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Rings of h-deformed differential operators

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In memory of Petr Kulish

Abstract

We describe the center of the ring $\mathrm{Diff}_{\mathbf{h}}(n)$ of $\mathbf{h}$-deformed differential operators of type A. We establish an isomorphism between certain localizations of $\mathrm{Diff}_{\mathbf{h}}(n)$ and the Weyl algebra W_n extended by n indeterminates.

1 Introduction

The ring $\mathrm{Diff}_{\mathbf{h}}(n)$ of $\mathbf{h}$-deformed differential operators of type A appears in the theory of reduction algebras. A reduction algebra $\mathrm{R}_{\mathfrak{g}}^{\mathcal{A}}$ provides a tool to study decompositions of representations of an associative algebra $\mathcal{A}$ with respect to its subalgebra in the situation when this subalgebra is the universal enveloping algebra of a reductive Lie algebra $\mathfrak{g}$ [M, AST]. We refer to [T, Zh] for the general theory and uses of reduction algebras.

Decompositions of tensor products of representations of a reductive Lie algebra $\mathfrak{g}$ is a particular case of a restriction problem, associated to the diagonal embedding of $\mathrm{U}(\mathfrak{g})$ into $\mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g})$. The corresponding reduction algebra, denoted $\mathcal{D}(\mathfrak{g})$, is called "diagonal reduction algebra" [KO2]. A description of the diagonal reduction algebra $\mathcal{D}(\mathbf{gl}_n)$ in terms of generators and (ordering) defining relations was given in [KO2, KO3].

The diagonal reduction algebra $\mathcal{D}(\mathbf{gl}_n)$ admits an analogue of the "oscillator realization", in the rings $\mathrm{Diff}_{\mathbf{h}}(n, N)$, $N = 1, 2, 3, \ldots$, of $\mathbf{h}$-deformed differential operators, see [KO5]. The ring $\mathrm{Diff}_{\mathbf{h}}(n, N)$ can be obtained by the reduction of the ring of differential operators in nN variables (that is, of the Weyl algebra $\mathrm{W}_{nN} = \mathrm{W}_n^{\otimes N}$) with respect to the natural action of $\mathbf{gl}_n$. Similarly to the ring of q-differential operators [WZ], the algebra $\mathrm{Diff}_{\mathbf{h}}(n, N)$ can be described in the R-matrix formalism. The R-matrix, needed here, is a solution of the so-called dynamical Yang–Baxter equation (we refer to [F, GV, ES] for different aspects of the dynamical Yang–Baxter equation and its solutions).

The ring $\mathrm{Diff}_{\mathbf{h}}(n, N)$ is formed by N copies of the ring $\mathrm{Diff}_{\mathbf{h}}(n) = \mathrm{Diff}_{\mathbf{h}}(n, 1)$. The aim of the present article is to investigate the structure of the ring $\mathrm{Diff}_{\mathbf{h}}(n)$. Our first result is the description of the center of $\mathrm{Diff}_{\mathbf{h}}(n)$: it is a ring of polynomials in n generators.

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As follows from the results of [KO4], the ring $\mathrm{Diff}_{\mathbf{h}}(n)$ is a noetherian Ore domain. It is therefore natural to investigate its field of fractions and test the validity of the Gelfand–Kirillov-like conjecture [GK]. For the ring of q-differential operators, the isomorphism (up to a certain localization and completion) with the Weyl algebra was given in [O]. The second result of the present article consists in a construction of an isomorphism between certain localizations of $\mathrm{Diff}_{\mathbf{h}}(n)$ and the Weyl algebra W_n extended by n indeterminates. In particular, our isomorphism implies the isomorphism of the corresponding fields of fractions.

According to the general theory of reduction algebras, the ring $\mathrm{Diff}_{\mathbf{h}}(n)$ admits the action of Zhelobenko operators [Zh, KO1] by automorphisms. The Zhelobenko operators generate the action of the braid group B_n. However the Weyl algebra W_n admits the action of the symmetric group S_n by automorphisms. As a by-product of our construction we define the action of the symmetric group by automorphisms on the ring $\mathrm{Diff}_{\mathbf{h}}(n)$. Moreover these formulas can be generalized to produce the action of the symmetric group by automorphisms on the rings $\mathrm{Diff}_{\mathbf{h}}(n, N)$ for any N, and on the diagonal reduction algebra $\mathcal{D}(\mathbf{gl}_n)$. We conjecture that the general reduction algebra $\mathrm{R}_{\mathfrak{g}}^{\mathcal{A}}$ admits an action, by automorphisms, of the Weyl group of $\mathfrak{g}$.

Section 2 contains the definition of the ring of $\mathbf{h}$-deformed differential operators and some of their properties used in the sequel. In Section 3 we present a family of n quadratic central elements. We then describe an n-parametric family of "highest weight" representations of $\mathrm{Diff}_{\mathbf{h}}(n)$ and calculate values of the quadratic central elements in these representations. In Section 4 we introduce the necessary localizations of $\mathrm{Diff}_{\mathbf{h}}(n)$ and of the Weyl algebra, check the Ore conditions and establish the above mentioned isomorphism of the localized rings. In Section 5 we prove the completeness of the family of central elements constructed in Section 3. Then we describe the action of the symmetric group on $\mathrm{Diff}_{\mathbf{h}}(n, N)$ and $\mathcal{D}(\mathbf{gl}_n)$ as well as the action of the braid group, generated by Zhelobenko operators, on a localization of the Weyl algebra. Also, we present a $2n$-parametric family of representations of the algebra $\mathrm{Diff}_{\mathbf{h}}(n)$ implied by our construction.

Notation

Throughout the paper, $\mathfrak{k}$ denotes the ground ring of characteristic zero.

The symbol s_i stands for the transposition $(i, i+1)$.

We denote by $\mathrm{U}(\mathbf{h})$ the free commutative $\mathfrak{k}$-algebra in generators $\tilde{h}_i$, $i = 1, \ldots, n$. Set $\tilde{h}_{ij} = \tilde{h}_i - \tilde{h}_j \in \mathbf{h}$. We define $\bar{\mathrm{U}}(\mathbf{h})$ to be the ring of fractions of $\mathrm{U}(\mathbf{h})$ with respect to the multiplicative set of denominators, generated by the elements $(h_{ij} + k)^{-1}$, $k \in \mathbb{Z}$. Let

$$\psi_i := \prod_{k:k>i} \tilde{h}_{ik} \ , \psi_i' := \prod_{k:k<i} \tilde{h}_{ik} \ \text{ and } \ \chi_i := \psi_i \psi_i' \ , \ i = 1, \ldots, n \ . \tag{1}$$

Let ε_j, $j = 1, \ldots, n$, be the elementary translations of the generators of $\mathrm{U}(\mathbf{h})$, $\varepsilon_j : \tilde{h}_i \mapsto \tilde{h}_i + \delta_i^j$. For an element $p \in \bar{\mathrm{U}}(\mathbf{h})$ we denote $\varepsilon_j(p)$ by $p[\varepsilon_j]$.

2 Definition and properties of rings of h-deformed differential operators

Let $\hat{\mathrm{R}} = \{\hat{\mathrm{R}}_{ij}^{kl}\}_{i,j,k,l=1}^{n}$ be a matrix of elements of $\bar{\mathrm{U}}(\mathbf{h})$, with nonzero entries

$$\hat{\mathrm{R}}_{ij}^{ij} = \frac{1}{\tilde{h}_{ij}} \ , \quad i \neq j \ , \qquad \text{and} \qquad \hat{\mathrm{R}}_{ji}^{ij} = \begin{cases} \dfrac{\tilde{h}_{ij}^2 - 1}{\tilde{h}_{ij}^2} \ , & i < j, \\ 1 \ , & i \geq j \ . \end{cases} \tag{2}$$

The matrix $\hat{\mathrm{R}}$ is the standard solution of the dynamical Yang–Baxter equation

$$\sum_{a,b,u} \hat{\mathrm{R}}_{ab}^{ij} \hat{\mathrm{R}}_{ur}^{bk}[-\varepsilon_a] \hat{\mathrm{R}}_{mn}^{au} = \sum_{a,b,u} \hat{\mathrm{R}}_{ab}^{jk}[-\varepsilon_i] \hat{\mathrm{R}}_{mu}^{ia} \hat{\mathrm{R}}_{nr}^{ub}[-\varepsilon_m] \tag{3}$$

of type A.

The ring $\mathrm{Diff}_{\mathbf{h}}(n)$ of $\mathbf{h}$-deformed differential operators of type A is a $\bar{\mathrm{U}}(\mathbf{h})$-bimodule with the generators x^j and $\bar{\partial}_j$, $j = 1, \ldots, n$. The ring $\mathrm{Diff}_{\mathbf{h}}(n)$ is free as a one-sided $\bar{\mathrm{U}}(\mathbf{h})$-module; the left and right $\bar{\mathrm{U}}(\mathbf{h})$-module structures are related by

$$\tilde{h}_i x^j = x^j(\tilde{h}_i + \delta_i^j) \ , \ \tilde{h}_i \bar{\partial}_j = \bar{\partial}_j(\tilde{h}_i - \delta_i^j) \ . \tag{4}$$

The defining relations for the generators x^j and $\bar{\partial}_j$, $j = 1, \ldots, n$, read (see [KO5])

$$x^i x^j = \sum_{k,l} \hat{\mathrm{R}}_{kl}^{ij} x^k x^l \ , \qquad \bar{\partial}_i \bar{\partial}_j = \sum_{k,l} \hat{\mathrm{R}}_{ji}^{lk} \bar{\partial}_k \bar{\partial}_l \ , \qquad x^i \bar{\partial}_j = \sum_{k,l} \hat{\mathrm{R}}_{lj}^{ki}[\varepsilon_k] \bar{\partial}_k x^l - \delta_j^i \ , \tag{5}$$

or, in components,

$$x^i x^j = \frac{\tilde{h}_{ij} + 1}{\tilde{h}_{ij}} x^j x^i \ , \ i < j \ , \tag{6}$$

$$\bar{\partial}_i \bar{\partial}_j = \frac{\tilde{h}_{ij} - 1}{\tilde{h}_{ij}} \bar{\partial}_j \bar{\partial}_i \ , \ i < j \ , \tag{7}$$

$$x^i \bar{\partial}_j = \begin{cases} \bar{\partial}_j x^i & , \ i < j \ , \\ \dfrac{\tilde{h}_{ij}(\tilde{h}_{ij} - 2)}{(\tilde{h}_{ij} - 1)^2} \bar{\partial}_j x^i & , \ i > j \ , \end{cases} \tag{8}$$

$$x^i \bar{\partial}_i = \sum_j \frac{1}{1 - \tilde{h}_{ij}} \bar{\partial}_j x^j - 1 \ . \tag{9}$$

The ring $\mathrm{Diff}_{\mathbf{h}}(n)$ admits Zhelobenko automorphisms $\check{\mathrm{q}}_i$, $i=1,\dots,n-1$, given by (see [KO5])

$$\begin{aligned}
&\check{\mathrm{q}}_i(x^i) = -x^{i+1}\frac{\tilde{h}_{i,i+1}}{\tilde{h}_{i,i+1}-1}, \qquad \check{\mathrm{q}}_i(x^{i+1}) = x^i, \quad \check{\mathrm{q}}_i(x^j) = x^j, \quad j\neq i,i+1\ ,\\
&\check{\mathrm{q}}_i(\bar{\partial}_i) = -\frac{\tilde{h}_{i,i+1}-1}{\tilde{h}_{i,i+1}}\bar{\partial}_{i+1}, \qquad \check{\mathrm{q}}_i(\bar{\partial}_{i+1}) = \bar{\partial}_i, \quad \check{\mathrm{q}}_i(\bar{\partial}_j) = \bar{\partial}_j, \quad j\neq i,i+1\ ,\\
&\check{\mathrm{q}}_i(\tilde{h}_j) = \tilde{h}_{s_i(j)}\ .
\end{aligned} \tag{10}$$

The operators $\check{\mathrm{q}}_i$, $i=1,\dots,n-1$, generate the action of the braid group, see [Zh, KO1].

The ring $\mathrm{Diff}_{\mathbf{h}}(n)$ admits an involutive anti-automorphism ϵ, defined by

$$\epsilon(\tilde{h}_i)=\tilde{h}_i, \epsilon(\bar{\partial}_i)=\varphi_i x^i, \epsilon(x^i)=\bar{\partial}_i\varphi_i^{-1}, \text{ where } \varphi_i := \frac{\psi_i}{\psi_i[-\varepsilon_i]} = \prod_{k:k>i}\frac{\tilde{h}_{ik}}{\tilde{h}_{ik}-1},\ i=1,\dots,n, \tag{11}$$

The proof reduces to the formula

$$\frac{\varphi_i[-\epsilon_j]}{\varphi_i} = \frac{\tilde{h}_{ij}^2-1}{\tilde{h}_{ij}^2} \quad \text{for} \quad 1\le i<j\le n\ .$$

The construction of central elements in the next Section uses the elements

$$\Gamma_i := \bar{\partial}_i x^i \quad \text{for} \quad i=1,\dots,n\ . \tag{12}$$

We collect some properties of these elements.

Lemma 1. We have

(i) $\Gamma_i x^j = \dfrac{\tilde{h}_{ij}+1}{\tilde{h}_{ij}}x^j\Gamma_i$ and $\Gamma_i\bar{\partial}_j = \dfrac{\tilde{h}_{ij}-1}{\tilde{h}_{ij}}\bar{\partial}_j\Gamma_i$ for $i\neq j$, $i,j=1,\dots,n$.

(ii) $\check{\mathrm{q}}_i(\Gamma_j)=\Gamma_{s_i(j)}$ for $i=1,\dots,n-1$ and $j=1,\dots,n$.

(iii) $\Gamma_i\Gamma_j=\Gamma_j\Gamma_i$ for $i,j=1,\dots,n$.

Proof. Formulas (i) and (ii) are obtained by a direct calculation; (iii) follows from (i). □

We will use the following technical Lemma.

Lemma 2. Let $\mathfrak{A}$ be an associative algebra. Assume that elements $\breve{h}_i$, $\breve{Z}_i, \breve{Z}_i\in\mathfrak{A}$, $i=1,\dots,n$, satisfy

$$\breve{h}_i\breve{h}_j=\breve{h}_j\breve{h}_i\ ,\ \breve{h}_i\breve{Z}^j=\breve{Z}^j(\breve{h}_i+\delta_i^j)\ ,\ \breve{h}_i\breve{Z}_j=\breve{Z}_j(\breve{h}_i-\delta_i^j)\ ,\ i,j=1,\dots,n\ .$$

Let $\breve{h}_{ij} := \breve{h}_i - \breve{h}_j$ and

$$\breve{\psi}_i := \prod_{k:k>i} \breve{h}_{ik} \ , \breve{\psi}'_i := \prod_{k:k<i} \breve{h}_{ik} \ , \ i = 1, \dots, n \ .$$

Assume that the elements $\breve{h}_{ij}$ are invertible. Then

(i) the elements $\breve{Z}^i$ satisfy

$$\breve{Z}^i \breve{Z}^j = \frac{\breve{h}_{ij} + 1}{\breve{h}_{ij}} \breve{Z}^j \breve{Z}^i \ \text{ for } \ i < j \ , \ i, j = 1, \dots, n$$

if and only if any of the two families $\{\breve{Z}^{\circ i}\}_{i=1}^n$ or $\{\breve{Z}'^{\circ i}\}_{i=1}^n$ where

$$\breve{Z}^{\circ i} := \psi_i \breve{Z}^i \ , \ \breve{Z}'^{\circ i} := \breve{Z}^i \psi'_i \tag{13}$$

is commutative;

(ii) the elements $\breve{Z}_i$ satisfy

$$\breve{Z}_i \breve{Z}_j = \frac{\breve{h}_{ij} - 1}{\breve{h}_{ij}} \breve{Z}_j \breve{Z}_i \ \text{ for } \ i < j \ , \ i, j = 1, \dots, n$$

if and only if any of the two families $\{\breve{Z}_i^{\circ}\}_{i=1}^n$ or $\{\breve{Z}_i'^{\circ}\}_{i=1}^n$ where

$$\breve{Z}_i^{\circ} := \psi_i \breve{Z}_i \ , \ \breve{Z}_i'^{\circ} := \breve{Z}_i \psi'_i \tag{14}$$

is commutative.

Proof. A direct calculation. □

3 Quadratic central elements

Let $e_k := \sum_{i_1 < \dots < i_k} \tilde{h}_{i_1} \dots \tilde{h}_{i_k}$, $k = 0, \dots, n$, be the elementary symmetric functions in the variables $\tilde{h}_1, \dots, \tilde{h}_n$. Set

$$c_k := \sum_j \frac{\partial e_k}{\partial \tilde{h}_j} \Gamma_j - e_k \ ,$$

where Γ_j, $j = 1, \dots, n$, are the elements defined in (12).

It follows from Lemma 1 that $\check{\mathrm{q}}_j(c_k) = c_k$ for all $j = 1, \dots, n-1$ and $k = 1, \dots, n$.

Proposition 3. The elements c_k, $k = 1, \dots, n$, belong to the center of the ring $\mathrm{Diff}_{\mathbf{h}}(n)$.

Proof. We shall use the generating functions

$$e(t) := \sum_{k=0}^{n} e_k t^k = \prod_i (1+\tilde{h}_i t)$$

and

$$c(t) := \sum_{k=1}^{n} c_k t^k = u(t)e(t) + 1 \quad \text{with} \quad u(t) := t \sum_i \frac{1}{1+\tilde{h}_i t}\Gamma_i - 1 \ .$$

The expression $u(t)$ is introduced for convenience; the denominator $1+\tilde{h}_i t$, which is not defined in the ring $\mathrm{Diff}_{\mathbf{h}}(n)$, vanishes in the combination $u(t)e(t)$.

We shall check that the polynomial $c(t)$ is central. We have

$$x^j e(t) = \frac{1+(\tilde{h}_j-1)t}{1+\tilde{h}_j t} e(t) x^j \ . \tag{15}$$

Next, it follows from Lemma 1 that

$$x^j u(t) = \Bigg(\sum_{k:k\neq j} \frac{t}{1+\tilde{h}_k t} \frac{\tilde{h}_{kj}}{\tilde{h}_{kj}+1} \Gamma_k + \frac{t}{1+(\tilde{h}_j-1)t} \Big(\sum_k \frac{1}{1-\tilde{h}_{jk}} \Gamma_k - 1 \Big) - 1 \Bigg) x^j \ .$$

The coefficient of Γ_k in this expression is equal to $\frac{1+\tilde{h}_j}{1+(\tilde{h}_j-1)t}$ for both $k \neq j$ and $k = j$. Therefore,

$$x^j u(t) = \frac{1+\tilde{h}_j t}{1+(\tilde{h}_j-1)t} u(t) x^j \ . \tag{16}$$

Combining (15) and (16) we find that $c(t)$ commutes with x^j, $j = 1, \ldots, n$. For $\bar{\partial}_j$ one can either make a parallel calculation or use the anti-automorphism (11). □

Lemma 4. (i) The matrix V, defined by $V_j^k := \frac{\partial e_j}{\partial \tilde{h}_k}$, is invertible. Its inverse is

$$(V^{-1})_i^j = \frac{(-1)^{j-1}\tilde{h}_i^{n-j}}{\chi_i} \ ,$$

where the elements χ_i are defined in (1).

(ii) We have

$$\chi_j \Gamma_j = \tilde{h}_j^n - \tilde{h}_j^n c(-\tilde{h}_j^{-1}) \ . \tag{17}$$

Proof. (i) See, e.g. [OP], Proposition 4.

(ii) Rewrite the equality $c_k = \sum_j V_k^j \Gamma_j - e_k$ in the form

$$\Gamma_j = \sum_k (V^{-1})_j^k (c_k + e_k) = \frac{1}{\chi_j} \sum_k (-1)^{k-1} \tilde{h}_j^{n-k} (c_k + e_k) = -\frac{\tilde{h}_j^n}{\chi_j} (c(-\tilde{h}_j^{-1}) + e(-\tilde{h}_j^{-1}) - 1) \ .$$

Since $e(-\tilde{h}_j^{-1}) = 0$, we obtain (17). □

Highest weight representations. The ring $\mathrm{Diff}_{\mathbf{h}}(n)$ admits an n-parametric family of "highest weight" representations. To define them, let $\mathfrak{D}_n$ be an $\bar{\mathrm{U}}(\mathbf{h})$-subring of $\mathrm{Diff}_{\mathbf{h}}(n)$ generated by $\{\bar{\partial}_i\}_{i=1}^n$. Let $\vec{\lambda} := \{\lambda_1, \dots, \lambda_n\}$ be a sequence of length n of complex numbers such that $\lambda_i - \lambda_j \notin \mathbb{Z}$ for all $i, j = 1, \dots, n$, $i \neq j$. Denote by $M_{\vec{\lambda}}$ the one-dimensional $\mathfrak{k}$-vector space with the basis vector $|\,\rangle$. Under the specified conditions on $\vec{\lambda}$ the formulas

$$\tilde{h}_i \colon |\,\rangle \mapsto \lambda_i |\,\rangle \ , \ \bar{\partial}_i \colon |\,\rangle \mapsto 0 \ , \ i = 1, \dots, n \ ,$$

define the $\mathfrak{D}_n$-module structure on $M_{\vec{\lambda}}$. We shall call the induced representation $\mathrm{Ind}_{\mathfrak{D}_n}^{\mathrm{Diff}_{\mathbf{h}}(n)} M_{\vec{\lambda}}$ the "highest weight representation" of highest weight $\vec{\lambda}$.

Lemma 5. The central operator c_k, $k = 1, \dots, n$, acts on the module $\mathrm{Ind}_{\mathfrak{D}_n}^{\mathrm{Diff}_{\mathbf{h}}(n)} M_{\vec{\lambda}}$ by scalar multiplication on $-e_k|_{\tilde{h}_i \mapsto \lambda_i - 1}$, the evaluation of the symmetric function $-e_k$ on the shifted vector $\{\lambda_1 - 1, \dots, \lambda_n - 1\}$.

Proof. It is sufficient to calculate the value of c_k on the highest weight vector $|\,\rangle$. In terms of generating functions we have to check that

$$e(t)u(t) \colon |\,\rangle \mapsto -\prod_i (1 + (\lambda_i - 1)t) \ |\,\rangle \ .$$

It follows from [KO5], section 3.3, that

$$\Gamma_j \ |\,\rangle = \frac{\chi_j[\varepsilon_j]}{\chi_j} \ |\,\rangle \ , \ j = 1, \dots, n \ .$$

Therefore we have to check that

$$e(t) \left(t \sum_i \frac{1}{1 + \tilde{h}_i t} \frac{\chi_j[\varepsilon_j]}{\chi_j} - 1 \right) : |\,\rangle \mapsto -\prod_i (1 + (\lambda_i - 1)t) \ |\,\rangle \ . \tag{18}$$

We use another formula from [KO5] (Note 3 after the proof of Proposition 4.3 in Section 4.2)

$$\prod_l \frac{\tilde{h}_0 - \tilde{h}_l - 1}{\tilde{h}_0 - \tilde{h}_l} + \sum_j \frac{1}{\tilde{h}_0 - \tilde{h}_j} \frac{\chi_j[-\varepsilon_j]}{\chi_j} = 1 \ ,$$

where $\tilde{h}_0$ is an indeterminate. After the replacements $\tilde{h}_0 \to t^{-1}$ and $\tilde{h}_j \to -\tilde{h}_j$, $j = 1, \dots, n$, this formula becomes

$$\frac{e(t)[-\varepsilon]}{e(t)} + t \sum_j \frac{1}{1 + \tilde{h}_j t} \frac{\chi_j[\varepsilon_j]}{\chi_j} = 1 \ ,$$

where $\varepsilon = \varepsilon_1 + \dots + \varepsilon_n$, which implies (18). $\square$

4 Isomorphism between rings of fractions

It follows from the results of [KO4] that the ring $\mathrm{Diff}_{\mathbf{h}}(n)$ has no zero divisors. Let S_x be the multiplicative set generated by x^j, $j = 1, \dots, n$. The set S_x satisfies both left and right Ore conditions (see, e.g., [A] for definitions): say, for the left Ore conditions we have to check only that for any x^k and a monomial $m = \bar{\partial}_{i_1} \dots \bar{\partial}_{i_A} x^{j_1} \dots x^{j_B}$ there exist $\tilde{s} \in \mathrm{S}_x$ and $\tilde{m} \in \mathrm{Diff}_{\mathbf{h}}(n)$ such that $\tilde{s}m = \tilde{m}x^k$. The structure of the commutation relations (6-9) shows that one can choose $\tilde{s} = (x^k)^\nu$ with sufficiently large ν. Denote by $\mathrm{S}_x^{-1}\mathrm{Diff}_{\mathbf{h}}(n)$ the localization of the ring $\mathrm{Diff}_{\mathbf{h}}(n)$ with respect to the set S_x.

Let W_n be the Weyl algebra, the algebra with the generators X^j, D_j, $j = 1, \dots, n$, and the defining relations

$$X^i X^j = X^j X^i \ , \ D_i D_j = D_j D_i \ , \ D_i X^j = \delta_i^j + X^j D_i \ , \quad i, j = 1, \dots, n \ .$$

Let T be the multiplicative set generated by $X^j D_j - X^k D_k + \ell$, $1 \le j < k \le n$, $\ell \in \mathbb{Z}$, and X^j, $j = 1, \dots, n$. The set T satisfies left and right Ore conditions (see [KO4], Appendix). Denote by $\mathrm{T}^{-1}\mathrm{W}_n$ the localization of W_n relative to the set T.

Let $a_1, \dots, a_n$ be a family of commuting variables. We shall use the following notation:

$$\mathcal{H}_j := D_j X^j \ , \ \mathcal{H}_{jk} := \mathcal{H}_j - \mathcal{H}_k \ ,$$

$$\Psi'_j := \prod_{k:k<j} \mathcal{H}_{jk} \ , \ \Psi_j := \prod_{k:k>j} \mathcal{H}_{jk} \ ,$$

$$\mathbf{C}(t) := \sum_{k=1}^{n} a_k t^k \ , \ \Upsilon_i := \mathcal{H}_i^n \left(1 - \mathbf{C}(-\mathcal{H}_i^{-1})\right) \ .$$

The polynomial $\mathbf{C}$ has degree n so the element Υ_i is a polynomial in $\mathcal{H}_i$, $i = 1, \dots, n$.

Theorem 6. The ring $\mathrm{S}_x^{-1}\mathrm{Diff}_{\mathbf{h}}(n)$ is isomorphic to the ring $\mathfrak{k}[a_1, \dots, a_n] \otimes \mathrm{T}^{-1}\mathrm{W}_n$.

Proof. The knowledge of the central elements (Proposition 3) allows to exhibit a generating set of the ring $\mathrm{S}_x^{-1}\mathrm{Diff}_{\mathbf{h}}(n)$ in which the required isomorphism is quite transparent.

In the localized ring $\mathrm{S}_x^{-1}\mathrm{Diff}_{\mathbf{h}}(n)$ we can use the set of generators $\{\tilde{h}_i, x^i, \Gamma_i\}_{i=1}^n$ instead of $\{\tilde{h}_i, x^i, \bar{\partial}_i\}_{i=1}^n$. By Lemma 4 (ii), $\{\tilde{h}_i, x^i, c_i\}_{i=1}^n$ is also a generating set. Finally, $\mathfrak{B}_{\mathrm{D}} := \{\tilde{h}_i, x^{\circ i}, c_i\}_{i=1}^n$, where $x'^{\circ i} := x^i \psi'_i$, $i = 1, \dots, n$, is a generating set of the localized ring $\mathrm{S}_x^{-1}\mathrm{Diff}_{\mathbf{h}}(n)$ as well. It follows from Lemma 2 that the family $\{x'^{\circ i}\}_{i=1}^n$ is commutative. The complete set of the defining relations for the generators from the set $\mathfrak{B}_{\mathrm{D}}$ reads

$$\begin{aligned} &\tilde{h}_i \tilde{h}_j = \tilde{h}_j \tilde{h}_i \ , \ \tilde{h}_i x'^{\circ j} = x'^{\circ j}(\tilde{h}_i + \delta_i^j) \ , \ x'^{\circ i} x'^{\circ j} = x'^{\circ j} x'^{\circ i} \ , \ i, j = 1, \dots, n \ , \\ &c_i \text{ are central} \ , \ i = 1, \dots, n \ . \end{aligned} \tag{19}$$

In the localized ring $\mathfrak{k}[a_1,\dots,a_n]\otimes \mathrm{T}^{-1}\mathrm{W}_n$ we can pass to the set of generators $\mathfrak{B}_{\mathrm{W}} := \{\mathcal{H}_i, X^i, a_i\}_{i=1}^n$ with the defining relations

$$\begin{aligned} &\mathcal{H}_i\mathcal{H}_j = \mathcal{H}_j\mathcal{H}_i \ , \ \mathcal{H}_i X^j = X^j(\mathcal{H}_i + \delta_i^j) \ , \ X^iX^j = X^jX^i \ , \ i,j = 1,\dots,n \ , \\ &a_i \text{ are central} \ , \ i = 1,\dots,n \ . \end{aligned} \tag{20}$$

The comparison of (19) and (20) shows that we have the isomorphism

$$\mu\colon \mathfrak{k}[a_1,\dots,a_n]\otimes \mathrm{T}^{-1}\mathrm{W}_n \to \mathrm{S}_x^{-1}\mathrm{Diff}_{\mathbf{h}}(n)$$

given on our generating sets $\mathfrak{B}_{\mathrm{D}}$ and $\mathfrak{B}_{\mathrm{W}}$ by

$$\mu\colon X^i \mapsto x'^{\circ i} \ , \ \mathcal{H}_i \mapsto \tilde{h}_i \ , \ a_i \mapsto c_i \ , \ i = 1,\dots,n \ . \tag{21}$$

The proof is completed. □

We shall now rewrite the formulas for the isomorphism μ in terms of the original generators of the rings $\mathrm{S}_x^{-1}\mathrm{Diff}_{\mathbf{h}}(n)$ and $\mathfrak{k}[a_1,\dots,a_n]\otimes \mathrm{T}^{-1}\mathrm{W}_n$.

Lemma 7. We have

$$\mu\colon X^i \mapsto x^i\psi_i' \ , \ D_i \mapsto (\psi_i')^{-1}\tilde{h}_i(x^i)^{-1} \ , \ a_i \mapsto c_i \ , \ i = 1,\dots,n \ . \tag{22}$$

and

$$\mu^{-1}\colon \tilde{h}_i \mapsto \mathcal{H}_i \ , \ x^i \mapsto X^i\frac{1}{\Psi_i'} \ , \ \bar{\partial}_i \mapsto \frac{\Upsilon_i}{\Psi_i}(X^i)^{-1} \ , \ i = 1,\dots,n \ . \tag{23}$$

Proof. We shall comment only on the last formula in (23). Lemma 4 part (ii) implies that $\mu^{-1}(\chi_i\Gamma_i) = \Upsilon_i$ and the formula for $\mu^{-1}(\bar{\partial}_i)$ follows since $\bar{\partial}_i = \Gamma_i(x^i)^{-1}$. □

5 Comments

We shall now establish several corollaries of our construction.

1. We can now give the description of the center of the ring $\mathrm{Diff}_{\mathbf{h}}(n)$.

Lemma 8. The center of the ring $\mathrm{Diff}_{\mathbf{h}}(n)$ is formed by polynomials in the elements $\{c_i\}_{i=1}^n$.

Proof. This is a direct consequence of the defining relations (19) for the generating set $\mathfrak{B}_{\mathrm{D}}$. Indeed, any central element ζ must have $\mathbf{h}$-weight zero, so it belongs to a subring generated by c_i and $\tilde{h}_i$, $i = 1,\dots,n$. Interpret ζ as a rational function in $\tilde{h}_i$, $i = 1,\dots,n$. Since ζ commutes with $x'^{\circ i}$, $i = 1,\dots,n$, this rational function is periodic, with period 1, with respect to any $\tilde{h}_i$, $i = 1,\dots,n$. Therefore ζ belongs to the subring generated by c_i, $i = 1,\dots,n$, as stated.

Another proof consists in using the isomorphism μ and the triviality of the center of the Weyl algebra. □

2. The symmetric group S_n acts by automorphisms on the algebra W_n,

$$\pi(X^j) = X^{\pi(j)} \ , \ \pi(D_j) = D_{\pi(j)} \ \text{ for } \ \pi \in \mathrm{S}_n \ .$$

The isomorphism μ translates this action to the action of S_n on the ring $\mathrm{S}_x^{-1}\mathrm{Diff}_{\mathbf{h}}(n)$. It turns out that the subring $\mathrm{Diff}_{\mathbf{h}}(n)$ is preserved by this action. We present the formulas for the action of the generators s_i of S_n.

$$\begin{aligned} &s_i(x^i) = -x^{i+1}\tilde{h}_{i,i+1} \ , \ s_i(x^{i+1}) = x^i\frac{1}{\tilde{h}_{i,i+1}} \ , \ s_i(x^j) = x^j \text{ for } j \neq i, i+1 \ , \\ &s_i(\bar{\partial}_i) = -\frac{1}{\tilde{h}_{i,i+1}}\bar{\partial}_{i+1} \ , \ s_i(\bar{\partial}_{i+1}) = \tilde{h}_{i,i+1}\bar{\partial}_i \ , \ s_i(\bar{\partial}_j) = \bar{\partial}_j \text{ for } j \neq i, i+1 \ , \\ &s_i(\tilde{h}_j) = \tilde{h}_{s_i(j)} \ . \end{aligned} \tag{24}$$

3a. For the R-matrix description of the diagonal reduction algebra $\mathcal{D}(\mathbf{gl}_n)$ in [KO5] we used the ring $\mathrm{Diff}_{\mathbf{h}}(n, N)$ formed by N copies of the ring $\mathrm{Diff}_{\mathbf{h}}(n)$. We do not know an analogue of the isomorphism μ for the ring $\mathrm{Diff}_{\mathbf{h}}(n, N)$. However a straightforward analogue of the formulas (24) provides an action of S_n by automorphisms on the ring $\mathrm{Diff}_{\mathbf{h}}(n, N)$.

We recall that the ring $\mathrm{Diff}_{\mathbf{h}}(n, N)$ is a $\bar{\mathrm{U}}(\mathbf{h})$-bimodule with the generators $x^{j,\alpha}$ and $\bar{\partial}_{j,\alpha}$, $j = 1, \ldots, n$, $\alpha = 1, \ldots, N$. The ring $\mathrm{Diff}_{\mathbf{h}}(n, N)$ is free as a one-sided $\bar{\mathrm{U}}(\mathbf{h})$-module; the left and right $\bar{\mathrm{U}}(\mathbf{h})$-module structures are related by

$$\tilde{h}_i x^{j,\alpha} = x^{j,\alpha}(\tilde{h}_i + \delta_i^j) \ , \ \tilde{h}_i\bar{\partial}_{j,\alpha} = \bar{\partial}_{j,\alpha}(\tilde{h}_i - \delta_i^j) \ . \tag{25}$$

The defining relations for the generators $x^{j,\alpha}$ and $\bar{\partial}_{j,\alpha}$, $j = 1, \ldots, n$, $\alpha = 1, \ldots, N$, read

$$x^{i,\alpha}x^{j,\beta} = \sum_{k,l} \hat{\mathrm{R}}_{kl}^{ij} x^{k,\beta}x^{l,\alpha} \ , \ \bar{\partial}_{i,\alpha}\bar{\partial}_{j,\beta} = \sum_{k,l} \hat{\mathrm{R}}_{ji}^{lk} \bar{\partial}_{k,\beta}\bar{\partial}_{l,\alpha} \ , \ x^{i,\alpha}\bar{\partial}_{j,\beta} = \sum_{k,l} \hat{\mathrm{R}}_{lj}^{ki}[\varepsilon_k]\bar{\partial}_{k,\beta}x^{l,\alpha} - \delta_\beta^\alpha\delta_j^i \ , \tag{26}$$

or, in components,

$$x^{i,\alpha}x^{j,\beta} = \frac{1}{\tilde{h}_{ij}}x^{i,\beta}x^{j,\alpha} + \frac{\tilde{h}_{ij}^2 - 1}{\tilde{h}_{ij}^2}x^{j,\beta}x^{i,\alpha} \ , \ x^{j,\alpha}x^{i,\beta} = -\frac{1}{\tilde{h}_{ij}}x^{j,\beta}x^{i,\alpha} + x^{i,\beta}x^{j,\alpha} \ , \ 1 \leq i < j \leq n \ , \tag{27}$$

$$\bar{\partial}_{i,\alpha}\bar{\partial}_{j,\beta} = -\frac{1}{\tilde{h}_{ij}}\bar{\partial}_{i,\beta}\bar{\partial}_{j,\alpha} + \frac{\tilde{h}_{ij}^2 - 1}{\tilde{h}_{ij}^2}\bar{\partial}_{j,\beta}\bar{\partial}_{i,\alpha} \ , \ \bar{\partial}_{j,\alpha}\bar{\partial}_{i,\beta} = \frac{1}{\tilde{h}_{ij}}\bar{\partial}_{j,\beta}\bar{\partial}_{i,\alpha} + \bar{\partial}_{i,\beta}\bar{\partial}_{j,\alpha} \ , \ 1 \leq i < j \leq n \ , \tag{28}$$

$$x^{i,\alpha}\bar{\partial}_{j,\beta} = \bar{\partial}_{j,\beta}x^{i,\alpha} \ , \ x^{j,\alpha}\bar{\partial}_{i,\beta} = \frac{\tilde{h}_{ij}(\tilde{h}_{ij} + 2)}{(\tilde{h}_{ij} + 1)^2}\bar{\partial}_{i,\beta}x^{j,\alpha} \ , \ 1 \leq i < j \leq n \ , \tag{29}$$

$$x^{i,\alpha}\bar{\partial}_{i,\beta} = \sum_{k=1}^{n} \frac{1}{1 - \tilde{h}_{ik}}\bar{\partial}_{k,\beta}x^{k,\alpha} - \delta_\beta^\alpha \ , \ 1 \leq i \leq n. \tag{30}$$

Lemma 9. The maps s_i, $i = 1, \dots, n-1$, defined on the generators of $\mathrm{Diff}_{\mathbf{h}}(n, N)$ by

$$\begin{aligned} & s_i(x^{i,\alpha}) = -x^{i+1,\alpha}\tilde{h}_{i,i+1} \ , \ s_i(x^{i+1,\alpha}) = x^{i,\alpha}\frac{1}{\tilde{h}_{i,i+1}} \ , \ s_i(x^{j,\alpha}) = x^{j,\alpha} \text{ for } j \neq i, i+1 \ , \\ & s_i(\bar{\partial}_{i,\alpha}) = -\frac{1}{\tilde{h}_{i,i+1}}\bar{\partial}_{i+1,\alpha} \ , \ s_i(\bar{\partial}_{i+1,\alpha}) = \tilde{h}_{i,i+1}\bar{\partial}_{i,\alpha} \ , \ s_i(\bar{\partial}_{j,\alpha}) = \bar{\partial}_{j,\alpha} \text{ for } j \neq i, i+1 \ , \\ & s_i(\tilde{h}_j) = \tilde{h}_{s_i(j)} \ , \end{aligned} \tag{31}$$

extend to automorphisms of the ring $\mathrm{Diff}_{\mathbf{h}}(n, N)$. Moreover, these automorphisms satisfy the Artin relations and therefore give the action of the symmetric group S_n by automorphisms.

Proof. After the formulas (31) are written down, the verification is a direct calculation. □

3b. The operators $s'_i := \epsilon s_i \epsilon$, where ϵ is the anti-automorphism (11), generate the action of the symmetric group S_n by automorphisms as well. The action of the automorphism s'_i, $i = 1, \dots, n-1$, involves only the element $\tilde{h}_{i,i+1}$ (as the action of the automorphism s_i) and is given by

$$\begin{aligned} & s'_i(x^{i,\alpha}) = -\frac{1}{\tilde{h}_{i,i+1}}x^{i+1,\alpha} \ , \ s'_i(x^{i+1,\alpha}) = \tilde{h}_{i,i+1}x^{i,\alpha} \ , \ s'_i(x^{j,\alpha}) = x^{j,\alpha} \text{ for } j \neq i, i+1 \ , \\ & s'_i(\bar{\partial}_{i,\alpha}) = -\bar{\partial}_{i+1,\alpha}\tilde{h}_{i,i+1} \ , \ s'_i(\bar{\partial}_{i+1,\alpha}) = \bar{\partial}_{i,\alpha}\frac{1}{\tilde{h}_{i,i+1}} \ , \ s'_i(\bar{\partial}_{j,\alpha}) = \bar{\partial}_{j,\alpha} \text{ for } j \neq i, i+1 \ , \\ & s_i(\tilde{h}_j) = \tilde{h}_{s_i(j)} \ . \end{aligned} \tag{32}$$

4. The diagonal reduction algebra $\mathcal{D}(\mathbf{gl}_n)$ is a $\bar{\mathrm{U}}(\mathbf{h})$-bimodule with the generators L_i^j, $i, j = 1, \dots, n$. The defining relations of $\mathcal{D}(\mathbf{gl}_n)$ are given by the reflection equation, see [KO5]

$$\hat{\mathrm{R}}_{12}\,\mathrm{L}_1\,\hat{\mathrm{R}}_{12}\,\mathrm{L}_1 - \mathrm{L}_1\,\hat{\mathrm{R}}_{12}\,\mathrm{L}_1\,\hat{\mathrm{R}}_{12} = \hat{\mathrm{R}}_{12}\,\mathrm{L}_1 - \mathrm{L}_1\,\hat{\mathrm{R}}_{12} \ ,$$

where $\mathrm{L} = \{\mathrm{L}_i^j\}_{i,j=1}^n$ is the matrix of generators (we refer to [C, S, RS, KS, IO, IOP, IMO1, IMO2] for various aspects and applications of the reflection equation).

For each N there is a homomorphism ([KO5], Section 4.1)

$$\tau_N \colon \mathcal{D}(\mathbf{gl}_n) \to \mathrm{Diff}_{\mathbf{h}}(n, N) \ \text{ defined by } \ \tau_N(\mathrm{L}_i^j) = \sum_\alpha x^{j,\alpha}\bar{\partial}_{i,\alpha} \ .$$

Moreover τ_N is an embedding for $N \geq n$.

The formulas (31) show that the image of τ_N is preserved by the automorphisms s_i.

The element $s_i(\tau_N(\mathrm{L}_k^j))$ can be written by the same formula for all N. Since τ_N is injective for $N \geq n$ we conclude that the formulas (31) induce the action of the symmetric group S_n on the diagonal reduction algebra $\mathcal{D}(\mathbf{gl}_n)$ by automorphisms.

The resulting formulas for the action of the automorphisms s_i, $i = 1, \dots, n-1$, on the generators L_j^k, $j, k = 1, \dots, n$, read

$$
\begin{aligned}
& s_i(\mathrm{L}_j^i) = -\,\mathrm{L}_j^{i+1}\,\tilde{h}_{i,i+1}\ ,\ s_i(\mathrm{L}_j^{i+1}) = \mathrm{L}_j^i\,\frac{1}{\tilde{h}_{i,i+1}}\ ,\ j \neq i, i+1\ ,\\
& s_i(\mathrm{L}_i^j) = -\frac{1}{\tilde{h}_{i,i+1}}\,\mathrm{L}_{i+1}^j\ ,\ s_i(\mathrm{L}_{i+1}^j) = \tilde{h}_{i,i+1}\,\mathrm{L}_j^i\ ,\ j \neq i, i+1\ ,\\
& s_i(\mathrm{L}_i^i) = \mathrm{L}_{i+1}^{i+1}\ ,\ s_i(\mathrm{L}_{i+1}^i) = -\,\mathrm{L}_i^{i+1}(\tilde{h}_{i,i+1}-1)^2\ ,\\
& s_i(\mathrm{L}_i^{i+1}) = -\,\mathrm{L}_{i+1}^i\,\frac{1}{(\tilde{h}_{i,i+1}+1)^2}\ ,\ s_i(\mathrm{L}_{i+1}^{i+1}) = \mathrm{L}_i^i\ ,\\
& s_i(\mathrm{L}_j^k) = \mathrm{L}_j^k\ ,\ k \neq i, i+1 \text{ and } j \neq i, i+1\ .
\end{aligned}
$$

5. The isomorphism μ can be also used to translate the action (10) of the braid group by Zhelobenko operators to the action of the braid group by automorphisms on the ring $\mathfrak{k}[a_1, \dots, a_n] \otimes \mathrm{T}^{-1}\mathrm{W}_n$. It turns out that this action preserves the subring $\mathrm{T}^{-1}\mathrm{W}_n$. Moreover, let T_0 be the multiplicative set of T generated by $X^jD_j - X^kD_k + \ell$, $1 \leq j < k \leq n$, $\ell \in \mathbb{Z}$. Then the action of the operators $\check{\mathrm{q}}_i$, $i = 1, \dots, n-1$, preserves the subring $\mathrm{T}_0^{-1}\mathrm{W}_n$. We present the formulas for the action of the operators $\check{\mathrm{q}}_i$, $i = 1, \dots, n-1$:

$$
\begin{aligned}
& \check{\mathrm{q}}_i(X^i) = \frac{1}{\mathcal{H}_{i,i+1}}X^{i+1}\ ,\ \check{\mathrm{q}}_i(X^{i+1}) = X^i\mathcal{H}_{i,i+1}\ ,\ \check{\mathrm{q}}_i(X^j) = X^j \text{ for } j \neq i, i+1\ ,\\
& \check{\mathrm{q}}_i(D_i) = D_{i+1}\mathcal{H}_{i,i+1}\ ,\ \check{\mathrm{q}}_i(D_{i+1}) = \frac{1}{\mathcal{H}_{i,i+1}}D_i\ ,\ \check{\mathrm{q}}_i(D_j) = D_j \text{ for } j \neq i, i+1\ .
\end{aligned}
$$

6. The isomorphism (23) allows to construct a $2n$-parametric family of $\mathrm{Diff}_{\mathbf{h}}(n)$-modules different from the highest weight representations. Let $\vec{\gamma} := \{\gamma_1, \dots, \gamma_n\}$ be a sequence of length n of complex numbers such that $\gamma_i - \gamma_j \notin \mathbb{Z}$ for all $i, j = 1, \dots, n$, $i \neq j$. Let $V_{\vec{\gamma}}$ be the vector space with the basis

$$
v_{\vec{j}} := (X^1)^{j_1+\gamma_1}(X^2)^{j_2+\gamma_2}\dots(X^n)^{j_n+\gamma_n}\ ,\ \text{where } \vec{j} := \{j_1, \dots, j_n\}\ ,\ j_1, \dots, j_n \in \mathbb{Z}\ .
$$

Under the conditions on $\vec{\gamma}$, $V_{\vec{\gamma}}$ is naturally a $\mathrm{T}^{-1}\mathrm{W}_n$-module. Define the action of the elements a_k on the space $V_{\vec{\gamma}}$ by $a_k\colon v_{\vec{j}} \mapsto A_k v_{\vec{j}}$ where $\vec{A} := \{A_1, \dots, A_n\}$ is another sequence of length n of complex numbers. Then $V_{\vec{\gamma}}$ becomes an $\mathfrak{k}[a_1, \dots, a_n] \otimes \mathrm{T}^{-1}\mathrm{W}_n$-module and therefore $\mathrm{Diff}_{\mathbf{h}}(n)$-module which we denote by $V_{\vec{\gamma},\vec{A}}$. The central operator c_k acts on $V_{\vec{\gamma},\vec{A}}$ by scalar multiplication on A_k.

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