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# Double-sided probing by map of Asplünd's distances using Logarithmic Image Processing in the framework of Mathematical Morphology

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**Abstract.** We establish the link between Mathematical Morphology and the map of Asplünd's distances between a probe and a grey scale function, using the Logarithmic Image Processing scalar multiplication. We demonstrate that the map is the logarithm of the ratio between a dilation and an erosion of the function by a structuring function: the probe. The dilations and erosions are mappings from the lattice of the images into the lattice of the positive functions. Using a flat structuring element, the expression of the map of Asplünd's distances can be simplified with a dilation and an erosion of the image; these mappings stay in the lattice of the images. We illustrate our approach by an example of pattern matching with a non-flat structuring function.

**Keywords:** Map of Asplünd's distances, Mathematical Morphology, dilation, erosion, Logarithmic Image Processing, Asplünd's metric, double-sided probing, pattern recognition

## 1 Introduction

Asplünd's metric is a useful method of pattern matching based on a double-sided probing, i.e. a probing by a greatest lower bound probe and a least upper bound probe. It was originally defined for binary shapes, or sets [1, 7], by using the smallest homothetic shape (probe) containing the shape to be analysed and the greatest homothetic probe contained by the shape. Jourlin et al. [13, 14] have extended this metric to functions and to grey-level images in the framework of the Logarithmic Image Processing (LIP) [15, 16] using a multiplicative or an additive LIP law [11]. Then, Asplünd's metric has been extended to colour and multivariate images by a marginal approach in [23, 11] or by a spatio-colour (i.e. vectorial [21, 22]) approach in [24].

Other approaches of double-sided probing have been previously defined in the framework of Mathematical Morphology [19, 27]. The well-known hit-or-miss transform [27] allows to extract all the pixels such that the first set of a structuring element fits the object while the second set misses it (i.e. fits its background). An extension based on two operations of dilation (for grey level images) has been

proposed in [18]. It consists of a unique structuring element, which is used in the two dilations in order to match the signal from above and from below.

Banon et al. [2] use two structuring elements obtained by two translations of a unique template along the grey level axis. They use an erosion and an anti-dilation to count the pixels whose values are in between the two structuring elements.

Odone et al. [25] use an approach inspired by the computation of the Hausdorff distance. They consider a grey level image as a tridimensional (3D) graph. They dilate by a 3D ball a template in order to compute a 3D “interval”. Then, for any point of the image, they translate vertically the “interval” in order to contain the maximum number of points of the function and they count this number.

Barat et al. [4] present a unified framework for these last three methods. They show that the previous methods correspond to a neighbourhood of functions (i.e. a tolerance tube) with a different metric for each method. Their topological approach is named virtual double-sided image probing (VDIP) and they defined it as a difference between a grey-scale dilation and an erosion. For pattern matching, only the patterns which are in the tolerance tube are selected. It is a metric defined on the equivalence class of functions according to an additive grey level shift.

In [13], Jourlin et al. have introduced the logarithmic homothetic defined according to the LIP multiplication. This makes a compensation of the lighting variation due to a multiplicative effect, i.e. a thickening or a thinning of the object crossed by the light.

In the current paper, the important novelty introduced is the link between the map of Asplünd’s metrics defined in the LIP multiplicative framework and the operations of Mathematical Morphology. We will show that the map of Asplünd’s distances in the LIP multiplicative framework is the logarithm of the ratio between an anti-dilation and an anti-erosion.

This gives access to many other notions well defined in the corpus of Mathematical Morphology.

The paper is organised as follows: 1) a reminder of the main notions (LIP, Asplünd’s metrics, fundamental operations and framework of Mathematical Morphology), 2) the demonstration of the link between the map of Asplünd’s distances and Mathematical Morphology for flat structuring element (se) and for non-flat ones and 3) an illustration of pattern matching with Asplünd’s metric.

## 2 Prerequisites

In the current section, we remind the different mathematical notions and frameworks to be used: LIP model, Asplünd’s metric and the basis of Mathematical Morphology.

## 2.1 Logarithmic Image Processing (LIP)

The LIP model, created by Joulain et al. [15, 17, 16, 10], is a mathematical framework for image processing based on the physical law of transmittance. It is perfectly suited to process images acquired with transmitted light (when the object is located between the source and the sensor) but also with reflected light, due to the consistency of the model with human vision [6]. The mathematical operations performed using the LIP model are consistent with the physical principles of image formation. Therefore the values of an images defined in  $[0, M[$  stay in this bounded domain. For 8 bits images  $M = 256$  and the 256 grey levels are in the range of integers  $[0, \dots, 255]$ .

A grey scale image  $f$  is a function defined on a domain  $D \subset \mathbb{R}^N$  with values in  $\mathcal{T} = [0, M[, M \in \mathbb{R}$ .  $f$  is a member of the space  $\mathcal{I} = \mathcal{T}^D$ .

Due to the link with the transmittance law:  $T_f = 1 - f/M$ , the grey-scale is inverted in the LIP framework, 0 corresponds to the white extremity of the grey scale, when no obstacle is located between the source and the sensor, while the other extremity  $M$  corresponds to the black value, when the source cannot be transmitted through the obstacle.

In the LIP sense, the addition of two images corresponds to the superposition of two obstacles (objects) generating  $f$  and  $g$ :

$$f \triangle g = f + g - \frac{f \cdot g}{M} \quad (1)$$

From this law, we deduce the LIP multiplication of  $f$  by a scalar  $\lambda \in \mathbb{R}$ :

$$\lambda \triangle f = M - M \left( 1 - \frac{f}{M} \right)^\lambda \quad (2)$$

It corresponds to a thickness change of the observed object in the ratio  $\lambda$ . If  $\lambda > 1$ , the thickness is increased and the image becomes darker than  $f$ , while if  $\lambda \in [0, 1]$ , the thickness is decreased and the image becomes brighter than  $f$ .

The LIP laws satisfy strong mathematical properties. Let  $\mathcal{F}(D, [-\infty, M[)$  be the set of functions defined on  $D$  with values in  $] - \infty, M[$ . We equipped it with the two logarithmic laws and  $(\mathcal{F}(D, [-\infty, M[), \triangle, \triangle)$  becomes a real vector space.  $(\mathcal{I}, \triangle, \triangle)$  is the positive cone of this vector space [16].

There exists a colour version of the LIP model [12].

The LIP framework has been successfully applied to numerous problems for industry, medical applications, digital photography, etc. It gives access to new notions of contrast and metrics which take into account the variation of light, for example the Asplünd's metric for functions.

## 2.2 Asplünd's metric for functions using the LIP multiplicative law

Let us remind the novel notion of Asplünd's metric defined in [13, 14] for functions in place of sets. It consists of using the logarithmic homothetic  $\lambda \triangle f$ .

Let  $\mathcal{T}^* = ]0, M[$  and the space of positive images be  $\mathcal{I}^* = \mathcal{T}^{*D}$ .

**Definition 1. Asplünd's metric** Given two images  $f, g \in \mathcal{I}^*$ ,  $g$  is chosen as the probing function for example, and we define the two numbers:  $\lambda = \inf \{\alpha, f \leq \alpha \triangle g\}$  and  $\mu = \sup \{\beta, \beta \triangle g \leq f\}$ . The corresponding “functional Asplünd's metric”  $d_{As}^\triangle$  (with the LIP multiplication) is:

$$d_{As}^\triangle(f, g) = \ln(\lambda/\mu) \quad (3)$$

From a mathematical point of view [11],  $d_{As}^\triangle$  is a metric if the images  $f, g \in \mathcal{I}^*$  are replaced by their equivalence classes  $f^\triangle = \{g/\exists k > 0, k \triangle g = f\}$  and  $g^\triangle$ .

The relation  $(\exists k > 0, k \triangle g = f)$  is clearly an equivalence relation written  $fRg$ , because it satisfies the three properties: i) reflexivity  $\forall f \in \mathcal{I}, fRf$ , ii) symmetry  $\forall (f, g) \in \mathcal{I}^2, fRg \Leftrightarrow gRf$  and iii) transitivity  $\forall (f, g, h) \in \mathcal{I}^3, (fRg \text{ and } gRh) \Rightarrow fRh$ . Let us now give a rigorous definition of the multiplicative Asplünd's metric using the space of equivalence classes  $\mathcal{I}^\triangle$

$$\forall (f^\triangle, g^\triangle) \in (I^\triangle)^2, \quad d_{As}^\triangle(f^\triangle, g^\triangle) = d_{As}^\triangle(f_1, g_1) \quad (4)$$

$d_{As}^\triangle(f_1, g_1)$  is defined by eq. 3 between two elements  $f_1$  and  $g_1$  of the equivalence classes  $f^\triangle$  and  $g^\triangle$ .

The demonstration of the metric properties are in the appendix (section 6).

Several examples have shown the interest of using Asplünd's metric for pattern matching between a template function  $t : D_t \rightarrow \mathcal{T}$  and the function  $f$ . For each point  $x$  of  $D$ , the distance  $d_{As}^\triangle(f|_{D_t(x)}, t)$  is computed in the neighbourhood  $D_t(x)$  centred in  $x$ , with  $f|_{D_t(x)}$  being the restriction of  $f$  to  $D_t(x)$ . Therefore, one can define a map of Asplünd's distances [23].

**Definition 2. Map of Asplünd's distances** Given a grey-level image  $f \in \mathcal{I}^*$  and a probe  $t \in (\mathcal{T}^*)^{D_t}$ ,  $t > 0$ , their map of Asplünd's distances is:

$$As_t^\triangle f : \begin{cases} \mathcal{I}^* \times (\mathcal{T}^*)^{D_t} \rightarrow (\mathbb{R}^+)^D \\ (f, t) \rightarrow As_t^\triangle f(x) = d_{As}^\triangle(f|_{D_t(x)}, t) \end{cases} \quad (5)$$

$D_t(x)$  is the neighbourhood associated to  $D_t$  centred in  $x \in D$ .

One can notice that the template  $t$  is acting like a structuring element.

### 2.3 Short reminder on Mathematical Morphology

In this subsection we give a reminder of the basis notions used in Mathematical Morphology (MM) [19]. MM is defined in complete lattices [27, 9, 3, 5].

**Definition 3. Complete lattice** Given a set  $\mathcal{L}$  and a binary relation  $\leq$  defining a partial order on  $\mathcal{L}$ , we say that  $(\mathcal{L}, \leq)$  is a partially ordered set or poset.  $\mathcal{L}$  is a complete lattice if any non empty subset  $\mathcal{X}$  of  $\mathcal{L}$  has a supremum (a least

upper bound) and an infimum (a greatest lower bound). The infimum and the supremum will be denoted, respectively, by  $\wedge \mathcal{X}$  and  $\vee \mathcal{X}$ . Two elements of the complete lattice  $\mathcal{L}$  are important: the least element  $O$  and the greatest element  $I$ .

The set of images from  $D$  to  $[0, M]$ ,  $\bar{\mathcal{I}} = [0, M]^D$ , is a complete lattice with the partial order relation  $\leq$ , by inheritance of the complete lattice structure of  $[0, M]$ . The least and greatest elements are the constant functions  $f_0$  and  $f_M$  whose values are equal respectively to 0 and  $M$  for all elements of  $D$ . The supremum and infimum are respectively, for any  $\mathcal{X} \subset \bar{\mathcal{I}}$

$$\begin{aligned} (\wedge_{\bar{\mathcal{I}}} \mathcal{X})(x) &= \wedge_{[0, M]} \{f(x) : f \in \mathcal{X}\} \quad (x \in D) \\ (\vee_{\bar{\mathcal{I}}} \mathcal{X})(x) &= \vee_{[0, M]} \{f(x) : f \in \mathcal{X}\} \quad (x \in D) \end{aligned} \quad (6)$$

The set of functions  $\bar{\mathbb{R}}^D$  is also a complete lattice with  $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$  with the usual order  $\leq$ , like the set of all subsets of  $D$ , written  $\mathcal{P}(D)$ , with the set inclusion  $\subset$ .

**Definition 4. Erosion, dilation, anti-erosion, anti-dilation [3]** Given  $\mathcal{L}_1$  and  $\mathcal{L}_2$  two complete lattices, a mapping  $\psi \in \mathcal{L}_2^{\mathcal{L}_1}$  is

1. an erosion iff  $\forall \mathcal{X} \subset \mathcal{L}_1, \psi(\wedge \mathcal{X}) = \wedge \psi(\mathcal{X})$ , then we write  $\varepsilon = \psi$ ;
2. a dilation iff  $\forall \mathcal{X} \subset \mathcal{L}_1, \psi(\vee \mathcal{X}) = \vee \psi(\mathcal{X})$ , then we write  $\delta = \psi$ ;
3. an anti-erosion iff  $\forall \mathcal{X} \subset \mathcal{L}_1, \psi(\wedge \mathcal{X}) = \vee \psi(\mathcal{X})$ , then we write  $\varepsilon^a = \psi$ ;
4. an anti-dilation iff  $\forall \mathcal{X} \subset \mathcal{L}_1, \psi(\vee \mathcal{X}) = \wedge \psi(\mathcal{X})$ , then we write  $\delta^a = \psi$ .

As the definitions of these mappings apply even to the empty subset of  $\mathcal{L}_1$ , we have:  $\varepsilon(I) = I$ ,  $\delta(O) = O$ ,  $\varepsilon^a(I) = O$  and  $\delta^a(O) = I$ .

Erosions and dilations are increasing mappings:  $\forall \mathcal{X}, \mathcal{Y} \subset \mathcal{L}_1, \mathcal{X} \leq \mathcal{Y} \Rightarrow \psi(\mathcal{X}) \leq \psi(\mathcal{Y})$  while anti-erosions and anti-dilations are decreasing mappings:  $\forall \mathcal{X}, \mathcal{Y} \subset \mathcal{L}_1, \mathcal{X} \leq \mathcal{Y} \Rightarrow \psi(\mathcal{Y}) \leq \psi(\mathcal{X})$ .

**Definition 5. Structuring element [26, 9, 29]** Let us define a pulse function  $i_{x,t} \in \bar{\mathcal{I}}$  of level  $t$  at the point  $x$ :

$$i_{x,t}(x) = t; \quad i_{x,t}(y) = 0 \text{ if } x \neq y. \quad (7)$$

The function  $f$  can be decomposed into the supremum of its pulses  $f = \vee \{i_{x,f(x)}, x \in D\}$ . It is easy to define dilations and erosions which are not translation-invariant (in the domain  $D$ ). Let  $W$  be a map  $\bar{\mathcal{I}} \rightarrow \bar{\mathcal{I}}$  associating to each pulse function  $i_{x,t} \in \bar{\mathcal{I}}$  a (functional) “window”  $W(i_{x,t})$ . Then the operator  $\delta_W : \bar{\mathcal{I}} \rightarrow \bar{\mathcal{I}}$  defined by:

$$\delta_W(f) = \vee \{W(i_{x,f(x)}), x \in D\} \quad (8)$$

is a dilation. When all “windows”  $W(i_{x,f(x)})$  are translation invariant (in  $D$ ), they take the form  $W(i_{x,f(x)}) = B(x)$  with  $B(x) = B_x$  being a structuring element (or structuring function).

In this case the previously defined dilation  $\delta$  and erosion  $\varepsilon$ , in the same lattice  $(\overline{\mathcal{I}}, \leq)$ , can be simplified:

$$\begin{aligned} (\delta_B(f))(x) &= \vee \{f(x-h) + B(h), h \in D_B\} = (f \oplus B)(x) \\ (\varepsilon_B(f))(x) &= \wedge \{f(x+h) - B(h), h \in D_B\} = (f \ominus B)(x) \end{aligned} \quad (9)$$

$D_B \subset D$  is the definition domain of the structuring function  $B : D_B \rightarrow \overline{\mathcal{T}}$ .  $\oplus$  and  $\ominus$  are the extension of Minkowski operations [20, 8] between sets to functions [27].

### 3 Map of Asplünd's distances and mathematical morphology

We now link the map of Asplünd's distances with Mathematical Morphology.

Given  $\overline{\mathbb{R}}^+ = [0, +\infty]$  a complete lattice with the natural order  $\leq$ , the map of the least upper bounds  $\lambda_B$  between the probe  $B \in (\mathcal{T}^*)^{D_B}$  and the function  $f \in \overline{\mathcal{I}}$  is defined as:

$$\lambda_B f : \begin{cases} \overline{\mathcal{I}} \times (\mathcal{T}^*)^{D_B} \rightarrow & (\overline{\mathbb{R}}^+)^D \\ (f, B) \rightarrow \lambda_B f(x) = \wedge \{ \alpha(x), f(x+h) \leq \alpha(x) \triangle B(h), h \in D_B \} \end{cases} \quad (10)$$

The map of the greatest lower bounds  $\mu_B$  between the probe  $B \in (\mathcal{T}^*)^{D_B}$  and the function  $f \in \overline{\mathcal{I}}$  is defined as:

$$\mu_B f : \begin{cases} \overline{\mathcal{I}} \times (\mathcal{T}^*)^{D_B} \rightarrow & (\overline{\mathbb{R}}^+)^D \\ (f, B) \rightarrow \mu_B f(x) = \vee \{ \beta(x), \beta(x) \triangle B(h) \leq f(x+h), h \in D_B \} \end{cases} \quad (11)$$

The two mappings  $\lambda_B$  and  $\mu_B$  are defined between two complete lattices  $\mathcal{L}_1 = (\overline{\mathcal{I}}, \leq)$  and  $\mathcal{L}_2 = ((\overline{\mathbb{R}}^+)^D, \leq)$  with the natural order  $\leq$ . Therefore, the least element of  $(\mathcal{L}_1, \leq)$  corresponds to the constant function equal to zero,  $O = f_0$  and the greatest element is the constant function equal to  $M$ ,  $I = f_M$ .

Using the equations 10 and 11 the map of Asplünd's distances (eq. 5) can be simplified:

$$As_B^{\triangle} f = \ln \left( \frac{\lambda_B f}{\mu_B f} \right), \text{ with } f > 0. \quad (12)$$

In addition,  $\forall x \in D, \forall h \in D_B, \forall \alpha \in \mathbb{R}^+$ , we have:

$$\begin{aligned} \alpha(x) \triangle B_0 \geq f(x+h) &\Leftrightarrow M - M(1 - B(h)/M)^{\alpha(x)} \geq f(x+h) \quad (\text{from eq. 2}) \\ &\Leftrightarrow \alpha(x) \geq \frac{\ln(1 - f(x+h)/M)}{\ln(1 - B(x)/M)} \text{ because } (1 - B(x)/M) \in ]0, 1[. \end{aligned} \quad (13)$$

We assume that  $\tilde{f} = \ln(1 - f/M)$ . Using equation 13, equation 10 becomes:

$$\lambda_B f = \wedge \left\{ \alpha(x), \alpha(x) \geq \frac{\tilde{f}(x+h)}{\tilde{B}(x)}, h \in D_B \right\} = \vee \left\{ \frac{\tilde{f}(x+h)}{\tilde{B}(x)}, h \in D_B \right\} \quad (14)$$

In a similar way:

$$\mu_B f = \vee \left\{ \beta(x), \beta(x) \leq \frac{\tilde{f}(x+h)}{\tilde{B}(x)}, h \in D_B \right\} = \wedge \left\{ \frac{\tilde{f}(x+h)}{\tilde{B}(x)}, h \in D_B \right\} \quad (15)$$

### 3.1 Case of a flat structuring element

In the case of a flat structuring element  $B = B_0 \in \mathcal{T}^*$  ( $\forall x \in D_B$ ,  $B(x) = B_0$ ), the equations 14 and 15 can be simplified.

$$\begin{aligned} \lambda_{B_0} f &= \frac{1}{B_0} \wedge \left\{ \tilde{f}(x+h), h \in D_B \right\}, \quad \text{because } \tilde{B}_0 < 0 \\ &= \frac{1}{B_0} \ln \left( 1 - \frac{\vee \{f(x-h), -h \in D_B\}}{M} \right) \\ &= \frac{1}{B_0} \ln \left( 1 - \frac{\delta_{\tilde{D}_B} f}{M} \right) \end{aligned} \quad (16)$$

Notice: the infimum  $\wedge$  is changed into an supremum  $\vee$  because the function  $\tilde{f} : x \rightarrow \ln(1 - x/M)$  is a continuous decreasing mapping. The reflected (or transposed) domain  $\tilde{D}_B$  is  $\tilde{D}_B = \{-h, h \in D_B\}$  and the reflected structuring function  $\tilde{B}$  is defined by the reflection of its definition domain  $\forall x \in \tilde{D}_B$ ,  $\tilde{B}(x) = B(-x)$  [28].

Similarly

$$\begin{aligned} \mu_{B_0} f &= \frac{1}{B_0} \vee \left\{ \tilde{f}(x+h), h \in D_B \right\}, \quad \text{because } \tilde{B}_0 < 0 \\ &= \frac{1}{B_0} \ln \left( 1 - \frac{\wedge \{f(x+h), h \in D_B\}}{M} \right) \\ &= \frac{1}{B_0} \ln \left( 1 - \frac{\varepsilon_{D_B} f}{M} \right) \end{aligned} \quad (17)$$

With the equations 16 and 17, the map of Asplünd's distances (eq. 12) becomes:

$$As_{B_0}^{\triangle} f = \ln \left[ \frac{\ln \left( 1 - \frac{\delta_{\tilde{D}_B} f}{M} \right)}{\ln \left( 1 - \frac{\varepsilon_{D_B} f}{M} \right)} \right], \quad \text{with } f > 0. \quad (18)$$

This important result shows that, with a flat probe, the map of Asplünd's distances can be computed using logarithms and operations of morphological erosion and dilation of an image. From an implementation point of view, the programming of the map of Asplünd's distances becomes easier, because the majority of image processing libraries contains morphological operations.

Notice: by replacing the dilation and erosion by rank-filters [26] one can compute the map of Asplünd's distances with a tolerance [14, 23].

### 3.2 General case: a structuring function

Using a general structuring function in the equations 14 and 15, the map of Asplünd's distances is expressed as:

$$As_B^{\triangle} f = \ln \left( \frac{\lambda_B f}{\mu_B f} \right) = \ln \left( \frac{\vee \left\{ \frac{\tilde{f}(x+h)}{\tilde{B}(x)}, h \in D_B \right\}}{\wedge \left\{ \frac{\tilde{f}(x+h)}{\tilde{B}(x)}, h \in D_B \right\}} \right), \quad \text{with } f > 0. \quad (19)$$

Let us study the properties of mappings  $\lambda_B, \mu_B \in \mathcal{L}_2^{\mathcal{L}_1}, \forall f, g \in \overline{\mathcal{I}}$

$$\begin{aligned}
\lambda_B(f \vee g) &= \vee \left\{ \frac{\widetilde{f \vee g}(x+h)}{\widetilde{B}(x)}, h \in D_B \right\} \\
&= \vee \left\{ \frac{\widetilde{f}(x+h) \wedge \widetilde{g}(x+h)}{\widetilde{B}(x)}, h \in D_B \right\}, \text{ because } \widetilde{f} \text{ is decreasing} \\
&= \vee \left\{ \frac{\widetilde{f}(x+h)}{\widetilde{B}(x)} \vee \frac{\widetilde{g}(x+h)}{\widetilde{B}(x)}, h \in D_B \right\}, \text{ because } \widetilde{B}(x) < 0 \\
&= \left[ \vee \left\{ \frac{\widetilde{f}(x+h)}{\widetilde{B}(x)}, h \in D_B \right\} \right] \vee \left[ \vee \left\{ \frac{\widetilde{g}(x+h)}{\widetilde{B}(x)}, h \in D_B \right\} \right] \\
&= \lambda_B(f) \vee \lambda_B(g)
\end{aligned} \tag{20}$$

According to definition 4, 2 (p. 5),  $\lambda_B$  is a dilation. In addition,

$$\lambda_B(O) = \lambda_B(f_0) = \wedge \left\{ \alpha(x), \alpha(x) \geq \frac{\widetilde{0}(x+h)}{\widetilde{B}(x)}, h \in D_B \right\} = 0 = O. \tag{21}$$

Similarly, we have:

$$\begin{aligned}
\mu_B(f \wedge g) &= \wedge \left\{ \frac{\widetilde{f \wedge g}(x+h)}{\widetilde{B}(x)}, h \in D_B \right\} \\
&= \wedge \left\{ \frac{\widetilde{f}(x+h) \vee \widetilde{g}(x+h)}{\widetilde{B}(x)}, h \in D_B \right\}, \text{ because } \widetilde{f} \text{ is decreasing} \\
&= \left[ \wedge \left\{ \frac{\widetilde{f}(x+h)}{\widetilde{B}(x)}, h \in D_B \right\} \right] \wedge \left[ \wedge \left\{ \frac{\widetilde{g}(x+h)}{\widetilde{B}(x)}, h \in D_B \right\} \right], \text{ because } \widetilde{B}(x) < 0 \\
&= \mu_B(f) \wedge \mu_B(g)
\end{aligned} \tag{22}$$

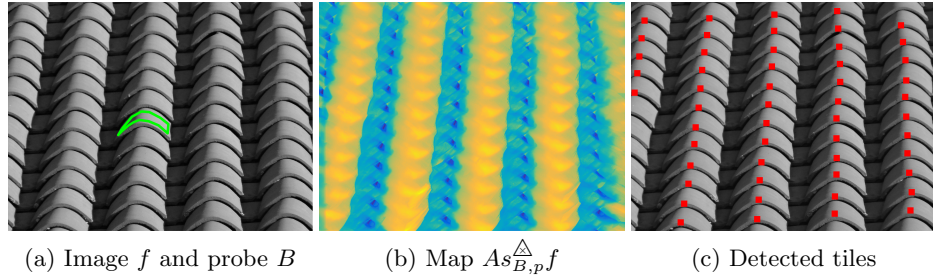
According to definition 4, 1 (p. 5),  $\mu_B$  is an erosion. In addition,

$$\mu_B(I) = \mu_B(f_M) = \vee \left\{ \beta(x), \beta(x) \leq \frac{\widetilde{M}(x+h)}{\widetilde{B}(x)}, h \in D_B \right\} = +\infty = I. \tag{23}$$

Therefore, the map of Asplünd's distances is the logarithm of the ratio between a dilation and an erosion of the function  $f$  by the structuring function  $B$ . The map of the least upper bounds  $\lambda_B$  is an anti-dilation and the map of the greatest lower bounds  $\mu_B$  is an anti-erosion. The two maps are defined from the mapping  $(\mathcal{L}_1 = \overline{\mathcal{I}}, \leq)$  and the mapping  $(\mathcal{L}_2 = (\overline{\mathbb{R}^+})^D, \leq)$  with their respective natural orders.

## 4 Illustration

We illustrate the map of Asplünd's distances (fig 1 (a)) by extracting a tile (i.e. the probe or the structuring function) and looking for the similar ones. Importantly, the probe has a non convex domain shape and is not flat. We compute the map of Asplünd's distances between the probe  $B$  and the image  $f$  with a tolerance,  $As_{B,p}^{\triangle} f$ , as introduced in [14, 23]. This is a metric which is robust to noise and computed by discarding a percentage  $p$  of points which are the closest to the least upper bounds and to the greatest lower bounds. The tiles are located at the local minima extracted by a threshold (0.7) of the map of distances (fig. 1 (b)). The tiles similar to the probe, according to the Asplünd's distance, are correctly detected (fig. 1 (c)).



**Fig. 1.** Detection of tiles using the map of Asplünd's distances  $As_{B,p}^{\Delta} f$  with a tolerance  $p = 30\%$ . (a) The probe  $B$  (in green) is extracted in the image  $f$ . (b) The minima (in blue) of the map of distances are extracted with a threshold of 0.7. (c) Detected tiles (red dots).

## 5 Conclusion

In the current paper, we have shown that the map of Asplünd's distances between a probe and a function using the LIP multiplication is linked with morphological operations. The probe corresponds to a structuring function and the map of Asplünd's distances is the logarithm of the ratio between a dilation and an erosion of the function by the structuring function into the lattice of positive functions  $((\mathbb{R}^+)^D, \leq)$ . The dilation is the map of the least upper bounds  $\lambda_B f$ , between the function  $f$  and the probe  $B$ , while the erosion is the map of the greatest lower bounds  $\mu_B f$ .

The dilation and the erosion are mappings between the complete lattices of the images  $(\bar{\mathcal{I}}, \leq)$  and the lattice  $((\mathbb{R}^+)^D, \leq)$  with the natural order. When using a flat structuring element, the expression of the map of Asplünd's distances can be simplified with a dilation and an erosion of the image into the same lattice of the images  $(\bar{\mathcal{I}}, \leq)$ . An example of pattern matching has been presented with a non-flat structuring function.

The obtained results set the pattern matching approach by Asplünd's distances in the well established framework of Mathematical Morphology. The current reasoning can be extended to the double-sided probing by Asplünd's distances for colour and multivariate images using the LIP multiplicative or the LIP additive framework [23, 11, 24]. This will be presented in a coming paper.

## 6 Appendix

Let us demonstrate that the Asplünd's metric  $d_{As}^{\Delta}$  is a metric in the space of equivalence classes  $\mathcal{I}^{\Delta}$ . In order to be a metric on  $(\mathcal{I}^{\Delta} \times \mathcal{I}^{\Delta}) \rightarrow \mathbb{R}^+$ ,  $d_{As}^{\Delta}$  must satisfy the four following properties:

1. (positivity):  $\forall f^\Delta \neq g^\Delta \in \mathcal{I}^\Delta, \forall x \in D, \lambda \triangle g^\Delta(x) > \mu \triangle g^\Delta(x)$  (def. 1, p. 4), because  $g^\Delta > 0$   
 $\Rightarrow \lambda > \mu$  because  $\mathcal{I}^\Delta$  is an ordered set with the order  $\leq$   
 $\Rightarrow \forall f^\Delta \neq g^\Delta \in \mathcal{I}^\Delta, d_{As}^\Delta(f^\Delta, g^\Delta) > 0$ .
2. (Axiom of separation):

$$\left. \begin{aligned} d_{As}^\Delta(f^\Delta, g^\Delta) = 0 &\Rightarrow \lambda = \mu \\ (\text{def. 1}) &\Rightarrow \lambda \triangle g^\Delta \geq f^\Delta \geq \mu \triangle g^\Delta \end{aligned} \right\} \Rightarrow \lambda \triangle g^\Delta = f^\Delta \Rightarrow f^\Delta = g^\Delta \text{ in } \mathcal{I}^\Delta \quad (24)$$

Reciprocally:

$$\left. \begin{aligned} \forall f^\Delta, g^\Delta \in \mathcal{I}^\Delta, f^\Delta = g^\Delta &\Rightarrow \lambda \triangle g^\Delta = f^\Delta \\ (\text{def. 1}) &\lambda \triangle g^\Delta \geq f^\Delta \geq \mu \triangle g^\Delta \end{aligned} \right\} \Rightarrow \lambda \triangle g^\Delta = f^\Delta = \mu \triangle g^\Delta$$

$$\Rightarrow \lambda = \mu \Rightarrow d_{As}^\Delta(f^\Delta, g^\Delta) = 0 \quad (25)$$

Eq. 24 and 25  $\Rightarrow \left\{ \forall f^\Delta, g^\Delta \in \mathcal{I}^\Delta, d_{As}^\Delta(f^\Delta, g^\Delta) = 0 \Leftrightarrow f^\Delta = g^\Delta \right\}$ .

3. (Triangle inequality): Let us define:  $d_{As}^\Delta(f^\Delta, g^\Delta) = \ln(\lambda_1/\mu_1)$ ,  $d_{As}^\Delta(g^\Delta, h^\Delta) = \ln(\lambda_2/\mu_2)$  and  $d_{As}^\Delta(f^\Delta, h^\Delta) = \ln(\lambda_3/\mu_3)$ . We have

$$d_{As}^\Delta(f^\Delta, g^\Delta) + d_{As}^\Delta(g^\Delta, h^\Delta) = \ln\left(\frac{\lambda_1 \lambda_2}{\mu_1 \mu_2}\right) \quad (26)$$

$$\text{Def. 1} \Rightarrow \left\{ \begin{aligned} \lambda_1 &= \inf \left\{ k_1 : \forall x, k_1 \triangle g^\Delta(x) \geq f^\Delta(x) \right\} \\ \lambda_2 &= \inf \left\{ k_2 : \forall x, k_2 \triangle h_j^\Delta(x) \geq g^\Delta(x) \right\} \\ \lambda_3 &= \inf \left\{ k_3 : \forall x, k_3 \triangle h_j^\Delta(x) \geq f^\Delta(x) \right\} \end{aligned} \right.$$

$$\begin{aligned} \Rightarrow \lambda_1 \lambda_2 &\leq \inf_{k_1} \left\{ \inf_{k_2} \left\{ \forall x, k_1 \triangle (k_2 \triangle h_j^\Delta(x)) \geq k_1 \triangle g^\Delta(x) \right\} \geq f^\Delta(x) \right\} \\ &\leq \inf \left\{ k' : \forall x, k' \triangle h_j^\Delta(x) \geq f^\Delta(x) \right\} \text{ with } k' = k_1 \times k_2 \\ \Rightarrow \lambda_1 \lambda_2 &\leq \lambda_3 \text{ with } \lambda_1, \lambda_2, \lambda_3 > 0 \end{aligned} \quad (27)$$

In the same way:

$$\mu_1 \mu_2 \geq \mu_3 \text{ with } \mu_1, \mu_2, \mu_3 > 0 \quad (28)$$

$$\begin{aligned} \text{Eq. 26, 27 28} &\Rightarrow \frac{\lambda_1 \lambda_2}{\mu_1 \mu_2} \geq \frac{\lambda_3}{\mu_3} \\ \Rightarrow \forall f^\Delta, g^\Delta, h^\Delta &\in \mathcal{I}^\Delta, d_{As}^\Delta(f^\Delta, h^\Delta) \leq d_{As}^\Delta(f^\Delta, g^\Delta) + d_{As}^\Delta(g^\Delta, h^\Delta). \end{aligned}$$

4. (Axiom of symmetry): Let us define:  $d_{As}^\Delta(f^\Delta, g^\Delta) = \ln(\lambda_1/\mu_1)$ ,  $d_{As}^\Delta(g^\Delta, f^\Delta) = \ln(\lambda_2/\mu_2)$ .

$$\text{Def. 1} \Rightarrow \lambda_1 = \inf \left\{ k : \forall x, g^\Delta \geq \frac{1}{k} \triangle f^\Delta \right\} \text{ because } k > 0$$

$$\Rightarrow \frac{1}{\lambda_1} = \sup \left\{ k' : \forall x, g^\Delta \geq k' \triangle f^\Delta \right\} \Rightarrow \frac{1}{\lambda_1} = \mu_2.$$

In the same way, we have  $\frac{1}{\mu_1} = \lambda_2$ .

Therefore,  $\forall f^\Delta, g^\Delta \in \mathcal{I}^\Delta, d_{As}^\Delta(f^\Delta, g^\Delta) = \ln(\lambda_1/\mu_1) = \ln(\lambda_2/\mu_2) = d_{As}^\Delta(g^\Delta, f^\Delta)$ .

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