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# Modal Analysis of Liquid-Structure Interaction

Roger Ohayon and Jean-Sébastien Schotté

**Abstract** The computation of the linear vibrations of structures partially filled with liquids is of prime importance in various industries such as aerospace, naval, civil and nuclear engineering. It is proposed here to present a formulation for the modal analysis of incompressible liquids in elastic tanks taking into account sloshing, pressurization and elastogravity effects obtained through an adapted procedure of linearization. A corresponding Finite Element formulation is then presented.

## 1 Introduction

The case of structures partially filled with liquids (propellants, cooling liquids, liquid natural gas, etc.) is very common and many industrial domains are interested in this issue especially in the transport and aerospace industries (for instance liquid propelled launchers and satellites). Computational aspects of fluid-structure interaction and sloshing (with or without surface tension effects) may be found in [1, 4, 5, 6, 7, 9, 13, 14, 15, 21]. We are here interested in the linear vibrational response of elastic structures partially filled with a liquid. As we are primarily concerned with the low frequency domain, the liquid may be considered as inviscid. As a result, the liquid small motions are irrotational. This constraint can be taken into account through various procedures [2, 8]. Using a scalar description for the fluid, alternative symmetric variational formulations have been derived [10, 13].

In the present work, through a specific derivation of the linearized equations of the coupled system, particular phenomena, such as tank ullage gas pressure and

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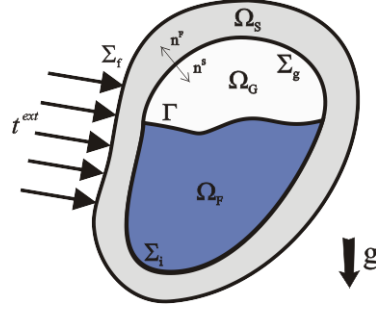
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elastogravity stiffness effects, are analyzed [17, 18, 20]. Firstly, the nonlinear equations are posed, followed by an adapted linearization procedure. Secondly a finite element discretization, leading to a symmetric matrix system, is presented through an appropriate variational formulation.

## 2 Local equations in the deformed configuration

We suppose that the referential is a Galilean frame. Thermo-mechanical coupling is not considered here, therefore all mechanical parameters of the system are supposed given at a nominal temperature and are considered constant during the observation time. The dynamic state of this coupled problem is characterized by the structure displacement field  $\mathbf{U}^S(\mathbf{x}, t)$ , the fluid velocity field  $\mathbf{v}^F(\mathbf{x}, t)$ , the fluid pressure field  $P(\mathbf{x}, t)$  and the tank ullage gas pressure  $P^G(\mathbf{x}, t)$ , which are the unknown fields of the problem. The description of the system is given by Fig. 1. Each domain will be designated with a ' when it refers to its time-dependent position.



**Fig. 1** Structure partially filled with liquid

### 2.1 Local equations of the fluid

We consider an inviscid heavy fluid with a free surface  $\Gamma$  inside a deformable structure. The tank ullage is filled by an inert pressurized gas (no phase or chemical transformations are taken into account). The local equations are written on the deformed configuration, denoted by '.

$$\rho^F \frac{d\mathbf{v}^F}{dt} = -\nabla_{\mathbf{x}} P + \rho^F \mathbf{g} \quad \text{in } \Omega'_F \quad (1a)$$

$$\frac{d\rho^F}{dt} + \rho^F \nabla_{\mathbf{x}} \cdot \mathbf{v}^F = 0 \quad \text{in } \Omega'_F \quad (1b)$$

$$P = P^G \quad \text{on } \Gamma' \quad (1c)$$

$$\mathbf{v}^F \cdot \mathbf{n}'_F + \mathbf{v}^S \cdot \mathbf{n}'_S = 0 \quad \text{on } \Sigma'_i \quad (1d)$$

Equation (1a) is the balance momentum equation of the inviscid heavy fluid and equation (1b) expresses the mass conservation of the fluid (continuity equation). Equation (1c) is the equilibrium equation of the fluid free surface when capillarity forces are neglected. Equation (1d) yields the sliding condition on the structure wall for an inviscid fluid. In addition to this set of differential equations, Cauchy's initial conditions have to be added.

## 2.2 Local equations of the structure

The structure is supposed elastic ( $\mathbf{H}$  denotes the tensor of elastic coefficients) and submitted to external surface forces, including the external and internal pressures.

$$\text{Div}_{\mathbf{x}} \boldsymbol{\sigma} + \rho^S \mathbf{g} = \rho^S \mathbf{a} \quad \text{in } \Omega'_S \quad (2a)$$

$$\boldsymbol{\sigma} = \mathbf{H} \boldsymbol{\varepsilon} \quad \text{in } \Omega'_S \quad (2b)$$

$$\boldsymbol{\varepsilon} = \frac{1}{2} (\mathbf{D}_{\mathbf{x}}(\mathbf{U}^S) + {}^t\mathbf{D}_{\mathbf{x}}(\mathbf{U}^S)) \quad \text{in } \Omega'_S \quad (2c)$$

$$\boldsymbol{\sigma} \mathbf{n}'_S = \mathbf{t}^{\text{ext}} - P^{\text{ext}} \mathbf{n}'_S \quad \text{on } \Sigma'_f \quad (2d)$$

$$\boldsymbol{\sigma} \mathbf{n}'_S = -P^G \mathbf{n}'_S \quad \text{on } \Sigma'_g \quad (2e)$$

$$\boldsymbol{\sigma} \mathbf{n}'_S = -P \mathbf{n}'_S \quad \text{on } \Sigma'_i \quad (2f)$$

Equation (2a) is the local balance of momentum (in a fixed frame of reference) where  $\mathbf{a}$  is the structure acceleration. Equations (2b) and (2c) yield the classical definition of the Cauchy's stress tensor  $\boldsymbol{\sigma}$  and strain tensor  $\boldsymbol{\varepsilon}$ , and their relation through Hooke's law. In equation (2d),  $P^{\text{ext}}$  is the external pressure (the atmospheric pressure for instance) and  $\mathbf{t}^{\text{ext}}$  are additional local surface loads (thrust or aerodynamic forces for instance).

## 2.3 Local equations in the tank ullage gas

The tank ullage being supposed to be pressurized by a gas, a simple gas behavior model will be used to represent its effect on the dynamics of the fluid-structure system. As the acoustic phenomena in the gas enclosed in the tank ullage are generally decoupled from the low frequency fluid-structure vibrations which are the frequency

range of interest of the present paper, the acoustic field in the gas will be indeed represented by an uniform value  $P^G(t)$  in  $\Omega_G$  and the isentropic law of transformation of a perfect gas will be used [12, 18] :

$$P^G |\Omega'_G|^\gamma = P_0^G |\Omega_G|^\gamma \quad \text{in } \Omega'_G \quad (3)$$

where  $\gamma$  is the heat capacity ratio of the gas.

### 3 Reference equilibrium configuration for the coupled system

The geometry of the reference state will be described by the variable  $\mathbf{X}$ . In the reference configuration, the structure and fluid are supposed at rest. The local equations defining this reference state are then :

$$\mathbf{v}_0^F = \mathbf{0} \quad \text{in } \Omega_F \quad (4a)$$

$$\nabla_{\mathbf{X}} P_0 = \rho_0^F \mathbf{g} \quad \text{in } \Omega_F \quad (4b)$$

$$P_0 = P_0^G \quad \text{on } \Gamma \quad (4c)$$

$$\mathbf{v}_0^S = \mathbf{0} \quad \text{in } \Omega_S \quad (4d)$$

$$\text{Div}_{\mathbf{X}} \boldsymbol{\sigma}_0 + \rho_0^S \mathbf{g} = \mathbf{0} \quad \text{in } \Omega_S \quad (4e)$$

$$\boldsymbol{\sigma}_0 = \mathbf{H} \boldsymbol{\varepsilon}_{\mathbf{X}}(\mathbf{u}_0^S) \quad \text{in } \Omega_S \quad (4f)$$

$$\boldsymbol{\varepsilon}_{\mathbf{X}}(\mathbf{u}_0^S) = \frac{1}{2} (\mathbf{D}_{\mathbf{X}}(\mathbf{u}_0^S) + {}^t\mathbf{D}_{\mathbf{X}}(\mathbf{u}_0^S)) \quad \text{in } \Omega_S \quad (4g)$$

$$\boldsymbol{\sigma}_0 \mathbf{n}_S = \mathbf{t}_0^{\text{ext}} - P^{\text{ext}} \mathbf{n}_S \quad \text{on } \Sigma_f \quad (4h)$$

$$\boldsymbol{\sigma}_0 \mathbf{n}_S = -P_0^G \mathbf{n}_S \quad \text{on } \Sigma_g \quad (4i)$$

$$\boldsymbol{\sigma}_0 \mathbf{n}_S = -P_0 \mathbf{n}_S \quad \text{on } \Sigma_i \quad (4j)$$

where  $\mathbf{t}_0^{\text{ext}}$  is the stationary part of the external loads. The external pressure  $P^{\text{ext}}$  is supposed to be a constant field. To solve this problem, a nonlinear iterative procedure is generally needed since these equations are written on domains whose position depends on the unknowns  $(\mathbf{u}_0^S, \mathbf{u}_0^F)$  [16].

### 4 Linearized local equations

The solution of the static problem (4) is supposed known and we are interested here in the small amplitude response of the coupled system around this static reference state (described by the variable  $\mathbf{X}$ ). Using the procedure developed in [20], a first order Taylor expansion of the nonlinear dynamic equations (1,2,3) around the equilibrium position yield the expression of the linearized equations. In the following,  $\delta[h](\mathbf{u})$  will denote the Fréchet derivative of a function  $h$  with respect to the displacement  $\mathbf{U}$ , at the static position  $\mathbf{U} = \mathbf{0}$  and in direction  $\mathbf{u}$ .

#### 4.1 Linearization of the fluid equations

To linearize equation (1a), we first need to differentiate the gradient term  $\nabla_{\mathbf{x}}P$  which is a spatial derivative with respect to the coordinates  $\mathbf{x}$  of the deformed fluid domain, depending, by definition, on the displacement  $\mathbf{U}$  since  $\mathbf{x} = \mathbf{X} + \mathbf{U}$ . Using the following relation :

$$\delta[\nabla_{\mathbf{x}}P](\mathbf{u}^F) = \nabla_{\mathbf{x}}\delta[P](\mathbf{u}^F) - \nabla_{\mathbf{x}}\left(\nabla_{\mathbf{x}}P_0 \cdot \mathbf{u}^F\right) \quad (5)$$

and the value of  $\nabla_{\mathbf{x}}P_0$  given by (4b), we finally obtain:

$$\delta[\nabla_{\mathbf{x}}P](\mathbf{u}^F) = \nabla_{\mathbf{x}}p_{\mathcal{L}}(\mathbf{u}^F) - \nabla_{\mathbf{x}}(\rho_0^F \mathbf{g} \cdot \mathbf{u}^F) \quad (6)$$

where  $p_{\mathcal{L}}(\mathbf{u}^F) = \delta[P](\mathbf{u}^F)$  is the Lagrangian pressure fluctuation in the liquid, i.e. the variation of the pressure observed when following a fluid particle.

Using the classical linearization of the acceleration term (the convection term disappears) and the fact that the Lagrangian density fluctuation  $\delta[\rho^F](\mathbf{u}^F)$  in an incompressible liquid is null, the linearized version of Euler's equation (1a) can be written:

$$\rho_0^F \frac{\partial^2 \mathbf{u}^F}{\partial t^2} = \nabla_{\mathbf{x}}\left(\rho_0^F \mathbf{g} \cdot \mathbf{u}^F - p_{\mathcal{L}}(\mathbf{u}^F)\right) \quad (7)$$

Using the classical expression of the Lagrangian derivative, the mass conservation equation (1b) can be written

$$\frac{\partial \rho^F}{\partial t} + \mathbf{v}^F \cdot \nabla_{\mathbf{x}}\rho^F + \rho^F \nabla_{\mathbf{x}} \cdot \mathbf{v}^F = 0 \quad (8)$$

The linearization of this equation can be achieved with the same approach as previously. Considering that the liquid is at rest in its equilibrium position ( $\mathbf{v}_0^F = \mathbf{0}$ ), that it is supposed incompressible ( $\delta[\rho^F](\mathbf{u}^F) = 0$ ) and that its density in the reference state is homogeneous ( $\nabla_{\mathbf{x}}\rho_0^F = 0$ ), then the linearized form of the mass conservation equation (8) can be written

$$\frac{\partial}{\partial t}\left(\rho_0^F \nabla_{\mathbf{x}} \cdot \mathbf{u}^F\right) = 0 \quad (9)$$

The same approach applied to equation (1d) yields the following linearized equation:

$$\frac{\partial}{\partial t}\left(\mathbf{u}^F \cdot \mathbf{n}_F + \mathbf{u}^S \cdot \mathbf{n}_S\right) = 0 \quad (10)$$

We deduce therefore that the continuity of the normal displacement is satisfied at any time  $t$  as soon as it is satisfied by the initial conditions. The linearized equations of the fluid around the reference state are finally ( $\forall t \in \mathbb{R}^+$ ):

$$\rho_0^F \frac{\partial^2 \mathbf{u}^F}{\partial t^2} = \nabla_{\mathbf{x}} \left( \rho_0^F \mathbf{g} \cdot \mathbf{u}^F - p_{\mathcal{L}}(\mathbf{u}^F) \right) \quad \text{in } \Omega_F \quad (11a)$$

$$\nabla_{\mathbf{x}} \cdot \mathbf{u}^F = 0 \quad \text{in } \Omega_F \quad (11b)$$

$$p_{\mathcal{L}} = p^G \quad \text{on } \Gamma \quad (11c)$$

$$\mathbf{u}^F \cdot \mathbf{n}_F + \mathbf{u}^S \cdot \mathbf{n}_S = 0 \quad \text{on } \Sigma_i \quad (11d)$$

where  $p^G = \delta[P^G](\mathbf{u}^F, \mathbf{u}^S)$  is the pressure fluctuation in the gas when the fluid and structure are deformed. To these equations, we add initial Cauchy conditions. Equation (9) has been simplified by supposing that the initial condition satisfies this mass conservation equation.

## 4.2 Linearization of the structural equation

The local equations of the structure (2) are now linearized using the same approach. However, to simplify the derivation of the equations, before the differentiation, we use a Lagrangian transformation to write those equations in the reference configuration, as follows:

$$\text{Div}_{\mathbf{x}}(\mathbf{F}\mathbf{S}) = \rho_0^S \frac{d^2 \mathbf{U}^S}{dt^2} - \rho_0^S \mathbf{g} \quad \text{in } \Omega_S \quad (12a)$$

$$\mathbf{S}(\mathbf{U}^S) = \mathbf{H}\mathbf{E}(\mathbf{u}_0^S + \mathbf{U}^S) \quad (12b)$$

$$\mathbf{F}\mathbf{S}\mathbf{n}_S = \mathbf{t}^{\text{ext}} \det(\mathbf{F}) \|\mathbf{F}^{-1}\mathbf{n}_S\| - P^{\text{ext}} \det(\mathbf{F}) \mathbf{F}^{-1}\mathbf{n}_S \quad \text{on } \Sigma_f \quad (12c)$$

$$\mathbf{F}\mathbf{S}\mathbf{n}_S = -P^G \det(\mathbf{F}) \mathbf{F}^{-1}\mathbf{n}_S \quad \text{on } \Sigma_g \quad (12d)$$

$$\mathbf{F}\mathbf{S}\mathbf{n}_S = -P \det(\mathbf{F}) \mathbf{F}^{-1}\mathbf{n}_S \quad \text{on } \Sigma_i \quad (12e)$$

In these equations,  $\mathbf{E}$  is the (nonlinear) Green-Lagrange strain tensor and  $\mathbf{F}$  is the transformation gradient tensor, defined by  $\mathbf{F} = \mathbf{D}_{\mathbf{x}}(\mathbf{x})$ . The linearization of (12a) is classical and gives

$$\text{Div}_{\mathbf{x}} \left( \mathbf{D}_{\mathbf{x}}(\mathbf{u}^S) \boldsymbol{\sigma}_0 + \mathbf{H} \boldsymbol{\varepsilon}_0(\mathbf{u}^S) \right) = \rho_0^S \frac{\partial^2 \mathbf{u}^S}{\partial t^2} \quad (13)$$

where

$$\boldsymbol{\sigma}_0 = \mathbf{H}\mathbf{E}(\mathbf{u}_0^S) \quad (14)$$

$$\boldsymbol{\varepsilon}_0(\mathbf{u}^S) = \delta[\mathbf{E}](\mathbf{u}^S) \quad (15)$$

For the differentiation of equation (12e), the following term has to be computed:

$$\delta[P \det(\mathbf{F}) \mathbf{F}^{-1}](\mathbf{u}^S) = \delta[P](\mathbf{u}^S) \mathbf{I} + P_0 \delta[\det(\mathbf{F}) \mathbf{F}^{-1}](\mathbf{u}^S) \quad (16)$$

In the first term of this expression, we recognize a Lagrangian fluctuation of the pressure on the fluid-structure interface which is denoted by  $p_{\mathcal{L}}^S(\mathbf{u}^S)$ . Let us remark

that this term is defined as a pressure fluctuation related to a structural point and then is different from the Lagrangian pressure fluctuation  $p_{\mathcal{L}}(\mathbf{u}^F)$  defined previously in equation (6) for a fluid particle. This difficulty is a consequence of the sliding of the inviscid liquid on the structure wall.

To explicit the second term of (16), we use the following expressions:

$$\delta[\det(\mathbf{F})](\mathbf{u}) = \text{Tr}[\mathbf{D}_{\mathbf{X}}\mathbf{u}] \mathbf{I} \quad (17)$$

$$\delta[\mathbf{F}^{-1}](\mathbf{u}) = -\mathbf{D}_{\mathbf{X}}\mathbf{u} \quad (18)$$

Then, we obtain

$$\frac{d}{d\mathbf{U}^S} \left( \det(\mathbf{F}) \mathbf{F}^{-1} \right) \Big|_0 (\mathbf{u}^S) = (\nabla_{\mathbf{X}} \cdot \mathbf{u}^S) \mathbf{I} - \mathbf{D}_{\mathbf{X}}(\mathbf{u}^S) \quad (19)$$

Using this relation in (16) yields the linearized expression of (12e):

$$\left( \mathbf{D}_{\mathbf{X}}(\mathbf{u}^S) \boldsymbol{\sigma}_0 + \mathbf{H} \boldsymbol{\varepsilon}_0(\mathbf{u}^S) \right) \mathbf{n}_S = -p_{\mathcal{L}}^S(\mathbf{u}^S) \mathbf{n}_S - P_0 \boldsymbol{\tau}(\mathbf{u}^S) \quad (20)$$

where the vector  $\boldsymbol{\tau}$  is defined as

$$\boldsymbol{\tau}(\mathbf{u}^S) = (\nabla_{\mathbf{X}} \cdot \mathbf{u}^S) \mathbf{n}_S - \mathbf{D}_{\mathbf{X}}(\mathbf{u}^S) \mathbf{n}_S \quad (21)$$

The comparison of this expression with the following one, that we would have obtained by a direct differentiation of the right-hand term of equation (2f):

$$\delta[-P \mathbf{n}'_S d\Sigma'](\mathbf{u}^S) = -p_{\mathcal{L}}^S(\mathbf{u}^S) \mathbf{n}_S d\Sigma - P_0 \delta[\mathbf{n}'_S d\Sigma'](\mathbf{u}^S) \quad (22)$$

shows that  $\boldsymbol{\tau}$  can be interpreted as the linearization of the normal vector rotation:

$$\boldsymbol{\tau}(\mathbf{u}^S) d\Sigma = \delta[\mathbf{n}'_S d\Sigma'](\mathbf{u}^S) \quad (23)$$

The same treatment can be applied to equation (12d). However, for equation (12c), we have to deal with the external force term  $\mathbf{t}^{\text{ext}}$ . The differentiation of this term yields:

$$\delta \left[ \mathbf{t}^{\text{ext}} \det(\mathbf{F}) \|\mathbf{F}^{-1} \mathbf{n}_S\| \right] (\mathbf{u}^S) = \delta[\mathbf{t}^{\text{ext}}](\mathbf{u}^S) + \mathbf{t}_0^{\text{ext}} \delta \left[ \det(\mathbf{F}) \|\mathbf{F}^{-1} \mathbf{n}_S\| \right] (\mathbf{u}^S) \quad (24)$$

A direct computation of the second term can be avoided if we remember that

$$\delta \left[ \det(\mathbf{F}) \|\mathbf{F}^{-1} \mathbf{n}_S\| \right] (\mathbf{u}^S) dS = \delta[dS'](\mathbf{u}^S) \quad (25)$$

where  $dS$  and  $dS'$  are respectively the elementary surfaces on the reference and the actual geometries. Since, from (23), we have  $\boldsymbol{\tau}(\mathbf{u}^S) dS = \delta[\mathbf{n}'_S dS'](\mathbf{u}^S) = \delta[\mathbf{n}'](\mathbf{u}^S) dS + \mathbf{n} \delta[dS'](\mathbf{u}^S)$  and  $\mathbf{n} \cdot \delta[\mathbf{n}'](\mathbf{u}^S) = 0$ , we can deduce from (21) that

$$\delta \left[ \det(\mathbf{F}) \|\mathbf{F}^{-1} \mathbf{n}_S\| \right] (\mathbf{u}^S) = \mathbf{n}_S \cdot \boldsymbol{\tau}(\mathbf{u}^S) = \nabla_{\mathbf{X}} \cdot \mathbf{u}^S - \mathbf{n}_S \cdot \mathbf{D}_{\mathbf{X}}(\mathbf{u}^S) \mathbf{n}_S \quad (26)$$



Finally, the linearized expression of (12c) can be written

$$\left( \mathbf{D}_X(\mathbf{u}^S) \boldsymbol{\sigma}_0 + \mathbf{H} \boldsymbol{\varepsilon}_0(\mathbf{u}^S) \right) \mathbf{n}_S = \mathbf{t}^d + \tilde{\mathbf{t}} + \mathbf{t}_0^{\text{ext}} \mathbf{n}_S \cdot \boldsymbol{\tau}(\mathbf{u}^S) - P^{\text{ext}} \boldsymbol{\tau}(\mathbf{u}^S) \quad (27)$$

where we have distinguished two terms in the variation of the external surface loads  $\delta[\mathbf{t}^{\text{ext}}](\mathbf{u}^S)$ : a dynamic time-depend part  $\mathbf{t}^d$  and a displacement dependant part  $\tilde{\mathbf{t}}(\mathbf{u}^S)$ . Let us verify that if the external surface load  $\mathbf{t}^{\text{ext}}$  is a pressure, i.e.  $\mathbf{t}^{\text{ext}} = -P_t \mathbf{n}_S$ , then equations (24) and (25) lead to coherent expressions of (27) and (20):

$$\begin{aligned} \delta[\mathbf{t}^{\text{ext}}] d\Sigma + \mathbf{t}_0^{\text{ext}} \mathbf{n}_S \cdot \boldsymbol{\tau} d\Sigma &= \delta[-P_t \mathbf{n}_S'] d\Sigma - P_{t0} \mathbf{n}_S \delta[d\Sigma'] \\ &= -\delta[P_t] \mathbf{n}_S d\Sigma - P_{t0} \delta[\mathbf{n}_S' d\Sigma'] \end{aligned} \quad (28)$$

By using (23), we denote then that expressions (20) and (27) are equivalent.

The linearized equations of the structure established in this section are summarized hereafter:

$$\text{Div}_X \mathbf{s}(\mathbf{u}^S) = \rho_0^S \frac{\partial^2 \mathbf{u}^S}{\partial t^2} \quad \text{in } \Omega_S \quad (29a)$$

$$\mathbf{s}(\mathbf{u}^S) = \mathbf{D}_X(\mathbf{u}^S) \boldsymbol{\sigma}_0 + \mathbf{H} \boldsymbol{\varepsilon}_0(\mathbf{u}^S) \quad \text{in } \Omega_S \quad (29b)$$

$$\mathbf{s} \mathbf{n}_S = \mathbf{t}^d + \tilde{\mathbf{t}}(\mathbf{u}^S) + \mathbf{t}_0^{\text{ext}} \mathbf{n}_S \cdot \boldsymbol{\tau}(\mathbf{u}^S) - P^{\text{ext}} \boldsymbol{\tau}(\mathbf{u}^S) \quad \text{on } \Sigma_f \quad (29c)$$

$$\mathbf{s} \mathbf{n}_S = -p^G(\mathbf{u}^S, \mathbf{u}^F) \mathbf{n}_S - P_0^G \boldsymbol{\tau}(\mathbf{u}^S) \quad \text{on } \Sigma_g \quad (29d)$$

$$\mathbf{s} \mathbf{n}_S = -p^S_{\mathcal{L}}(\mathbf{u}^S) \mathbf{n}_S - P_0 \boldsymbol{\tau}(\mathbf{u}^S) \quad \text{on } \Sigma_i \quad (29e)$$

To these equations, Cauchy conditions have to be added.

### 4.3 Linearization of the gas equations

The differentiation of the gas equation (3) yields

$$\delta[P^G](\mathbf{u}^S, \mathbf{u}^F) |\Omega_G|^\gamma + P_0^G \gamma |\Omega_G|^{\gamma-1} \delta[|\Omega_G'|](\mathbf{u}^S, \mathbf{u}^F) = 0 \quad (30)$$

Since the gas volume  $|\Omega_G'|$  is defined by

$$|\Omega_G'| = \int_{\Omega_G'} d\Omega' = \int_{\Omega_G} \det(\mathbf{F}) d\Omega \quad (31)$$

Using the relation (17) and Stokes theorem, we obtain

$$\delta[|\Omega_G'|](\mathbf{u}^S, \mathbf{u}^F) = \int_{\Omega_G} \nabla_X \cdot \mathbf{u} d\Omega = - \int_{\Sigma_g} \mathbf{u}^S \cdot \mathbf{n}_S d\Sigma - \int_{\Gamma} \mathbf{u}^F \cdot \mathbf{n}_F d\Gamma \quad (32)$$

Finally the linearized equation of the gas can be written

$$p^G(\mathbf{u}^S, \mathbf{u}^F) = \frac{P_0^G \gamma}{|\Omega_G|} \left( \int_{\Sigma_g} \mathbf{u}^S \cdot \mathbf{n}_S d\Sigma + \int_{\Gamma} \mathbf{u}^F \cdot \mathbf{n}_F d\Gamma \right) \quad \text{in } \Omega_G \quad (33)$$

Equations (11), (29) and (33) form the coupled problem we want to solve. In these equations, all the spatial derivatives are carried out with respect to the coordinate vector  $\mathbf{X}$  and then, for sake of convenience, this subscript  $\mathbf{X}$  will be omitted in the following.

## 5 Resolution of the coupled fluid-structure problem

### 5.1 Variational formulation for a potential incompressible fluid

Taking the curl of equation (11a) shows that the small motions  $\mathbf{u}^F$  of an incompressible fluid are irrotational and then can be represented by the gradient of a displacement potential  $\varphi$  defined by

$$\nabla \wedge \mathbf{u}^F = 0 \quad \implies \quad \mathbf{u}^F = \nabla \varphi \quad (34)$$

However, this relation does not ensure the uniqueness of  $\varphi$ . Therefore, it is necessary to specify an unicity condition on  $\varphi$ , generically written  $\ell(\varphi) = 0$ , where  $\ell$  is a linear form such that  $\ell(1) \neq 0$ . This unicity condition will be chosen explicitly in the following.

This new variable  $\varphi$  can be substituted to  $\mathbf{u}^F$  in the fluid equations, in particular in (11a) which yields, after integration with respect to  $\mathbf{X}$ :

$$p_{\mathcal{L}} = \rho_0^F \mathbf{g} \cdot \nabla \varphi - \rho_0^F \frac{\partial^2 \varphi}{\partial t^2} + \pi \quad (35)$$

where  $\pi$  is an integration constant which represents a uniform pressure in the fluid whose value depends on the choice of the unicity condition  $\ell(\varphi)$ . The expression (35) of  $p_{\mathcal{L}}$  as a function of  $\varphi$  allows to eliminate  $p_{\mathcal{L}}$  from the equations. However, before establishing the variational formulation, we choose to introduce a scalar variable  $\eta$  to describe the normal component of the fluid free surface displacement:

$$\eta = \mathbf{u}^F \cdot \mathbf{n}_F = \nabla \varphi \cdot \mathbf{n}_F \quad \text{on } \Gamma \quad (36)$$

Since  $\mathbf{n}_F = \mathbf{i}_z$  on  $\Gamma$ , the local equations of the boundary problem (11) are now written for an incompressible fluid in terms of  $(\varphi, \eta, \pi, p^G)$  as follows:

$$\Delta \varphi = 0 \quad \text{in } \Omega_F \quad (37a)$$

$$\mathbf{n}_F \cdot \nabla \varphi = \mathbf{u}^s \cdot \mathbf{n}_F \quad \text{on } \Sigma_i \quad (37b)$$

$$\mathbf{n}_F \cdot \nabla \varphi = \eta \quad \text{on } \Gamma \quad (37c)$$

$$\rho_0^F g \eta = -\rho_0^F \frac{\partial^2 \varphi}{\partial t^2} + \pi - p^G \quad \text{on } \Gamma \quad (37d)$$

$$p^G = \frac{P_0^G \gamma}{|\Omega_G|} \int_{\Sigma_i \cup \Sigma_g} \mathbf{u}^s \cdot \mathbf{n}_s \, d\Sigma \quad \text{in } \Omega_G \quad (37e)$$

$$\ell(\varphi) = 0 \quad (37f)$$

$$(37g)$$

where equation (37e) has been obtained from (33) by considering the following relation given by the integration of property (11b) on the fluid domain:

$$0 = \int_{\Sigma_i} \mathbf{u}^F \cdot \mathbf{n}_F \, d\Sigma + \int_{\Gamma} \mathbf{u}^F \cdot \mathbf{n}_F \, d\Gamma \quad (38)$$

The equation (37e) shows that, due to the incompressibility of the fluid,  $p^G$  is now a function depending only of the tank wall deformation  $\mathbf{u}^s$ .

From the local equations (37a), (37b) and (37c), we can write the following variational formulation for the liquid, where  $\mathcal{C}_\varphi^\ell$  is the admissible space of solutions  $\varphi$  defined by

$$\mathcal{C}_\varphi^\ell = \{ \varphi \in \mathcal{H}^1(\Omega_F) / \ell(\varphi) = 0 \} \quad (39)$$

for  $t \in \mathbb{R}^+$ ,  $\exists \varphi \in \mathcal{C}_\varphi^\ell$  such that  $\forall \delta \varphi \in \mathcal{C}_\varphi^\ell$

$$\mathcal{F}(\varphi, \delta \varphi) - \mathcal{B}(\eta, \delta \varphi) = \mathcal{C}(\delta \varphi, \mathbf{u}^s) \quad (40)$$

$$\text{with } \mathcal{F}(\varphi, \delta \varphi) = \rho_0^F \int_{\Omega_F} \nabla \varphi \cdot \nabla \delta \varphi \, d\Omega \quad (41)$$

$$\mathcal{B}(\eta, \delta \varphi) = \rho_0^F \int_{\Gamma} \eta \delta \varphi \, d\Gamma \quad (42)$$

$$\mathcal{C}(\delta \varphi, \mathbf{u}^s) = \rho_0^F \int_{\Sigma_i} \mathbf{u}^s \cdot \mathbf{n}_F \delta \varphi \, d\Sigma \quad (43)$$

In this expression,  $\mathcal{F}$  is the symmetric positive semi-definite bilinear form associated with the fluid kinetic energy and  $\mathcal{C}$  is the classical fluid-structure coupling operator.

We also consider the following relation equivalent to (38), which expresses the incompressibility of the fluid:

$$-\mathcal{B}(\eta, 1) = \mathcal{C}(1, \mathbf{u}^s) \quad (44)$$

The variational formulation corresponding to the local equation (37d) is written

for  $t \in \mathbb{R}^+$ ,  $\exists \eta \in \mathcal{C}_\eta$  such that  $\forall \delta \eta \in \mathcal{C}_\eta$ ,

$$\mathcal{S}_g(\eta, \delta \eta) + \mathcal{B}\left(\delta \eta, \frac{\partial^2 \varphi}{\partial t^2}\right) - \frac{\pi}{\rho_0^F} \mathcal{B}(\delta \eta, 1) + \frac{p^G}{\rho_0^F} \mathcal{B}(\delta \eta, 1) = 0 \quad (45)$$

$$\text{with } \mathcal{S}_g(\eta, \delta \eta) = \rho_0^F g \int_\Gamma \eta \delta \eta \, d\Gamma \quad (46)$$

where  $\mathcal{C}_\eta$  is the admissible space for  $\eta$ , defined as  $\mathcal{C}_\eta = \{\eta \in \mathcal{L}^2(\Gamma)\}$  and  $\mathcal{S}_g$  is the symmetric positive definite bilinear form associated with the fluid gravity potential energy.

A final equation is obtained from the expression (37e) of  $p^G$ :

$$\chi^G \frac{p^G}{\rho_0^F} - \mathcal{B}(\eta, 1) = \mathcal{C}_g(1, \mathbf{u}^S) \quad (47)$$

$$\text{with } \mathcal{C}_g(\delta \varphi, \mathbf{u}^S) = \rho_0^F \int_{\Sigma_g} \mathbf{u}^S \cdot \mathbf{n}_s \delta \varphi \, d\Sigma \quad (48)$$

$$\text{and } \chi^G = \frac{\rho_0^{F^2} |\Omega_G|}{P_0^G \gamma} \quad (49)$$

## 5.2 Variational formulation for the structure

From the local equations of the structure given in (29), a variational formulation for the structure can be written, where  $\mathcal{C}_u = \{\mathbf{u}^S \in \mathcal{H}^1(\Omega_s)^3\}$  denotes the space of the kinematically admissible solutions. In the following, the fluid pressure  $P_0$  in the reference configuration will be replaced by its expression obtained from (4b) and (4c):

$$P_0(\mathbf{X}) = P_0^G + \rho_0^F \mathbf{g} \cdot \mathbf{X} \quad (50)$$

where  $O$  is a point of the liquid free surface.

Then the variational principle for the structure is written

$$\begin{aligned} \exists \mathbf{u}^S \in \mathcal{C}_u \text{ such that } \forall \delta \mathbf{u} \in \mathcal{C}_u, \\ (\mathcal{K}_E + \mathcal{K}_G + \mathcal{K}_P + \mathcal{K}_I)(\mathbf{u}^S, \delta \mathbf{u}) + \mathcal{M}\left(\frac{\partial^2 \mathbf{u}^S}{\partial t^2}, \delta \mathbf{u}\right) + \frac{p^G}{\rho_0^F} \mathcal{C}_g(1, \delta \mathbf{u}) \dots \\ + \int_{\Sigma_i} p^S_{\mathcal{L}}(\mathbf{u}^S) \mathbf{n}_s \cdot \delta \mathbf{u} + (\rho_0^F \mathbf{g} \cdot \mathbf{X}) \tau(\mathbf{u}^S) \cdot \delta \mathbf{u} \, d\Sigma = f^{\text{ext}}(\delta \mathbf{u}) \end{aligned} \quad (51)$$

$$\text{with } \mathcal{K}_E(\mathbf{u}^S, \delta \mathbf{u}) = \int_{\Omega_S} \mathbf{H} \boldsymbol{\varepsilon}_0(\mathbf{u}^S) : \mathbf{D}_\mathbf{x}(\delta \mathbf{u}) \, d\Omega \quad (52)$$

$$\mathcal{K}_G(\mathbf{u}^S, \delta \mathbf{u}) = \int_{\Omega_S} \mathbf{D}_\mathbf{x}(\mathbf{u}^S) \boldsymbol{\sigma}_0 : \mathbf{D}_\mathbf{x}(\delta \mathbf{u}) \, d\Omega \quad (53)$$

$$\mathcal{K}_P(\mathbf{u}^S, \delta \mathbf{u}) = \int_{\Sigma_f} P^{\text{ext}} \boldsymbol{\tau}(\mathbf{u}^S) \cdot \delta \mathbf{u} \, d\Sigma + \int_{\Sigma_g \cup \Sigma_i} P_0^G \boldsymbol{\tau}(\mathbf{u}^S) \cdot \delta \mathbf{u} \, d\Sigma \quad (54)$$

$$\mathcal{K}_t(\mathbf{u}^S, \delta \mathbf{u}) = - \int_{\Sigma_f} \tilde{\mathbf{t}}(\mathbf{u}^S) \cdot \delta \mathbf{u} + \mathbf{n}_S \cdot \boldsymbol{\tau}(\mathbf{u}^S) \mathbf{t}_0^{\text{ext}} \cdot \delta \mathbf{u} \, d\Sigma \quad (55)$$

$$\mathcal{M}(\mathbf{u}^S, \delta \mathbf{u}) = \int_{\Omega_S} \rho_0^S \mathbf{u}^S \cdot \delta \mathbf{u} \, d\Omega \quad (56)$$

$$f^{\text{ext}}(\delta \mathbf{u}) = \int_{\Sigma_f} \mathbf{t}^d \cdot \delta \mathbf{u} \, d\Sigma \quad (57)$$

where  $\mathcal{K}_E$  is the elastic stiffness operator of the structure deformed in its static configuration. If these initial deformations  $\mathbf{u}_0^S$  are small,  $\boldsymbol{\varepsilon}_0$  tends to  $\boldsymbol{\varepsilon}$  and  $\mathcal{K}_E$  is then the standard symmetric positive stiffness operator.  $\mathcal{M}$  is the classical symmetric positive definite operator associated to the mass of the structure and  $\mathcal{K}_G$  is the symmetric operator, due to the initial prestress  $\boldsymbol{\sigma}_0$  in the structure, classically known as the geometric stiffness.  $\mathcal{K}_P$  is the stiffness operator arising from the follower force effect of the initial pressures applied on the structure external and internal walls (the external pressure  $P^{\text{ext}}$  and the pressurization  $P_0^G$ ).  $\mathcal{K}_t$  is a similar term due to the presence of a static external surface load  $\mathbf{t}^{\text{ext}}$  which can be also a follower force.  $f^{\text{ext}}$  is a linear form representing the dynamic part of this load.

We need now an expression of  $p_{\mathcal{L}}^S(\mathbf{u}^S)$  as a function of the unknowns  $\mathbf{u}^S$  and  $\mathbf{u}^F$ . This expression is obtained by considering a fluid particle in contact with a structural point of the fluid interface in the reference configuration. At time  $t$ , the pressure at this structural point is  $P(\mathbf{U}^S)$  and the pressure of the same fluid particle is  $P(\mathbf{U}^F)$ . The difference can be written as follows:

$$P(\mathbf{U}^S) - P(\mathbf{U}^F) = \nabla_\mathbf{x} P \cdot (\mathbf{U}^S - \mathbf{U}^F) \quad (58)$$

By differentiating this expression at the reference configuration and in the direction  $(\mathbf{u}^S, \mathbf{u}^F)$ , we obtain:

$$p_{\mathcal{L}}^S(\mathbf{u}^S) - p_{\mathcal{L}}(\mathbf{u}^F) = \nabla_\mathbf{x} P_0 \cdot (\mathbf{u}^S - \mathbf{u}^F) = \rho_0^F \mathbf{g} \cdot (\mathbf{u}^S - \mathbf{u}^F) \quad (59)$$

Since the expression of  $p_{\mathcal{L}}$  is given by (35), we can write the following expression:

$$p_{\mathcal{L}}^S(\mathbf{u}^S) = \rho_0^F \mathbf{g} \cdot \mathbf{u}^S - \rho_0^F \frac{\partial^2 \phi}{\partial t^2} + \pi \quad (60)$$

By introducing this relation into the previously given variational formulation of the structure, we obtain:

$$\exists \mathbf{u}^s \in \mathcal{C}_u \text{ such that } \forall \delta \mathbf{u} \in \mathcal{C}_u, \quad (61)$$

$$\mathcal{K}_T(\mathbf{u}^s, \delta \mathbf{u}) + \frac{p^G}{\rho_0^F} \mathcal{C}_g(1, \delta \mathbf{u}) - \frac{\pi}{\rho_0^F} \mathcal{C}(1, \delta \mathbf{u}) + \mathcal{M}\left(\frac{\partial^2 \mathbf{u}^s}{\partial t^2}, \delta \mathbf{u}\right) + \mathcal{C}\left(\frac{\partial^2 \varphi}{\partial t^2}, \delta \mathbf{u}\right) = f^{\text{ext}}(\delta \mathbf{u})$$

$$\text{with } \mathcal{K}_T(\mathbf{u}^s, \delta \mathbf{u}) = (\mathcal{K}_E + \mathcal{K}_G + \mathcal{K}_P + \mathcal{K}_t + \mathcal{K}_g)(\mathbf{u}^s, \delta \mathbf{u}) \quad (62)$$

$$\mathcal{K}_g(\mathbf{u}^s, \delta \mathbf{u}) = \int_{\Sigma_i} (\rho_0^F \mathbf{g} \cdot \mathbf{u}^s) \mathbf{n}_s \cdot \delta \mathbf{u} + (\rho_0^F \mathbf{g} \cdot \mathbf{X}) \tau(\mathbf{u}^s) \cdot \delta \mathbf{u} \, d\Sigma \quad (63)$$

where  $\mathcal{K}_g$  is the stiffness operator related to the follower force effect of the fluid hydrostatic pressure on the structure wall (it can be shown that this operator is symmetric [18]).

### 5.3 Finite element discretization and matrix form

The classical Rayleigh-Ritz finite element method can be applied to discretize the space part of the unknown fields. We consider here compatible meshes for the structure and the fluid. Since  $\varphi$  belongs to  $\mathcal{C}_\varphi^\ell$ , the condition  $\ell(\varphi) = 0$  have to be taken into account in strong form. One way to do is to choose  $\ell(\varphi)$  such that one component  $i$  of  $\varphi$  is equal to 0 ( $\ell(\varphi) = \varphi_i = 0$ ). Then, all the row and columns associated to these zero values are removed from the matrix system. To simplify the notations, vectors of non-discarded nodal values will be denoted by the same letter as the associated continuous variable  $\mathbf{u}^s(\mathbf{x}, \omega)$ ,  $\varphi(\mathbf{x}, \omega)$  and  $\eta(\mathbf{x}, \omega)$ . In the same way, the (non-discarded part of) matrix operators associated to the sesquilinear forms will be denoted by the same capital letter, and the linear and antilinear forms by a lower case letter, as for example

$$\mathcal{B}(\delta \eta, \varphi) \implies \delta \eta^* \mathbf{B} \varphi \quad \text{and} \quad \mathcal{B}(\delta \eta, 1) \implies \delta \eta^* \mathbf{b} \quad (64)$$

The fluid equation (40) shows that  $\varphi$  can be expressed as a function of  $\eta$  and  $\mathbf{u}^s$ . After a finite element discretization of this equation, since  $\mathbf{F}$  is invertible [13], we obtain the following expression:

$$\varphi = \mathbf{F}^{-1} {}^t \mathbf{C} \mathbf{u}^s + \mathbf{F}^{-1} \mathbf{B} \eta \quad (65)$$

After a Fourier transform and a finite element discretization of equations (61), (45), (44) and (47), we introduce this relation to eliminate the unknown  $\varphi$  and obtain the following symmetric matrix system:

$$\left( \begin{bmatrix} \mathbf{K}_T & \mathbf{0} & -\mathbf{c} & \mathbf{c}_g \\ \mathbf{0} & \mathbf{S}_g & -\mathbf{b} & \mathbf{b} \\ -{}^t \mathbf{c} & -{}^t \mathbf{b} & \mathbf{0} & \mathbf{0} \\ {}^t \mathbf{c}_g & {}^t \mathbf{b} & \mathbf{0} & -\chi^G \end{bmatrix} - \omega^2 \begin{bmatrix} \mathbf{M} + \mathbf{C} \mathbf{F}^{-1} {}^t \mathbf{C} & \mathbf{C} \mathbf{F}^{-1} \mathbf{B} & \mathbf{0} & \mathbf{0} \\ {}^t \mathbf{B} \mathbf{F}^{-1} {}^t \mathbf{C} & {}^t \mathbf{B} \mathbf{F}^{-1} \mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \begin{pmatrix} \mathbf{u}^s \\ \eta \\ \pi / \rho_0^F \\ p^G / \rho_0^F \end{pmatrix} = \begin{pmatrix} \mathbf{f}^{\text{ext}} \\ \mathbf{0} \\ 0 \\ 0 \end{pmatrix} \quad (66)$$

## 6 Applications

In order to compute the vibrations of a pressurized elastic structure containing an inviscid incompressible liquid with a free surface, one has to solve equation (66) in the frequency domain (and then obtain the time response by an inverse Fourier transform). To reduce the computing times, a modal synthesis method could be used [20]. It is based on the projection of the liquid equation onto its sloshing eigenmodes which are the eigensolutions of the liquid to equations (37) for a rigid motionless cavity ( $\mathbf{u}^s = \mathbf{0}$ ). Equation (37e) shows that, if the structure is undeformed,  $p^G = 0$ . Then, the sloshing modes  $(\eta_\alpha, \pi_\alpha) \cos(\omega_\alpha t)$  are solutions of:

$$\left( \begin{bmatrix} \mathbf{S}_g & -\mathbf{b} \\ -\mathbf{b}^t & \mathbf{0} \end{bmatrix} - \omega_\alpha^2 \begin{bmatrix} {}^t\mathbf{B}\mathbf{F}^{-1}\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \begin{pmatrix} \eta_\alpha \\ \pi_\alpha / \rho_0^F \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix} \quad (67)$$

If the matrix  ${}^t\mathbf{B}\mathbf{F}^{-1}\mathbf{B}$  is denoted by  $\mathbf{M}_F$ , we deduce that the sloshing eigenmodes of the liquid are only described by their free surface normal displacement  $\eta_\alpha$  which are the solutions of the eigenvalue problem  $\mathbf{S}_g \eta_\alpha = \omega_\alpha^2 \mathbf{M}_F \eta_\alpha$  with  ${}^t\mathbf{b} \eta_\alpha = 0$ , where  $\mathbf{S}_g$  is the gravity potential energy operator,  $\mathbf{M}_F$  is the mass operator of the liquid in a rigid motionless cavity, condensed on its free surface, and  $\pi$  has the role of a Lagrange multiplier associated to the incompressibility condition  ${}^t\mathbf{b} \eta_\alpha = 0$  (which represents the fluid volume invariance). Since  $\mathbf{S}_g$  and  $\mathbf{M}_F$  are symmetric, the eigenmodes  $\eta_\alpha$  are real, orthogonal and form a basis for the liquid free surface deformations. Let us also remark that  $\mathbf{S}_g$  being non singular, there is no zero-frequency sloshing modes.

Several examples of application of the linearized fluid-structure formulation presented here have been published during the past few years. The possibility to predict the coupling between the vibrations of the structure and the liquid sloshing using this formulation has been illustrated on test cases and on some more realistic examples [7, 19] and then validated by comparison with experimental results [18]. An application to the specific case of free structures containing liquids and a validation by comparison with some benchmark results have been published in [17]. The nonlinear effect of the hydrostatic pressure on the hydroelastic vibrations of a plate has also been analyzed in [16].

## 7 Conclusions and perspectives

This paper presents a new way of deriving the linearized variational formulation for fluid-structure problems involving a deformable structure and an internal inviscid liquid with a free surface. Damping phenomena in the fluid and at the interfaces (fluid-structure and free surface) are of prime importance and are under analysis [11, 22]. From numerical point of view, isogeometric analysis [3] could be worthwhile of investigation for tanks of complex geometry.

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