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Rings of \mathbf{h} -deformed differential operators

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In memory of Petr Kulish

Abstract

We describe the center of the ring $\text{Diff}_{\mathbf{h}}(n)$ of \mathbf{h} -deformed differential operators of type A. We establish an isomorphism between certain localizations of $\text{Diff}_{\mathbf{h}}(n)$ and the Weyl algebra W_n extended by n indeterminates.

1 Introduction

The ring $\text{Diff}_{\mathbf{h}}(n)$ of \mathbf{h} -deformed differential operators of type A appears in the theory of reduction algebras. A reduction algebra $R_{\mathfrak{g}}^A$ provides a tool to study decompositions of representations of an associative algebra \mathcal{A} with respect to its subalgebra in the situation when this subalgebra is the universal enveloping algebra of a reductive Lie algebra \mathfrak{g} [M, AST]. We refer to [T, Zh] for the general theory and uses of reduction algebras.

Decompositions of tensor products of representations of a reductive Lie algebra \mathfrak{g} is a particular case of a restriction problem, associated to the diagonal embedding of $U(\mathfrak{g})$ into $U(\mathfrak{g}) \otimes U(\mathfrak{g})$. The corresponding reduction algebra, denoted $\mathcal{D}(\mathfrak{g})$, is called “diagonal reduction algebra” [KO2]. A description of the diagonal reduction algebra $\mathcal{D}(\mathfrak{gl}_n)$ in terms of generators and (ordering) defining relations was given in [KO2, KO3].

The diagonal reduction algebra $\mathcal{D}(\mathfrak{gl}_n)$ admits an analogue of the “oscillator realization”, in the rings $\text{Diff}_{\mathbf{h}}(n, N)$, $N = 1, 2, 3, \dots$, of \mathbf{h} -deformed differential operators, see [KO5]. The ring $\text{Diff}_{\mathbf{h}}(n, N)$ can be obtained by the reduction of the ring of differential operators in nN variables (that is, of the Weyl algebra $W_{nN} = W_n^{\otimes N}$) with respect to the natural action of \mathfrak{gl}_n . Similarly to the ring of q -differential operators [WZ], the algebra $\text{Diff}_{\mathbf{h}}(n, N)$ can be described in the R-matrix formalism. The R-matrix, needed here, is a solution of the so-called dynamical Yang–Baxter equation (we refer to [F, GV, ES] for different aspects of the dynamical Yang–Baxter equation and its solutions).

The ring $\text{Diff}_{\mathbf{h}}(n, N)$ is formed by N copies of the ring $\text{Diff}_{\mathbf{h}}(n) = \text{Diff}_{\mathbf{h}}(n, 1)$. The aim of the present article is to investigate the structure of the ring $\text{Diff}_{\mathbf{h}}(n)$. Our first result is the description of the center of $\text{Diff}_{\mathbf{h}}(n)$: it is a ring of polynomials in n generators.

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As follows from the results of [KO4], the ring $\text{Diff}_{\mathbf{h}}(n)$ is a noetherian Ore domain. It is therefore natural to investigate its field of fractions and test the validity of the Gelfand–Kirillov-like conjecture [GK]. For the ring of q -differential operators, the isomorphism (up to a certain localization and completion) with the Weyl algebra was given in [O]. The second result of the present article consists in a construction of an isomorphism between certain localizations of $\text{Diff}_{\mathbf{h}}(n)$ and the Weyl algebra W_n extended by n indeterminates. In particular, our isomorphism implies the isomorphism of the corresponding fields of fractions.

According to the general theory of reduction algebras, the ring $\text{Diff}_{\mathbf{h}}(n)$ admits the action of Zhelobenko operators [Zh, KO1] by automorphisms. The Zhelobenko operators generate the action of the braid group B_n . However the Weyl algebra W_n admits the action of the symmetric group S_n by automorphisms. As a by-product of our construction we define the action of the symmetric group by automorphisms on the ring $\text{Diff}_{\mathbf{h}}(n)$. Moreover these formulas can be generalized to produce the action of the symmetric group by automorphisms on the rings $\text{Diff}_{\mathbf{h}}(n, N)$ for any N , and on the diagonal reduction algebra $\mathcal{D}(\mathbf{gl}_n)$. We conjecture that the general reduction algebra $R_{\mathfrak{g}}^A$ admits an action, by automorphisms, of the Weyl group of \mathfrak{g} .

Section 2 contains the definition of the ring of \mathbf{h} -deformed differential operators and some of their properties used in the sequel. In Section 3 we present a family of n quadratic central elements. We then describe an n -parametric family of “highest weight” representations of $\text{Diff}_{\mathbf{h}}(n)$ and calculate values of the quadratic central elements in these representations. In Section 4 we introduce the necessary localizations of $\text{Diff}_{\mathbf{h}}(n)$ and of the Weyl algebra, check the Ore conditions and establish the above mentioned isomorphism of the localized rings. In Section 5 we prove the completeness of the family of central elements constructed in Section 3. Then we describe the action of the symmetric group on $\text{Diff}_{\mathbf{h}}(n, N)$ and $\mathcal{D}(\mathbf{gl}_n)$ as well as the action of the braid group, generated by Zhelobenko operators, on a localization of the Weyl algebra. Also, we present a $2n$ -parametric family of representations of the algebra $\text{Diff}_{\mathbf{h}}(n)$ implied by our construction.

Notation

Throughout the paper, \mathfrak{k} denotes the ground ring of characteristic zero.

The symbol s_i stands for the transposition $(i, i + 1)$.

We denote by $U(\mathbf{h})$ the free commutative \mathfrak{k} -algebra in generators \tilde{h}_i , $i = 1, \dots, n$. Set $\tilde{h}_{ij} = \tilde{h}_i - \tilde{h}_j \in \mathbf{h}$. We define $\bar{U}(\mathbf{h})$ to be the ring of fractions of $U(\mathbf{h})$ with respect to the multiplicative set of denominators, generated by the elements $(h_{ij} + k)^{-1}$, $k \in \mathbb{Z}$. Let

$$\psi_i := \prod_{k:k>i} \tilde{h}_{ik}, \psi'_i := \prod_{k:k<i} \tilde{h}_{ik} \text{ and } \chi_i := \psi_i \psi'_i, \quad i = 1, \dots, n. \quad (1)$$

Let ε_j , $j = 1, \dots, n$, be the elementary translations of the generators of $U(\mathbf{h})$, $\varepsilon_j : \tilde{h}_i \mapsto \tilde{h}_i + \delta_i^j$. For an element $p \in \bar{U}(\mathbf{h})$ we denote $\varepsilon_j(p)$ by $p[\varepsilon_j]$.

2 Definition and properties of rings of \mathbf{h} -deformed differential operators

Let $\hat{\mathbf{R}} = \{\hat{\mathbf{R}}_{ij}^{kl}\}_{i,j,k,l=1}^n$ be a matrix of elements of $\bar{\mathbf{U}}(\mathbf{h})$, with nonzero entries

$$\hat{\mathbf{R}}_{ij}^{ij} = \frac{1}{\tilde{h}_{ij}} , \quad i \neq j , \quad \text{and} \quad \hat{\mathbf{R}}_{ji}^{ij} = \begin{cases} \frac{\tilde{h}_{ij}^2 - 1}{\tilde{h}_{ij}^2} , & i < j , \\ 1 , & i \geq j . \end{cases} \quad (2)$$

The matrix $\hat{\mathbf{R}}$ is the standard solution of the dynamical Yang–Baxter equation

$$\sum_{a,b,u} \hat{\mathbf{R}}_{ab}^{ij} \hat{\mathbf{R}}_{ur}^{bk} [-\varepsilon_a] \hat{\mathbf{R}}_{mn}^{au} = \sum_{a,b,u} \hat{\mathbf{R}}_{ab}^{jk} [-\varepsilon_i] \hat{\mathbf{R}}_{mu}^{ia} \hat{\mathbf{R}}_{nr}^{ub} [-\varepsilon_m] \quad (3)$$

of type A.

The ring $\text{Diff}_{\mathbf{h}}(n)$ of \mathbf{h} -deformed differential operators of type A is a $\bar{\mathbf{U}}(\mathbf{h})$ -bimodule with the generators x^j and $\bar{\partial}_j$, $j = 1, \dots, n$. The ring $\text{Diff}_{\mathbf{h}}(n)$ is free as a one-sided $\bar{\mathbf{U}}(\mathbf{h})$ -module; the left and right $\bar{\mathbf{U}}(\mathbf{h})$ -module structures are related by

$$\tilde{h}_i x^j = x^j (\tilde{h}_i + \delta_i^j) , \quad \tilde{h}_i \bar{\partial}_j = \bar{\partial}_j (\tilde{h}_i - \delta_i^j) . \quad (4)$$

The defining relations for the generators x^j and $\bar{\partial}_j$, $j = 1, \dots, n$, read (see [KO5])

$$x^i x^j = \sum_{k,l} \hat{\mathbf{R}}_{kl}^{ij} x^k x^l , \quad \bar{\partial}_i \bar{\partial}_j = \sum_{k,l} \hat{\mathbf{R}}_{ji}^{lk} \bar{\partial}_k \bar{\partial}_l , \quad x^i \bar{\partial}_j = \sum_{k,l} \hat{\mathbf{R}}_{lj}^{ki} [\varepsilon_k] \bar{\partial}_k x^l - \delta_j^i , \quad (5)$$

or, in components,

$$x^i x^j = \frac{\tilde{h}_{ij} + 1}{\tilde{h}_{ij}} x^j x^i , \quad i < j , \quad (6)$$

$$\bar{\partial}_i \bar{\partial}_j = \frac{\tilde{h}_{ij} - 1}{\tilde{h}_{ij}} \bar{\partial}_j \bar{\partial}_i , \quad i < j , \quad (7)$$

$$x^i \bar{\partial}_j = \begin{cases} \bar{\partial}_j x^i , & i < j , \\ \frac{\tilde{h}_{ij}(\tilde{h}_{ij} - 2)}{(\tilde{h}_{ij} - 1)^2} \bar{\partial}_j x^i , & i > j , \end{cases} \quad (8)$$

$$x^i \bar{\partial}_i = \sum_j \frac{1}{1 - \tilde{h}_{ij}} \bar{\partial}_j x^j - 1 . \quad (9)$$

The ring $\text{Diff}_{\mathbf{h}}(n)$ admits Zhelobenko automorphisms \check{q}_i , $i = 1, \dots, n-1$, given by (see [KO5])

$$\begin{aligned} \check{q}_i(x^i) &= -x^{i+1} \frac{\tilde{h}_{i,i+1}}{\tilde{h}_{i,i+1} - 1}, & \check{q}_i(x^{i+1}) &= x^i, & \check{q}_i(x^j) &= x^j, & j &\neq i, i+1, \\ \check{q}_i(\bar{\partial}_i) &= -\frac{\tilde{h}_{i,i+1} - 1}{\tilde{h}_{i,i+1}} \bar{\partial}_{i+1}, & \check{q}_i(\bar{\partial}_{i+1}) &= \bar{\partial}_i, & \check{q}_i(\bar{\partial}_j) &= \bar{\partial}_j, & j &\neq i, i+1, \\ \check{q}_i(\tilde{h}_j) &= \tilde{h}_{s_i(j)}. \end{aligned} \quad (10)$$

The operators \check{q}_i , $i = 1, \dots, n-1$, generate the action of the braid group, see [Zh, KO1].

The ring $\text{Diff}_{\mathbf{h}}(n)$ admits an involutive anti-automorphism ϵ , defined by

$$\epsilon(\tilde{h}_i) = \tilde{h}_i, \epsilon(\bar{\partial}_i) = \varphi_i x^i, \epsilon(x^i) = \bar{\partial}_i \varphi_i^{-1}, \text{ where } \varphi_i := \frac{\psi_i}{\psi_i[-\varepsilon_i]} = \prod_{k:k>i} \frac{\tilde{h}_{ik}}{\tilde{h}_{ik} - 1}, i = 1, \dots, n, \quad (11)$$

The proof reduces to the formula

$$\frac{\varphi_i[-\varepsilon_j]}{\varphi_i} = \frac{\tilde{h}_{ij}^2 - 1}{\tilde{h}_{ij}^2} \text{ for } 1 \leq i < j \leq n.$$

The construction of central elements in the next Section uses the elements

$$\Gamma_i := \bar{\partial}_i x^i \text{ for } i = 1, \dots, n. \quad (12)$$

We collect some properties of these elements.

Lemma 1. We have

- (i) $\Gamma_i x^j = \frac{\tilde{h}_{ij} + 1}{\tilde{h}_{ij}} x^j \Gamma_i$ and $\Gamma_i \bar{\partial}_j = \frac{\tilde{h}_{ij} - 1}{\tilde{h}_{ij}} \bar{\partial}_j \Gamma_i$ for $i \neq j$, $i, j = 1, \dots, n$.
- (ii) $\check{q}_i(\Gamma_j) = \Gamma_{s_i(j)}$ for $i = 1, \dots, n-1$ and $j = 1, \dots, n$.
- (iii) $\Gamma_i \Gamma_j = \Gamma_j \Gamma_i$ for $i, j = 1, \dots, n$.

Proof. Formulas (i) and (ii) are obtained by a direct calculation; (iii) follows from (i). \square

We will use the following technical Lemma.

Lemma 2. Let \mathfrak{A} be an associative algebra. Assume that elements $\check{h}_i, \check{Z}_i, \check{Z}_i \in \mathfrak{A}$, $i = 1, \dots, n$, satisfy

$$\check{h}_i \check{h}_j = \check{h}_j \check{h}_i, \check{h}_i \check{Z}^j = \check{Z}^j (\check{h}_i + \delta_i^j), \check{h}_i \check{Z}_j = \check{Z}_j (\check{h}_i - \delta_i^j), i, j = 1, \dots, n.$$

Let $\check{h}_{ij} := \check{h}_i - \check{h}_j$ and

$$\check{\psi}_i := \prod_{k:k>i} \check{h}_{ik}, \check{\psi}'_i := \prod_{k:k<i} \check{h}_{ik}, \quad i = 1, \dots, n.$$

Assume that the elements \check{h}_{ij} are invertible. Then

(i) the elements \check{Z}^i satisfy

$$\check{Z}^i \check{Z}^j = \frac{\check{h}_{ij} + 1}{\check{h}_{ij}} \check{Z}^j \check{Z}^i \quad \text{for } i < j, \quad i, j = 1, \dots, n$$

if and only if any of the two families $\{\check{Z}^{\circ i}\}_{i=1}^n$ or $\{\check{Z}'^{\circ i}\}_{i=1}^n$ where

$$\check{Z}^{\circ i} := \psi_i \check{Z}^i, \quad \check{Z}'^{\circ i} := \check{Z}^i \psi'_i \quad (13)$$

is commutative;

(ii) the elements \check{Z}_i satisfy

$$\check{Z}_i \check{Z}_j = \frac{\check{h}_{ij} - 1}{\check{h}_{ij}} \check{Z}_j \check{Z}_i \quad \text{for } i < j, \quad i, j = 1, \dots, n$$

if and only if any of the two families $\{\check{Z}_i^{\circ}\}_{i=1}^n$ or $\{\check{Z}_i'^{\circ}\}_{i=1}^n$ where

$$\check{Z}_i^{\circ} := \psi_i \check{Z}_i, \quad \check{Z}_i'^{\circ} := \check{Z}_i \psi'_i \quad (14)$$

is commutative.

Proof. A direct calculation. □

3 Quadratic central elements

Let $e_k := \sum_{i_1 < \dots < i_k} \check{h}_{i_1} \dots \check{h}_{i_k}$, $k = 0, \dots, n$, be the elementary symmetric functions in the variables $\check{h}_1, \dots, \check{h}_n$. Set

$$c_k := \sum_j \frac{\partial e_k}{\partial \check{h}_j} \Gamma_j - e_k,$$

where Γ_j , $j = 1, \dots, n$, are the elements defined in (12).

It follows from Lemma 1 that $\check{q}_j(c_k) = c_k$ for all $j = 1, \dots, n-1$ and $k = 1, \dots, n$.

Proposition 3. The elements c_k , $k = 1, \dots, n$, belong to the center of the ring $\text{Diff}_{\mathbf{h}}(n)$.

Proof. We shall use the generating functions

$$e(t) := \sum_{k=0}^n e_k t^k = \prod_i (1 + \tilde{h}_i t)$$

and

$$c(t) := \sum_{k=1}^n c_k t^k = u(t)e(t) + 1 \quad \text{with} \quad u(t) := t \sum_i \frac{1}{1 + \tilde{h}_i t} \Gamma_i - 1 .$$

The expression $u(t)$ is introduced for convenience; the denominator $1 + \tilde{h}_i t$, which is not defined in the ring $\text{Diff}_{\mathbf{h}}(n)$, vanishes in the combination $u(t)e(t)$.

We shall check that the polynomial $c(t)$ is central. We have

$$x^j e(t) = \frac{1 + (\tilde{h}_j - 1)t}{1 + \tilde{h}_j t} e(t) x^j . \quad (15)$$

Next, it follows from Lemma 1 that

$$x^j u(t) = \left(\sum_{k:k \neq j} \frac{t}{1 + \tilde{h}_k t} \frac{\tilde{h}_{kj}}{\tilde{h}_{kj} + 1} \Gamma_k + \frac{t}{1 + (\tilde{h}_j - 1)t} \left(\sum_k \frac{1}{1 - \tilde{h}_{jk}} \Gamma_k - 1 \right) - 1 \right) x^j .$$

The coefficient of Γ_k in this expression is equal to $\frac{1 + \tilde{h}_j}{1 + (\tilde{h}_j - 1)t}$ for both $k \neq j$ and $k = j$. Therefore,

$$x^j u(t) = \frac{1 + \tilde{h}_j t}{1 + (\tilde{h}_j - 1)t} u(t) x^j . \quad (16)$$

Combining (15) and (16) we find that $c(t)$ commutes with x^j , $j = 1, \dots, n$. For $\bar{\partial}_j$ one can either make a parallel calculation or use the anti-automorphism (11). \square

Lemma 4. (i) The matrix V , defined by $V_j^k := \frac{\partial e_j}{\partial h_k}$, is invertible. Its inverse is

$$(V^{-1})_i^j = \frac{(-1)^{j-1} \tilde{h}_i^{n-j}}{\chi_i} ,$$

where the elements χ_i are defined in (1).

(ii) We have

$$\chi_j \Gamma_j = \tilde{h}_j^n - \tilde{h}_j^n c(-\tilde{h}_j^{-1}) . \quad (17)$$

Proof. (i) See, e.g. [OP], Proposition 4.

(ii) Rewrite the equality $c_k = \sum_j V_k^j \Gamma_j - e_k$ in the form

$$\Gamma_j = \sum_k (V^{-1})_j^k (c_k + e_k) = \frac{1}{\chi_j} \sum_k (-1)^{k-1} \tilde{h}_j^{n-k} (c_k + e_k) = -\frac{\tilde{h}_j^n}{\chi_j} (c(-\tilde{h}_j^{-1}) + e(-\tilde{h}_j^{-1}) - 1) .$$

Since $e(-\tilde{h}_j^{-1}) = 0$, we obtain (17). \square

Highest weight representations. The ring $\text{Diff}_{\mathbf{h}}(n)$ admits an n -parametric family of “highest weight” representations. To define them, let \mathfrak{D}_n be an $\bar{\mathbf{U}}(\mathbf{h})$ -subring of $\text{Diff}_{\mathbf{h}}(n)$ generated by $\{\bar{\partial}_i\}_{i=1}^n$. Let $\vec{\lambda} := \{\lambda_1, \dots, \lambda_n\}$ be a sequence of length n of complex numbers such that $\lambda_i - \lambda_j \notin \mathbb{Z}$ for all $i, j = 1, \dots, n, i \neq j$. Denote by $M_{\vec{\lambda}}$ the one-dimensional \mathfrak{k} -vector space with the basis vector $|\rangle$. Under the specified conditions on $\vec{\lambda}$ the formulas

$$\tilde{h}_i: |\rangle \mapsto \lambda_i |\rangle, \quad \bar{\partial}_i: |\rangle \mapsto 0, \quad i = 1, \dots, n,$$

define the \mathfrak{D}_n -module structure on $M_{\vec{\lambda}}$. We shall call the induced representation $\text{Ind}_{\mathfrak{D}_n}^{\text{Diff}_{\mathbf{h}}(n)} M_{\vec{\lambda}}$ the “highest weight representation” of highest weight $\vec{\lambda}$.

Lemma 5. The central operator c_k , $k = 1, \dots, n$, acts on the module $\text{Ind}_{\mathfrak{D}_n}^{\text{Diff}_{\mathbf{h}}(n)} M_{\vec{\lambda}}$ by scalar multiplication on $-e_k|_{\tilde{h}_i \mapsto \lambda_i - 1}$, the evaluation of the symmetric function $-e_k$ on the shifted vector $\{\lambda_1 - 1, \dots, \lambda_n - 1\}$.

Proof. It is sufficient to calculate the value of c_k on the highest weight vector $|\rangle$. In terms of generating functions we have to check that

$$e(t)u(t): |\rangle \mapsto - \prod_i (1 + (\lambda_i - 1)t) |\rangle.$$

It follows from [KO5], section 3.3, that

$$\Gamma_j |\rangle = \frac{\chi_j[\varepsilon_j]}{\chi_j} |\rangle, \quad j = 1, \dots, n.$$

Therefore we have to check that

$$e(t) \left(t \sum_i \frac{1}{1 + \tilde{h}_i t} \frac{\chi_j[\varepsilon_j]}{\chi_j} - 1 \right) : |\rangle \mapsto - \prod_i (1 + (\lambda_i - 1)t) |\rangle. \quad (18)$$

We use another formula from [KO5] (Note 3 after the proof of Proposition 4.3 in Section 4.2)

$$\prod_l \frac{\tilde{h}_0 - \tilde{h}_l - 1}{\tilde{h}_0 - \tilde{h}_l} + \sum_j \frac{1}{\tilde{h}_0 - \tilde{h}_j} \frac{\chi_j[-\varepsilon_j]}{\chi_j} = 1,$$

where \tilde{h}_0 is an indeterminate. After the replacements $\tilde{h}_0 \rightarrow t^{-1}$ and $\tilde{h}_j \rightarrow -\tilde{h}_j$, $j = 1, \dots, n$, this formula becomes

$$\frac{e(t)[- \varepsilon]}{e(t)} + t \sum_j \frac{1}{1 + \tilde{h}_j t} \frac{\chi_j[\varepsilon_j]}{\chi_j} = 1,$$

where $\varepsilon = \varepsilon_1 + \dots + \varepsilon_n$, which implies (18). □

4 Isomorphism between rings of fractions

It follows from the results of [KO4] that the ring $\text{Diff}_{\mathbf{h}}(n)$ has no zero divisors. Let S_x be the multiplicative set generated by x^j , $j = 1, \dots, n$. The set S_x satisfies both left and right Ore conditions (see, e.g., [A] for definitions): say, for the left Ore conditions we have to check only that for any x^k and a monomial $m = \bar{\partial}_{i_1} \dots \bar{\partial}_{i_A} x^{j_1} \dots x^{j_B}$ there exist $\tilde{s} \in S_x$ and $\tilde{m} \in \text{Diff}_{\mathbf{h}}(n)$ such that $\tilde{s}m = \tilde{m}x^k$. The structure of the commutation relations (6-9) shows that one can choose $\tilde{s} = (x^k)^\nu$ with sufficiently large ν . Denote by $S_x^{-1}\text{Diff}_{\mathbf{h}}(n)$ the localization of the ring $\text{Diff}_{\mathbf{h}}(n)$ with respect to the set S_x .

Let W_n be the Weyl algebra, the algebra with the generators X^j, D_j , $j = 1, \dots, n$, and the defining relations

$$X^i X^j = X^j X^i, \quad D_i D_j = D_j D_i, \quad D_i X^j = \delta_i^j + X^j D_i, \quad i, j = 1, \dots, n.$$

Let T be the multiplicative set generated by $X^j D_j - X^k D_k + \ell$, $1 \leq j < k \leq n$, $\ell \in \mathbb{Z}$, and X^j , $j = 1, \dots, n$. The set T satisfies left and right Ore conditions (see [KO4], Appendix). Denote by $T^{-1}W_n$ the localization of W_n relative to the set T .

Let a_1, \dots, a_n be a family of commuting variables. We shall use the following notation:

$$\begin{aligned} \mathcal{H}_j &:= D_j X^j, \quad \mathcal{H}_{jk} := \mathcal{H}_j - \mathcal{H}_k, \\ \Psi'_j &:= \prod_{k:k < j} \mathcal{H}_{jk}, \quad \Psi_j := \prod_{k:k > j} \mathcal{H}_{jk}, \\ \mathbf{C}(t) &:= \sum_{k=1}^n a_k t^k, \quad \Upsilon_i := \mathcal{H}_i^n (1 - \mathbf{C}(-\mathcal{H}_i^{-1})). \end{aligned}$$

The polynomial \mathbf{C} has degree n so the element Υ_i is a polynomial in \mathcal{H}_i , $i = 1, \dots, n$.

Theorem 6. The ring $S_x^{-1}\text{Diff}_{\mathbf{h}}(n)$ is isomorphic to the ring $\mathfrak{k}[a_1, \dots, a_n] \otimes T^{-1}W_n$.

Proof. The knowledge of the central elements (Proposition 3) allows to exhibit a generating set of the ring $S_x^{-1}\text{Diff}_{\mathbf{h}}(n)$ in which the required isomorphism is quite transparent.

In the localized ring $S_x^{-1}\text{Diff}_{\mathbf{h}}(n)$ we can use the set of generators $\{\tilde{h}_i, x^i, \Gamma_i\}_{i=1}^n$ instead of $\{\tilde{h}_i, x^i, \bar{\partial}_i\}_{i=1}^n$. By Lemma 4 (ii), $\{\tilde{h}_i, x^i, c_i\}_{i=1}^n$ is also a generating set. Finally, $\mathfrak{B}_D := \{\tilde{h}_i, x^{oi}, c_i\}_{i=1}^n$, where $x^{oi} := x^i \psi'_i$, $i = 1, \dots, n$, is a generating set of the localized ring $S_x^{-1}\text{Diff}_{\mathbf{h}}(n)$ as well. It follows from Lemma 2 that the family $\{x^{oi}\}_{i=1}^n$ is commutative. The complete set of the defining relations for the generators from the set \mathfrak{B}_D reads

$$\begin{aligned} \tilde{h}_i \tilde{h}_j &= \tilde{h}_j \tilde{h}_i, \quad \tilde{h}_i x'^{oj} = x'^{oj} (\tilde{h}_i + \delta_i^j), \quad x'^{oi} x'^{oj} = x'^{oj} x'^{oi}, \quad i, j = 1, \dots, n, \\ c_i &\text{ are central, } i = 1, \dots, n. \end{aligned} \tag{19}$$

In the localized ring $\mathfrak{k}[a_1, \dots, a_n] \otimes T^{-1}W_n$ we can pass to the set of generators $\mathfrak{B}_W := \{\mathcal{H}_i, X^i, a_i\}_{i=1}^n$ with the defining relations

$$\begin{aligned} \mathcal{H}_i \mathcal{H}_j &= \mathcal{H}_j \mathcal{H}_i, \quad \mathcal{H}_i X^j = X^j (\mathcal{H}_i + \delta_i^j), \quad X^i X^j = X^j X^i, \quad i, j = 1, \dots, n, \\ a_i &\text{ are central, } i = 1, \dots, n. \end{aligned} \quad (20)$$

The comparison of (19) and (20) shows that we have the isomorphism

$$\mu: \mathfrak{k}[a_1, \dots, a_n] \otimes T^{-1}W_n \rightarrow S_x^{-1} \text{Diff}_{\mathbf{h}}(n)$$

given on our generating sets \mathfrak{B}_D and \mathfrak{B}_W by

$$\mu: X^i \mapsto x'^{oi}, \quad \mathcal{H}_i \mapsto \tilde{h}_i, \quad a_i \mapsto c_i, \quad i = 1, \dots, n. \quad (21)$$

The proof is completed. \square

We shall now rewrite the formulas for the isomorphism μ in terms of the original generators of the rings $S_x^{-1} \text{Diff}_{\mathbf{h}}(n)$ and $\mathfrak{k}[a_1, \dots, a_n] \otimes T^{-1}W_n$.

Lemma 7. We have

$$\mu: X^i \mapsto x^i \psi'_i, \quad D_i \mapsto (\psi'_i)^{-1} \tilde{h}_i (x^i)^{-1}, \quad a_i \mapsto c_i, \quad i = 1, \dots, n. \quad (22)$$

and

$$\mu^{-1}: \tilde{h}_i \mapsto \mathcal{H}_i, \quad x^i \mapsto X^i \frac{1}{\Psi'_i}, \quad \bar{\partial}_i \mapsto \frac{\Upsilon_i}{\Psi_i} (X^i)^{-1}, \quad i = 1, \dots, n. \quad (23)$$

Proof. We shall comment only on the last formula in (23). Lemma 4 part (ii) implies that $\mu^{-1}(\chi_i \Gamma_i) = \Upsilon_i$ and the formula for $\mu^{-1}(\bar{\partial}_i)$ follows since $\bar{\partial}_i = \Gamma_i (x^i)^{-1}$. \square

5 Comments

We shall now establish several corollaries of our construction.

1. We can now give the description of the center of the ring $\text{Diff}_{\mathbf{h}}(n)$.

Lemma 8. The center of the ring $\text{Diff}_{\mathbf{h}}(n)$ is formed by polynomials in the elements $\{c_i\}_{i=1}^n$.

Proof. This is a direct consequence of the defining relations (19) for the generating set \mathfrak{B}_D . Indeed, any central element ζ must have \mathbf{h} -weight zero, so it belongs to a subring generated by c_i and \tilde{h}_i , $i = 1, \dots, n$. Interpret ζ as a rational function in \tilde{h}_i , $i = 1, \dots, n$. Since ζ commutes with x'^{oi} , $i = 1, \dots, n$, this rational function is periodic, with period 1, with respect to any \tilde{h}_i , $i = 1, \dots, n$. Therefore ζ belongs to the subring generated by c_i , $i = 1, \dots, n$, as stated.

Another proof consists in using the isomorphism μ and the triviality of the center of the Weyl algebra. \square

2. The symmetric group S_n acts by automorphisms on the algebra W_n ,

$$\pi(X^j) = X^{\pi(j)} , \quad \pi(D_j) = D_{\pi(j)} \quad \text{for } \pi \in S_n .$$

The isomorphism μ translates this action to the action of S_n on the ring $S_x^{-1}\text{Diff}_{\mathbf{h}}(n)$. It turns out that the subring $\text{Diff}_{\mathbf{h}}(n)$ is preserved by this action. We present the formulas for the action of the generators s_i of S_n .

$$\begin{aligned} s_i(x^i) &= -x^{i+1}\tilde{h}_{i,i+1} , \quad s_i(x^{i+1}) = x^i \frac{1}{\tilde{h}_{i,i+1}} , \quad s_i(x^j) = x^j \text{ for } j \neq i, i+1 , \\ s_i(\bar{\partial}_i) &= -\frac{1}{\tilde{h}_{i,i+1}}\bar{\partial}_{i+1} , \quad s_i(\bar{\partial}_{i+1}) = \tilde{h}_{i,i+1}\bar{\partial}_i , \quad s_i(\bar{\partial}_j) = \bar{\partial}_j \text{ for } j \neq i, i+1 , \\ s_i(\tilde{h}_j) &= \tilde{h}_{s_i(j)} . \end{aligned} \tag{24}$$

3a. For the R-matrix description of the diagonal reduction algebra $\mathcal{D}(\mathbf{gl}_n)$ in [KO5] we used the ring $\text{Diff}_{\mathbf{h}}(n, N)$ formed by N copies of the ring $\text{Diff}_{\mathbf{h}}(n)$. We do not know an analogue of the isomorphism μ for the ring $\text{Diff}_{\mathbf{h}}(n, N)$. However a straightforward analogue of the formulas (24) provides an action of S_n by automorphisms on the ring $\text{Diff}_{\mathbf{h}}(n, N)$.

We recall that the ring $\text{Diff}_{\mathbf{h}}(n, N)$ is a $\bar{U}(\mathbf{h})$ -bimodule with the generators $x^{j,\alpha}$ and $\bar{\partial}_{j,\alpha}$, $j = 1, \dots, n$, $\alpha = 1, \dots, N$. The ring $\text{Diff}_{\mathbf{h}}(n, N)$ is free as a one-sided $\bar{U}(\mathbf{h})$ -module; the left and right $\bar{U}(\mathbf{h})$ -module structures are related by

$$\tilde{h}_i x^{j,\alpha} = x^{j,\alpha}(\tilde{h}_i + \delta_i^j) , \quad \tilde{h}_i \bar{\partial}_{j,\alpha} = \bar{\partial}_{j,\alpha}(\tilde{h}_i - \delta_i^j) . \tag{25}$$

The defining relations for the generators $x^{j,\alpha}$ and $\bar{\partial}_{j,\alpha}$, $j = 1, \dots, n$, $\alpha = 1, \dots, N$, read

$$x^{i,\alpha} x^{j,\beta} = \sum_{k,l} \hat{R}_{kl}^{ij} x^{k,\beta} x^{l,\alpha} , \quad \bar{\partial}_{i,\alpha} \bar{\partial}_{j,\beta} = \sum_{k,l} \hat{R}_{ji}^{lk} \bar{\partial}_{k,\beta} \bar{\partial}_{l,\alpha} , \quad x^{i,\alpha} \bar{\partial}_{j,\beta} = \sum_{k,l} \hat{R}_{lj}^{ki} [\varepsilon_k] \bar{\partial}_{k,\beta} x^{l,\alpha} - \delta_{\beta}^{\alpha} \delta_j^i , \tag{26}$$

or, in components,

$$x^{i,\alpha} x^{j,\beta} = \frac{1}{\tilde{h}_{ij}} x^{i,\beta} x^{j,\alpha} + \frac{\tilde{h}_{ij}^2 - 1}{\tilde{h}_{ij}^2} x^{j,\beta} x^{i,\alpha} , \quad x^{j,\alpha} x^{i,\beta} = -\frac{1}{\tilde{h}_{ij}} x^{j,\beta} x^{i,\alpha} + x^{i,\beta} x^{j,\alpha} , \quad 1 \leq i < j \leq n , \tag{27}$$

$$\bar{\partial}_{i,\alpha} \bar{\partial}_{j,\beta} = -\frac{1}{\tilde{h}_{ij}} \bar{\partial}_{i,\beta} \bar{\partial}_{j,\alpha} + \frac{\tilde{h}_{ij}^2 - 1}{\tilde{h}_{ij}^2} \bar{\partial}_{j,\beta} \bar{\partial}_{i,\alpha} , \quad \bar{\partial}_{j,\alpha} \bar{\partial}_{i,\beta} = \frac{1}{\tilde{h}_{ij}} \bar{\partial}_{j,\beta} \bar{\partial}_{i,\alpha} + \bar{\partial}_{i,\beta} \bar{\partial}_{j,\alpha} , \quad 1 \leq i < j \leq n , \tag{28}$$

$$x^{i,\alpha} \bar{\partial}_{j,\beta} = \bar{\partial}_{j,\beta} x^{i,\alpha} , \quad x^{j,\alpha} \bar{\partial}_{i,\beta} = \frac{\tilde{h}_{ij}(\tilde{h}_{ij} + 2)}{(\tilde{h}_{ij} + 1)^2} \bar{\partial}_{i,\beta} x^{j,\alpha} , \quad 1 \leq i < j \leq n , \tag{29}$$

$$x^{i,\alpha} \bar{\partial}_{i,\beta} = \sum_{k=1}^n \frac{1}{1 - \tilde{h}_{ik}} \bar{\partial}_{k,\beta} x^{k,\alpha} - \delta_{\beta}^{\alpha} , \quad 1 \leq i \leq n . \tag{30}$$

Lemma 9. The maps s_i , $i = 1, \dots, n-1$, defined on the generators of $\text{Diff}_{\mathbf{h}}(n, N)$ by

$$\begin{aligned} s_i(x^{i,\alpha}) &= -x^{i+1,\alpha} \tilde{h}_{i,i+1}, \quad s_i(x^{i+1,\alpha}) = x^{i,\alpha} \frac{1}{\tilde{h}_{i,i+1}}, \quad s_i(x^{j,\alpha}) = x^{j,\alpha} \text{ for } j \neq i, i+1, \\ s_i(\bar{\partial}_{i,\alpha}) &= -\frac{1}{\tilde{h}_{i,i+1}} \bar{\partial}_{i+1,\alpha}, \quad s_i(\bar{\partial}_{i+1,\alpha}) = \tilde{h}_{i,i+1} \bar{\partial}_{i,\alpha}, \quad s_i(\bar{\partial}_{j,\alpha}) = \bar{\partial}_{j,\alpha} \text{ for } j \neq i, i+1, \\ s_i(\tilde{h}_j) &= \tilde{h}_{s_i(j)}, \end{aligned} \quad (31)$$

extend to automorphisms of the ring $\text{Diff}_{\mathbf{h}}(n, N)$. Moreover, these automorphisms satisfy the Artin relations and therefore give the action of the symmetric group S_n by automorphisms.

Proof. After the formulas (31) are written down, the verification is a direct calculation. \square

3b. The operators $s'_i := \epsilon s_i \epsilon$, where ϵ is the anti-automorphism (11), generate the action of the symmetric group S_n by automorphisms as well. The action of the automorphism s'_i , $i = 1, \dots, n-1$, involves only the element $\tilde{h}_{i,i+1}$ (as the action of the automorphism s_i) and is given by

$$\begin{aligned} s'_i(x^{i,\alpha}) &= -\frac{1}{\tilde{h}_{i,i+1}} x^{i+1,\alpha}, \quad s'_i(x^{i+1,\alpha}) = \tilde{h}_{i,i+1} x^{i,\alpha}, \quad s'_i(x^{j,\alpha}) = x^{j,\alpha} \text{ for } j \neq i, i+1, \\ s'_i(\bar{\partial}_{i,\alpha}) &= -\bar{\partial}_{i+1,\alpha} \tilde{h}_{i,i+1}, \quad s'_i(\bar{\partial}_{i+1,\alpha}) = \bar{\partial}_{i,\alpha} \frac{1}{\tilde{h}_{i,i+1}}, \quad s'_i(\bar{\partial}_{j,\alpha}) = \bar{\partial}_{j,\alpha} \text{ for } j \neq i, i+1, \\ s_i(\tilde{h}_j) &= \tilde{h}_{s_i(j)}. \end{aligned} \quad (32)$$

4. The diagonal reduction algebra $\mathcal{D}(\mathbf{gl}_n)$ is a $\bar{\mathbf{U}}(\mathbf{h})$ -bimodule with the generators L_i^j , $i, j = 1, \dots, n$. The defining relations of $\mathcal{D}(\mathbf{gl}_n)$ are given by the reflection equation, see [KO5]

$$\hat{R}_{12} L_1 \hat{R}_{12} L_1 - L_1 \hat{R}_{12} L_1 \hat{R}_{12} = \hat{R}_{12} L_1 - L_1 \hat{R}_{12},$$

where $L = \{L_i^j\}_{i,j=1}^n$ is the matrix of generators (we refer to [C, S, RS, KS, IO, IOP, IMO1, IMO2] for various aspects and applications of the reflection equation).

For each N there is a homomorphism ([KO5], Section 4.1)

$$\tau_N: \mathcal{D}(\mathbf{gl}_n) \rightarrow \text{Diff}_{\mathbf{h}}(n, N) \quad \text{defined by} \quad \tau_N(L_i^j) = \sum_{\alpha} x^{j,\alpha} \bar{\partial}_{i,\alpha}.$$

Moreover τ_N is an embedding for $N \geq n$.

The formulas (31) show that the image of τ_N is preserved by the automorphisms s_i .

The element $s_i(\tau_N(L_k^j))$ can be written by the same formula for all N . Since τ_N is injective for $N \geq n$ we conclude that the formulas (31) induce the action of the symmetric group S_n on the diagonal reduction algebra $\mathcal{D}(\mathbf{gl}_n)$ by automorphisms.

The resulting formulas for the action of the automorphisms s_i , $i = 1, \dots, n-1$, on the generators L_j^k , $j, k = 1, \dots, n$, read

$$\begin{aligned}
s_i(L_j^i) &= -L_j^{i+1} \tilde{h}_{i,i+1}, \quad s_i(L_j^{i+1}) = L_j^i \frac{1}{\tilde{h}_{i,i+1}}, \quad j \neq i, i+1, \\
s_i(L_i^j) &= -\frac{1}{\tilde{h}_{i,i+1}} L_{i+1}^j, \quad s_i(L_{i+1}^j) = \tilde{h}_{i,i+1} L_j^i, \quad j \neq i, i+1, \\
s_i(L_i^i) &= L_{i+1}^{i+1}, \quad s_i(L_{i+1}^i) = -L_i^{i+1} (\tilde{h}_{i,i+1} - 1)^2, \\
s_i(L_i^{i+1}) &= -L_{i+1}^i \frac{1}{(\tilde{h}_{i,i+1} + 1)^2}, \quad s_i(L_{i+1}^{i+1}) = L_i^i, \\
s_i(L_j^k) &= L_j^k, \quad k \neq i, i+1 \text{ and } j \neq i, i+1.
\end{aligned}$$

5. The isomorphism μ can be also used to translate the action (10) of the braid group by Zhelobenko operators to the action of the braid group by automorphisms on the ring $\mathfrak{k}[a_1, \dots, a_n] \otimes T^{-1}W_n$. It turns out that this action preserves the subring $T^{-1}W_n$. Moreover, let T_0 be the multiplicative set of T generated by $X^j D_j - X^k D_k + \ell$, $1 \leq j < k \leq n$, $\ell \in \mathbb{Z}$. Then the action of the operators \check{q}_i , $i = 1, \dots, n-1$, preserves the subring $T_0^{-1}W_n$. We present the formulas for the action of the operators \check{q}_i , $i = 1, \dots, n-1$:

$$\begin{aligned}
\check{q}_i(X^i) &= \frac{1}{\mathcal{H}_{i,i+1}} X^{i+1}, \quad \check{q}_i(X^{i+1}) = X^i \mathcal{H}_{i,i+1}, \quad \check{q}_i(X^j) = X^j \text{ for } j \neq i, i+1, \\
\check{q}_i(D_i) &= D_{i+1} \mathcal{H}_{i,i+1}, \quad \check{q}_i(D_{i+1}) = \frac{1}{\mathcal{H}_{i,i+1}} D_i, \quad \check{q}_i(D_j) = D_j \text{ for } j \neq i, i+1.
\end{aligned}$$

6. The isomorphism (23) allows to construct a $2n$ -parametric family of $\text{Diff}_{\mathbf{h}}(n)$ -modules different from the highest weight representations. Let $\vec{\gamma} := \{\gamma_1, \dots, \gamma_n\}$ be a sequence of length n of complex numbers such that $\gamma_i - \gamma_j \notin \mathbb{Z}$ for all $i, j = 1, \dots, n$, $i \neq j$. Let $V_{\vec{\gamma}}$ be the vector space with the basis

$$v_{\vec{j}} := (X^1)^{j_1+\gamma_1} (X^2)^{j_2+\gamma_2} \dots (X^n)^{j_n+\gamma_n}, \quad \text{where } \vec{j} := \{j_1, \dots, j_n\}, \quad j_1, \dots, j_n \in \mathbb{Z}.$$

Under the conditions on $\vec{\gamma}$, $V_{\vec{\gamma}}$ is naturally a $T^{-1}W_n$ -module. Define the action of the elements a_k on the space $V_{\vec{\gamma}}$ by $a_k: v_{\vec{j}} \mapsto A_k v_{\vec{j}}$ where $\vec{A} := \{A_1, \dots, A_n\}$ is another sequence of length n of complex numbers. Then $V_{\vec{\gamma}}$ becomes an $\mathfrak{k}[a_1, \dots, a_n] \otimes T^{-1}W_n$ -module and therefore $\text{Diff}_{\mathbf{h}}(n)$ -module which we denote by $V_{\vec{\gamma}, \vec{A}}$. The central operator c_k acts on $V_{\vec{\gamma}, \vec{A}}$ by scalar multiplication on A_k .

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