeed equal to the credible set with j loans left and therefore the functions $\widehat{\mathfrak{U}}_j$ and $\widehat{\mathfrak{L}}_j$ correspond to its upper and lower boundaries.

Proposition 4.3. *For every $1 \leq j \leq I$, $\widehat{\mathcal{C}}_j = \mathcal{C}_j$.*

5 Optimal contracts

In this section we study two kind of contracts that the investor can offer to the bank, the shutdown contract, which corresponds to a single contract designed to be accepted only for the good bank and the screening contract, corresponding to a menu of contracts, one for each type of agent, providing incentives to the bank to accept the contract designed for her true type.

5.1 Shutdown contract

In the so–called shutdown contract, the investor designs a contract $\Psi_g = (k^g, D^g, \theta^g)$ only for the good bank and makes sure that the bad bank will not accept it. Under these conditions the utility of the investor at time $t = 0$ is

$$v_0^{\mathrm{g,Shut}}(\Psi_g) = p_g\mathbb{E}^{\mathbb{P}^{k^g}}\left[\int_0^\tau \mu(I-N_s)ds - dD^g_s\right]. \quad (5.1)$$

So the investor will offer a contract which maximises (5.1) subject to the constraints

$$u^g_0(k^g,\theta^g,D^g) \geq R_0 \geq \sup_{k\in\mathfrak{K}} u^b_0(k,\theta^g,D^g), \quad (5.2)$$

$$u^g_0(k^g,\theta^g,D^g) = \sup_{k\in\mathfrak{K}} u^g_0(k,\theta^g,D^g). \quad (5.3)$$

Recalling the dynamics (4.5)–(4.6), we can rewrite the investor's maximisation problem as follows

$$v_0^{\mathrm{Shut}} := \sup_{(\theta^g,D^g)\in\mathcal{A}^g_{\mathrm{Shut}}} p_g\mathbb{E}^{\mathbb{P}^{k^\star,g}(\theta^g,D^g)}\left[\int_0^\tau \mu(I-N_s)ds - dD^g_s\right],$$

where

$$\mathcal{A}^g_{\mathrm{Shut}} := \left\{(\theta^g,D^g)\in\Theta\times\mathcal{D},\ U_0^{b,c}(\theta^g,D^g) \leq R_0 \leq U^g_0(\theta^g,D^g)\right\}.$$

Remark 5.1. *We will use the notation $U^{b,c}(\theta^g, D^g)$ for the value function that the bad bank gets if she does not reveal her true type and accepts the contract designed for the good bank. We make a distinction between this process and $U^b(\theta^b, D^b)$, which corresponds to the value function that the bad bank obtain if she accepts the contract designed for her by the investor. We make the same distinction between the associated processes $h^{1,b,c}(\theta, D)$, $h^{2,b,c}(\theta, D)$ and $h^{1,b}(\theta, D)$, $h^{2,b}(\theta, D)$.*

As in the previous section, we will define a simple set of contracts and consider the value functions of the agents as diffussion processes controlled by $(D, \theta, h^{1,g}, h^{2,g}, h^{1,b,c}, h^{2,b,c})$. As explained before, by doing so we will look at a larger class of "contracts". Nonetheless, we will prove later that under reasonable assumption the solution of the problem we consider do coincide with the optimal shutdown contract.

Define for any $(t, u^g, u^{b,c}) \in [0, +\infty) \times \mathcal{C}_{I-N_t}$

$$\widehat{\mathcal{A}}^g(t, u^g, u^{b,c}) := \Bigg\{ \Psi_g = (D^g, \theta^g, h^{1,g}, h^{2,g}, h^{1,b,c}, h^{2,b,c}) \in \mathcal{D} \times \Theta \times \mathcal{H}^4, \text{ such that } \forall s \in [t, \tau],$$
$$U^g_{s^-}(\Psi_g) = h^{1,g}_s + h^{2,g}_s,\ U^g_{s^-}(\Psi_g) - h^{1,g}_s \geq \frac{B(I - N_s)}{r + \lambda^{I-N_s}_s},\ U^g_t(\Psi_g) = u^g$$
$$U^{b,c}_{s^-}(\Psi_g) = h^{1,b,c}_s + h^{2,b,c}_s,\ U^{b,c}_{s^-}(\Psi_g) - h^{1,b,c}_s \geq \frac{B(I - N_s)}{r + \lambda^{I-N_s}_s},\ U^{b,c}_t(\Psi_g) = u^{b,c} \Bigg\}.$$

We will also consider in the sequel the following standard control problem, for any $(u^{b,c}, u^g) \in \mathcal{C}_I$

$$\widehat{v}^g_0(u^{b,c}, u^g) := \sup_{\Psi_g \in \widehat{\mathcal{A}}^g(0, u^g, u^{b,c})} p_g \mathbb{E}^{\mathbb{P}^{k^\star, g(\Psi_g)}} \left[\int_0^\tau \mu(I - N_s) ds - dD^g_s \right].$$

We abuse notations and also call elements of $\widehat{\mathcal{A}}^g(t, u^g, u^{b,c})$ contracts.

5.1.1 Value function of the investor

In this section, we characterise the value function of the investor when he offers only shutdown contracts. We will start by computing the value function on the boundaries of the credible set, before explaining how it can be characterised by a specific HJB equation in the interior of the credible set, under reasonable assumptions.

5.1.1.1 Value function of the investor on the lower boundary

Recall the lower boundary with j loans left

$$\widehat{\mathfrak{L}}_j(u^{b,c}) = \begin{cases} u^{b,c},\ c(j,1) \leq u^{b,c} \leq C(j), \\ \dfrac{\rho_g}{\rho_b} u^{b,c} - \dfrac{(\rho_g - \rho_b)}{\rho_b} C(j),\ C(j) \leq u^{b,c} < \infty. \end{cases}$$

Consider any starting time $t \geq 0$. For $u^{b,c} \in \mathcal{C}_{I-N_t}$, we denote by $V^{\mathfrak{L},g}(u^{b,c})$ the value function of the investor in the lower boundary, that is

$$V^{\mathfrak{L},g}_t(u^{b,c}) := \operatorname*{ess\,sup}_{\Psi_g \in \widehat{\mathcal{A}}^g(t, \widehat{\mathfrak{L}}_{I-N_t}(u^{b,c}), u^{b,c})} \mathbb{E}^{\mathbb{P}^{k^\star, g(\Psi_g)}} \left[\int_t^\tau (\mu(I - N_s) ds - dD^g_s) \middle| \mathcal{G}_t \right]. \tag{5.4}$$

The following two propositions are proved in Appendix G and give explicitly the value of $V^{\mathfrak{L},g}_t(u^{b,c})$.

Proposition 5.1. *For every $u^{b,c} \in \mathcal{C}_{I-N_t}$, if $u^{b,c} \geq C(I-N_t)$ then the value function of the investor in the lower boundary is given by*

$$V_t^{\mathfrak{L},g}(u^{b,c}) = \sum_{i=N_t}^{I-1} \frac{\mu(I-i)}{\widehat{\lambda}_{I-i}^{SH}} - \left(\frac{u^{b,c} - C(I-N_t)}{\rho_b}\right).$$

Proposition 5.2. *Fix some $t \geq 0$. For every $u^{b,c} \in \mathcal{C}_{I-N_t}$, with $c(I-N_t,1) \leq u^{b,c} < C(I-N_t)$, let $\nu(u^{b,c})$ be the unique solution of the following equation in ν*

$$\left(\frac{B(I-N_t)}{r+\widehat{\lambda}_{I-N_t}^{SH}} - u^{b,c}\right) + \sum_{i=N_t+1}^{I-1} \int_{s_i(\nu)}^{\infty} \left(\frac{B(I-i)}{r+\widehat{\lambda}_{I-i}^{SH}} e^{-rx}\right) f_{\tau_i}(x)dx = 0,$$

where f_{τ_i} is the density of the law of τ_i under $\mathbb{P}^{k^{SH}}$ and where

$$s_i(\nu) := \begin{cases} 0, \ \nu \leq \dfrac{\mu(r+\widehat{\lambda}_{I-i}^{SH})}{B\widehat{\lambda}_{I-i}^{SH}}, \\ \dfrac{1}{r}\ln\left(\dfrac{\nu B\widehat{\lambda}_{I-i}^{SH}}{\mu(r+\widehat{\lambda}_{I-i}^{SH})}\right), \ \nu \geq \dfrac{\mu(r+\widehat{\lambda}_{I-i}^{SH})}{B\widehat{\lambda}_{I-i}^{SH}}. \end{cases}$$

Then the value function of the investor in the lower boundary is given by

$$V_t^{\mathfrak{L},g}(u^{b,c}) = \frac{\mu(I-N_t)}{\widehat{\lambda}_{I-N_t}^{SH}} + \sum_{i=N_t+1}^{I-1} \int_{s_i(\nu(u^{b,c}))}^{\infty} \frac{\mu(I-i)}{\widehat{\lambda}_{I-i}^{SH}} f_{\tau_i}(x) \ dx.$$

Remark 5.2. *Observe that the function $V_t^{\mathfrak{L},g}$ computed in Propositions 5.1 and 5.2 depends on t only through the quantity $I-N_t$. Define, for any $j = 1,\ldots,J$ the map*

$$\widehat{V}_j^{\mathfrak{L},g}(u^{b,c}) := \begin{cases} \displaystyle\sum_{i=1}^{j} \frac{\mu i}{\widehat{\lambda}_i^{SH}} - \left(\frac{u^{b,c} - C(j)}{\rho_b}\right), \ u^{b,c} \geq C(j), \\ \dfrac{\mu j}{\widehat{\lambda}_j^{SH}} + \displaystyle\sum_{i=1}^{j-1} \int_{s_{I-j}(\nu(u^{b,c}))}^{\infty} \frac{\mu i}{\widehat{\lambda}_i^{SH}} f_{\tau_{I-i}}(x) \ dx, \ u^{b,c} \in (c(j,1), C(j)). \end{cases}$$

We have therefore, that $V_t^{\mathfrak{L},g}(u^{b,c}) = \widehat{V}_{I-N_t}^{\mathfrak{L},g}(u^{b,c})$.

5.1.1.2 Value function of the investor on the upper boundary

The next proposition states that the upper boundary of the credible set is absorbing in the following sense: if under any contract the pair of value functions of the banks reaches the upper boundary at some time, the pair will stay on the upper boundary until the pool is liquidated.

Proposition 5.3. *Consider $(t, u^g, u^{b,c}) \in [0,+\infty) \times \mathcal{C}_{I-N_t}$ such that $u^g = \widehat{\mathfrak{U}}_{I-N_t}(u^{b,c})$ and any contract $\Psi_g = (D^g, \theta^g, h^{1,g}, h^{2,g}, h^{1,b,c}, h^{2,b,c}) \in \widehat{\mathcal{A}}^g(t, u^g, u^{b,c})$. Then $U_s^g(\Psi_g) = \widehat{\mathfrak{U}}_{I-N_s}(U_s^{b,c}(\Psi_g))$ for every $s \in [t,\tau)$.*

The next proposition states an important property satisfied by the contracts which make the continuation utilities of the banks lie in the upper boundary of the credible set.

Proposition 5.4. *Consider $(t, u^g, u^{b,c}) \in [0,+\infty) \times \mathcal{C}_{I-N_t}$ such that $u^g = \widehat{\mathfrak{U}}_{I-N_t}(u^{b,c})$ and any contract $\Psi_g = (D^g, \theta^g, h^{1,g}, h^{2,g}, h^{1,b,c}, h^{2,b,c}) \in \widehat{\mathcal{A}}^g(t, u^g, u^{b,c})$. Then*

(*i*) $\theta_s^g = 0$ *for every* $s \in [t,\tau)$ *such that* $U_s^{b,c}(\Psi_g) < b_s$.

(*ii*) *If* $U_{s_0}^{b,c}(\Psi_g) \geq b_{s_0}$ *for some* $s_0 \in [t,\tau)$ *then* $k_s^{\star,b,c}(\Psi_g) = 0$ *and* $U_s^{b,c}(\Psi_g) \geq b_s$ *for every* $s \in [s_0,\tau)$.

We are now ready to give the value function of the investor on the upper boundary of the credible set, under the assumptions of Theorem 3.1.

Proposition 5.5. *Under Assumption 3.1, we have that for any* $t \geq 0$ *and any* $u^{b,c} \in \widehat{\mathcal{V}}_{I-N_t}$, *the value function of the investor on the upper boundary, defined by*

$$V_t^{\mathfrak{U},g}(u^{b,c}) := \operatorname*{ess\,sup}_{\Psi_g \in \widehat{\mathcal{A}}^g(t,\widehat{\mathfrak{U}}_{I-N_t}(u^{b,c}),u^{b,c})} \mathbb{E}^{\mathbb{P}^{k^{\star,g}(\Psi_g)}}\left[\int_t^\tau (\mu(I-N_s)ds - dD_s^g)\middle|\mathcal{G}_t\right], \tag{5.5}$$

verifies $V_t^{\mathfrak{U},g}(u^{b,c}) = \widehat{V}_{I-N_t}^{\mathfrak{U},g}(u^{b,c})$, *where for any* $j = 1,\cdots,I$

$$\widehat{V}_j^{\mathfrak{U},g}(u^{b,c}) := \begin{cases} \dfrac{\mu j}{\widehat{\lambda}_j^{SH}} + \widehat{C}^j\left(u^{b,c} - \dfrac{Bj}{r+\widehat{\lambda}_j^{SH}}\right)^{\frac{\widehat{\lambda}_j^{SH}}{r+\widehat{\lambda}_j^{SH}}}, \ u^{b,c} < x_j^\star, \\ \dfrac{\mu j}{\widehat{\lambda}_j^0} + \left(v_j^b(\widehat{b}_j) - \dfrac{\mu j}{\widehat{\lambda}_j^0}\right)\left(\widehat{b}_j\dfrac{r+\widehat{\lambda}_j^0}{r+\widehat{\lambda}_j^{SH}}\right)^{-\frac{\widehat{\lambda}_j^0}{r+\widehat{\lambda}_j^{SH}}}\left(u^{b,c} - \dfrac{Bj}{r+\widehat{\lambda}_j^{SH}}\right)^{\frac{\widehat{\lambda}_j^0}{r+\widehat{\lambda}_j^{SH}}}, \ u^{b,c} \in \left[x_j^\star, \widehat{b}_j\right), \\ v_j^b(u^{b,c}), \ u^{b,c} \geq \widehat{b}_j, \end{cases}$$

with

$$\widehat{C}^j := \left(\frac{\mu j}{\widehat{\lambda}_j^0} - \frac{\mu j}{\widehat{\lambda}_j^{SH}} + \left(\frac{\rho_b}{\rho_g}\right)^{\frac{\widehat{\lambda}_j^0}{r+\widehat{\lambda}_j^0}}\left(v_j^b(\widehat{b}_j) - \frac{\mu j}{\widehat{\lambda}_j^0}\right)\right)\left(\frac{\rho_b}{\rho_g}\right)^{-\frac{\widehat{\lambda}_j^{SH}}{r+\widehat{\lambda}_j^0}}\left(\frac{\widehat{b}_j(r+\widehat{\lambda}_j^0)}{r+\widehat{\lambda}_j^{SH}}\right)^{-\frac{\widehat{\lambda}_j^{SH}}{r+\widehat{\lambda}_j^{SH}}}.$$

5.1.1.3 Value function of the investor in the credible set

We define, for any $t \geq 0$ and any $(u^{b,c},u^g) \in \widehat{\mathcal{C}}_{I-N_t}$, the value function of the investor in the credible set by

$$V_t^g(u^{b,c},u^g) := \operatorname*{ess\,sup}_{\Psi_g \in \widehat{\mathcal{A}}^g(t,u^g,u^{b,c})} \mathbb{E}^{\mathbb{P}^{k^{\star,g}(\Psi_g)}}\left[\int_t^\tau (\mu(I-N_s)ds - dD_s^g)\middle|\mathcal{G}_t\right]. \tag{5.6}$$

The system of HJB equations associated to this control problem is given by $\widehat{V}_0^g \equiv 0$, and for any $1 \leq j \leq I$

$$\min\left\{-\sup_{\overline{C}^j}\left\{\begin{array}{l} \partial_{u^{b,c}}\widehat{V}_j^g(u^{b,c},u^g)\left(ru^{b,c} - Bk^{b,c} + [h^{1,b,c} + (1-\theta)h^{2,b,c}]\widehat{\lambda}_j^{k^{b,c}}\right) \\ +\partial_{u^g}\widehat{V}_j^g(u^{b,c},u^g)\left(ru^g - Bk^g + [h^{1,g} + (1-\theta)h^{2,g}]\widehat{\lambda}_j^{k^g}\right) \\ +[\widehat{V}_{j-1}^g(u^{b,c} - h^{1,b,c}, u^g - h^{1,g}) - \widehat{V}_j^g(u^{b,c},u^g)]\widehat{\lambda}_j^{k^g} \\ -\widehat{V}_{j-1}^g(u^{b,c} - h^{1,b,c}, u^g - h^{1,g})(1-\theta)\widehat{\lambda}_j^{k^g} + \mu j \end{array}\right\}\right.$$
$$\left., \rho_b\partial_{u^{b,c}}\widehat{V}_j^g(u^{b,c},u^g) + \rho_g\partial_{u^g}\widehat{V}_j^g(u^{b,c},u^g) + 1\right\} = 0. \tag{5.7}$$

With $k^{b,c} = j\cdot 1_{\{h^{1,b,c}+(1-\theta)h^{2,b,c} < \widehat{b}_j\}}$, $k^g = j\cdot 1_{\{h^{1,g}+(1-\theta)h^{2,g} < \widehat{b}_j\}}$ and the set of constraints

$$\overline{C}^j = \left\{(\theta,h^{1,b,c},h^{2,b,c},h^{1,g},h^{2,g}), \theta \in [0,1], u^g = h^{1,g}+h^{2,g}, u^{b,c} = h^{1,b,c}+h^{2,b,c}, h^{2,g}; h^{2,b,c} \geq \frac{B(j-1)}{r+\widehat{\lambda}_{j-1}^{SH}}\right\}.$$

The boundary conditions of (5.7) are given by

$$\widehat{V}_j^g(u^{b,c}, \widehat{\mathfrak{U}}_j(u^{b,c})) = \widehat{V}_j^{\mathcal{U},g}(u^{b,c}), \text{ for every } u^{b,c} \in \widehat{\mathcal{V}}_j,$$
$$\widehat{V}_j^g(u^{b,c}, \widehat{\mathfrak{L}}_j(u^{b,c})) = \widehat{V}_j^{\mathcal{L},g}(u^{b,c}), \text{ for every } u^{b,c} \in \widehat{\mathcal{V}}_j.$$

The last step would now be to make a rigorous link between a solution in an appropriate sense to the above system and the value function V^g. We have then two possibilities at hand.

(*i*) First, we can use classical arguments to prove that $\widehat{V}_j^g$ is a viscosity solution of the above PDE for every $j = 1, \dots, I$, a result we should then complement with a comparison theorem ensuring uniqueness of the viscosity solution. This would provide a complete characterisation of the value function of the investor, and more importantly would make the problem amenable to numerical computations, using for instance classical finite difference methods. As for the optimal contract, it will correspond to the maximisers in the Hamiltonian above, and therefore would require that we prove that $\widehat{V}_j^g$ is at least weakly differentiable (for instance if $\widehat{V}_j^g$ is concave or Lipschitz continuous, which we expect from the form of the problem) to be well defined. This program can in principle be carried out using standard arguments in viscosity theory of Hamilton–Jacobi equations. However, given the length of the paper, we believe that it would not serve a specific purpose and decided to leave these arguments out.

(*ii*) Another possibility would be to show existence of a smooth solution to the PDE, and prove a comparison theorem similar to Theorem 4.1. However, since the above recursive system involves elliptic PDEs in dimension 2 in a non–trivial domain, we do not expect to be able to obtain explicit solutions in general, which means that existence would have to be proved through abstract arguments. Once again, we believe that such considerations are outside the scope of the paper. We will therefore simply state without proof (since it would be extremely similar to that of Theorem 4.1) a verification theorem adapted to our framework.

Theorem 5.1. *Assume that the system* (5.7) *has a* C^1-*solution and that the supremum in the Hamiltonian is attained at some* $(\theta_i^\star(u^{b,c}, u^g), h_i^{\star,1,b,c}(u^{b,c}, u^g), h_i^{\star,2,b,c}(u^{b,c}, u^g), h_i^{\star,1,g}(u^{b,c}, u^g), h_i^{\star,2,g}(u^{b,c}, u^g))$. *Define then*

$$\begin{aligned}\delta_s^{\star,g}(u^{b,c}, u^g) := \frac{1}{\rho_g}\Big(ru^g - Bk_s^{\star,g} + \widehat{\lambda}_{I-N_s}^{k_s^{\star,g}} h_{I-N_s}^{\star,1,g}(u^{b,c}, u^g) + (1 - \theta_{I-N_s}^\star(u^{b,c}, u^g))\widehat{\lambda}_{I-N_s}^{k_s^{\star,g}} h_{I-N_s}^{\star,2,g}(u^{b,c}, u^g)\Big)\\ \times \mathbf{1}_{\{\rho_b \partial_{u^{b,c}} \widehat{V}_{I-N_s}^g(u^{b,c},u^g) + \rho_g \partial_{u^g} \widehat{V}_{I-N_s}^g(u^{b,c},u^g) + 1\}},\end{aligned}$$

where

$$\begin{aligned}k_s^{\star,g}(u^{b,c}, u^g) &:= (I - N_s)\mathbf{1}_{\{h_{I-N_s}^{\star,1,g}(u^{b,c},u^g) + (1-\theta_{I-N_s}^{\star,g}(u^{b,c},u^g))h_{I-N_s}^{\star,2,g}(u^{b,c},u^g) < \widehat{b}_{I-N_s}\}},\\ k_s^{\star,b,c}(u^{b,c}, u^g) &:= (I - N_s)\mathbf{1}_{\{h_{I-N_s}^{\star,1,b,c}(u^{b,c},u^g) + (1-\theta_{I-N_s}^{\star,g}(u^{b,c},u^g))h_{I-N_s}^{\star,2,b,c}(u^{b,c},u^g) < \widehat{b}_{I-N_s}\}}.\end{aligned}$$

If the corresponding contract is admissible

$$\Psi^{\star,g} := \left(\left(\delta_{I-N.}^{\star,g}, \theta_{I-N.}^\star, h_{I-N.}^{\star,1,b,c}, h_{I-N.}^{\star,2,b,c}, h_{I-N.}^{\star,1,g}, h_{I-N.}^{\star,2,g}\right)(U^{\star,b,c}, U^{\star,g})\right),$$

where $(U^{\star,b,c}, U^{\star,g})$ *are weak solutions to the corresponding SDEs*

$$\begin{aligned}dU_s^{\star,g} = &\left(rU_s^{\star,g} - Bk_s^{\star,g}(U_s^{\star,b,c}, U_s^{\star,g}) - \rho_g \delta_{I-N_s}^{\star,g}(U_s^{\star,b,c}, U_s^{\star,g})\right)ds\\ &- h_{I-N_s}^{\star,1,g}(U_s^{\star,b,c}, U_s^{\star,g})\left(dN_s - \lambda_s^{k^{\star,g}((U^{\star,b,c},U^{\star,g}))}ds\right)\\ &- h_{I-N_s}^{\star,2,g}(U_s^{\star,b,c}, U_s^{\star,g})\left(dH_s - (1 - \theta_{I-N_s}^{\star,g}(U_s^{\star,b,c}, U_s^{\star,g}))\lambda_s^{k^{\star,g}((U^{\star,b,c},U^{\star,g}))}ds\right),\end{aligned}$$

$$
\begin{aligned}
dU_s^{\star,b,c} = & \left(rU_s^{\star,b,c} - Bk_s^{\star,b,c}(U_s^{\star,b,c}, U_s^{\star,g}) - \rho_b \delta_{I-N_s}^{\star,g}(U_s^{\star,b,c}, U_s^{\star,g})\right) ds \\
& - h_{I-N_s}^{\star,1,b,c}(U_s^{\star,b,c}, U_s^{\star,g}) \left(dN_s - \lambda_s^{k^{\star,b,c}((U^{\star,b,c}, U^{\star,g}))} ds\right) \\
& - h_{I-N_s}^{\star,2,b,c}(U_s^{\star,b,c}, U_s^{\star,g}) \left(dH_s - (1-\theta_s^{\star,g}(U_s^{\star,b,c}, U_s^{\star,g}))\lambda_s^{k^{\star,b,c}((U^{\star,b,c}, U^{\star,g}))} ds\right),
\end{aligned}
$$

then we have

$$
v_0^{\text{Shut}} = \sup_{u^{b,c} \le R_0 \le u^g} \widehat{v}_0^g(u^{b,c}, u^g) = \sup_{u^{b,c} \le R_0 \le u^g} p_g \widehat{V}_I^g(u^{b,c}, u^g),
$$

and $\Psi^{\star,g}$ is an optimal contract for the investor.

5.2 Screening contract

Recall that in the screening contract the investor designs a menu of contracts, one for each agent, and his expected utility is given by

$$
v_0\left((\Psi_i)_{i\in\{g,b\}}\right) = \sum_{i\in\{g,b\}} p_i \mathbb{E}^{\mathbb{P}^{k^i}}\left[\int_0^\tau (I - N_s)\,\mu ds - dD_s^i\right]. \tag{5.8}
$$

In this case, we will have to keep track of the continuation utilities of both banks, when they choose the contract designed for them, as well as when they do not truthfully reveal their type. We will denote by v_0 the maximal utility that the investor can get out of the screening contract.

$$
v_0 := \sup_{(\theta^g,\theta^b,D^g,D^b)\in\mathcal{A}_{\text{Scr}}} p_g \mathbb{E}^{\mathbb{P}^{k^{\star,g}(\theta^g,D^g)}}\left[\int_0^\tau \mu(I-N_s)ds - dD_s^g\right] + p_b \mathbb{E}^{\mathbb{P}^{k^{\star,b}(\theta^b,D^b)}}\left[\int_0^\tau \mu(I-N_s)ds - dD_s^b\right],
$$

where

$$
\mathcal{A}_{\text{Scr}} := \left\{(\theta^g,\theta^b,D^g,D^b) \in \Theta^2\times\mathcal{D}^2, U_0^b(\theta^b,D^b) \ge R_0, U_0^j(\theta^j,D^j) \ge U_0^{j,c}(\theta^i,D^i), (i,j)\in\{g,b\}^2,\ i\neq j\right\}.
$$

Remark 5.3. *Observe that we can omit the condition $U_0^g(\theta^g,D^g) \ge R_0$ in the definition of $\mathcal{A}_{\text{Scr}}$. Indeed, it is implied by the inequality $U_0^{g,c}(\theta^b,D^b) \ge U_0^b(\theta^b,D^b)$, which follows from Lemma 4.1.*

Different from the study of the shutdown contract, where the investor contracts only the good bank, in order to obtain the optimal screening contract we need to characterise also the value function of the investor when he contracts the bad bank. We will therefore follow Section 5.1.1, but by replacing the good bank by the bad bank. Hence, we define for any $(t,u^b,u^{g,c}) \in [0,+\infty)\times\mathcal{C}_{I-N_t}$ the set

$$
\begin{aligned}
\widehat{\mathcal{A}}^b(t,u^{g,c},u^b) := \Bigg\{ & \Psi_b = (D^b,\theta^b,h^{1,g,c},h^{2,g,c},h^{1,b},h^{2,b}) \in \mathcal{D}\times\Theta\times\mathcal{H}^4, \text{ such that } \forall s\in[t,\tau], \\
& U_{s^-}^b(\Psi_b) = h_s^{1,b} + h_s^{2,b},\ U_{s^-}^b(\Psi_b) - h_s^{1,b} \ge \frac{B(I-N_s)}{r+\lambda_s^{I-N_s}},\ U_t^b(\Psi_b) = u^b, \\
& U_{s^-}^{g,c}(\Psi_b) = h_s^{1,g,c} + h_s^{2,g,c},\ U_{s^-}^{g,c}(\Psi_b) - h_s^{1,g,c} \ge \frac{B(I-N_s)}{r+\lambda_s^{I-N_s}},\ U_t^{g,c}(\Psi_b) = u^{g,c} \Bigg\}.
\end{aligned}
$$

We also introduce the following stochastic control problem for any $(u^b,u^{g,c}) \in \mathcal{C}_I$

$$
\widehat{v}_0^{\text{b}}(u^b,u^{g,c}) := \sup_{\Psi_b\in\widehat{\mathcal{A}}^b(0,u^{g,c},u^b)} p_b \mathbb{E}^{\mathbb{P}^{k^{\star,b}(\Psi_b)}}\left[\int_0^\tau \mu(I-N_s)ds - dD_s^b\right].
$$

The aim of the next sections is to compute the function $\widehat{v}_0^{\text{b}}(u^{g,c},u^b)$, representing the utility of the investor when hiring the bad bank. We start by studying it on the boundary of the credible set.

5.2.1 Boundary study

We denote by $V^{\mathfrak{L},b}(u^{g,c})$ the value function of the investor in the lower boundary, when hiring the bad bank, defined by

$$V_t^{\mathfrak{L},b}(u^b) := \operatorname*{ess\,sup}_{\Psi_b \in \widehat{\mathcal{A}}^b(t,\widehat{\mathfrak{L}}_{I-N_t}(u^b),u^b)} \mathbb{E}^{\mathbb{P}^{k^\star,b}(\Psi_b)}\left[\int_t^\tau \left(\mu(I-N_s)ds - dD_s^b\right)\middle| \mathcal{G}_t\right]. \tag{5.9}$$

The first result is that the value function of the investor on the lower boundary of the credible set is the same when hiring either the bad or the good bank. This is mainly due to the fact that both banks shirk on the lower boundary.

Proposition 5.6. *For every $u^b \in \mathcal{C}_{I-N_t}$, we have $V_t^{\mathfrak{L},b}(u^b) = V_t^{\mathfrak{L},g}(u^b)$.*

Proof. By definition we have the set equality $\widehat{\mathcal{A}}^g(t,\widehat{\mathfrak{L}}_{I-N_t}(u^b),u^b) = \widehat{\mathcal{A}}^b(t,\widehat{\mathfrak{L}}_{I-N_t}(u^b),u^b)$. From Lemmas E.1 and E.2 we know that for every $\Psi_b \in \widehat{\mathcal{A}}^b(t,\widehat{\mathfrak{L}}_{I-N_t}(u^b),u^b)$, both agents always shirk under Ψ_b, therefore the objective functions in the definitions of $V_t^{\mathfrak{L},g}(u^b)$ and $V_t^{\mathfrak{L},b}(u^b)$ are also the same and equality holds. □

Let us now consider the upper boundary. We denote by $V^{\mathfrak{U},b}(u^b)$ the value function of the investor on the upper boundary when hiring the bad agent.

$$V_t^{\mathfrak{U},b}(u^b) := \operatorname*{ess\,sup}_{\Psi_b \in \widehat{\mathcal{A}}^b(t,\widehat{\mathfrak{U}}_{I-N_t}(u^b),u^b)} \mathbb{E}^{\mathbb{P}^{k^\star,b}(\Psi_b)}\left[\int_t^\tau \left(\mu(I-N_s)ds - dD_s^b\right)\middle| \mathcal{G}_t\right]. \tag{5.10}$$

We have the following result.

Proposition 5.7. *Under the assumptions of Theorem 3.1, for any $t \geq 0$ and any $u^b \in \widehat{\mathcal{V}}_{I-N_t}$, we have that $V_t^{\mathfrak{U},b}(u^b) = \widehat{V}_{I-N_t}^{\mathfrak{U},b}(u^b)$, where for any $j = 1, \cdots, I$*

$$\widehat{V}_j^{\mathfrak{U},b}(u^b) := \begin{cases} \dfrac{\mu j}{\widehat{\lambda}_j^{SH}} + \tilde{C}^j \left(u^b - \dfrac{Bj}{r+\widehat{\lambda}_j^{SH}}\right)^{\frac{\widehat{\lambda}_j^{SH}}{r+\widehat{\lambda}_j^{SH}}}, & u^b < \widehat{b}_j, \\ v_j^b(u^b), & u^b \geq \widehat{b}_j, \end{cases}$$

with

$$\tilde{C}^j = \left(v_j^b(\widehat{b}_j) - \frac{\mu j}{\widehat{\lambda}_j^{SH}}\right)\left(\frac{\widehat{b}_j(r+\widehat{\lambda}_j^0)}{r+\widehat{\lambda}_j^{SH}}\right)^{\frac{-\widehat{\lambda}_j^{SH}}{r+\widehat{\lambda}_j^{SH}}}.$$

Proof. The proof is identical to the proof of Proposition 5.5, with the only difference that since the principal is hiring the bad agent, for $u^b < \widehat{b}_j$ the ODE associated to the value function is

$$0 = \widehat{\mathcal{V}}_j'(u^b)\left(\left(r+\widehat{\lambda}_j^{SH}\right)u^b - Bj\right) - \widehat{\mathcal{V}}_j(u^b)\widehat{\lambda}_j^{SH} + \mu j,$$

with the boundary condition $\widehat{\mathcal{V}}_j\left(\frac{Bj}{r+\widehat{\lambda}_j^{SH}}\right) = \frac{\mu j}{\widehat{\lambda}_j^{SH}}$. □

5.2.2 Study of the credible set

We define, for any $t \geq 0$ and any $(u^b, u^{g,c}) \in \widehat{\mathcal{C}}_{I-N_t}$, the value function of the investor in the credible set when hiring the bad bank by

$$V_t^b(u^b, u^{g,c}) := \operatorname*{ess\,sup}_{\Psi_b \in \widehat{\mathcal{A}}^b(t,u^{g,c},u^b)} \mathbb{E}^{\mathbb{P}^{k^{\star,b}(\Psi_b)}} \left[\int_t^\tau \left(\mu(I - N_s)ds - dD_s^b \right) \middle| \mathcal{G}_t \right]. \tag{5.11}$$

The system of HJB equations associated to this control problem is given by $\widehat{V}_0^b \equiv 0$, and for any $1 \leq j \leq I$

$$\min \left\{ - \sup_{\overline{C}^j} \left\{ \begin{array}{l} \partial_{u^b} \widehat{V}_j^b(u^b, u^{g,c}) \left(ru^b - Bk^b + [h^{1,b} + (1-\theta)h^{2,b}]\widehat{\lambda}_j^{k^b} \right) \\ + \partial_{u^{g,c}} \widehat{V}_j^b(u^b, u^{g,c}) \left(ru^{g,c} - Bk^{g,c} + [h^{1,g,c} + (1-\theta)h^{2,g,c}]\widehat{\lambda}_j^{k^{g,c}} \right) \\ + [\widehat{V}_{j-1}^b(u^b - h^{1,b}, u^{g,c} - h^{1,g,c}) - \widehat{V}_j^b(u^b, u^{g,c})]\widehat{\lambda}_j^{k^b} \\ - \widehat{V}_{j-1}^b(u^b - h^{1,b}, u^{g,c} - h^{1,g,c})(1-\theta)\widehat{\lambda}_j^{k^b} + \mu j \end{array} \right\} \right.$$
$$\left. , \rho_b \partial_{u^b} \widehat{V}_j^b(u^b, u^{g,c}) + \rho_g \partial_{u^{g,c}} \widehat{V}_j^b(u^b, u^{g,c}) + 1 \right\} = 0. \tag{5.12}$$

With $k^b = j \cdot 1_{\{h^{1,b} + (1-\theta)h^{2,b} < \widehat{b}_j\}}$, $k^{g,c} = j \cdot 1_{\{h^{1,g,c} + (1-\theta)h^{2,g,c} < \widehat{b}_j\}}$ and the same set of constraints $\overline{C}^j$ as in the system of HJB equations associated to the functions $\widehat{V}_j^g(u^{b,c}, u^g)$. The boundary conditions of (5.12) are given by

$$\widehat{V}_j^b(u^b, \widehat{\mathfrak{U}}_j(u^b)) = \widehat{V}_j^{\mathcal{U},b}(u^b), \text{ for every } u^{b,c} \in \widehat{\mathcal{V}}_j,$$
$$\widehat{V}_j^b(u^b, \widehat{\mathfrak{L}}_j(u^b)) = \widehat{V}_j^{\mathcal{L},g}(u^b), \text{ for every } u^{b,c} \in \widehat{\mathcal{V}}_j.$$

Theorem 5.2. *Assume that the conditions of Theorem 5.1 hold, that* (5.12) *admits a* C^1-*solution and that the supremum in the Hamiltonian is attained for* $(\theta_i^\star, h_i^{\star,1,g,c}, h_i^{\star,2,g,c}, h_i^{\star,1,b}, h_i^{\star,2,b})(u^{g,c}, u^b)$. *Define then*

$$\delta_s^{\star,b}(u^{g,c}, u^b) := \frac{1}{\rho_b} \left(ru^b - Bk_s^{\star,b} + \widehat{\lambda}_{I-N_s}^{k_s^{\star,b}} h_{I-N_s}^{\star,1,b}(u^{g,c}, u^b) + (1 - \theta_{I-N_s}^\star(u^{g,c}, u^b)) \widehat{\lambda}_{I-N_s}^{k_s^{\star,b}} h_{I-N_s}^{\star,2,b}(u^{g,c}, u^b) \right)$$
$$\times \mathbf{1}_{\{\rho_g \partial_{u^{g,c}} \widehat{V}_{I-N_s}^b(u^{g,c},u^b) + \rho_b \partial_{u^b} \widehat{V}_{I-N_s}^b(u^{g,c},u^b) + 1\}},$$

where

$$k_s^{\star,b}(u^{g,c}, u^b) := (I - N_s) \cdot 1_{\{h_{I-N_s}^{\star,1,b}(u^{g,c},u^b) + (1-\theta_{I-N_s}^{\star,b}(u^{g,c},u^b)) h_{I-N_s}^{\star,2,b}(u^{g,c},u^b) < \widehat{b}_{I-N_s}\}},$$
$$k_s^{\star,g,c}(u^{g,c}, u^b) := (I - N_s) \cdot 1_{\{h_{I-N_s}^{\star,1,g,c}(u^{g,c},u^b) + (1-\theta_{I-N_s}^{\star,b}(u^{g,c},u^b)) h_{I-N_s}^{\star,2,g,c}(u^{g,c},u^b) < \widehat{b}_{I-N_s}\}}.$$

If the corresponding contract is admissible

$$\Psi^{\star,b} := \left(\left(\delta_{I-N_\cdot}^{\star,b}, \theta_{I-N_\cdot}^\star, h_{I-N_\cdot}^{\star,1,g,c}, h_{I-N_\cdot}^{\star,2,g,c}, h_{I-N_\cdot}^{\star,1,b}, h_{I-N_\cdot}^{\star,2,b} \right) (U^{\star,g,c}, U^{\star,b}) \right),$$

where $(U^{\star,g,c}, U^{\star,b})$ *are weak solutions to the corresponding SDEs*

$$\begin{aligned} dU_s^{\star,b} = & \left(rU_s^{\star,b} - Bk_s^{\star,b}(U_s^{\star,g,c}, U_s^{\star,b}) - \rho_b \delta_{I-N_s}^{\star,b}(U_s^{\star,g,c}, U_s^{\star,b}) \right) ds \\ & - h_{I-N_s}^{\star,1,b}(U_s^{\star,g,c}, U_s^{\star,b}) \left(dN_s - \lambda_s^{k^{\star,b}((U^{\star,g,c},U^{\star,b}))} ds \right) \\ & - h_{I-N_s}^{\star,2,b}(U_s^{\star,g,c}, U_s^{\star,b}) \left(dH_s - (1 - \theta_{I-N_s}^{\star,b}(U_s^{\star,g,c}, U_s^{\star,b})) \lambda_s^{k^{\star,b}((U^{\star,g,c},U^{\star,b}))} ds \right), \end{aligned}$$

$$
\begin{aligned}
dU_s^{\star,g,c} = & \left(rU_s^{\star,g,c} - Bk_s^{\star,g,c}(U_s^{\star,g,c}, U_s^{\star,b}) - \rho_g \delta^{\star,b}_{I-N_s}(U_s^{\star,g,c}, U_s^{\star,b})\right) ds \\
& - h^{\star,1,g,c}_{I-N_s}(U_s^{\star,g,c}, U_s^{\star,b}) \left(dN_s - \lambda_s^{k^{\star,g,c}((U^{\star,g,c},U^{\star,b}))} ds\right) \\
& - h^{\star,2,g,c}_{I-N_s}(U_s^{\star,g,c}, U_s^{\star,b}) \left(dH_s - (1-\theta^{\star,b}_{I-N_s}(U_s^{\star,g,c}, U_s^{\star,b}))\lambda_s^{k^{\star,g,c}((U^{\star,g,c},U^{\star,b}))} ds\right),
\end{aligned}
$$

then $(\Psi^{\star,g}, \Psi^{\star,b})$ *is an optimal menu of contracts for the investor, and we have*

$$
\begin{aligned}
v_0 &= \sup_{\{R_0 \le u^b, u^{b,c} \le u^b, u^{g,c} \le u^g\}} \widehat{v}_0^g(u^{b,c}, u^g) + \widehat{v}_0^{\mathrm{b}}(u^b, u^{g,c}) \\
&= \sup_{\{R_0 \le u^b, u^{b,c} \le u^b, u^{g,c} \le u^g\}} p_g \widehat{V}_I^g(u^{b,c}, u^g) + p_b \widehat{V}_I^b(u^b, u^{g,c}).
\end{aligned}
$$

5.3 Description of the optimal contracts

In this section we describe the optimal contracts for the investor when he designs a contract for the good or the bad bank. We explain in detail the optimal contracts on the boundaries of the credible set, which can be obtained explicitly from the value function of the investor. In the interior of the credible set, we discuss the properties we expect the optimal contracts to have when the verification theorems 5.1 and 5.2 hold.

5.3.1 Optimal contracts on the boundaries of the credible set

We start with the upper boundary of the credible set. The following result is a direct consequence of the proofs of Proposition 5.5 and 5.7, and the optimal contract for the pure moral hazard case described in Theorem 3.1.

Proposition 5.8. *Under Assumption 3.1, consider for any* $t \ge 0$ *and* $(u^b, \mathfrak{U}_t(u^b)) \in \mathcal{C}_{I-N_t}$ *the process* $(u_s^b)_{s \ge t}$ *as the solution of the following SDE on* $[t, \tau)$

$$
du_s^b = \left((ru_s^b - Bk_s^{b,\star} + \lambda_s^{k^{b,\star}}(h_s^{1,b,\star} + (1-\theta_s^\star)h_s^{2,b,\star})\right) ds - \rho_b dD_s^\star - h_s^{1,b,\star} dN_s - h_s^{2,b,\star} dH_s, \qquad (5.13)
$$

with initial value u^b *at* t*, and with*

$$
\begin{aligned}
D_s^\star &:= \mathbf{1}_{\{s=t\}} \frac{(u^b - \gamma^b_{I-N_t})^+}{\rho_b} + \int_t^s \delta^{I-N_r}(u_r^b) dr, \;\; \theta_s^\star := \theta^{I-N_s}(u_s^b), \\
h_s^{1,b,\star} &:= h^{1,b,I-N_s}(u_s^b), \;\; h_s^{2,b,\star} := h^{2,b,I-N_s}(u_s^b), k_s^{b,\star} := k^{b,j}(u_s^b),
\end{aligned}
$$

for $s \in [t, \tau)$ *and* $j = 1, \dots, I$*, where*

$$
\begin{aligned}
\delta^j(u) &:= \mathbf{1}_{\{u=\gamma_j^b\}} \frac{\widehat{\lambda}_j^0 \widehat{b}_j + r\gamma_j^b}{\rho_i}, \;\; \theta^j(u) := \mathbf{1}_{\{u \in [\widehat{b}_j, \widehat{b}_{j-1} + \widehat{b}_j)\}} \frac{u - \widehat{b}_j}{\widehat{b}_{j-1}} + \mathbf{1}_{\{u \in [\widehat{b}_j + \widehat{b}_{j-1}, \gamma_j^b)\}}, \\
h^{1,b,j}(u) &:= \mathbf{1}_{\{u \in [c(j,1), \widehat{b}_j)\}} u + \mathbf{1}_{\{u \in [\widehat{b}_j, \widehat{b}_{j-1} + \widehat{b}_j)\}} (u - \widehat{b}_{j-1}) + \mathbf{1}_{\{u \in [\widehat{b}_j + \widehat{b}_{j-1}, \gamma_j^b)\}} \widehat{b}_j, \\
h^{2,b,j}(u) &:= u - h^{1,b,j}(u), \;\; k^{b,j}(u) = j \mathbf{1}_{\{h^{1,b,j}(u) + (1-\theta^j(u))h^{2,b,j}(u) < \widehat{b}_j\}}.
\end{aligned}
$$

Then, the contract $\Psi^\star = (D^\star, \theta^\star, h^{1,b,\star}, h^{2,b,\star})$ *is the unique solution of problems* (5.5) *and* (5.10)*.*

Let us comment the optimal contract for the investor on the upper boundary of the credible set. It is the same if he designs a contract for the good or the bad bank. The state process $(u_s^b)_{s \ge t}$ defined by (5.13) corresponds to the value function of the bad bank under the optimal contract. The optimal

contract offers no payments to the banks when u_s^b is smaller than $\gamma^b_{I-N_s}$. In this case the continuation utility of the bad bank is an increasing process and eventually reaches the value $\gamma^b_{I-N_s}$, if no default happens in the meantime. Payments are postponed until this moment. If the initial value for the bad agent u^b is greater than $\gamma^b_{I-N_t}$, a lump-sum payment is made at t^- in order to have $u_t = \gamma^b_{I-N_t}$. When $u_s^b = \gamma^b_{I-N_s}$, the banks receive constant payments which keep the value function of the bad bank constant at this level. Concerning the liquidation of the project, if at the default time τ_j, it holds that $u^b_{\tau_j} < \widehat{b}_j$ the project is liquidated. In case $u^b_{\tau_j} \in [\widehat{b}_j + \widehat{b}_{j-1}, \gamma^b_j)$, the project will continue with probability $\theta_j \in (0,1)$ which will be closer to one as $u^b_{\tau_j}$ gets closer to γ^b_j. If $u^b_{\tau_j} \geq \gamma^b_j$, the project will be maintained. Finally, the bad bank will monitor all the loans only when her value function is greater than $\widehat{b}_{I-N_s}$, whereas the good bank will monitor when the value of the bad bank is greater than $x^\star_{I-N_s}$. Figure 2 depicts the optimal contract of the investor on the upper boundary of the credible set, denoting $\widehat{B}_j := \widehat{b}_j + \widehat{b}_{j-1}$.

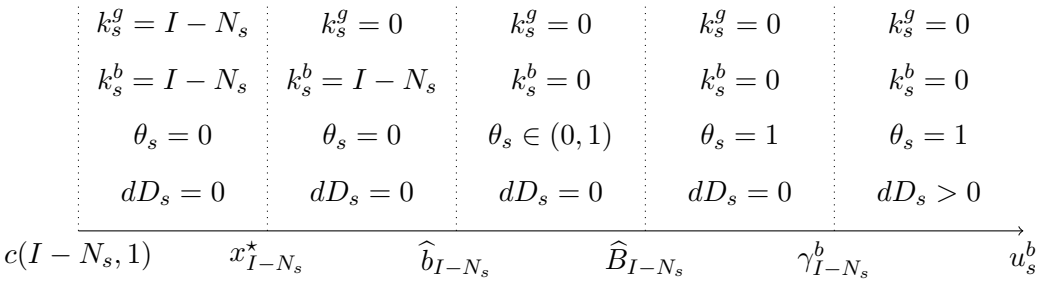

Figure 2: Optimal contract on the upper boundary.

For the lower boundary of the credible set, we have the following result.

Proposition 5.9. *Under Assumption 3.1, consider for any $t \geq 0$ and $(u^b, \mathfrak{L}_t(u^b)) \in \mathcal{C}_{I-N_t}$ the process $(u_s^b)_{s\geq t}$ as the solution of the following SDE on $[t, \tau)$*

$$du_s^b = \Big((r u_s^b - B k_s^{b,\star} + \lambda_s^{k^{b,\star}}(h_s^{1,b,\star} + (1-\theta_s^\star)h_s^{2,b,\star})\Big)\, ds - \rho_b dD_s^\star - h_s^{1,b,\star} dN_s - h_s^{2,b,\star} dH_s, \tag{5.14}$$

with initial value u^b at t, and with

$$D_s^\star := \mathbf{1}_{\{s=t\}} \frac{(u^b - C(I-N_s))^+}{\rho_b},\ \theta_s^\star := \mathbf{1}_{\{u_s^b \geq C(I-N_s)\}},$$

$$h_s^{1,b,\star} := u_s^b - C(I - N_s - 1)\mathbf{1}_{\{u_s^b \geq C(I-N_s)\}},\ h_s^{2,b,\star} := C(I-N_s-1)\mathbf{1}_{\{u_s^b \geq C(I-N_s)\}},$$

$$k_s^{b,\star} = (I-N_s)\mathbf{1}_{\{h_s^{1,b,\star} + (1-\theta_s^\star)h_s^{2,b,\star} < b_s\}},$$

for $s \in [t, \tau)$. Then, the contract $\Psi^\star = (D^\star, \theta^\star, h^{1,b,\star}, h^{2,b,\star})$ is the unique solution of (5.4) and (5.9).

Proof. The payments and the value of $\theta^\star$ in the case $u^b \geq C(I - N_t)$ are a direct consequence of the proof of Proposition 5.1. From the proof of Proposition 5.2 we have that if $u^b < C(I-N_t)$ then

$$\theta_s^\star = \mathbf{1}_{\left\{s - t > \frac{1}{r}\ln\left(\frac{\nu(u^b) B \widehat{\lambda}^{SH}_{I-N_t}}{\mu(r + \widehat{\lambda}^{SH}_{I-N_t})}\right)\right\}},$$

where $\nu(u^b)$ the solution of the associated dual problem. Since the quantity inside of the logarithm decreases with time, we have that $\theta^\star$ is a process which starts at zero, jumps to one at some instant and keeps constant afterwards. This means that if $\theta^\star$ jumps to one at some time s and the project is still running, necessarily the continuation utility of the bad agent is equal to $C(I-N_s)$ because the project will continue until the last default. □

On the lower boundary of the credible set, the optimal contract for the investor also does not depend on the type of the bank. If the initial value of the bad bank u^b is greater than $C(I - N_t)$, the banks receive a lump-sum payment such that $u^b_{t+} = C(I - N_t)$. This is the only payment offered by the contract. If there is a default at some time s such that $u^b_s < C(I - N_s)$, the project is liquidated. When $u^b_s = C(I - N_s)$ the contract maintains the project until the last default. Since the optimal contract does not provides incentives to the banks to monitor the loans, the good and the bad bank shirk until the liquidation of the project. Figure 3 depicts the optimal contract of the investor on the lower boundary of the credible set.

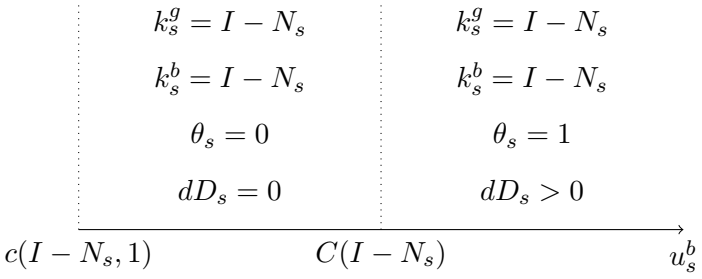

Figure 3: Optimal contract on the lower boundary.

5.3.2 Discussion about the optimal contracts in the interior of the credible set

Figure 4 represents the optimal contracts on the boundaries of the credible set as well as the movements of the values of the banks along these curves. The green zone corresponds to the region where the contract offers payments to the agents and the project is maintained if there is a default. The red zone corresponds to the region where there are no payments and the project is liquidated immediately after a default. Intermediate situations correspond to the yellow zone. We remark that the banks are paid only on the green zone.

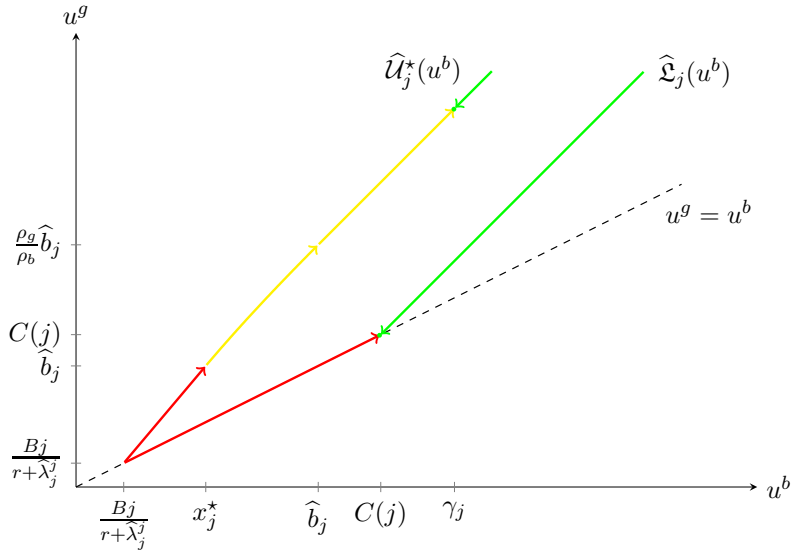

Figure 4: Optimal contract on the boundaries of the credible set.

Let us now consider the whole credible set and explain how we expect the green and red zones on the boundaries to propagate towards the interior region. If the verification theorems 5.1 and 5.2 hold, then

the optimal contracts for problems (5.6) and (5.11) correspond to the maximisers in the Hamiltonian of the systems (5.7) and (5.12). Moreover, payments only take place when the value function of the investor saturates the gradient constraint. Therefore, it is natural to expect that if at some point of the credible set the banks are paid, this will also be the case under movements in the direction (ρ_b, ρ_g). The interpretation of this property is that the green region, where the banks are paid and the project is maintained after a default, is formed by the points where the banks have a good performance and they are rewarded. A movement in the direction (ρ_b, ρ_g) correspond to a better performance of both banks, so it seems unnatural to deprive them of the reward. We can do the opposite interpretation for the red region, consisting of the points where the banks receive no payments and the project is liquidated after a default. In consequence, we expect that under the optimal contracts, it will be possible to identify red and green areas in the credible set, where the characteristics described in the boundaries will remain, and that will be delimited by some curves similar to those shown in figure 5 below.

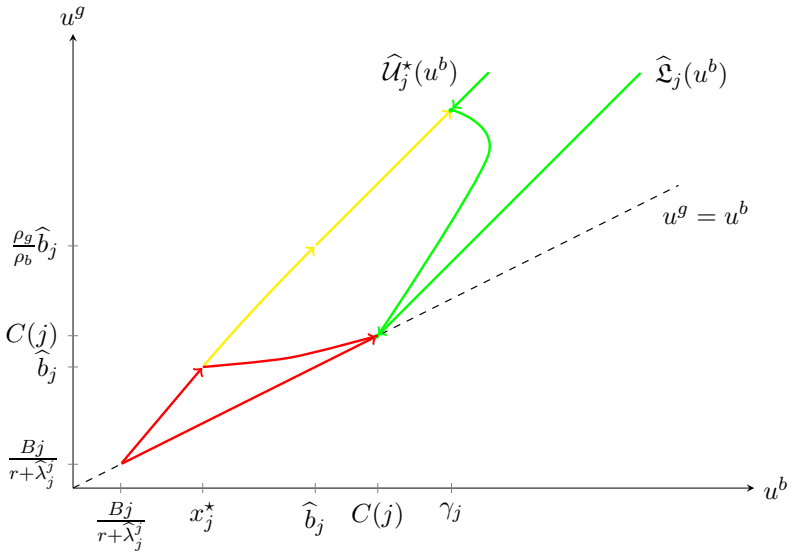

Figure 5: Optimal contract on the credible set.

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A Proofs for the pure moral hazard case

We provide in this section all the proofs of the results of Section 3. We start with the

Proof. [Proof of Proposition 3.1] Using the martingale representation theorem[8] (recall that D is supposed to be integrable and that k is bounded by definition), we deduce that for any $k \in \mathfrak{K}$ there exist $\mathbb{G}-$predictable processes $h^{1,i,k}$ and $h^{2,i,k}$ such that

$$\begin{aligned} du_t^i(k,\theta^i,D^i) = &\left(ru_t^i(k,D^i,\theta^i) - Bk_t\right)dt - \rho_i dD_t^i - h_t^{1,i,k}\left(dN_t - \lambda_t^k dt\right) \\ &- h_t^{2,i,k}\left(dH_t - (1-\theta_t^i)\lambda_t^k dt\right),\ 0 \le t < \tau,\ \mathbb{P}-a.s. \end{aligned} \tag{A.1}$$

Let us then define

$$\begin{aligned} Y_t^{i,k} &:= u_t^i(k,\theta^i,D^i),\ Z_t^{i,k} := (h_t^{1,i,k}, h_t^{2,i,k})^\top,\ M_t := (N_t, H_t)^\top, \\ \widetilde{M}_t^i &:= M_t - \int_0^t \lambda_s^0 (1, 1-\theta_s^i)^\top ds,\ K_t^i := \rho_i D_t^i, \end{aligned}$$

so that we can rewrite (A.1) as follows

$$Y_t^{i,k} = 0 - \int_t^\tau f^i(s,k_s,Y_s^{i,k},Z_s^{i,k})ds + \int_t^\tau Z_s^{i,k}\cdot d\widetilde{M}_s^i + \int_t^\tau dK_s^i,\ 0 \le t \le \tau,\ \mathbb{P}-a.s.,$$

[8] We emphasise that since the filtration $\mathbb{G}$ is augmented and generated by inhomogeneous Poisson processes, the predictable martingale representation holds for any of the probability measures $(\mathbb{P}^k)_{k\in\mathfrak{K}}$.

where

$$f^i(t,k,y,z) := ry - Bk + k\alpha_{I-N_t}\varepsilon z \cdot (1, 1-\theta_t^i)^\top.$$

In other words, $(Y^{i,k}, Z^{i,k})$ appears as a (super–)solution to a BSDE with (finite) random terminal time, as studied for instance by Peng [43] or Darling and Pardoux [13]. Following then Hamadène and Lepeltier [22] and El Karoui and Quenez [15]. By direct computations, it is easy to see that g^i satisfies, for any $(t,y,y',z,z') \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$

$$\left|g^i(t,y,z) - g^i(t,y',z)\right| = r\left|y-y'\right|,$$

$$g^i(t,y,z) - g^i(t,y,z') \leq \sup_{0\leq k\leq I-N_t}\left\{k\alpha_{I-N_t}\varepsilon(z-z')\cdot(1,1-\theta_t^i)^\top\right\} = \gamma_t(z,z')\lambda_t^0(z-z')\cdot(1,1-\theta_t^i)^\top,$$

where $\gamma_t(z,z') := \varepsilon \mathbf{1}_{\{(z-z')\cdot(1,1-\theta_t^i)^\top>0\}}$, verifies $0 \leq \gamma_t(z,z') \leq \varepsilon$. In particular, this means that the generator g^i satisfies the classical sufficient condition, introduced by Royer [49, Condition (A_γ)], ensuring that a comparison theorem holds for the corresponding BSDE with jumps (see [49, Theorem 2.5]). Moreover, since the intensity of the Poisson process M under $\mathbb{P}$ is bounded, it is clear that τ has exponential moments of any order. Since in addition we have $g^i(t,0,0) = -B(I-N_t)$, it is clear that the generator and the terminal condition of the BSDE (3.2) admit moments of any order and thus satisfy all the requirements ensuring wellposedness. Therefore, we deduce immediately that for any $k \in \mathfrak{K}$

$$Y_t^{i,k} \leq Y_t^i = Y_t^{i,k^{\star,i}},\ \mathbb{P}-a.s.,$$

where we defined

$$k_t^{\star,i} := (I-N_t)\mathbf{1}_{\{Z_t^i\cdot(1,1-\theta_t^i)^\top < b_t\}},\ \text{and } b_t := \frac{B}{\alpha_{I-N_t}\varepsilon},\ t\geq 0.$$

This means that Y^i is the value function of the bank, and that her optimal response given $(\theta^i, D^i) \in \Theta \times \mathcal{D}$ is $k^{\star,i}$. □

We continue with the

Proof. [Proof of Lemma 3.1] First of all, it is clear that the bank of type ρ_i can get arbitrarily large levels of utility (it suffices for the investor to set $dD_s^i := nds$ for n large enough, starting from time t). The bank's maximal level of utility is therefore $+\infty$, which corresponds to a utility equal to $-\infty$ for the investor. Then, coming back to the definition of the bank's problem, or to the BSDE (3.2), it is clear, for instance by using the comparison theorem for super solutions to (3.2) (see [49, Theorem 2.5]), that in order to minimise the utility that the bank obtains, the investor has to set $D^i = 0$. Moreover, since by definition we must always have $Y_t^i \geq 0$ and $Y_\tau^i = 0$, and since the totally inaccessible jumps of Y (recall that D is assumed to be predictable) are given by $\Delta Y_t^i = -Z_t^i \cdot \Delta M_t$, we must have that

$$Y_{t^-}^i = Z_t^i\cdot(1,1)^\top,\ \text{and } Y_{t^-}^i \geq Z_t^i\cdot(1,0)^\top,\ t>0,\ \mathbb{P}-a.s., \tag{A.2}$$

Indeed, the support of the laws of τ and the τ^j under $\mathbb{P}$ is $[0,+\infty)$. This implies in particular that we must have $Z_t^i\cdot(0,1)^\top \geq 0$, which in turn implies that the generator g^i is then non–increasing with respect to θ^i, and thus that the minimal utility for the bank is attained, as expected, when $\theta^i = 0$. Then, if $(\theta^i, D^i) = (0,0)$ (which is obviously in $\Theta \times \mathcal{D}$) starting from time t, it is clear that the bank

will never monitor and will obtain

$$U_t^i(0,0) = B(I-N_t)\mathbb{E}^{\mathbb{P}^{I-N.}}\left[\int_t^\tau e^{-r(s-t)}ds\middle|\mathcal{G}_t\right] = \frac{B(I-N_t)}{r}\left(1-\mathbb{E}^{\mathbb{P}^{I-N.}}\left[e^{-r(\tau-t)}\middle|\mathcal{G}_t\right]\right)$$
$$= \frac{B(I-N_t)}{r}\left(1-\int_0^{+\infty}\lambda_t^{I-N_t}e^{-x(r+\lambda_t^{I-N_t})}dx\right)$$
$$= \frac{B(I-N_t)}{r+\lambda_t^{I-N_t}}.$$

Notice that this corresponds to the investor getting

$$\mu(I-N_t)\mathbb{E}^{\mathbb{P}^{I-N.}}[\tau-t|\,\mathcal{G}_t] = \frac{\mu(I-N_t)}{\lambda_t^{I-N_t}}.$$

□

We finish with the

Proof. [Proof of Lemma 3.2] Let us show that for any $(\theta^i, D^i) \in \Theta \times \mathcal{D}$ enforcing $k=0$ from time t, we have $U_t^i(\theta^i, D^i) \geq b_t$. With such a contract, we must have

$$Z_s^i \cdot (1, 1-\theta_s^i)^\top \geq b_s,\ s \geq t.$$

By (A.2), this implies that for $s \geq t$, $Y_{s^-}^i \geq b_s$, which, by right-continuity at time t leads to the desired result. Notice also that this result implies the so–called *limited liability* property of the bank, which reads

$$Y_{t^-}^i - Z_t^i \cdot (1,0)^\top \geq b_t.$$

Now, in order for the investor to ensure that $U_t^i(\theta^i, D^i) = b_t$, it suffices for him, after time t, to offer the optimal contract derived in [42] (with initial condition b_t at time t), which we recall below (see Theorem 3.1). By [42, Proposition 3.16], the utility of the bank will then be b_t. □

B Utility of not monitoring

In this section we compute the utilities that the banks get from always shirking (without considering the payments) under contracts which liquidates the pool after some fixed number of defaults. Observe first that we have

$$\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[e^{-r(\tau_{N_t+1}-t)}\middle|\mathcal{G}_t\right] = \int_0^\infty e^{-rx}\lambda_{I-N_t}^{SH}e^{-\lambda_{I-N_t}^{SH}x}\,dx = \frac{\lambda_{I-N_t}^{SH}}{r+\lambda_{I-N_t}^{SH}},$$

and for any $l \in \{N_t+1, \ldots, I-1\}$

$$\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[e^{-r(\tau_{l+1}-\tau_l)}\middle|\mathcal{G}_t\right] = \int_0^\infty e^{-rx}\lambda_{I-l}^{SH}e^{-\lambda_{I-l}^{SH}x}\,dx = \frac{\lambda_{I-l}^{SH}}{r+\lambda_{I-l}^{SH}}.$$

For $m \in \{2, \ldots, I-N_t\}$, consider $\theta \in \Theta$ given by

$$\theta_s = \begin{cases} 1,\ t \leq s \leq \tau_{N_t+m}, \\ 0,\ s > \tau_{N_t+m}. \end{cases}$$

It means that the pool will be liquidated exactly after the following m defaults, so that the utility that the bank gets from shirking is

$$\begin{aligned}
u_t(k^{SH},\theta,0) &= \mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^\tau e^{-r(s-t)}B(I-N_s)\,ds\middle|\mathcal{G}_t\right] \\
&= \mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^{\tau_{N_t+1}} e^{-r(s-t)}B(I-N_t)\,ds + \sum_{i=N_t+1}^{N_t+m-1}\int_{\tau_i}^{\tau_{i+1}} e^{-r(s-t)}B(I-i)\,ds\middle|\mathcal{G}_t\right] \\
&= \frac{B(I-N_t)}{r}\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[1-e^{-r(\tau_{N_t+1}-t)}\middle|\mathcal{G}_t\right] \\
&\quad + \sum_{i=N_t+1}^{N_t+m-1}\frac{B(I-i)}{r}\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[e^{-r(\tau_i-t)}-e^{-r(\tau_{i+1}-t)}\middle|\mathcal{G}_t\right] \\
&= \frac{B(I-N_t)}{r+\lambda^{SH}_{I-N_t}} + \sum_{i=N_t+1}^{N_t+m-1}\frac{B(I-i)}{r}\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\left(1-e^{-r(\tau_{i+1}-\tau_i)}\right)\prod_{l=N_t}^{i-1}e^{-r(\tau_{l+1}-\tau_l)}\middle|\mathcal{G}_t\right].
\end{aligned}$$

Therefore, by independence we have

$$\begin{aligned}
u_t(k^{SH},\theta,0) &= \frac{B(I-N_t)}{r+\lambda^{SH}_{I-N_t}} + \sum_{i=N_t+1}^{N_t+m-1}\frac{B(I-i)}{r+\lambda^{SH}_{I-i}}\prod_{l=N_t}^{i-1}\frac{\lambda^{SH}_{I-l}}{r+\lambda^{SH}_{I-l}} \\
&= \frac{B(I-N_t)}{r+\lambda^{SH}_{I-N_t}} + \sum_{i=I-N_t-m+1}^{I-N_t-1}\frac{Bi}{r+\lambda^{SH}_{i}}\prod_{l=i+1}^{I-N_t}\frac{\lambda^{SH}_{l}}{r+\lambda^{SH}_{l}}.
\end{aligned}$$

C Short–term contracts with constant payment

In this section we analyse the optimal responses and the value functions of the banks at a starting time $t \geq 0$, under contracts with constant payments of the form $dD_s = cds$, where c is any $\mathcal{G}_t$-measurable random variable, and with $\theta \equiv 0$, so that the pool is liquidated immediately after the first default.

C.1 Optimal responses and feasible set

In this section we compute the optimal responses of the agents to the described contracts, depending on the value of c. We also show that for this class of contracts the set of expected payoff of the agents, starting of time t, is exactly $\mathcal{V}_t = \left[B(I-N_t)/(r+\lambda_t^{k^{SH}}),\infty\right)$.

(i) Let $k^0 := 0$. If the bank of type ρ_i always monitors, we have

$$u_t^i(k^0,\theta,D) = \mathbb{E}^{\mathbb{P}^0}\left[\int_t^\tau e^{-r(s-t)}\rho_i c ds\middle|\mathcal{G}_t\right] = \frac{\rho_i c}{r+\lambda_t^{k^0}}.$$

Hence, the continuation utility is constant in time and if the payment c is exactly equal to $u^i(r+\lambda_t^{k^0})/\rho_i$, then the bank receives exactly u^i. In this case, k^0 is incentive compatible if and only if $u^i \geq b_{I-N_t}$. The minimum payment such that the bank of type ρ_i will always work is therefore

$$\underline{c_i} = \frac{b_{I-N_t}(r+\lambda_t^{k^0})}{\rho_i}.$$

(ii) If the bank of type ρ_i always shirks, her continuation utility is constant and equal to

$$u_t^i(k^{SH},\theta,D) = \mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^\tau e^{-r(s-t)}(\rho_i c+B)ds\middle|\mathcal{G}_t\right] = \frac{\rho_i c + B(I-N_t)}{r+\lambda_t^{k^{SH}}}.$$

Then, if one takes c equal

$$\frac{u^i(r+\lambda_t^{k^{SH}}) - B(I-N_t)}{\rho_i},$$

the bank receives u^i. Therefore k^{SH} is incentive compatible if and only if $u^i < b_{I-N_t}$. Nevertheless, since the payment c must be positive, u^i must be greater than $B(I-N_t)/(r+\lambda_t^{k^{SH}})$. The supremum of the payments such that the bank of type ρ_i will always shirk is therefore equal to

$$\overline{c_i} = \frac{b_{I-N_t}(r+\lambda_t^{k^{SH}}) - B(I-N_t)}{\rho_i} = \frac{b_{I-N_t}(r+\lambda_t^{k^0})}{\rho_i} = \underline{c_i}.$$

Therefore the set of expected payoff under this class of contracts is $\mathcal{V}_t$. Let us summarise our findings.

Response of the bank of type ρ_i to the contract $\theta \equiv 0,\ dD_s = cds$, after time t:

With $\overline{c_i} = \dfrac{b_{I-N_t}(r+\lambda_t^{k^0})}{\rho_i}$,

- if $c \leq \overline{c_i} \implies k^{\star,i}(\theta, D) = k^{SH},\ U_t^i(\theta, D) = \dfrac{\rho_i c + B(I-N_t)}{r+\lambda_t^{k^{SH}}}$.
- if $c \geq \overline{c_i} \implies k^{\star,i}(\theta, D) = k^0,\ U_t^i(\theta, D) = \dfrac{\rho_i c}{r+\lambda_t^{k^0}}$.

C.2 Credible region under short–term contracts with constant payments

Once we know the optimal responses of the good and the bad bank for every payment c, we can study the relationship between their value functions for any short–term contract with constant payments.

(i) Suppose $c \in [0, \overline{c_g})$. Since $\overline{c_g} < \overline{c_b}$, we have that $k^{\star,b}(\theta, D) = k^{\star,g}(\theta, D) = k^{SH}$ and

$$U_t^g(\theta, D) = \frac{B(I-N_t)}{r+\lambda_t^{k^{SH}}} + \frac{\rho_g c}{r+\lambda_t^{k^{SH}}},\ U_t^b(\theta, D) = \frac{B(I-N_t)}{r+\lambda_t^{k^{SH}}} + \frac{\rho_b c}{r+\lambda_t^{k^{SH}}}.$$

Thus, the value functions verify the following equation

$$U_t^g(\theta, D) = \frac{\rho_g}{\rho_b} U_t^b(\theta, D) + \frac{B(I-N_t)}{r+\lambda_t^{k^{SH}}}\left(1 - \frac{\rho_g}{\rho_b}\right),$$

as well as

$$U_t^g(\theta, D) \in \left[\frac{B(I-N_t)}{r+\lambda_t^{k^{SH}}}, b_{I-N_t}\right),\ U_t^b(\theta, D) \in \left[\frac{B(I-N_t)}{r+\lambda_t^{k^{SH}}}, \frac{\rho_b}{\rho_g} b_{I-N_t} + \frac{B(I-N_t)}{r+\lambda_t^{k^{SH}}}\left(1 - \frac{\rho_b}{\rho_g}\right)\right).$$

(ii) If $c \in [\overline{c_g}, \overline{c_b})$, then $k^{\star,g}(\theta, D) = k^0$, $k^{\star,b}(\theta, D) = k^{SH}$ and the value functions of the banks are

$$U_t^g(\theta, D) = \frac{\rho_g c}{r+\lambda_t^{k^0}},\ U_t^b(\theta, D) = \frac{B(I-N_t)}{r+\lambda_t^{k^{SH}}} + \frac{\rho_b c}{r+\lambda_t^{k^{SH}}}.$$

Hence, they verify

$$U_t^g(\theta, D) = \frac{\rho_g}{\rho_b}\left(\frac{r+\lambda_t^{k^{SH}}}{r+\lambda_t^{k^0}}\right) U_t^b(\theta, D) - \frac{\rho_g}{\rho_b}\frac{B(I-N_t)}{r+\lambda_t^{k^0}},$$

with

$$U_t^g(\theta, D) \in \left[b_{I-N_t}, \frac{\rho_g}{\rho_b} b_{I-N_t}\right),\ U_t^b(\theta, D) \in \left[\frac{\rho_b}{\rho_g} b_{I-N_t} + \frac{B(I-N_t)}{r+\lambda_t^{k^{SH}}}\left(1 - \frac{\rho_b}{\rho_g}\right), b_{I-N_t}\right).$$

(*iii*) Finally, if $c \in [\overline{c_b}, \infty)$ then $k^{\star,b}(\theta, D) = k^{\star,g}(\theta, D) = k^0$ and

$$U_t^g(\theta, D) = \frac{\rho_g c}{r + \lambda_t^{k^0}},\ U_t^b(\theta, D) = \frac{\rho_b c}{r + \lambda_t^{k^{SH}}}.$$

Hence

$$U_t^g(\theta, D) = \frac{\rho_g}{\rho_b} U_t^b(\theta, D),$$

with

$$U_t^g(\theta, D) \in \left[\frac{\rho_g}{\rho_b} b_{I-N_t}, \infty\right),\ U_t^b(\theta, D) \in [b_{I-N_t}, \infty).$$

Figure 6 shows the pair of values of the banks that can be obtained using contracts with constant payments. For simplicity, u^g denotes the value function of the good bank and u^b that of the bad bank, and $j := I - N_t$. Depending on the payments, the values of the banks belong to one of the three lines represented, the last one being unbounded.

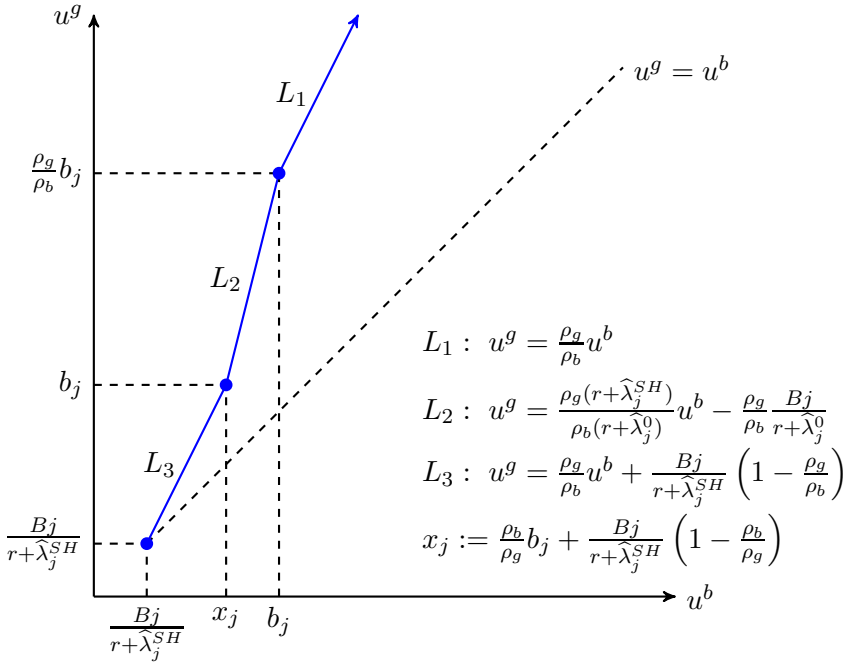

Figure 6: Credible region under short-term contracts with constant payments.

C.3 Initial lump–sum payment

Take any point $(u^b, u^g) \in L_1 \cup L_2 \cup L_3$. We know that there exists a contract $\theta \equiv 0$, $dD_s = cds$, starting from time t, such that $U_t^b(\theta, D) = u^b$ and $U_t^g(\theta, D) = u^g$. Consider the payments D^ℓ which differ from D only at time t, where a lump-sum payment of size $\ell > 0$ is made. This added lump-sum payment will not change the banks' incentives and the new value functions at time t will be

$$U_t^g(\theta, D^\ell) = u^g + \rho_g \ell,\ U_t^b(\theta, D^\ell) = u^b + \rho_b \ell.$$

Hence, the new pair of values of the banks belong to the line with slope $\frac{\rho_g}{\rho_b}$ which passes through the point (u^b, u^g). Since in our setting there is no upper bound on the payment, by increasing the value of ℓ it is possible to reach every point of the ray which starts at (u^b, u^g) and goes in the positive direction.

The subregion of the credible set that can be obtained by short-term contracts with constant payments and initial lump–sum payments is shown in Figure 7, with the same conventions as in Figure 6.

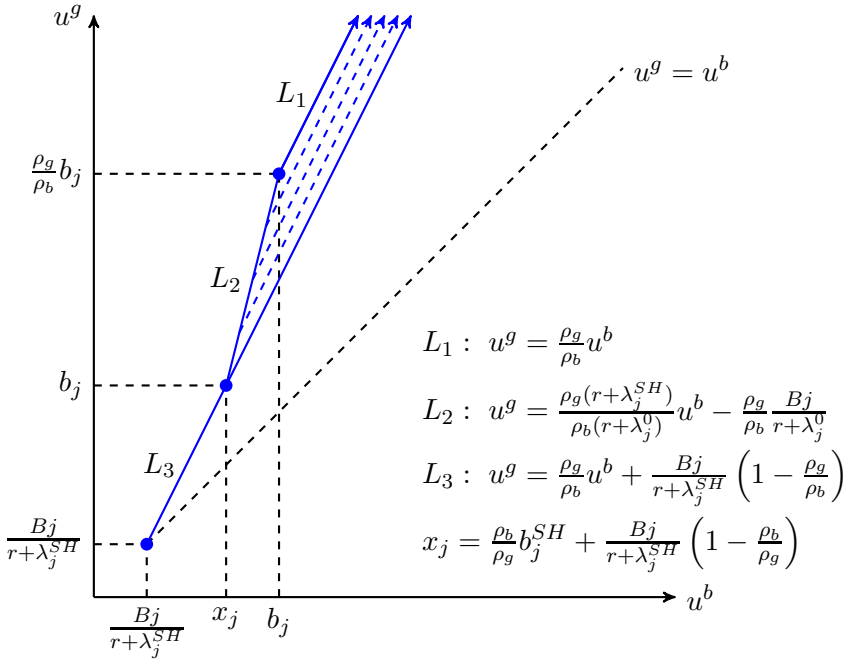

Figure 7: Credible region under short-term contracts with constant payment and lump-sum payments.

D Short-term contracts with delay

In this section we study the optimal responses of the banks and their value functions at a starting time $t \geq 0$, under contracts with constant payment after a certain time $t^\star > t$, and $\theta \equiv 0$. The case $t^\star = t$ corresponds to the situation of Appendix C.

D.1 Optimal responses and feasible set

In this section we compute the optimal responses of the agents to the described contracts, depending on the values of c and $t^\star$. We also show that under this class of contracts the set of expected payoff of the agents, starting at time t, is exactly $\mathcal{V}_t = \left[B(I - N_t)/(r + \lambda_t^{k^{SH}}), \infty\right)$.

(i) If the bank of type ρ_i always works, at any time $t \leq s < t^\star$, her continuation utility is, noticing that since $\theta = 0$, we have that $(\lambda_u^{k^0})_{u \geq t}$ is constant,

$$u_s^i(k^0, \theta, D) = \mathbb{E}^{\mathbb{P}^0}\left[\int_{t^\star \wedge \tau}^{\tau} e^{-r(u-s)} \rho_i c du \middle| \mathcal{G}_s\right] = \frac{e^{-(r+\lambda_t^{k^0})(t^\star - s)} \rho_i c}{r + \lambda_t^{k^0}} = u_t^i(k^0, \theta, D) e^{(r+\lambda_t^{k^0})(s-t)}.$$

Therefore, at $s = t^\star$ the continuation utility of the bank is $u_{t^\star}^i(k^0, \theta, D) = u_t^i(k^0, \theta, D) e^{(r+\lambda_t^{k^0})(t^\star - t)}$. Next, for any $s > t^\star$, the continuation utility of the bank will be

$$u_s^i(k^0, \theta, D) = \mathbb{E}^{\mathbb{P}^0}\left[\int_{s}^{\tau} e^{-r(u-s)} \rho_i c ds \middle| \mathcal{G}_s\right] = \frac{\rho_i c}{r + \lambda_t^{k^0}}.$$

Then, we see that once the bank starts being paid, her continuation utility becomes constant and it must be equal to $u^i_{t^\star}(k^0,\theta,D)$. Then, if for some $u^i \geq 0$, one chooses c equal to

$$\frac{u^i e^{(r+\lambda_t^{k^0})t^\star}(r+\lambda_t^{k^0})}{\rho_i}, \tag{D.1}$$

the continuation utility of the bank will be an increasing process with initial value u^i. Therefore, k^0 is incentive compatible if and only if $u^i \geq b_{I-N_t}$. The minimum payment and delay such that the bank always works are $t^\star = 0$ and

$$\underline{c_i} = \frac{b_{I-N_t}(r+\lambda_t^{k^0})}{\rho_i}.$$

(ii) If the bank of type ρ_i always shirks, at any time $t \leq s < t^\star$, her continuation utility is

$$u^i_s(k^{SH},\theta,D) = \mathbb{E}^{\mathbb{P}^{SH}}\left[\int_{t^\star\wedge\tau}^{\tau} e^{-r(u-s)}\rho_i c du + \int_s^\tau B du \middle| \mathcal{G}_s\right] = \frac{e^{-(r+\lambda_t^{k^{SH}})(t^\star-s)}\rho_i c}{r+\lambda_t^{k^{SH}}} + \frac{B(I-N_t)}{r+\lambda_t^{k^{SH}}}.$$

Therefore

$$u^i_s(k^{SH},\theta,D) = e^{(r+\lambda_t^{k^{SH}})(s-t)}\left(u^i_t(k^{SH},\theta,D) - \frac{B(I-N_t)}{r+\lambda_t^{k^{SH}}}\right) + \frac{B(I-N_t)}{r+\lambda_t^{k^{SH}}},$$

and the continuation utility is an increasing process. Recall that k^{SH} is incentive compatible if and only if $u^i_s(k^{SH},\theta,D) < b_{I-N_t}$ for every $s \geq t$. However, if $t^\star$ is large, there will exist t_w such that $u^i_{t_w}(k^{SH},\theta,D) = b_{I-N_t}$ and the bank will start to work. More precisely, t_w depends on the initial value $u^i_t(k^{SH},\theta,D)$ and is given by

$$t_w = t + \frac{1}{r+\lambda_t^{k^{SH}}}\log\left(\frac{b_{I-N_t}(r+\lambda_t^{k^{SH}}) - B(I-N_t)}{u^i_t(k^{SH},\theta,D)(r+\lambda_t^{k^{SH}}) - B(I-N_t)}\right).$$

Notice that $t_w \geq t$ if and only if $b_{I-N_t} \geq u^i_t(k^{SH},\theta,D)$. Therefore, k^{SH} is incentive compatible if and only if $t^\star < t_w$. Under this condition, at $t = t^\star$ the continuation utility of the bank is

$$u^i_{t^\star}(k^{SH},\theta,D) = e^{(r+\lambda_t^{k^{SH}})(t^\star-t)}\left(u^i_t(k^{SH},\theta,D) - \frac{B(I-N_t)}{r+\lambda_t^{k^{SH}}}\right) + \frac{B(I-N_t)}{r+\lambda_t^{k^{SH}}} < b_{I-N_t}.$$

Once the bank starts being paid her continuation utility is constant and equal to

$$u^i_s(k^{SH},\theta,D) = \mathbb{E}^{\mathbb{P}^{SH}}\left[\int_s^\tau e^{-r(u-s)}(\rho_i c + B(I-N_t))ds\right] = \frac{\rho_i c + B(I-N_t)}{r+\lambda_t^{k^{SH}}}.$$

So if the payment c is equal to

$$\frac{e^{(r+\lambda_j^{SH})(t^\star-t)}\left(u^i(r+\lambda_t^{k^{SH}}) - B(I-N_t)\right)}{\rho_i}, \tag{D.2}$$

the expected payoff of the bank at time t is u^i. The supremum of the delays and payments such that the bank always shirks are t_w and

$$\overline{c_i} = \frac{e^{(r+\lambda_t^{k^{SH}})(t_w-t)}\left[b_{I-N_t}(r+\lambda_t^{k^{SH}}) - B(I-N_t)\right]}{\rho_i} = \frac{b_{I-N_t}(r+\lambda_t^{k^0})}{\rho_i} = \underline{c_i}.$$

(*iii*) Finally, consider the case when $t^\star$ is greater than t_w. Under this contract, the bank will shirk until time t_w and will work afterwards. Indeed, from the previous analysis we know that this strategy is incentive compatible. At time t_w we have that $u^i_{t_w}(k^{SH},\theta,D) = b_{I-N_t}$ and for $s \in [t_w, t^\star)$ the continuation utility is given by

$$u^i_s(k^0,\theta,D) = \mathbb{E}^{\mathbb{P}^0}\left[\int_{t^\star\wedge\tau}^{\tau} e^{-r(u-s)}\rho_i c du \middle| \mathcal{G}_s\right] = \frac{e^{-(r+\lambda_t^{k^0})(t^\star-s)}\rho_i c}{r+\lambda_t^{k^0}}$$

$$= e^{(r+\lambda_t^{k^0})(s-t_w)} u^i_{t_w}(k^{SH},\theta,D) = b_{I-N_t} e^{(r+\lambda_t^{k^0})(s-t_w)}.$$

Therefore, at $t = t^\star$ the continuation utility of the bank is

$$u^i_{t^\star}(k^0,\theta,D) = b_{I-N_t} e^{(r+\lambda_t^{k^0})(t^\star-t_w)},$$

and for any $s > t^\star$, the continuation utility of the bank is constant and equal to

$$u^i_s(k^0,\theta,D) = \mathbb{E}^{\mathbb{P}^0}\left[\int_{s}^{\tau} e^{-r(u-s)}\rho_i c du \middle| \mathcal{G}_s\right] = \frac{\rho_i c}{r+\lambda_t^{k^0}}.$$

So if the payment c is equal to

$$\frac{b_{I-N_t}(r+\lambda_t^{k^0})e^{(r+\lambda_t^{k^0})(t^\star-t)}}{\rho_i}\left(\frac{u^i(r+\lambda_t^{k^{SH}}) - B(I-N_t)}{b_{I-N_t}(r+\lambda_t^{k^0})}\right)^{\frac{r+\lambda_t^{k^0}}{r+\lambda_t^{k^{SH}}}}, \tag{D.3}$$

the expected payoff of the bank at time t is u^i. The minimum payment and delay such that the bank shirks first and works afterwards are $t^\star = t_w$ and

$$\underline{c_i} = \frac{b_{I-N_t}(r+\lambda_t^{k^0})}{\rho_i} = \overline{c_i}.$$

The following box summarizes our findings in this case. Here, $\overline{t_i}(c)$ is the corresponding expression for t_w as a function of the payments c.

Response of the bank of type ρ_i to the contract $\theta \equiv 0$, $dD_s = 1_{\{s\geq t^\star\}} c ds$ after t:

Let $\overline{c_i} = \dfrac{b_{I-N_t}(r+\lambda_t^{k^0})}{\rho_i}$, $\overline{t_i}(c) := t + \dfrac{1}{r+\lambda_t^{k^0}}\log\left(\dfrac{\rho_i c}{b_{I-N_t}(r+\lambda_t^{k^0})}\right)$.

- If $c \leq \overline{c_i} \implies k^{\star,i}(\theta,D) = k^{SH},\ U^i_t(\theta,D) = e^{-(r+\lambda_t^{k^{SH}})(t^\star-t)}\dfrac{\rho_i c}{r+\lambda_t^{k^{SH}}} + \dfrac{B(I-N_t)}{r+\lambda_t^{k^{SH}}}$.
- If $c > \overline{c_i},\ t^\star \leq \overline{t_i}(c) \implies k^{\star,i}(\theta,D) = k^0,\ U^i_t(\theta,D) = e^{-(r+\lambda_t^{k^0})(t^\star-t)}\dfrac{\rho_i c}{r+\lambda_t^{k^0}}$.
- If $c > \overline{c_i},\ t^\star > \overline{t_i}(c) \implies k^{\star,i}_s(\theta,D) = k^{SH}_s \mathbf{1}_{\{s<\overline{t_j}(c)\}} + k^0_s \mathbf{1}_{\{s\geq\overline{t_j}(c)\}}$ and

$$U^i_t(\theta,D) = e^{-(r+\lambda_t^{k^{SH}})(t^\star-t)}\left[\frac{\rho_i c}{b_{I-N_t}(r+\lambda_t^{k^0})}\right]^{\frac{r+\lambda_t^{k^{SH}}}{r+\lambda_t^{k^0}}}\frac{b_{I-N_t}(r+\lambda_t^{k^0})}{r+\lambda_t^{k^{SH}}} + \frac{B(I-N_t)}{r+\lambda_t^{k^{SH}}}.$$

D.2 The upper boundary can be reached with contracts with delay

In this section we show that in some cases the short-term contracts with delay provide to the agents a pair of value functions lying in the upper boundary of the credible set.

(*i*) Let $c > \overline{c_b} > \overline{c_g}$ and $t^\star \leq \overline{t_b}(c) < \overline{t_g}(c)$. Then $k^{\star,b}(\theta, D) = k^{\star,g}(\theta, D) = k^0$ and the values of the banks are

$$U_t^g(\theta, D) = \frac{\rho_g c}{r + \lambda_t^{k^0}} e^{-(r+\lambda_t^{k^0})(t^\star - t)},\ U_t^b(\theta, D) = \frac{\rho_b c}{r + \lambda_t^{k^0}} e^{-(r+\lambda_t^{k^0})(t^\star - t)}.$$

Therefore the utilities satisfy

$$U_t^g(\theta, D) = \frac{\rho_g}{\rho_b} U_t^b(\theta, D),\ \text{with } U_t^g(\theta, D) \in \left[\frac{\rho_g}{\rho_b} b_{I-N_t}, \infty\right),\ U_t^b(\theta, D) \in [b_{I-N_t}, \infty)\,.$$

(*ii*) If $c > \overline{c_b}$ and $\overline{t_b}(c) < t^\star \leq \overline{t_g}(c)$, we have that the good bank will always work and the bad bank will start to work at time $\overline{t_b(c)}$. Their value functions are

$$U_t^g(\theta, D) = \frac{\rho_g c}{r + \lambda_t^{k^0}} e^{-(r+\lambda_t^{k^0})(t^\star - t)},$$

$$U_t^b(\theta, D) = e^{-(r+\lambda_t^{k^{SH}})(t^\star - t)} \left[\frac{\rho_b c}{b_{I-N_t}(r + \lambda_t^{k^0})}\right]^{\frac{r+\lambda_t^{k^{SH}}}{r+\lambda_t^{k^0}}} \frac{b_{I-N_t}(r + \lambda_t^{k^0})}{r + \lambda_t^{k^{SH}}} + \frac{B(I - N_t)}{r + \lambda_t^{k^{SH}}},$$

so they belong to the curve

$$U_t^g(\theta, D) = \frac{\rho_g}{\rho_b} b_{I-N_t}^{\frac{\lambda_t^{k^{SH}} - \lambda_t^{k^0}}{r+\lambda_t^{k^{SH}}}} \left(U_t^b(\theta, D) - \frac{B(I - N_t)}{r + \lambda_t^{k^{SH}}}\right)^{\frac{r+\lambda_t^{k^0}}{r+\lambda_t^{k^{SH}}}} \left(\frac{r + \lambda_t^{k^{SH}}}{r + \lambda_t^{k^0}}\right)^{\frac{r+\lambda_t^{k^0}}{r+\lambda_t^{k^{SH}}}},$$

and take values in the sets (recall the definition of $x_j^\star$ in proposition 4.2)

$$U_t^g(\theta, D) \in \left[b_{I-N_t}, \frac{\rho_g}{\rho_b} b_{I-N_t}\right),\ U_t^b(\theta, D) \in [x_{I-N_t}^\star, b_{I-N_t}).$$

(*iii*) If $c > \overline{c_b}$ and $\overline{t_g}(c) < t^\star$, the good bank will start to work at time $\overline{t_g(c)}$ and the bad bank will start to work at time $\overline{t_b(c)}$. Their value functions are

$$U_t^g(\theta, D) = e^{-(r+\lambda_t^{k^{SH}})(t^\star - t)} \left[\frac{\rho_g c}{b_{I-N_t}(r + \lambda_t^{k^0})}\right]^{\frac{r+\lambda_t^{k^{SH}}}{r+\lambda_t^{k^0}}} \frac{b_{I-N_t}(r + \lambda_t^{k^0})}{r + \lambda_t^{k^{SH}}} + \frac{B(I - N_t)}{r + \lambda_t^{k^{SH}}},$$

$$U_t^b(\theta, D) = e^{-(r+\lambda_t^{k^{SH}})(t^\star - t)} \left[\frac{\rho_b c}{b_{I-N_t}(r + \lambda_t^{k^0})}\right]^{\frac{r+\lambda_t^{k^{SH}}}{r+\lambda_t^{k^0}}} \frac{b_{I-N_t}(r + \lambda_t^{k^0})}{r + \lambda_t^{k^{SH}}} + \frac{B(I - N_t)}{r + \lambda_t^{k^{SH}}},$$

so they belong to the line

$$U_t^g(\theta, D) = \left(\frac{\rho_g}{\rho_b}\right)^{\frac{r+\lambda_t^{k^{SH}}}{r+\lambda_t^{k^0}}} \left(U_t^b(\theta, D) - \frac{B(I - N_t)}{r + \lambda_t^{k^{SH}}}\right) + \frac{B(I - N_t)}{r + \lambda_t^{k^{SH}}},$$

with

$$U_t^g(\theta, D) \in \left[\frac{B(I - N_t)}{r + \lambda_t^{k^{SH}}}, b_{I-N_t}\right),\ U_t^b(\theta, D) \in \left[\frac{B(I - N_t)}{r + \lambda_t^{k^{SH}}}, x_{I-N_t}^\star\right).$$

D.3 Credible region under contracts with delay

From the previous subsection we know that for every point (u^b, u^g) on the upper boundary there exists a pair $(c, t^\star)$, with $c > \overline{c_b}$, such that under the contract $(\theta \equiv 0, dD_s = c\mathbf{1}_{\{s \geq t^\star\}} ds)$ we have $U_t^b(\theta, D) = u^b$ and $U_t^g(\theta, D) = u^g$. As explained in C.3, if we consider the contract (θ, D^ℓ) with an additional initial lump-sum payment, the incentives of the banks will not change and the new value functions of the agents will be $U_t^b(\theta, D^\ell) = u^b + \rho_b \ell$, $U_t^g(\theta, D) = u^g + \rho_g \ell$. Therefore under short-term contracts with delay which reach the upper boundary and lump-sum payments, all the subregion of the credible set delimited by the lines shown in Figure 8 can be reached. We will not enter into details but it can be proved that under all the short-term contracts with delay (not only the ones who reach the upper boundary) and lump-sum payments, the subregion of the credible set which can be reached is exactly the same. When there is only one loan left, this region is equal to the whole credible set but when $j > 1$ the credible set is strictly bigger due to the pair of utilities that can be achieved in situations when $\theta \not\equiv 0$.

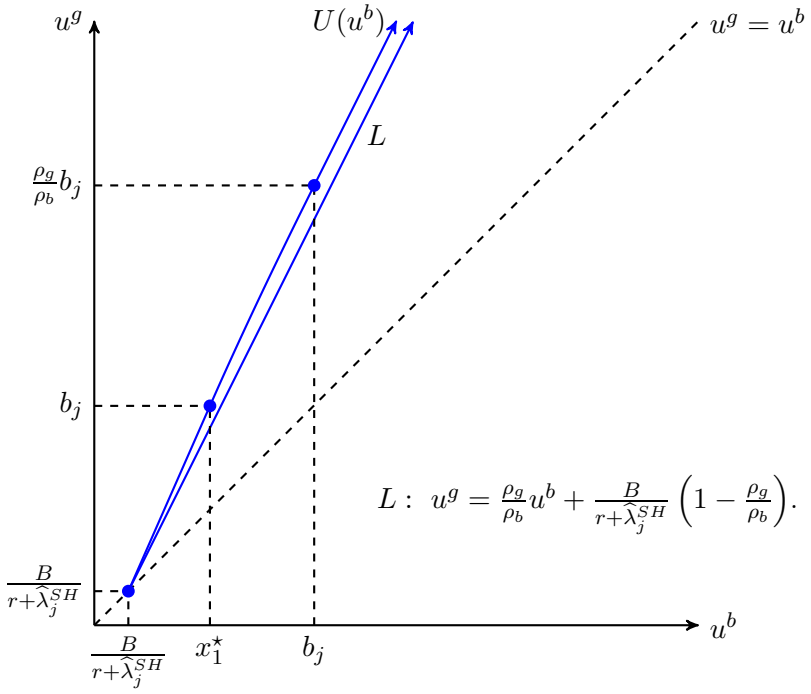

Figure 8: Credible region under short-term contracts with delay and lump-sum payment.

E Technical results for the lower boundary

We begin this section with the

Proof. [Proof of Lemma 4.1] The value functions of the banks under $\Psi := (\theta, D)$ are given by

$$U_t^g(\Psi) = \mathbb{E}^{\mathbb{P}^{k^\star, g}(\Psi)} \left[\int_t^\tau e^{-r(s-t)} (\rho_g dD_s + B k_s^{\star, g}(\Psi) ds) \middle| \mathcal{G}_t \right],$$
$$U_t^b(\Psi) = \mathbb{E}^{\mathbb{P}^{k^\star, b}(\Psi)} \left[\int_t^\tau e^{-r(s-t)} (\rho_b dD_s + B k_s^{\star, b}(\Psi) ds) \middle| \mathcal{G}_t \right].$$

Thus, we first have, $\mathbb{P}-a.s.$

$$\begin{aligned}U_t^g(\Psi) &\geq \mathbb{E}^{\mathbb{P}^{k^\star,b}(\Psi)}\left[\int_t^\tau e^{-r(s-t)}(\rho_g dD_s + Bk_s^{\star,b}(\Psi)ds)\middle|\mathcal{G}_t\right]\\ &\geq \mathbb{E}^{\mathbb{P}^{k^\star,b}(\Psi)}\left[\int_t^\tau e^{-r(s-t)}(\rho_b dD_s^g + Bk_s^{\star,b}(\Psi)ds)\middle|\mathcal{G}_t\right] = U_t^b(\Psi).\end{aligned}$$

But we also have

$$\begin{aligned}U_t^g(\Psi) &\geq \mathbb{E}^{\mathbb{P}^{k^\star,b}(\Psi)}\left[\int_t^\tau e^{-r(s-t)}(\rho_g dD_s + Bk_s^{\star,b}(\Psi)ds)\middle|\mathcal{G}_t\right]\\ &= U_t^b(\Psi) + (\rho_g-\rho_b)\mathbb{E}^{\mathbb{P}^{k^\star,b}(\Psi)}\left[\int_t^\tau e^{-r(s-t)}dD_s\middle|\mathcal{G}_t\right]\\ &= U_t^b(\Psi) + \frac{(\rho_g-\rho_b)}{\rho_b}\left(U_t^b(\Psi) - \mathbb{E}^{\mathbb{P}^{k^\star,b}(\Psi)}\left[\int_t^\tau e^{-r(s-t)}Bk_s^{\star,b}(\Psi)ds\middle|\mathcal{G}_t\right]\right)\\ &= \frac{\rho_g}{\rho_b}U_t^b(\Psi) - \frac{(\rho_g-\rho_b)}{\rho_b}\mathbb{E}^{\mathbb{P}^{k^\star,b}(\Psi)}\left[\int_t^\tau e^{-r(s-t)}Bk_s^{\star,b}(\Psi)ds\middle|\mathcal{G}_t\right].\end{aligned}$$

Observe next that

$$\sup_{k\in\mathfrak{K}}\mathbb{E}^{\mathbb{P}^k}\left[\int_t^\tau e^{-r(t-s)}Bk_s ds\middle|\mathcal{G}_t\right] = \mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^\tau e^{-r(t-s)}Bk_s^{SH}ds\middle|\mathcal{G}_t\right],$$

because the left–hand side is the value function of a bank who is offered a contract with no payments. Therefore, we have that

$$U_t^g(\Psi) \geq \frac{\rho_g}{\rho_b}U_t^b(\Psi) - \frac{(\rho_g-\rho_b)}{\rho_b}\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^\tau e^{-r(s-t)}Bk_s^{SH}ds\middle|\mathcal{G}_t\right] \geq \frac{\rho_g}{\rho_b}U_t^b(\Psi) - \frac{(\rho_g-\rho_b)}{\rho_b}C(I-N_t),$$

because the utility that the banks get from shirking is non–decreasing with respect to the process θ and its maximum value is equal to $C(I-N_t)$, attained when $\theta\equiv 1$ (see (4.2)). □

We continue this section with the

Proof. [Proof of Proposition 4.1] Due to Lemma 4.1, it suffices to prove the existence of contracts under which the value functions of the banks satisfy the equalities.

• **Step 1:** First, fix some $t\geq 0$, take any $u^b\in[c(I-N_t,1),C(I-N_t)]$ and fix $m\in\{1,\dots,I-N_t-1\}$ such that $c(I-N_t,m)\leq u^b\leq c(I-N_t,m+1)$. Next, take $\theta_t^0(u^b)\in[0,1]$ such that

$$u^b = c(I-N_t,m) + \theta_t^0(u^b)\left(c(I-N_t,m+1)-c(I-N_t,m)\right).$$

Then, there is a contract $(\theta,D)\in\Theta\times\mathcal{D}$ such that $U_t^g(\theta,D)=U_t^b(\theta,D)=u^b$. Such a contract can be defined as follows

$$dD_s := 0,\ \theta_s := \mathbf{1}_{\left\{t\leq s\leq\tau_{N_t+m}\right\}} + (1-\theta_t^0(u^b))\mathbf{1}_{\left\{\tau_{N_t+m}<s\leq\tau_{N_t+m+1}\right\}},\quad \text{for every } s\geq t.$$

The contract has no payments, it always maintains the pool after the first m defaults, maintains the pool with probability θ_0 after default $m+1$, and liquidates the pool at default $m+2$. It is clear that under this contract both banks always shirk in $[t,\tau]$, since they are not paid, and their value functions satisfy

$$\begin{aligned}U_t^g(\theta,D) = U_t^b(\theta,D) &= \mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^\tau e^{-r(s-t)}Bk_s^{SH}ds\middle|\mathcal{G}_t\right]\\ &= c(I-N_t,m) + \theta_t^0(u^b)\left(c(I-N_t,m+1)-c(I-N_t,m)\right) = u^b.\end{aligned}$$

• **Step 2:** Fix again some $t \geq 0$, and choose now any $u^b \geq C(I - N_t)$ and define

$$u^g := \frac{\rho_g}{\rho_b} u^b - \frac{(\rho_g - \rho_b)}{\rho_b} C(I - N_t).$$

Let $\ell_t := (u^b - C(I - N_t))/\rho_b$ and consider the admissible contract satisfying, $\theta_s = 1,\ dD_s = \ell_t \mathbf{1}_{\{s=t\}}$, for every $s \geq t$. The optimal strategy for both banks under this contract is to always shirk and then

$$U_t^b(\theta, D) = \mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^\tau e^{-r(s-t)}(\rho_b dD_s + Bk_s^{SH} ds)\middle| \mathcal{G}_t\right] = \rho_b \ell_t + C(I - N_t) = u^b,$$
$$U_t^g(\theta, D) = \mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^\tau e^{-r(s-t)}(\rho_g dD_s + Bk_s^{SH} ds)\middle| \mathcal{G}_t\right] = \rho_g \ell_t + C(I - N_t) = u^g.$$

□

We conclude this section by proving some useful results that will be used in Section 5.1.1 in the study of the value function of the investor on the lower boundary. We show that there are several ways of reaching the lower boundary and that all the contracts which can achieve it have the same structure as the ones used in the proof of Proposition 4.1.

Lemma E.1. *Consider any $(t, u^b, u^g) \in [0, \tau] \times \widehat{\mathcal{V}}_{I-N_t} \times \widehat{\mathcal{V}}_{I-N_t}$ such that in addition $u^b = u^g$. Any contract $\Psi = (\theta, D) \in \Theta \times D$ such that $U_t^b(\Psi) = u^b$ and $U_t^g(\Psi) = u^g$, has no payments on $[t, \tau]$ and consequently both banks always shirk under Ψ.*

Proof. Looking at the proof of (4.3) we deduce that necessarily

$$k_s^{\star,g}(\Psi) = k_s^{\star,b}(\Psi),\ dD_s = 0,\ \forall s \geq t.$$

Since there are no payments, we have that $k_s^{\star,g}(\Psi) = k_s^{\star,b}(\Psi) = k_s^{SH}$ for $s \in [t, \tau]$ and indeed have

$$U_t^g(\Psi) = U_t^b(\Psi) = \mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^\tau e^{-r(s-t)} B(I - N_s) ds\middle| \mathcal{G}_t\right].$$

□

Lemma E.2. *Consider any $(t, u^g, u^b) \in \mathbb{R}_+ \times \widehat{\mathcal{V}}_{I-N_t} \times \widehat{\mathcal{V}}_{I-N_t}$ such that in addition*

$$u^g = \frac{\rho_g}{\rho_b} u^b - \frac{(\rho_g - \rho_b)}{\rho_b} C(I - N_t).$$

Under any contract $\Psi = (\theta, D) \in \Theta \times D$ such that $U_t^b(\Psi) = u^b$ and $U_t^g(\Psi) = u^g$, the pool is not liquidated until the last default ($\tau = \tau^I$) and both banks always shirk on $[t, \tau]$.

Proof. Looking at the proof of (4.4), we deduce that necessarily $k_s^{\star,g}(\Psi) = k_s^{\star,b}(\Psi) = k_s^{SH},\ \theta_s = 1$, for every $s \geq t$. Thus, the value functions of the banks are given by

$$U_t^g(\Psi) = \rho_g \mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^{\tau^I} e^{-r(s-t)} dD_s\middle| \mathcal{G}_t\right] + C(I - N_t),$$
$$U_t^b(\Psi) = \rho_b \mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^{\tau^I} e^{-r(s-t)} dD_s\middle| \mathcal{G}_t\right] + C(I - N_t).$$

□

F Technical results for the upper boundary

Lemma F.1. *For every $j \geq 1$, $x_j^\star > \dfrac{\rho_b}{\rho_g} b_j$.*

Proof. For any $j \geq 1$, define the functions $g, h : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$g(x) := x^{\frac{r+\widehat{\lambda}_j^{SH}}{r+\widehat{\lambda}_j^0}} b_j \frac{r+\widehat{\lambda}_j^0}{r+\widehat{\lambda}_j^{SH}} + \frac{Bj}{r+\widehat{\lambda}_j^{SH}}, \; h(x) := b_j x.$$

Then g is strictly convex in $\mathbb{R}_+$ and we have that $g(1) = h(1) = b_j$ and $g'(1) = h'(1) = b_j$. Thus, h is the tangent line to g at $x = 1$ so $g(x) > h(x)$ for every $x \neq 1$ and therefore

$$x_j^\star = g\left(\frac{\rho_b}{\rho_g}\right) > h\left(\frac{\rho_b}{\rho_g}\right) = \frac{\rho_b}{\rho_g} b_j.$$

□

Proposition F.1. *For every $j \geq 1$, the function $\widehat{\mathcal{U}}_j^\star$ defined by (4.17) satisfies*

$$\frac{\widehat{\mathcal{U}}_j^\star(x)}{x} \leq \frac{\rho_g}{\rho_b}, \; \forall x \geq \frac{Bj}{r+\widehat{\lambda}_j^{SH}}.$$

Moreover, equality holds if and only if $x \geq \widehat{b}_j$.

Proof. Define $A(x) := \dfrac{\widehat{\mathcal{U}}_j^\star(x)}{x}$. If $x \geq \widehat{b}_{j-1}$ then $A(x) = \rho_g/\rho_b$. If now $x \in [x_j^\star, \widehat{b}_j)$, we have

$$A(x) = \frac{\rho_g}{\rho_b}(\widehat{b}_j)^{\frac{\widehat{\lambda}_j^{SH}-\widehat{\lambda}_j^0}{r+\widehat{\lambda}_j^{SH}}} \left(\frac{r+\widehat{\lambda}_j^{SH}}{r+\widehat{\lambda}_j^0}\right)^{\frac{r+\widehat{\lambda}_j^0}{r+\widehat{\lambda}_j^{SH}}} \frac{1}{x}\left(x - \frac{Bj}{r+\widehat{\lambda}_j^{SH}}\right)^{\frac{r+\widehat{\lambda}_j^0}{r+\widehat{\lambda}_j^{SH}}}.$$

This function is decreasing so that A reaches its maximum value over $[x_j^\star, \widehat{b}_j)$ at $x_j^\star$. Next, we have

$$A(x_j^\star) = \frac{\widehat{b}_j}{x_j^\star} < \frac{\rho_g}{\rho_b} \iff x_j^\star > \frac{\rho_b}{\rho_g} b_j,$$

and the last inequality holds as a consequence of Lemma F.1.

Finally, if $x \in \left[\frac{Bj}{r+\widehat{\lambda}_j^{SH}}, x_j^\star\right)$ then

$$A(x) = \frac{1}{x}\left(\frac{\rho_g}{\rho_b}\right)^{\frac{r+\widehat{\lambda}_j^{SH}}{r+\widehat{\lambda}_j^0}} \left(x - \frac{Bj}{r+\widehat{\lambda}_j^{SH}}\right) + \frac{1}{x}\frac{Bj}{r+\widehat{\lambda}_j^{SH}}.$$

This function is increasing, hence $A(x) \leq A(x_j^\star) < \frac{\rho_g}{\rho_b}, \; \forall x \in \left[\frac{Bj}{r+\widehat{\lambda}_j^{SH}}, x_j^\star\right]$. □

Corollary F.1. *Let $j \geq 2$ and $\widehat{\mathcal{U}}_j^\star, \widehat{\mathcal{U}}_{j-1}^\star$ defined by (4.17), and assume that $\widehat{\lambda}_j^{k^g} \leq \widehat{\lambda}_j^{k^b}$. Then, for any $u^b \geq h^{1,b} + \frac{B(j-1)}{r+\widehat{\lambda}_{j-1}^{SH}}$ we have*

$$\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^{1,b})\widehat{\lambda}_j^{k^g} - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\widehat{\lambda}_j^{k^b}(u^b - h^{1,b}) \leq 0.$$

Furthermore, equality holds if and only if $u^b - h^{1,b} \geq \widehat{b}_j$, $u^b \geq \widehat{b}_j$ and $\widehat{\lambda}_j^{k^b} = \widehat{\lambda}_j^{k^g}$.

Proof. Under the conditions of the corollary, the following allows us to conclude immediately

$$\frac{\widehat{\mathcal{U}}^\star_{j-1}(u^b - h^{1,b})}{u^b - h^{1,b}} \leq \frac{\rho_g}{\rho_b} \leq \left(\widehat{\mathcal{U}}^\star_j\right)'(u^b).$$

□

Corollary F.2. *For $j \geq 1$, let $\widehat{C}_j$ and $\widehat{\mathcal{U}}^\star_j$ be defined by (4.16) and (4.17) respectively. If $(\theta, h^{1,b}) \in \widehat{C}_j$ is such that $u^b - \theta(u^b - h^{1,b}) \geq \widehat{b}_j$ then $\widehat{\mathcal{U}}^\star_j(u^b) - \theta\widehat{\mathcal{U}}^\star_{j-1}(u^b - h^{1,b}) \geq \widehat{b}_j$. As a consequence, in the context of equation (4.15), for every $(\theta, h^{1,b}) \in \widehat{C}_j$ we have $k^g \leq k^b$ and $\widehat{\lambda}^{k^g}_j \leq \widehat{\lambda}^{k^b}_j$.*

Proof. First observe that $u^b - \theta(u^b - h^{1,b}) \geq \widehat{b}_j$ implies $u^b \geq \widehat{b}_j$. Then we have

$$\widehat{\mathcal{U}}^\star_j(u^b) - \widehat{b}_j \geq \frac{\rho_g}{\rho_b}(u^b - \widehat{b}_j) \geq \frac{\widehat{\mathcal{U}}^\star_{j-1}(u^b - h^{1,b})}{u^b - h^{1,b}}(u^b - \widehat{b}_j).$$

Also, $\theta \leq \dfrac{u^b - \widehat{b}_j}{u^b - h^{1,b}}$ and thus

$$\widehat{\mathcal{U}}^\star_j(u^b) - \theta\widehat{\mathcal{U}}^\star_{j-1}(u^b - h^{1,b}) \geq \widehat{\mathcal{U}}^\star_j(u^b) - \left(\frac{u^b - \widehat{b}_j}{u^b - h^{1,b}}\right)\widehat{\mathcal{U}}^\star_{j-1}(u^b - h^{1,b}) \geq \widehat{b}_j.$$

□

We now proceed with the

Proof. [Proof of Lemma 4.2] We start with the region $u^b < \widehat{b}_1$, $\widehat{\mathcal{U}}_1(u^b) < \widehat{b}_1$. For these points, we have that $k^b = k^g = 1$, so (4.11) can be solved easily and leads to, for some $C_1 \in \mathbb{R}$,

$$\widehat{\mathcal{U}}_1(u^b) = C_1\left(u^b - \frac{B}{r + \widehat{\lambda}^1_1}\right) + \frac{B}{r + \widehat{\lambda}^1_1}.$$

If $u^b < \widehat{b}_1$ and $\widehat{\mathcal{U}}_1(u^b) \geq \widehat{b}_1$, then $k^b = 1$, $k^g = 0$ and we can solve (4.11) to obtain for some $C_2 \in \mathbb{R}$

$$\widehat{\mathcal{U}}_1(u^b) = C_2\left(u^b - \frac{B}{r + \widehat{\lambda}^1_1}\right)^{\frac{r+\widehat{\lambda}^0_1}{r+\widehat{\lambda}^1_1}}.$$

Finally, when $u^b \geq \widehat{b}_1$ and $\widehat{\mathcal{U}}(u^b) \geq \widehat{b}_1$ the optimal strategies are $k^b = k^g = 0$ and we have for some $C_3 \in \mathbb{R}$, $\widehat{\mathcal{U}}_1(u^b) = C_3 u^b$. We are interested in smooth solutions of (4.11). Denote by $\widehat{\mathcal{U}}^{(1)}_1$, $\widehat{\mathcal{U}}^{(2)}_1$ and $\widehat{\mathcal{U}}^{(3)}_1$ the following functions

$$\widehat{\mathcal{U}}^{(1)}_1(u^b) := C_1\left(u^b - \frac{B}{r + \widehat{\lambda}^1_1}\right) + \frac{B}{r + \lambda^1_1},\ \widehat{\mathcal{U}}^{(2)}_1(u^b) := C_2\left(u^b - \frac{B}{r + \widehat{\lambda}^1_1}\right)^{\frac{r+\widehat{\lambda}^0_1}{r+\widehat{\lambda}^1_1}},\ \widehat{\mathcal{U}}^{(3)}_1(u^b) := C_3 u^b.$$

We will determine the relations between the constants which allow the smooth fitting of $\widehat{\mathcal{U}}_1$. First we impose $\widehat{\mathcal{U}}^{(2)}_1(\widehat{b}_1) = \widehat{\mathcal{U}}^{(3)}_1(\widehat{b}_1)$ and we get

$$C_2\left(\widehat{b}_1\frac{r + \widehat{\lambda}^0_1}{r + \widehat{\lambda}^1_1}\right)^{\frac{r+\widehat{\lambda}^0_1}{r+\widehat{\lambda}^1_1}} = C_3\widehat{b}_1.$$

It can be checked that this relation between C_1 and C_2 ensures also that $(\widehat{\mathcal{U}}_1^{(2)})'(\widehat{b}_1) = (\widehat{\mathcal{U}}_1^{(3)})'(\widehat{b}_1)$. Next, define x_1 as the point such that $\widehat{\mathcal{U}}_1^{(1)}(x_1) = \widehat{b}_1$, *i.e.*

$$x_1 = \frac{\widehat{b}_1}{C_1}\left(\frac{r+\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1}\right) + \frac{B}{r+\widehat{\lambda}_1^1}.$$

Also, define x_2 as the point such that $\widehat{\mathcal{U}}_1^{(2)}(x_2) = \widehat{b}_1$, *i.e.*

$$x_2 = \left(\frac{\widehat{b}_1}{C_2}\right)^{\frac{r+\widehat{\lambda}_1^1}{r+\widehat{\lambda}_1^0}} + \frac{B}{r+\widehat{\lambda}_1^1}.$$

We impose $x_1 = x_2$ and we get

$$\frac{\widehat{b}_1}{C_1}\left(\frac{r+\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1}\right) = \left(\frac{\widehat{b}_1}{C_2}\right)^{\frac{r+\widehat{\lambda}_1^1}{r+\widehat{\lambda}_1^0}},$$

and this relation ensures also that $(\widehat{\mathcal{U}}_1^{(1)})'(x_1) = (\widehat{\mathcal{U}}_1^{(2)})'(x_2)$. Expressing both C_1 and C_2 in terms of C_3 we get $\widehat{\mathcal{U}}_1^{(3)}(u^b) = C_3 u^b$, and

$$\widehat{\mathcal{U}}_1^{(1)}(u^b) = C_3^{\frac{r+\widehat{\lambda}_1^1}{r+\widehat{\lambda}_1^0}}\left(u^b - \frac{B}{r+\widehat{\lambda}_1^1}\right) + \frac{B}{r+\widehat{\lambda}_1^1},$$

$$\widehat{\mathcal{U}}_1^{(2)}(u^b) = C_3 \widehat{b}_1^{\frac{\widehat{\lambda}_1^1-\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1}}\left(\frac{r+\widehat{\lambda}_1^1}{r+\widehat{\lambda}_1^0}\right)^{\frac{r+\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1}}\left(u^b - \frac{B}{r+\widehat{\lambda}_1^1}\right)^{\frac{r+\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1}}.$$

□

We pursue with the

Proof. [Proof of Lemma 4.3] For $C > 0$, define the following modification $\widehat{\mathcal{U}}_1^{C,\star}$ of $\widehat{\mathcal{U}}_1^C$

$$\widehat{\mathcal{U}}_1^{C,\star}(u^b) := \begin{cases} \widehat{\mathcal{U}}_1^C(u^b),\ u^b \le x_1^{C,\star}, \\ \dfrac{\rho_g}{\rho_b}(u^b - x_1^{C,\star}) + \widehat{\mathcal{U}}_1^C(x_1^{C,\star}),\ u^b \ge x_1^{C,\star}, \end{cases}$$

where

$$x_1^{C,\star} := \inf\left\{u^b \in \left[\frac{B}{r+\widehat{\lambda}_1^1}, +\infty\right),\ \left(\widehat{\mathcal{U}}_1^C\right)'(u^b) \le \frac{\rho_g}{\rho_b}\right\}.$$

The function $\widehat{\mathcal{U}}_1^{C,\star}$ is continuously differentiable, solves the diffusion equation in $[B/(r+\widehat{\lambda}_1^1), x_1^{C,\star})$ and satisfies $\left(\widehat{\mathcal{U}}_1^{C,\star}\right)' = \rho_g/\rho_b$ in $(x_1^{C,\star}, \infty)$. In the following we will study for which values of C this function indeed solves the HJB equation.

– First of all, if $C^{\frac{r+\widehat{\lambda}_1^1}{r+\widehat{\lambda}_1^0}} \le \frac{\rho_g}{\rho_b}$, we have that

$$x_1^{C,\star} = \frac{B}{r+\widehat{\lambda}_1^1},\ \widehat{\mathcal{U}}_1^{C,\star}(u^b) = \frac{\rho_g}{\rho_b}\left(u^b - \frac{B}{r+\widehat{\lambda}_1^1}\right) + \frac{B}{r+\widehat{\lambda}_1^1},\ \left(\widehat{\mathcal{U}}_1^{C,\star}\right)'(u^b)\rho_b - \rho_g = 0,$$

so that we need to check that for every u^b in $[B/(r+\widehat{\lambda}_1^1), \infty)$

$$r\widehat{\mathcal{U}}_1^{C,\star}(u^b) - \left(\widehat{\mathcal{U}}_1^{C,\star}\right)'(u^b)\left(ru^b - Bk^b + u^b\widehat{\lambda}_1^{k^b}\right) + \widehat{\mathcal{U}}_1^{C,\star}(u^b)\widehat{\lambda}_1^{k^g} - Bk^g \ge 0.$$

Take $u^b > \widehat{b}_1$. Then $k^g = k^b = 0$, and we have

$$
\begin{aligned}
&r\widehat{\mathcal{U}}_1^{C,\star}(u^b) - \left(\widehat{\mathcal{U}}_1^{C,\star}\right)'(u^b)\left(ru^b - Bk^b + u^b\widehat{\lambda}_1^{k^b}\right) + \widehat{\mathcal{U}}_1^{C,\star}(u^b)\widehat{\lambda}_1^{k^g} - Bk^g \\
&= r\left[\frac{\rho_g}{\rho_b}\left(u^b - \frac{B}{r+\widehat{\lambda}_1^1}\right) + \frac{B}{r+\widehat{\lambda}_1^1}\right] - \frac{\rho_g}{\rho_b}\left[(r+\widehat{\lambda}_1^0)u^b\right] + \widehat{\lambda}_1^0\left[\frac{\rho_g}{\rho_b}\left(u^b - \frac{B}{r+\widehat{\lambda}_1^1}\right) + \frac{B}{r+\widehat{\lambda}_1^1}\right] \\
&= (r+\widehat{\lambda}_1^0)\frac{B}{r+\widehat{\lambda}_1^1}\left(1 - \frac{\rho_g}{\rho_b}\right) < 0.
\end{aligned}
$$

Hence $\widehat{\mathcal{U}}_1^{C,\star}$ is not a solution of (4.12).

– If $\left(\frac{\rho_g}{\rho_b}\right)^{\frac{r+\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1}} < C \leq \frac{\rho_g}{\rho_b}$, then $x_1^{C,\star} = \widehat{b}_1\frac{r+\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1}\left(C\frac{\rho_b}{\rho_g}\right)^{\frac{r+\widehat{\lambda}_1^1}{\widehat{\lambda}_1^1-\widehat{\lambda}_1^0}} + \frac{B}{r+\widehat{\lambda}_1^1}$. Take $u^b > \widehat{b}_1$, then $k^g = k^b = 0$ and

$$
\begin{aligned}
&r\widehat{\mathcal{U}}_1^{C,\star}(u^b) - \left(\widehat{\mathcal{U}}_1^{C,\star}\right)'(u^b)\left(ru^b - Bk^b + u^b\widehat{\lambda}_1^{k^b}\right) + \widehat{\mathcal{U}}_1^{C,\star}(u^b)\widehat{\lambda}_1^{k^g} - Bk^g \\
&= (r+\widehat{\lambda}_1^0)\left(\widehat{b}_1 C^{\frac{r+\widehat{\lambda}_1^1}{\widehat{\lambda}_1^1-\widehat{\lambda}_1^0}}\left(\frac{\rho_b}{\rho_g}\right)^{\frac{r+\widehat{\lambda}_1^0}{\widehat{\lambda}_1^1-\widehat{\lambda}_1^0}}\frac{\widehat{\lambda}_1^1-\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1} - \frac{\rho_g}{\rho_b}\frac{B}{r+\widehat{\lambda}_1^1}\right) \\
&\leq (r+\widehat{\lambda}_1^0)\left(\widehat{b}_1\left(\frac{\rho_g}{\rho_b}\right)^{\frac{r+\widehat{\lambda}_1^1}{\widehat{\lambda}_1^1-\widehat{\lambda}_1^0}}\left(\frac{\rho_b}{\rho_g}\right)^{\frac{r+\widehat{\lambda}_1^0}{\widehat{\lambda}_1^1-\widehat{\lambda}_1^0}}\frac{\widehat{\lambda}_1^1-\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1} - \frac{\rho_g}{\rho_b}\frac{B}{r+\widehat{\lambda}_1^1}\right) \\
&= (r+\widehat{\lambda}_1^0)\left(\widehat{b}_1\frac{\rho_g}{\rho_b}\frac{\widehat{\lambda}_1^1-\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1} - \frac{B}{r+\widehat{\lambda}_1^1}\frac{\rho_g}{\rho_b}\right) = 0.
\end{aligned}
$$

The inequality is strict if $C < \frac{\rho_g}{\rho_b}$ so the only value of C such that $\widehat{\mathcal{U}}_1^{C,\star}$ solves the HJB equation is $C = \frac{\rho_g}{\rho_b}$.

– For large values of C, *i.e.* $C > \frac{\rho_g}{\rho_b}$, we have that $x_1^{C,\star} = +\infty$ and then $\widehat{\mathcal{U}}_1^{C,\star} = \widehat{\mathcal{U}}_1^C$. We exclude this case because these functions do not satisfy condition (4.13). □

We end this section with the

Proof. [Proof of Proposition 4.2] The proof is by induction. For $j = 1$ the result is proved in Step 2, so we take any $j > 1$ and assume that $\widehat{\mathcal{U}}_{j-1}^\star$ solves its corresponding diffusion equation. We will need to consider three different cases to prove that $\widehat{\mathcal{U}}_j^\star$ solves the equation (4.15). In each one of them we prove that the supremum in the right–hand side of (4.15) is attained with $\theta = 0$, so therefore the diffusion equation takes the same form as the one in the case with one loan left. Then, it follows from the analysis in Step 2 that its solution satisfies also the variational inequality (4.9).

– **Case 1:** $u^b < \widehat{b}_j$, $\widehat{\mathcal{U}}_j^\star(u^b) < \widehat{b}_j$.

In this case for any $(\theta, h^1) \in \widehat{C}^j$, we have that $k^g = k^b = j$. To simplify the notations, let us define $c_j(u^b) := \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\left(ru^b - Bj + u^b\widehat{\lambda}_j^{SH}\right)$, then the term inside the sup in (4.15) becomes

$$
c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^{SH} + Bj + \theta\widehat{\lambda}_j^{SH}\left[\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1) - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)(u^b - h^1)\right],
$$

and the optimal choice of θ in this case is 0 (uniquely) because from Corollary F.1 we have

$$
\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1) - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)(u^b - h^1) < 0.
$$

– **Case 2:** $u^b < \widehat{b}_j, \widehat{\mathcal{U}}_j^\star(u^b) \geq \widehat{b}_j$.

In this case $k^b = j$ for every $(\theta, h^1) \in \widehat{C}^j$. The term inside the sup in (4.15) becomes

$$c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^{k^g} + Bk^g + \theta\left[\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1)\widehat{\lambda}_j^{k^g} - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\widehat{\lambda}_j^{SH}(u^b - h^1)\right].$$

Define the following sets

$$\widehat{C}_j^0 := \{(\theta, h^1) \in \widehat{C}^j,\ \widehat{\mathcal{U}}_j^\star(u^b) - \theta\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1) \geq \widehat{b}_j\},\ \widehat{C}_j^j := \{(\theta, h^1) \in \widehat{C}^j,\ \widehat{\mathcal{U}}_j^\star(u^b) - \theta\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1) < \widehat{b}_j\},$$

and note that $k^g = 0$ for every $(\theta, h^1) \in \widehat{C}_j^0$ and $k^g = j$ for every $(\theta, h^1) \in \widehat{C}_j^j$. Also, the pair $(0, h^1)$ belongs to $\widehat{C}_j^0$ for every feasible h^1.

- If $(\theta, h^1) \in \widehat{C}_j^0$ we have

$$\begin{aligned}
&c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^{k^g} + Bk^g + \theta\left[\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1)\widehat{\lambda}_j^{k^g} - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\widehat{\lambda}_j^{SH}(u^b - h^1)\right] \\
&= c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^0 + \theta\left[\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1)\widehat{\lambda}_j^0 - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\widehat{\lambda}_j^{SH}(u^b - h^1)\right] \leq c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^0,
\end{aligned}$$

where the inequality is due to Corollary F.1.

- If $(\theta, h^1) \in \widehat{C}_j^j$ we have

$$\begin{aligned}
&c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^{k^g} + Bk^g + \theta\left[\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1)\widehat{\lambda}_j^{k^g} - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\widehat{\lambda}_j^{SH}(u^b - h^1)\right] \\
&= c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^{SH} + Bj + \theta\left[\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1)\widehat{\lambda}_j^{SH} - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\widehat{\lambda}_j^{SH}(u^b - h^1)\right] \\
&< c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^{SH} + Bj \\
&= c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^{SH} + b_j(\widehat{\lambda}_j^{SH} - \widehat{\lambda}_j^0) \\
&\leq c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^{SH} + \widehat{\mathcal{U}}_j^\star(u^b)(\widehat{\lambda}_j^{SH} - \widehat{\lambda}_j^0) = c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^0,
\end{aligned}$$

where the first inequality is a consequence of Corollary F.1 and the second one holds because $\widehat{\mathcal{U}}_j^\star(u^b) \geq \widehat{b}_j$. So we conclude that the optimal value for θ in this case is also 0 (uniquely).

– **Case 3:** $u^b \geq \widehat{b}_j, \widehat{\mathcal{U}}_j^\star(u^b) \geq \widehat{b}_j$.

Thanks to Proposition F.2 , we know that there are only three possibilities for the value of (k^b, k^g). Define the sets

$$\begin{aligned}
\widehat{C}_j^{0,0} &:= \left\{(\theta, h^1) \in \widehat{C}^j,\ u^b - \theta(u^b - h^1) \geq \widehat{b}_j,\ \widehat{\mathcal{U}}_j^\star(u^b) - \theta\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1) \geq \widehat{b}_j\right\}, \\
\widehat{C}_j^{j,0} &:= \left\{(\theta, h^1) \in \widehat{C}^j,\ u^b - \theta(u^b - h^1) < \widehat{b}_j,\ \widehat{\mathcal{U}}_j^\star(u^b) - \theta\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1) \geq \widehat{b}_j\right\}, \\
\widehat{C}_j^{j,j} &:= \left\{(\theta, h^1) \in \widehat{C}^j,\ u^b - \theta(u^b - h^1) < \widehat{b}_j,\ \widehat{\mathcal{U}}_j^\star(u^b) - \theta\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1) < \widehat{b}_j\right\}.
\end{aligned}$$

Then, $(k^b, k^g) = (0, 0)$ for every $(\theta, h^1) \in \widehat{C}_j^{0,0}$, $(k^b, k^g) = (j, 0)$ for every $(\theta, h^1) \in \widehat{C}_j^{j,0}$ and $(k^b, k^g) = (j, j)$ for every $(\theta, h^1) \in \widehat{C}_j^{j,j}$. Also, $(0, h^1)$ belongs to $\widehat{C}_j^{0,0}$ for any feasible h^1.

- If $(\theta, h^1) \in \widehat{C}_j^{0,0}$ then the term inside the sup in (4.15) is, because of Corollary F.1, equal to

$$\begin{aligned}
&\left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)u^b\left(r + \widehat{\lambda}_j^0\right) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^0 + \theta\widehat{\lambda}_j^0\left[\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1) - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)(u^b - h^1)\right] \\
&\leq \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)u^b\left(r + \widehat{\lambda}_j^0\right) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^0,
\end{aligned}$$

• If $(\theta, h^1) \in \widehat{C}_j^{j,0}$, then $h^1 < \widehat{b}_j$ and $\frac{u^b - b_j}{u^b - h^1} < \theta \leq \frac{\widehat{\mathcal{U}}_j^\star(u^b) - \widehat{b}_j}{\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1)}$. The term in the sup in (4.15) is

$$
\begin{aligned}
&c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^0 + \theta\left[\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1)\widehat{\lambda}_j^0 - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\widehat{\lambda}_j^{SH}(u^b - h^1)\right] \\
&< c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^0 + \left(\frac{u^b - \widehat{b}_j}{u^b - h^1}\right)\left[\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1)\widehat{\lambda}_j^0 - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\widehat{\lambda}_j^{SH}(u^b - h^1)\right] \\
&\leq c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\lambda_j^0 + (u^b - \widehat{b}_j)\left[\left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\widehat{\lambda}_j^0 - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\widehat{\lambda}_j^{SH}\right] \\
&= \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\left(ru^b + u^b\widehat{\lambda}_j^0\right) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^0.
\end{aligned}
$$

Both inequalities are direct consequences of Corollary F.1.

• Finally, if $(\theta, h^1) \in \widehat{C}_j^{j,j}$, note that $h^1 < \widehat{b}_j$, $\widehat{\mathcal{U}}_j^\star(u^b) - \widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1) < \widehat{b}_j$ and

$$
\frac{u^b - \widehat{b}_j}{u^b - h^1} \leq \frac{\widehat{\mathcal{U}}_j^\star(u^b) - \widehat{b}_j}{\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1)} < \theta.
$$

Then, the term inside the sup in (4.15) becomes

$$
\begin{aligned}
&c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^{SH} + Bj + \theta\widehat{\lambda}_j^{SH}\left[\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1) - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)(u^b - h^1)\right] \\
&\leq c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^{SH} + Bj + \frac{\widehat{\mathcal{U}}_j^\star(u^b) - \widehat{b}_j}{\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1)}\widehat{\lambda}_j^{SH}\left(\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1) - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)(u^b - h^1)\right) \\
&\leq c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^{SH} + Bj + \widehat{\lambda}_j^{SH}\left(\widehat{\mathcal{U}}_j^\star(u^b) - \widehat{b}_j - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\frac{\widehat{\mathcal{U}}_j^\star(u^b) - \widehat{b}_j}{\frac{\rho_g}{\rho_b}}\right) \\
&= c_j(u^b) - \widehat{b}_j\widehat{\lambda}_j^0 + \widehat{\lambda}_j^{SH}\left(-\frac{\rho_b}{\rho_g}\left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\widehat{\mathcal{U}}_j^\star(u^b) + \frac{\rho_b}{\rho_g}\left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\widehat{b}_j\right) \\
&= \widehat{\lambda}_j^{SH}\left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\left(u^b - \frac{\rho_b}{\rho_g}\widehat{\mathcal{U}}_j^\star(u^b)\right) + \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\left(ru^b + \frac{\rho_b}{\rho_g}\widehat{\lambda}_j^{SH}\widehat{b}_j - Bj\right) - \widehat{\lambda}_j^0\widehat{b}_j.
\end{aligned}
$$

The first inequality is a consequence of Corollary F.1 and the second one of the fact that the function $h^1 \longmapsto \widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1)/(u^b - h^1)$ is non–decreasing and constant for large values of h^1, which implies that $\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1)/(u^b - h^1) \leq \rho_g/\rho_b$. Now we use the explicit form of $\widehat{\mathcal{U}}_j^\star$ and compute

$$
\begin{aligned}
&\widehat{\lambda}_j^{SH}\left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\left(u^b - \frac{\rho_b}{\rho_g}\widehat{\mathcal{U}}_j^\star(u^b)\right) + \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\left(ru^b + \frac{\rho_b}{\rho_g}\widehat{\lambda}_j^{SH}\widehat{b}_j - Bj\right) - \widehat{\lambda}_j^0\widehat{b}_j \\
&= \frac{\rho_g}{\rho_b}ru^b + \widehat{\lambda}_j^{SH}\widehat{b}_j - \frac{\rho_g}{\rho_b}Bj - \widehat{\lambda}_j^0\widehat{b}_j = \frac{\rho_g}{\rho_b}ru^b + Bj\left(1 - \frac{\rho_g}{\rho_b}\right) < \frac{\rho_g}{\rho_b}ru^b.
\end{aligned}
$$

The term in the last line corresponds to $\left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\left(ru^b + u^b\widehat{\lambda}_j^0\right) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^0$ and therefore the optimal θ in this case is also 0. Observe that in this case every $(\theta, h^1) \in \widehat{C}_j^{0,0}$ such that $u^b - h^1 \geq \widehat{b}_j$ is optimal. □

We next continue with the

Proof. [Proof of Theorem 4.1] We divide the proof in 3 steps.

• **Step 1:** Let us prove first that the SDE (4.19) has a unique solution, keeping in mind that $\Psi^\star$ liquidates the pool immediately after the first default. We consider two cases: if $u^b < \widehat{b}_{I-N_t}$, by

right–continuity we can find for every solution of (4.19) some $\varepsilon \in (0, \tau - t)$ such that $u^b_s < \widehat{b}_{I-N_t}$ for $s \in [t, t+\varepsilon]$. Consequently u^b solves the ODE

$$du^b_s = \Big[(r + \widehat{\lambda}^{SH}_{I-N_t})u^b_s - B(I - N_t)\Big]\, ds,\ s \in [t, t+\varepsilon],$$

whose unique solution is given by

$$u^b_s = e^{(r+\widehat{\lambda}^{SH}_{I-N_t})(s-t)}\left(u^b - \frac{B(I-N_t)}{r+\widehat{\lambda}^{SH}_{I-N_t}}\right) + \frac{B(I-N_t)}{r+\widehat{\lambda}^{SH}_{I-N_t}},\ s \in [t, t+\varepsilon].$$

So, as long as there is no default and the project keeps running u^b_s will be deterministic until it reaches the value $\widehat{b}_{I-N_t}$. That will eventually happen at time

$$t^\star(u^b) := t + \frac{1}{r+\widehat{\lambda}^{SH}_{I-N_t}} \log\left(\frac{\widehat{b}_{I-N_t}(r+\widehat{\lambda}^0_{I-N_t})}{u^b(r+\widehat{\lambda}^{SH}_{I-N_t}) - B(I-N_t)}\right),$$

and we see from (4.19) that at time $t^\star(u^b)$ we will have $du^b_s = 0$, so $u^b_s = \widehat{b}_{I-N_t}$ for every $s \in [t^\star(u^b), \tau)$. In the second case, if $u^b \geq \widehat{b}_{I-N_t}$ then (4.19) becomes $du^b_s = -u^b_{s-} dN_s,\ s \in [t, \tau]$, and necessarily $u^b_s = u^b$ for every $s \in [t, \tau)$. This proves the existence and uniqueness of the solution of (4.19) in both cases.

• **Step 2:** Now we turn to the values of the banks under $\Psi^\star$. If $u^b \geq \widehat{b}_{I-N_t}$, we know from the previous analysis that $u^b_s = u^b \geq \widehat{b}_{I-N_t}$ for every $s \in [t, \tau)$, so in this case $\Psi^\star$ is a short–term contract with constant payment, see Section C.1. Using the notations of this section, since $c \geq \overline{c_b} \geq \overline{c_g}$ both banks will always work, the value function of the bad bank is $U^b_t(\Psi^\star) = \rho_b c/(r + \widehat{\lambda}^0_{I-N_t}) = u^b$ and the one of the good bank is $U^g_t(\Psi^\star) = \rho_g c/(r+\widehat{\lambda}^0_{I-N_t}) = \rho_g/\rho_b u^b = \widehat{\mathcal{U}}^\star_{I-N_t}(u^b)$.

In the case where $u^b < \widehat{b}_{I-N_t}$, $\Psi^\star$ is a short–term contract with delay $t^\star(u^b)$ and constant payment, see Section D.1. Using the notations of this section, since $c = \overline{c_b}$ the bad bank will always shirk and her value function is

$$U^b_t(\Psi^\star) = \rho_b c \frac{e^{-(r+\widehat{\lambda}^{SH}_{I-N_t})t^\star(u^b)}}{r+\widehat{\lambda}^{SH}_{I-N_t}} + \frac{B}{r+\widehat{\lambda}^{SH}_{I-N_t}} = u^b.$$

For the good bank we have two sub-cases. First, if $u^b \in [x^\star_{I-N_t}, \widehat{b}_{I-N_t})$ then $\overline{t_g}(c) \geq t^\star(u^b)$, so the good bank will always work and her value function is

$$U^g_t(\Psi^\star) = \frac{\rho_g}{\rho_b}\widehat{b}_{I-N_t}^{\frac{\widehat{\lambda}^{SH}_{I-N_t} - \widehat{\lambda}^0_{I-N_t}}{r+\widehat{\lambda}^{SH}_{I-N_t}}} \left(\frac{r+\widehat{\lambda}^{SH}_{I-N_t}}{r+\widehat{\lambda}^0_{I-N_t}}\right)^{\frac{r+\widehat{\lambda}^0_{I-N_t}}{r+\widehat{\lambda}^{SH}_{I-N_t}}} \left(u^b - \frac{B(I-N_t)}{r+\widehat{\lambda}^{SH}_{I-N_t}}\right)^{\frac{r+\widehat{\lambda}^0_{I-N_t}}{r+\widehat{\lambda}^{SH}_{I-N_t}}} = \widehat{\mathcal{U}}^\star_{I-N_t}(u^b).$$

If $u^b \in \left[\frac{B}{r+\widehat{\lambda}^{SH}_{I-N_t}}, x^\star_{I-N_t}\right)$ then $\overline{t_g}(c) < t^\star(u^b)$, so the good bank will start working at time $t^\star(u^b)$ and her value function is

$$U^g_t(\Psi^\star) = \frac{\rho_g}{\rho_b}^{\frac{r+\widehat{\lambda}^{SH}_{I-N_t}}{r+\widehat{\lambda}^0_{I-N_t}}} \left(u^b - \frac{B(I-N_t)}{r+\widehat{\lambda}^{SH}_{I-N_t}}\right) + \frac{B(I-N_t)}{r+\widehat{\lambda}^{SH}_{I-N_t}} = \widehat{\mathcal{U}}^\star_{I-N_t}(u^b).$$

• **Step 3:** Since $U^b_t(\Psi^\star) = u^b$, it is trivial that $\Psi^\star \in \overline{\mathcal{A}}^b(t, u^b)$. Consider now any contract $\Psi = (D, \theta, h^{1,b}, h^{2,b}) \in \overline{\mathcal{A}}^b(t, u^b)$. We recall that the value function of the bad bank under Ψ satisfies

$$dU^b_s(\Psi) = \Big(rU^b_s(\Psi) - Bk^{\star,b}_s(\Psi) + [h^{1,b}_s + h^{2,b}_s(1-\theta_s)]\lambda^{k^{\star,b}(\Psi)}_s\Big)\, ds - \rho_b dD_s - h^{1,b}_s dN_s - h^{2,b}_s dH_s,$$

with $k_s^{\star,b}(\Psi) = \mathbf{1}_{\{h_s^{1,b}+(1-\theta_s)h_s^{2,b}<b_s\}}$. Define the process

$$G_w := \int_t^w e^{-r(s-t)}\left[\rho_g dD_s + k_s^{\star,g}(\Psi)Bds\right] + e^{-r(w-t)}\widehat{\mathcal{U}}^\star_{I-N_w}(U_w^b(\Psi)),\ w\in[t,\tau].$$

Observe we can rewrite the second term in the following form (with the convention $\tau_{N_t}=t$, $\tau_{N_w+1}=w$)

$$\begin{aligned}e^{-r(w-t)}\widehat{\mathcal{U}}^\star_{I-N_w}(U_w^b(\Psi)) &= \sum_{i=N_t}^{N_w} e^{-r(\tau_{i+1}-t)}\widehat{\mathcal{U}}^\star_{I-i}\left(U^b_{\tau_{i+1}^-}(\Psi)\right) - e^{-r(\tau_i-t)}\widehat{\mathcal{U}}^\star_{I-i}\left(U^b_{\tau_i}(\Psi)\right)\\ &+\sum_{i=N_t}^{N_w-1} e^{-r(\tau_{i+1}-t)}\left(\widehat{\mathcal{U}}^\star_{I-(i+1)}\left(U^b_{\tau_{i+1}}(\Psi)\right) - \widehat{\mathcal{U}}^\star_{I-i}\left(U^b_{\tau_{i+1}^-}(\Psi)\right)\right) + \widehat{\mathcal{U}}^\star_{I-N_t}\left(U_t^b(\Psi)\right).\end{aligned}$$

Since the functions $\widehat{\mathcal{U}}_j^\star$ are C^1, we can apply Itô's formula on the intervals $[\tau_i\wedge\tau, \tau_{i+1}\wedge\tau)$ with $i\in\{N_t,\dots,N_w\}$ to obtain an integral expression for the first sum. Regarding the second sum, observe that

$$\begin{aligned}&\widehat{\mathcal{U}}^\star_{I-(i+1)}\left(U^b_{\tau_{i+1}}(\Psi)\right) - \widehat{\mathcal{U}}^\star_{I-i}\left(U^b_{\tau_{i+1}^-}(\Psi)\right)\\ &= \left(\widehat{\mathcal{U}}^\star_{I-(i+1)}\left(U^b_{\tau_{i+1}^-}(\Psi)-h^{1,b}_{\tau_{i+1}}\right) - \widehat{\mathcal{U}}^\star_{I-i}\left(U^b_{\tau_{i+1}^-}(\Psi)\right)\right)\Delta N_{\tau_{i+1}} - \widehat{\mathcal{U}}^\star_{I-(i+1)}\left(U^b_{\tau_{i+1}^-}(\Psi)-h^{1,b}_{\tau_{i+1}}\right)\Delta H_{\tau_{i+1}}\\ &= \int_{\tau_i}^{\tau_{i+1}}\left(\widehat{\mathcal{U}}^\star_{I-(i+1)}\left(U^b_{s^-}(\Psi)-h^{1,b}_s\right) - \widehat{\mathcal{U}}^\star_{I-i}\left(U^b_{s^-}(\Psi)\right)\right)dN_s - \int_{\tau_i}^{\tau_{i+1}}\widehat{\mathcal{U}}^\star_{I-(i+1)}\left(U^b_{s^-}(\Psi)-h^{1,b}_s\right)dH_s.\end{aligned}$$

Hence

$$\begin{aligned}G_{\tau\wedge v} =& \widehat{\mathcal{U}}^\star_{I-N_t}(u^b) + \sum_{i=N_t}^{I-1}\int_{\tau_i\wedge v}^{\tau_{i+1}\wedge v} e^{-r(s-t)}\left(\rho_g - \rho_b\left(\widehat{\mathcal{U}}^\star_{I-i}\right)'\left(U_s^b(\Psi)\right)\right)dD_s\\ &+\sum_{i=N_t}^{I-1}\int_{\tau_i\wedge v}^{\tau_{i+1}\wedge v} e^{-r(s-t)}\left(k_s^{\star,g}(\Psi)B - r\widehat{\mathcal{U}}^\star_{I-i}\left(U_s^b(\Psi)\right)\right)ds\\ &+\sum_{i=N_t}^{I-1}\int_{\tau_i\wedge v}^{\tau_{i+1}\wedge v} e^{-r(s-t)}\lambda_s^{k^\star,g(\Psi)}\left(\theta_s\widehat{\mathcal{U}}^\star_{I-i-1}\left(U^b_{s^-}(\Psi)-h^{1,b}_s\right) - \widehat{\mathcal{U}}^\star_{I-i}\left(U_s^b(\Psi)\right)\right)ds\\ &+\sum_{i=N_t}^{I-1}\int_{\tau_i\wedge v}^{\tau_{i+1}\wedge v} e^{-r(s-t)}\widehat{\mathcal{U}}^{\star\prime}_{I-i}\Big(U_s^b(\Psi)\Big)\Big(rU_s^b(\Psi) - Bk_s^{\star,b}(\Psi) + \lambda_s^{k^\star,b(\Psi)}(h_s^{1,b}+(1-\theta_s)h_s^{2,b})\Big)ds\\ &+\sum_{i=N_t}^{I-1}\int_{\tau_i\wedge v}^{\tau_{i+1}\wedge v} e^{-r(s-t)}\left(\widehat{\mathcal{U}}^\star_{I-i-1}\left(U^b_{s^-}(\Psi)-h^{1,b}_s\right) - \widehat{\mathcal{U}}^\star_{I-i}\left(U^b_{s^-}(\Psi)\right)\right)(dN_s - \lambda_s^{k^\star,g(\Psi)}ds)\\ &-\sum_{i=N_t}^{I-1}\int_{\tau_i\wedge v}^{\tau_{i+1}\wedge v} e^{-r(s-t)}\widehat{\mathcal{U}}^\star_{I-i-1}\left(U^b_{s^-}(\Psi)-h^{1,b}_s\right)(dH_s - \lambda_s^{k^\star,g(\Psi)}(1-\theta_s)ds).\end{aligned}$$

We know that the derivative of every $\widehat{\mathcal{U}}_j^\star$ is greater than ρ_g/ρ_b by definition, and since D is non–decreasing, the first sum of integrals is non–positive. Also, the functions $\widehat{\mathcal{U}}_j^\star$ are solutions of the system of HJB equations, which implies that for any admissible contract the second and the third sum of integrals are also non–positive. We deduce

$$\begin{aligned}G_{\tau\wedge v} \le& \widehat{\mathcal{U}}^\star_{I-N_t}(u^b) + \sum_{i=N_t}^{I-1}\int_{\tau_i\wedge v}^{\tau_{i+1}\wedge v} e^{r(t-s)}\Big(\widehat{\mathcal{U}}^\star_{I-i-1}\Big(U^b_{s^-}(\Psi)-h^{1,b}_s\Big) - \widehat{\mathcal{U}}^\star_{I-i}\Big(U^b_{s^-}(\Psi)\Big)\Big)(dN_s - \lambda_s^{k^\star,g}ds)\\ &-\sum_{i=N_t}^{I-1}\int_{\tau_i\wedge v}^{\tau_{i+1}\wedge v} e^{-r(s-t)}\widehat{\mathcal{U}}^\star_{I-i-1}\left(U^b_{s^-}(\Psi)-h^{1,b}_s\right)(dH_s - \lambda_s^{k^\star,g(\Psi)}(1-\theta_s)ds).\end{aligned}\tag{F.1}$$

Next, for every i we have that, recalling that the functions $\widehat{\mathcal{U}}_j^\star$ are non–decreasing and null at 0

$$\begin{aligned}\mathbb{E}^{\mathbb{P}^{k^\star,g}}\left[\int_t^\tau e^{-r(s-t)}\left|\widehat{\mathcal{U}}_{I-i-1}^\star\left(U_s^b(\Psi)-h_s^{1,b}\right)\right|ds\middle|\mathcal{G}_t\right] &\leq \mathbb{E}^{\mathbb{P}^{k^\star,g}}\left[\int_t^\tau e^{-r(s-t)}\frac{\rho_g}{\rho_b}U_s^b(\Psi)ds\middle|\mathcal{G}_t\right]\\ &\leq \mathbb{E}^{\mathbb{P}^{k^\star,g}}\left[\int_t^\tau e^{-r(s-t)}\frac{\rho_g}{\rho_b}u^b e^{(r+\lambda)s}ds\middle|\mathcal{G}_t\right]<\infty,\end{aligned}$$

with $\lambda := \max_{1\leq j\leq I}\widehat{\lambda}_j^{SH}$. Indeed, we have between two consecutive jump times of N

$$\begin{aligned}dU_s^b(\Psi) &= \left(rU_s^b(\Psi)-Bk_s^{\star,b}(\Psi)+(h_s^{1,b}+(U_s^b(\Psi)-h_s^{1,b})(1-\theta_s))\lambda_s^{k^{\star,b}(\Psi)}\right)ds-\rho_b dD_s\\ &\leq \left(rU_s^b(\Psi)+h_s^{1,b}\lambda_s^{k^{\star,b}(\Psi)}+(U_s^b(\Psi)-h_s^{1,b})(1-\theta_s)\lambda_s^{k^{\star,b}(\Psi)}\right)ds\\ &= U_s^b(\Psi)\left(r+(1-\theta_s)\lambda_s^{k^{\star,b}(\Psi)}\right)ds+h_s^{1,b}\theta_s\lambda_s^{k^{\star,b}(\Psi)}ds\\ &\leq U_s^b(\Psi)\left(r+\lambda_s^{k^{\star,b}(\Psi)}\right)ds,\end{aligned}$$

where we used the facts that $h_s^{1,b}\in[0,U_s^b(\Psi)]$, the functions $\widehat{\mathcal{U}}_j^\star$ are non–decreasing and $U_s^b(\Psi)$ is bounded from below and has positive jumps. Similarly

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}^{k^\star,g}}\left[\int_t^\tau e^{-r(s-t)}\left|\widehat{\mathcal{U}}_{I-i-1}^\star\left(U_{s^-}^b(\Psi)-h_s^{1,b}\right)-\widehat{\mathcal{U}}_{I-i}^\star\left(U_{s^-}^b(\Psi)\right)\right|ds\middle|\mathcal{G}_t\right]\\ &\leq \mathbb{E}^{\mathbb{P}^{k^\star,g}}\left[\int_t^\tau e^{-r(s-t)}\left|\widehat{\mathcal{U}}_{I-i-1}^\star\left(U_{s^-}^b(\Psi)-h_s^{1,b}\right)\right|ds\middle|\mathcal{G}_t\right]+\mathbb{E}^{\mathbb{P}^{k^\star,g}}\left[\int_t^\tau e^{-r(s-t)}\left|\widehat{\mathcal{U}}_{I-i}^\star\left(U_{s^-}^b(\Psi)\right)\right|ds\middle|\mathcal{G}_t\right]\\ &\leq \mathbb{E}^{\mathbb{P}^{k^\star,g}}\left[\int_t^\tau e^{-r(s-t)}\frac{\rho_g}{\rho_b}U_s^b(\Psi)ds\middle|\mathcal{G}_t\right]+\mathbb{E}^{\mathbb{P}^{k^\star,g}}\left[\int_t^\tau e^{-r(s-t)}\frac{\rho_g}{\rho_b}U_s^b(\Psi)ds\middle|\mathcal{G}_t\right]\\ &\leq 2\mathbb{E}^{\mathbb{P}^{k^\star,g}}\left[\int_t^\tau e^{-r(s-t)}\frac{\rho_g}{\rho_b}u^b e^{(r+\lambda)s}ds\middle|\mathcal{G}_t\right]<\infty.\end{aligned}$$

Then, the stochastic integrals appearing above are martingales, and taking conditional expectation in (F.1) we get $\mathbb{E}^{\mathbb{P}^{k^\star,g}}[G_{\tau\wedge v}|\,\mathcal{G}_t]\leq\widehat{\mathcal{U}}_{I-N_t}^\star(u^b)$ and from Fatou's Lemma we obtain

$$\widehat{\mathcal{U}}_{I-N_t}^\star(u^b)\geq\varliminf_{v\to\infty}\mathbb{E}^{\mathbb{P}^{k^\star,g}}\left[G_{\tau\wedge v}|\,\mathcal{G}_t\right]\geq\mathbb{E}^{\mathbb{P}^{k^\star,g}}\left[\varliminf_{v\to\infty}G_{\tau\wedge v}\middle|\mathcal{G}_t\right]=U_t^g(\Psi),$$

where we used that, $\mathbb{P}^{k^\star,g}-a.s.$

$$\begin{aligned}\varliminf_{v\to\infty}G_{\tau\wedge v} &= \lim_{v\to\infty}\int_t^{\tau\wedge v}e^{-r(s-t)}\left[\rho_g dD_s+k_s^{\star,g}(\Psi)Bds\right]+1_{\{v<\tau\}}e^{-r(v-t)}\widehat{\mathcal{U}}_{I-N_v}^\star(U_v^b(\Psi))\\ &= \int_t^\tau e^{-r(s-t)}\left[\rho_g dD_s+k_s^{\star,g}(\Psi)Bds\right].\end{aligned}$$

□

We end this section with the

Proof. [Proof of Proposition 4.3] The definition of $\widehat{\mathcal{C}}_j$ does not necessarily match with the credible set $\mathcal{C}_j$, however we can notice that the inclusion $\mathcal{C}_j\subseteq\widehat{\mathcal{C}}_j$ holds and therefore we only need to prove that $\widehat{\mathcal{C}}_j\subseteq\mathcal{C}_j$. We will make use of contracts with lump–sum payments to prove that every point from $\widehat{\mathcal{C}}_j$ belongs to the credible set $\mathcal{C}_j$. We start by defining the line with slope ρ_g/ρ_b which passes through the point $(u^b,u^g)=\left(\frac{Bj}{r+\widehat{\lambda}_j^{SH}},\frac{Bj}{r+\widehat{\lambda}_j^{SH}}\right)$,

$$\widehat{\mathfrak{M}}_j(u^b):=\frac{\rho_g}{\rho_b}u^b+\frac{Bj}{r+\widehat{\lambda}_j^{SH}}\left(1-\frac{\rho_g}{\rho_b}\right),$$

and the sets

$$\widehat{\mathcal{C}_j^1} := \left\{(u^b, u^g) \in \widehat{\mathcal{V}}_j \times \widehat{\mathcal{V}}_j,\ \widehat{\mathfrak{M}}_j(u^b) \leq u^g \leq \widehat{\mathfrak{U}}_j(u^b)\right\},$$
$$\widehat{\mathcal{C}_j^2} := \left\{(u^b, u^g) \in \widehat{\mathcal{V}}_j \times \widehat{\mathcal{V}}_j,\ \widehat{\mathfrak{L}}_j(u^b) \leq u^g \leq \widehat{\mathfrak{M}}_j(u^b)\right\}.$$

From Section D.3 in the Appendix, we know that $\widehat{\mathcal{C}_j^1} \subseteq \mathcal{C}_j$. The reason of this is that every point from the upper boundary $\widehat{\mathfrak{U}}_j$ belongs to the credible set and that if we perturb a contract $\Psi = (\theta, D)$ only by adding a lump–sum payment ε at time t, that is $dD_s^{\Psi'} = \mathbf{1}_{\{s=t\}}\varepsilon + dD_s^{\Psi}$, then the values of the banks under Ψ' are $U_t^g(\Psi') = u^g + \varepsilon\rho_g$ and $U_t^b(\Psi') = u^b + \varepsilon\rho_b$, so $(U_t^b(\Psi'), U_t^g(\Psi')) = (u^b, u^g) + \varepsilon(\rho_b, \rho_g)$. We use this idea to prove also that $\widehat{\mathcal{C}_j^2} \subseteq \mathcal{C}_j$. From Proposition 4.1, we know that the graph of $\widehat{\mathfrak{L}}_j$ is contained in $\mathcal{C}_j$. Therefore any point of the following form belongs to $\mathcal{C}_j$

$$(\widehat{u}^b, \widehat{u}^g) = (u^b, u^g) + \ell(\rho_b, \rho_g),\ \ell \geq 0,\ u^g = \widehat{\mathfrak{L}}_j(u^b). \tag{F.2}$$

By the geometry of the lower boundary $\widehat{\mathfrak{L}}_j$, the set of points of the form (F.2) is exactly $\widehat{\mathcal{C}_j^2}$. □

G Principal's value function on the boundary of the credible set

We start this section with the

Proof. [Proof of Proposition 5.1] Consider any time $t \geq 0$ and take any $u^{b,c} \geq C(I - N_t)$, as well as some $\Psi_g \in \widehat{\mathcal{A}}^g(t, \widehat{\mathfrak{L}}_{I-N_t}(u^{b,c}), u^{b,c})$. From Lemma E.2, we know that the components of Ψ_g must satisfy $\theta^g \equiv 1$ and that both banks shirk under Ψ_g. The payments determine the utility of the banks and the following holds by definition

$$\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^{\tau^I} e^{-r(s-t)} dD_s^g \middle| \mathcal{G}_t\right] = \frac{u^{b,c} - C(I - N_t)}{\rho_b}.$$

Besides, the utility of the investor under the contract Ψ_g is

$$\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^{\tau^I} (\mu(I - N_s)ds - dD_s^g) \middle| \mathcal{G}_t\right] = \sum_{i=N_t}^{I-1} \frac{\mu(I-i)}{\widehat{\lambda}_{I-i}^{SH}} - \mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^{\tau^I} dD_s^g \middle| \mathcal{G}_t\right].$$

Now, observe that

$$\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^{\tau^I} dD_s^g \middle| \mathcal{G}_t\right] \geq \mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^{\tau^I} e^{-r(s-t)} dD_s^g \middle| \mathcal{G}_t\right] = \frac{u^{b,c} - C(I - N_t)}{\rho_b},$$

and the equality holds if and only if D^g has a jump at time t of size $\frac{u^{b,c} - C(I-N_t)}{\rho_b}$ and $dD_s^g = 0$ for every $s > t$. That means that it is optimal for the investor to use a contract with an initial lump–sum payment and to pay nothing afterwards. Consequently, the value function of the investor on the lower boundary is given by

$$V_t^{\mathfrak{L},g}(u^{b,c}) = \sum_{i=N_t}^{I-1} \frac{\mu(I-i)}{\widehat{\lambda}_{I-i}^{SH}} - \left(\frac{u^{b,c} - C(I - N_t)}{\rho_b}\right).$$

□

We continue this section with the

Proof. [Proof of Proposition 5.2] Consider any time $t \geq 0$. Take any $u^{b,c} \in [c(I-N_t,1), C(I-N_t))$, and $\Psi_g \in \widehat{\mathcal{A}}^g(t, u^{b,c}, u^{b,c})$. From Lemma E.1, we know that the components of Ψ_g must satisfy $dD_s^g = 0$ for all $s \geq t$ and that both banks will shirk under this contract. Then, θ^g determines the continuation utilities of the banks in the following way

$$u^{b,c} = \mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^\tau e^{-r(s-t)} B(I-N_s)ds \middle| \mathcal{G}_t\right],$$

so in this case, the problem (5.4) reduces to

$$(P) \quad \sup_{\theta\in\Theta} \mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^\tau \mu(I-N_s)ds \middle| \mathcal{G}_t\right], \text{ s.t } \mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^\tau e^{-r(s-t)} B(I-N_s)ds \middle| \mathcal{G}_t\right] = u^{b,c}.$$

Next, we rewrite the objective function in a more convenient way

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^\tau \mu(I-N_s)ds \middle| \mathcal{G}_t\right] \\
&= \mu(I-N_t)\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\tau_{N_t+1} - t | \mathcal{G}_t\right] + \sum_{i=N_t+1}^{I-1} \mu(I-i)\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\mathbf{1}_{\{\tau>\tau_i\}}(\tau_{i+1}-\tau_i)\middle| \mathcal{G}_t\right] \\
&= \frac{\mu(I-N_t)}{\widehat{\lambda}^{SH}_{I-N_t}} + \sum_{i=N_t+1}^{I-1} \mu(I-i)\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\mathbf{1}_{\{\tau>\tau_i\}}(\tau_{i+1}-\tau_i)\middle| \mathcal{G}_{\tau_i}\right]\middle| \mathcal{G}_t\right] \\
&= \frac{\mu(I-N_t)}{\widehat{\lambda}^{SH}_{I-N_t}} + \sum_{i=N_t+1}^{I-1} \mu(I-i)\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\mathbf{1}_{\{\tau>\tau_i\}}\middle| \mathcal{G}_{\tau_i}\right]\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\tau_{i+1}-\tau_i\middle| \mathcal{G}_{\tau_i}\right]\middle| \mathcal{G}_t\right] \\
&= \frac{\mu(I-N_t)}{\widehat{\lambda}^{SH}_{I-N_t}} + \sum_{i=N_t+1}^{I-1} \frac{\mu(I-i)}{\widehat{\lambda}^{SH}_{I-i}}\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\theta_{\tau_i}\middle| \mathcal{G}_t\right].
\end{aligned}$$

We do the same with the constraint

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^\tau e^{-r(s-t)} B(I-N_s)ds \middle| \mathcal{G}_t\right] \\
&= \mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^{\tau_{N_t+1}} B(I-N_t)e^{-r(s-t)}ds + \sum_{i=N_t+1}^{I-1} \mathbf{1}_{\{\tau>\tau_i\}}\int_{\tau_i}^{\tau_{i+1}} e^{-r(s-t)}B(I-i)ds \middle| \mathcal{G}_t\right] \\
&= \frac{B(I-N_t)}{r+\widehat{\lambda}^{SH}_{I-N_t}} + \sum_{i=N_t+1}^{I-1} \frac{B(I-i)}{r}\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\mathbf{1}_{\{\tau>\tau_i\}}\left(e^{-r(\tau_i-t)} - e^{-r(\tau_{i+1}-t)}\right)\middle| \mathcal{G}_{\tau_i}\right]\middle| \mathcal{G}_t\right] \\
&= \frac{B(I-N_t)}{r+\widehat{\lambda}^{SH}_{I-N_t}} + \sum_{i=N_t+1}^{I-1} \frac{B(I-i)}{r}\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[e^{-r(\tau_i-t)}\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\mathbf{1}_{\{\tau>\tau_i\}}\middle| \mathcal{G}_{\tau_i}\right]\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[1-e^{-r(\tau_{i+1}-\tau_i)}\middle| \mathcal{G}_{\tau_i}\right]\middle| \mathcal{G}_t\right] \\
&= \frac{B(I-N_t)}{r+\widehat{\lambda}^{SH}_{I-N_t}} + \sum_{i=N_t+1}^{I-1} \frac{B(I-i)}{r+\widehat{\lambda}^{SH}_{I-i}}\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\theta_{\tau_i}e^{-r(\tau_i-t)}\middle| \mathcal{G}_t\right].
\end{aligned}$$

So we obtain the following expression for our problem

$$(P)\begin{cases}
\sup\limits_{\theta\in\Theta} & \dfrac{\mu(I-N_t)}{\widehat{\lambda}^{SH}_{I-N_t}} + \displaystyle\sum_{i=N_t+1}^{I-1} \frac{\mu(I-i)}{\widehat{\lambda}^{SH}_{I-i}}\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\theta_{\tau_i}\middle| \mathcal{G}_t\right] \\
\text{s.t} & \dfrac{B(I-N_t)}{r+\widehat{\lambda}^{SH}_{I-N_t}} + \displaystyle\sum_{i=N_t+1}^{I-1} \frac{B(I-i)}{r+\widehat{\lambda}^{SH}_{I-i}}\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\theta_{\tau_i}e^{-r(\tau_i-t)}\middle| \mathcal{G}_t\right] = u^{b,c}.
\end{cases}$$

We do not know how to solve (P) directly, so we will define its dual problem, characterise its solution and show that the duality gap is zero. In order to do that, we define the Lagrangian function $L : \Theta \times \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$ as follows

$$\begin{aligned} L(\theta,\nu,\omega) := &- \frac{\mu(I-N_t(\omega))}{\widehat{\lambda}^{SH}_{I-N_t(\omega)}} - \sum_{i=N_t(\omega)+1}^{I-1} \frac{\mu(I-i)}{\widehat{\lambda}^{SH}_{I-i}} \mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\theta_{\tau_i} \middle| \mathcal{G}_t\right](\omega) \\ &+ \nu \left(\frac{B(I-N_t(\omega))}{r+\widehat{\lambda}^{SH}_{I-N_t(\omega)}} + \sum_{i=N_t(\omega)+1}^{I-1} \frac{B(I-i)}{r+\widehat{\lambda}^{SH}_{I-i}} \mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\theta_{\tau_i} e^{-r(\tau_i - t)} \middle| \mathcal{G}_t \right](\omega) - u^{b,c} \right), \end{aligned}$$

and also define the dual function and the dual problem respectively as

$$g(\nu,\omega) := \inf_{\theta\in\Theta} L(\theta,\nu,\omega), \ (D) \ \sup_{\nu\in\mathbb{R}} g(\nu,\omega) \ .$$

Then, we have the weak duality inequality (where val denotes the value of the optimisation problem)

$$-\mathrm{val}(P) = \inf_{\theta\in\Theta}\sup_{\nu\in\mathbb{R}} L(\theta,\nu,\omega) \geq \sup_{\nu\in\mathbb{R}}\inf_{\theta\in\Theta} L(\theta,\nu,\omega) = \mathrm{val}(D).$$

We rewrite the dual function as follows

$$\begin{aligned} g(\nu,\omega) = &- \frac{\mu(I-N_t(\omega))}{\widehat{\lambda}^{SH}_{I-N_t(\omega)}} + \nu \left(\frac{B(I-N_t(\omega))}{r+\widehat{\lambda}^{SH}_{I-N_t(\omega)}} - u^{b,c} \right) \\ &+ \inf_{\theta\in\Theta} \sum_{i=N_t(\omega)+1}^{I-1} \int_\Omega \theta_{\tau_i}(\widetilde{\omega}) \left(\nu \frac{B(I-i)}{r+\widehat{\lambda}^{SH}_{I-i}} e^{-r(\tau_i(\widetilde{\omega})-t)} - \frac{\mu(I-i)}{\widehat{\lambda}^{SH}_{I-i}} \right) d\mathbb{P}^{SH}_{t,\omega}(\widetilde{\omega}), \end{aligned}$$

where $\mathbb{P}^{SH}_{t,\omega}$ is a regular conditional probability distribution for the conditional expectation with respect to the raw (that is to say not completed) version of $\mathcal{G}_t$. We have easily that it is optimal to set the optimal control θ^ν to be $\theta^\nu_{\tau_i}(\widetilde{\omega}) := \mathbf{1}_{\widetilde{\omega}\in A^i_\nu}(\widetilde{\omega})$, where the set A^i_ν is defined by

$$A^i_\nu := \begin{cases} \Omega, \ \text{if } \nu < \dfrac{\mu}{B}\dfrac{r+\widehat{\lambda}^{SH}_{I-i}}{\widehat{\lambda}^{SH}_{I-i}}, \\ \left\{ \widetilde{\omega}, \ \tau_i(\widetilde{\omega}) - t > \dfrac{1}{r}\ln\left(\dfrac{\nu B \widehat{\lambda}^{SH}_{I-i}}{\mu(r+\widehat{\lambda}^{SH}_{I-i})} \right) \right\}, \ \text{if } \nu \geq \dfrac{\mu}{B}\dfrac{r+\widehat{\lambda}^{SH}_{I-i}}{\widehat{\lambda}^{SH}_{I-i}}. \end{cases}$$

Therefore, for any $\nu \in \mathbb{R}$ the dual function has the following form, using that the conditional law of $\tau_i - t$ given $\mathcal{G}_t$ is the same as the law of τ_i

$$\begin{aligned} g(\nu,\omega) = &- \frac{\mu(I-N_t(\omega))}{\widehat{\lambda}^{SH}_{I-N_t(\omega)}} + \nu \left(\frac{B(I-N_t(\omega))}{r+\widehat{\lambda}^{SH}_{I-N_t(\omega)}} - u^{b,c} \right) \\ &+ \sum_{i=N_t(\omega)+1}^{I-1} \int_{s_i(\nu)}^\infty \left(\frac{\nu B(I-i)e^{-rx}}{r+\widehat{\lambda}^{SH}_{I-i}} - \frac{\mu(I-i)}{\widehat{\lambda}^{SH}_{I-i}} \right) f_{\tau_i}(x)dx. \end{aligned} \tag{G.1}$$

It is not difficult to see that g is a continuous and differentiable function. As we want to maximise g in the dual problem, we compute its derivative with respect to ν and we get

$$g'(\nu,\omega) = \frac{B(I-N_t(\omega))}{r+\widehat{\lambda}^{SH}_{I-N_t}} - u^{b,c} + \sum_{i=N_t+1}^{I-1} \int_{s_i(\nu)}^\infty \frac{B(I-i)}{r+\widehat{\lambda}^{SH}_{I-i}} e^{-rx} f_{\tau_i}(x)dx.$$

Since $\nu \longmapsto s_i(\nu)$ is non–decreasing for any $i = 1, \dots, I$, g' is non–increasing in ν. Furthermore, since $u^{b,c} \geq c(I - N_t, 1)$, we have the limit at $+\infty$ of g' is non–positive, and that its value for small ν is positive because $u^{b,c} < C(I - N_t)$ and

$$\frac{B(I - N_t(\omega))}{r + \widehat{\lambda}^{SH}_{I-N_t}} + \sum_{i=N_t+1}^{I-1} \int_0^\infty \frac{B(I-i)}{r + \widehat{\lambda}^{SH}_{I-i}} e^{-rx} f_{\tau_i}(x) dx = C(I - N_t).$$

Therefore, there is a unique value of ν that makes g' equal to 0.

Now, we compute for any ν the value of the constraint from the primal problem for the control θ^ν

$$\sum_{i=N_t+1}^{I-1} \frac{B(I-i)}{r + \widehat{\lambda}^{SH}_{I-i}} \mathbb{E}^{\mathbb{P}^{k^{SH}}} \left[\theta^\nu_{\tau_i} e^{-r(\tau_i - t)} \Big| \mathcal{G}_t \right] = \sum_{i=N_t+1}^{I-1} \int_{s_i(\nu)}^\infty \frac{B(I-i)}{r + \widehat{\lambda}^{SH}_{I-i}} e^{-rx} f_{\tau_i}(x) dx,$$

so θ^ν is feasible in problem (P) if and only if $g'(\nu, \omega) = 0$. Next, we compute for θ^ν the value of the objective function in the primal (minimisation) problem

$$-\frac{\mu(I - N_t)}{\widehat{\lambda}^{SH}_{I-N_t}} - \sum_{i=N_t+1}^{I-1} \frac{\mu(I-i)}{\widehat{\lambda}^{SH}_{I-i}} \mathbb{E}_t^{\mathbb{P}^{k^{SH}}} \left[\theta^\nu_{\tau_i} \right] = -\frac{\mu(I - N_t)}{\widehat{\lambda}^{SH}_{I-N_t}} - \sum_{i=N_t+1}^{I-1} \int_{s_i(\nu)}^\infty \frac{\mu(I-i)}{\widehat{\lambda}^{SH}_{I-i}} f_{\tau_i}(x) dx.$$

If this quantity is equal to $g(\nu, \cdot)$, the duality gap is zero. From (G.1) we see that this happens if and only if

$$\nu \left(\frac{B(I - N_t)}{r + \widehat{\lambda}^{SH}_{I-N_t}} - u^{b,c} + \sum_{i=N_t+1}^{I-1} \int_{s_i(\nu)}^\infty \frac{B(I-i)}{r + \widehat{\lambda}^{SH}_{I-i}} e^{-rx} f_{\tau_i}(x) dx \right) = 0 \iff \nu g'(\nu, \cdot) = 0.$$

We conclude that if $\nu \in \mathbb{R}$ is such that $g'(\nu) = 0$ then the control θ^ν is optimal in the primal problem. □

We continue with the

Proof. [Proof of Proposition 5.3] Define the process $\ell_s = \widehat{\mathfrak{U}}_{I-N_s}(U^{b,c}_s(\Psi_g)) - U^g_s(\Psi_g)$ and note that $\ell_s \geq 0$ for every $s \geq 0$. We will prove that $\ell_t = 0$ implies $\ell_v = 0$ for every $v \geq t$. Assume thus that $\ell_t = 0$. Following the same idea as in the proof of Theorem 4.1, we have for $v \geq t$

$$\begin{aligned}
\ell_v = & \sum_{i=N_t}^{I-1} \int_{\tau_i \wedge v}^{\tau_{i+1} \wedge v} - \left(r U^g_s(\Psi_g) - B k^{\star,g}_s(\Psi_g) + [h^{1,g}_s + (1 - \theta^g_s) h^{2,g}_s] \lambda^{k^{\star,g}(\Psi_g)}_s \right) ds \\
& + \sum_{i=N_t}^{I-1} \int_{\tau_i \wedge v}^{\tau_{i+1} \wedge v} \widehat{\mathfrak{U}}'_{I-i}(U^{b,c}_s(\Psi_g)) \left(r U^{b,c}_s(\Psi_g) - B k^{\star,b,c}_s(\Psi_g) + \lambda^{k^{\star,b,c}(\Psi_g)}_{I-i} (h^{1,b,c}_s + (1 - \theta^g_s) h^{2,b,c}_s) \right) ds \\
& + \sum_{i=N_t}^{I-1} \int_{\tau_i \wedge v}^{\tau_{i+1} \wedge v} \left(h^{1,g}_s + \widehat{\mathfrak{U}}_{I-i-1}(U^{b,c}_{s^-}(\Psi_g) - h^{1,b,c}_s) - \widehat{\mathfrak{U}}_{I-i}(U^{b,c}_{s^-}(\Psi_g)) \right) dN_s \\
& + \sum_{i=N_t}^{I-1} \int_{\tau_i \wedge v}^{\tau_{i+1} \wedge v} \left(h^{2,g}_s - \widehat{\mathfrak{U}}_{I-i-1}(U^{b,c}_{s^-}(\Psi_g) - h^{1,b,c}_s) \right) dH_s + \left(\rho_g - \rho_b \widehat{\mathfrak{U}}'_{I-i}(U^{b,c}_s(\Psi_g)) \right) dD^g_s.
\end{aligned}$$

Since the functions $\widehat{\mathfrak{U}}_i$ solve the system of HJB equations (4.9), and $\left(\rho_g - \rho_b \widehat{\mathfrak{U}}'_i(U^{b,c}_s(\Psi_g)) \right) dD^g_s \leq 0$

for every s, we have

$$\begin{aligned}
\ell_v \leq & \sum_{i=N_t}^{I-1} \int_{\tau_i \wedge v}^{\tau_{i+1} \wedge v} \left(r \widehat{\mathfrak{U}}_{I-i}(U_s^{b,c}(\Psi_g)) - r U_s^g(\Psi_g) - [h_s^{1,g} + (1-\theta_s^g) h_s^{2,g}] \lambda_s^{k^{\star,g}(\Psi_g)} \right) ds \\
& - \sum_{i=N_t}^{I-1} \int_{\tau_i \wedge v}^{\tau_{i+1} \wedge v} \lambda_s^{k^{\star,g}(\Psi_g)} \left(\theta_s \widehat{\mathfrak{U}}_{I-i-1}(U_{s^-}^{b,c}(\Psi_g) - h_s^{1,b,c}) - \widehat{\mathfrak{U}}_{I-i}(U_s^{b,c}(\Psi_g)) \right) ds \\
& + \sum_{i=N_t}^{I-1} \int_{\tau_i \wedge v}^{\tau_{i+1} \wedge v} \left(h_s^{1,g} + \widehat{\mathfrak{U}}_{I-i-1}(U_{s^-}^{b,c}(\Psi_g) - h_s^{1,b,c}) - \widehat{\mathfrak{U}}_{I-i}(U_{s^-}^{b,c}(\Psi_g)) \right) dN_s \\
& + \sum_{i=N_t}^{I-1} \int_{\tau_i \wedge v}^{\tau_{i+1} \wedge v} \left(h_s^{2,g} - \widehat{\mathfrak{U}}_{I-i-1}(U_{s^-}^{b,c}(\Psi_g) - h_s^{1,b,c}) \right) dH_s \\
= & \sum_{i=N_t}^{I-1} \int_{\tau_i \wedge v}^{\tau_{i+1} \wedge v} (r + \lambda_s^{k^{\star,g}}) (\widehat{\mathfrak{U}}_{I-i}(U_s^{b,c}(\Psi_g)) - U_s^g(\Psi_g)) + (h_s^{2,g} - \widehat{\mathfrak{U}}_{I-i-1}(U_s^{b,c}(\Psi_g) - h_s^{1,b,c})) \theta_s^g \lambda_s^{k^{\star,g}} ds \\
& + \sum_{i=N_t}^{I-1} \int_{\tau_i \wedge v}^{\tau_{i+1} \wedge v} \left(h_s^{1,g} + \widehat{\mathfrak{U}}_{I-i-1}(U_{s^-}^{b,c}(\Psi_g) - h_s^{1,b,c}) - \widehat{\mathfrak{U}}_{I-i}(U_{s^-}^{b,c}(\Psi_g)) \right) dN_s \\
& + \sum_{i=N_t}^{I-1} \int_{\tau_i \wedge v}^{\tau_{i+1} \wedge v} \left(h_s^{2,g} - \widehat{\mathfrak{U}}_{I-i-1}(U_{s^-}^{b,c}(\Psi_g) - h_s^{1,b,c}) \right) dH_s.
\end{aligned}$$

Recall from Remark 4.2 that on the upper boundary, we have

$$h_s^{1,g} = \widehat{\mathcal{U}}_{I-N_{s^-}}(U_{s^-}^{b,c}(\Psi_g)) - \widehat{\mathcal{U}}_{I-N_{s^-}-1}(U_{s^-}^{b,c}(\Psi_g) - h_s^{1,b,c}(\Psi_g)),\ h_s^{2,g} = \widehat{\mathcal{U}}_{I-N_{s^-}-1}(U_{s^-}^{b,c}(\Psi_g) - h_s^{1,b,c}(\Psi_g)),$$

so that for $i = N_t$ the drift of the right–hand side is 0 in $[\tau_i, \tau_{i+1})$ and the jump at time τ_{i+1} is also 0. It is easy to see that the same happens for every $i \in \{N_t, \dots, I\}$ and therefore $\ell_v \leq 0$ for every $v \geq 0$ which means $\ell_v = 0$ for every $v \geq t$. □

We go on with the

Proof. [Proof of Proposition 5.4]

(i) We have from the proof of Proposition 5.3 that the processes $(\theta^g, h^{1,b,c}, h^{2,b,c})$ are necessarily maximisers of the system of HJB equations (4.9). We can go back to the proof of Proposition 4.2, which is based on Corollary F.1, to observe that for $u^{b,c} < \widehat{b}_j$ the optimal $\theta \in C^j$ is uniquely given by $\theta = 0$.

(ii) Observe that for every $(t, u^{b,c}, u^g) \in [0,\tau] \times \widehat{\mathcal{V}}_{I-N_t} \times \widehat{\mathcal{V}}_{I-N_t}$ and $\Psi_g \in \widehat{\mathcal{A}}^g(t, u^g, u^{b,c})$ we have

$$\begin{aligned}
U_t^{b,c}(\Psi_g) &\geq \mathbb{E}^{\mathbb{P}^{k^{\star,g}(\Psi_g)}} \left[\int_t^\tau e^{-r(s-t)} (\rho_b dD_s^g + B k_s^{\star,g}(\Psi_g) ds) \middle| \mathcal{G}_t \right] \\
&= \frac{\rho_b}{\rho_g} U_t^g(\Psi_g) + \mathbb{E}^{\mathbb{P}^{k^{\star,g}(\Psi_g)}} \left[\int_t^\tau e^{-r(s-t)} B k_s^{\star,g}(\Psi_g) ds \middle| \mathcal{G}_t \right] \left(1 - \frac{\rho_b}{\rho_g} \right) \geq \frac{\rho_b}{\rho_g} U_t^g(\Psi_g).
\end{aligned}$$

Then $U_{s_0}^{b,c}(\Psi_g) = \frac{\rho_g}{\rho_b} U_{s_0}^g(\Psi_g)$ implies that $k_s^{\star,g}(\Psi_g) = k_s^{\star,b,c}(\Psi_g) = 0$, for every $s \in [s_0, \tau)$, and in consequence

$$U_s^{b,c}(\Psi_g) = \frac{\rho_g}{\rho_b} U_s^g(\Psi_g) \geq b_s, \text{ for every } s \in [s_0, \tau).$$

□

We end this section with the

Proof. [Proof of Proposition 5.5] We divide the proof in 2 steps.

• **Step 1:** We start with the region $u^{b,c} > \widehat{b}_{I-N_t}$. Let $\Psi_g = (D^g, \theta^g, h^{1,b,c}, h^{2,b,c}) \in \overline{\mathcal{A}}^g(t, u^{b,c})$ be such that $U_t^{b,c}(\Psi_g) = u^{b,c} \geq \widehat{b}_{I-N_t}$, $U_t^g(\Psi_g) = \widehat{\mathfrak{U}}_{I-N_t}(u^{b,c})$. From Proposition 5.4 we know that

$$U_s^{b,c}(\Psi_g) \geq \widehat{b}_{I-N_s},\ k^{\star,b,c}(\Psi_g) = 0,\ s \in [t, \tau).$$

Therefore, Problem (5.5) is equivalent to

$$V_t^{\mathfrak{U},g}(u^{b,c}) = \sup_{\Psi_g \in \overline{\mathcal{A}}^g(t,u^{b,c})} \mathbb{E}^{\mathbb{P}^0}\left[\int_t^\tau \mu(I - N_s)ds - \int_t^\tau dD_s^g\right],\ \text{s.t}\ \begin{cases} U_s^{b,c}(\Psi_g) \geq \widehat{b}_{I-N_s}, s \in [t,\tau), \\ \mathbb{E}^{\mathbb{P}^0}\left[\int_t^\tau e^{-r(s-t)} dD_s^g\right] = \dfrac{u^{b,c}}{\rho_b}. \end{cases}$$

This is exactly the problem of [42], recalled in Section 3.2, so we conclude that $V_t^{\mathfrak{U},g}(u^{b,c}) = v^b_{I-N_t}(u^{b,c})$.

• **Step 2:** For the rest of the upper boundary, observe that the system of HJB equations associated to (5.5) is is given by $\widehat{\mathcal{V}}_0 \equiv 0$, and for any $1 \leq j \leq I$

$$\min\left\{ - \sup_{(\theta,h^1,h^2)\in C^{\mathfrak{U},j}} \left\{ \begin{array}{c} \widehat{\mathcal{V}}_j'(u^{b,c})\left(ru^{b,c} - Bk^{b,c} + [h^1 + (1-\theta)h^2]\widehat{\lambda}_j^{k^{b,c}}\right) \\ +\mu j + \widehat{\lambda}_j^{k^g}\theta\widehat{\mathcal{V}}_{j-1}(u^{b,c} - h^1) - \widehat{\lambda}_j^{k^g}\widehat{\mathcal{V}}_j(u^{b,c}) \end{array} \right\},\ \widehat{\mathcal{V}}_j'(u^{b,c}) + \frac{1}{\rho_b} \right\} = 0, \quad \text{(G.2)}$$

for every $u^{b,c} \geq \frac{Bj}{r+\widehat{\lambda}_j^{SH}}$, with the boundary condition $\widehat{\mathcal{V}}_j(Bj/(r+\widehat{\lambda}_j^{SH})) = \mu j/\widehat{\lambda}_j^{SH}$, and where

$$k^{b,c} := j\mathbf{1}_{\{h^1 + (1-\theta)h^2 < \widehat{b}_j\}},\ k^g := j\mathbf{1}_{\{\widehat{\mathcal{U}}_j^\star(u^{b,c}) - \theta\widehat{\mathcal{U}}_{j-1}^\star(u^{b,c}-h^1) < \widehat{b}_j\}},$$

and the set of constraints $C^{\mathfrak{U},j}$ determined by Proposition 5.4 is defined by

$$C^{\mathfrak{U},j} := \left\{(\theta, h^1, h^2) \in [0,1]\times\mathbb{R}_+^2, h^1 + h^2 = u^{b,c}, h^2 \geq \frac{B(j-1)}{r + \widehat{\lambda}_{j-1}^{SH}}, \theta\mathbf{1}_{\{u^{b,c} < \widehat{b}_j\}} = (k^{b,c} + k^g)\mathbf{1}_{\{u^{b,c} \geq \widehat{b}_j\}} = 0\right\}.$$

Then, for any $u^{b,c} < \widehat{b}_j$, the diffusion equation in (G.2) reduces to the ODE

$$0 = \widehat{\mathcal{V}}_j'(u^{b,c})\left(\left(r + \widehat{\lambda}_j^{SH}\right)u^{b,c} - Bj\right) - \widehat{\mathcal{V}}_j(u^{b,c})\widehat{\lambda}_j^{k^g} + \mu j, \quad \text{(G.3)}$$

with the boundary condition $\widehat{\mathcal{V}}_j\left(\frac{Bj}{r+\widehat{\lambda}_j^{SH}}\right) = \frac{\mu j}{\widehat{\lambda}_j^{SH}}$. If $u^{b,c} < x_j^\star$, we get that

$$\widehat{\mathcal{V}}_j(u^{b,c}) = \frac{\mu j}{\widehat{\lambda}_j^{SH}} + C_1\left(\left(\frac{r + \widehat{\lambda}_j^{SH}}{\widehat{\lambda}_j^{SH}}\right)u^{b,c} - \frac{Bj}{\widehat{\lambda}_j^{SH}}\right)^{\frac{\widehat{\lambda}_j^{SH}}{r+\widehat{\lambda}_j^{SH}}},$$

for some $C_1 \in \mathbb{R}$. If $u^{b,c} \in \left[x_j^\star, \widehat{b}_j\right)$, equation (G.3) is solved by

$$\widehat{\mathcal{V}}_j(u^{b,c}) = \frac{\mu j}{\widehat{\lambda}_j^0} + C_2\left(\left(\frac{r + \widehat{\lambda}_j^{SH}}{\widehat{\lambda}_j^0}\right)u^{b,c} - \frac{Bj}{\widehat{\lambda}_j^0}\right)^{\frac{\widehat{\lambda}_j^0}{r+\widehat{\lambda}_j^{SH}}},$$

for some $C_2 \in \mathbb{R}$. The values of C_1 and C_2 for which the solution of equation (G.3) is continuous are

$$C_1 = \frac{\frac{\mu j}{\widehat{\lambda}_j^0} - \frac{\mu j}{\widehat{\lambda}_j^{SH}} + \left(\frac{\rho_b}{\rho_g}\right)^{\frac{\widehat{\lambda}_j^0}{r+\widehat{\lambda}_j^0}}\left(v_j^b(\widehat{b}_j) - \frac{\mu j}{\widehat{\lambda}_j^0}\right)}{\left(\frac{\rho_b}{\rho_g}\right)^{\frac{\widehat{\lambda}_j^{SH}}{r+\widehat{\lambda}_j^0}}\left(\frac{\widehat{b}_j(r+\widehat{\lambda}_j^0)}{\widehat{\lambda}_j^{SH}}\right)^{\frac{\widehat{\lambda}_j^{SH}}{r+\widehat{\lambda}_j^{SH}}}},\ C_2 = \left(v_j^b(\widehat{b}_j) - \frac{\mu j}{\widehat{\lambda}_j^0}\right)\left(\widehat{b}_j\frac{r+\widehat{\lambda}_j^0}{\widehat{\lambda}_j^0}\right)^{-\frac{\widehat{\lambda}_j^0}{r+\widehat{\lambda}_j^{SH}}}.$$

It follows from the properties of the map v_j^b, that the resulting function $\widehat{\mathcal{V}}_j$ is a concave map with slope greater than $-\frac{1}{\rho_b}$ and therefore the family $\{\widehat{\mathcal{V}}_j\}_{1\leq j\leq I}$ is a solution of the system of HJB equations (G.2). It can be proved similarly as in the proof of Theorem 4.1 (see also Theorem 3.15 in [42]), that the verification result holds for this family of functions. We therefore omit the proof of this result. □

H Extension: unbounded relationship between utilities of the banks

One possible extension of our model could rely on a further differentiation between the work of the two banks, *i.e.* when both banks work, the good one would be more efficient in the sense that the associated default intensity is strictly smaller than that of the bad bank. We can do this by introducing an extra type variable with values m_g and m_b, with $m_g < m_b$ and modelling the hazard rate of a non-defaulted loan i at time t, when it is monitored by a bank of type j as $\alpha_t^{i,j} = \alpha_{I-N_t}(1 + e_t^{i,j} m_j + (1 - e_t^{i,j})\varepsilon)$. Then, if the banks fails to monitor k loans, the default intensity will be

$$\lambda_t^{k,j} = \alpha_{I-N_t}((I - N_t)(1 + m_j) + (\varepsilon - m_j)k_t).$$

We did not consider such a situation because it creates a degeneracy, in the sense that the credible set no longer has an upper boundary. Indeed, consider for simplicity the case $j = 1$ and take any $u_0^b \geq b_1^j$, $t^\star \geq 0$ and choose the corresponding payment

$$c(t^\star) := u_0^b \frac{e^{(r+\widehat{\lambda}_1^{0,b})t^\star}(r + \widehat{\lambda}_1^{0,b})}{\rho_b} \geq \frac{b_1^b(r + \widehat{\lambda}_1^{0,b})}{\rho_b} \geq \frac{b_1^g(r + \widehat{\lambda}_1^{0,g})}{\rho_g}.$$

Then, under the contract with delay and constant payments given by $dD_s = c(t^\star)1_{\{s>t^\star\}}ds$ the bad bank will always work and her value function will be equal to u_0^b (see section D.1). Notice that the optimal strategy for the good bank will be also to work at every time. Then, her value function is equal to

$$u_0^g := u_0^b \frac{\rho_g(r + \widehat{\lambda}_1^{0,b})}{\rho_b(r + \widehat{\lambda}_1^{0,g})} e^{(\widehat{\lambda}_1^{0,b} - \widehat{\lambda}_1^{0,g})t^\star}.$$

We see that by increasing $t^\star$, it is possible to make u_0^g as big as we want and keep fixed the value of the bad bank. This means that the credible set will have no upper boundary in the interval $[b_1^b, \infty)$. Moving to any $j > 1$ and considering short-term contracts with delay, with $\theta = 0$ and the analogous payments, we observe the same degeneracy and the credible set will have no upper boundary in the interval $[b_j^b, \infty)$.

One way out of this problem would be to consider different discount rates for the banks, r_b and r_g, and assume that the default intensities are such that $\lambda_t^{0,b} + r_b \leq \lambda_t^{0,g} + r_g$. However, this complicates things a lot because simple statements that we expect to be true are very difficult to prove or need assumptions on the parameters of the problem. For example the inequality $U_t^g(D, \theta) \geq U_t^b(D, \theta)$ is no longer clear at all. We therefore refrained from going into that direction, and leave it for potential future research.