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THE DENSITY OF SUPERCONDUCTIVITY IN THE BULK REGIME

BERNARD HELFFER AND AYMAN KACHMAR

ABSTRACT. In the asymptotic limit of a large Ginzburg-Landau parameter, we give a new asymptotic formula for the L^2 -norm of the Ginzburg-Landau order parameter. The formula is valid in the bulk regime where the intensity of the applied magnetic field is of the same order as the Ginzburg-Landau parameter and strictly below the second critical field. Our formula complements the celebrated one of Sandier-Serfaty for the L^4 -norm.

1. INTRODUCTION AND MAIN RESULTS

The Ginzburg-Landau model. The Ginzburg-Landau functional is defined as the sum of two functionals, the *energy of the order parameter* and the *magnetic energy*. It reads as follows,

$$\mathcal{E}_{\text{GL}}(\psi, \mathbf{A}) = \mathcal{E}_{\text{op}}(\psi, \mathbf{A}) + \mathcal{E}_{\text{mag}}(\mathbf{A}), \quad (1.1)$$

where

$$\begin{aligned} \mathcal{E}_{\text{op}}(\psi, \mathbf{A}) &= \int_{\Omega} \left(|(\nabla - i\kappa H \mathbf{A})\psi|^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2} |\psi|^4 \right) dx, \\ \mathcal{E}_{\text{mag}}(\mathbf{A}) &= \kappa^2 H^2 \int_{\Omega} |\text{curl } \mathbf{A} - 1|^2 dx. \end{aligned} \quad (1.2)$$

Here:

- $\Omega \subset \mathbb{R}^2$ is an open, bounded and simply connected set with a C^∞ boundary; Ω is the cross section of a cylindrical superconducting sample placed vertically.
- $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)$ describes the state of superconductivity as follows: $|\psi|^2$ measures the local density of the superconducting Cooper pairs and $\text{curl } \mathbf{A}$ measures the induced magnetic field in the sample.
- $\kappa > 0$ is the Ginzburg-Landau parameter, a material characteristic of the sample.
- $H > 0$ measures the intensity of the applied magnetic field.
- The applied magnetic field is $\kappa H \vec{e}$, where $\vec{e} = (0, 0, 1)$.

We introduce the ground state energy of the functional in (1.1):

$$E_{\text{gs}}(\kappa, h_{\text{ex}}) = \inf \{ \mathcal{E}_{\text{GL}}(\psi, \mathbf{A}) : (\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2) \}. \quad (1.3)$$

For a given (κ, H) , a configuration $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)$ satisfying $\mathcal{E}_{\text{GL}}(\psi, \mathbf{A}) = E_{\text{gs}}(\kappa, h_{\text{ex}})$ is called a minimizer of the functional \mathcal{E}_{GL} and we will denote it by $(\psi, \mathbf{A})_{\kappa, H}$ to emphasize its dependence on κ and H . Such a minimizer is a solution of the following Ginzburg-Landau equations (we use the notation $\nabla^\perp = (\partial_{x_2}, -\partial_{x_1})$)

$$\begin{cases} -(\nabla - i\kappa H \mathbf{A})^2 \psi &= \kappa^2 (1 - |\psi|^2) \psi & \text{in } \Omega, \\ -\nabla^\perp \text{curl } \mathbf{A} &= (\kappa H)^{-1} \text{Im}(\overline{\psi} (\nabla - i\kappa H \mathbf{A}) \psi) & \text{in } \Omega, \\ \nu \cdot (\nabla - i\kappa H \mathbf{A}) \psi &= 0 & \text{on } \partial\Omega, \\ \text{curl } \mathbf{A} &= B_0 & \text{on } \partial\Omega. \end{cases} \quad (1.4)$$

Gauge invariant quantities. The physically relevant quantities, *density*, *induced magnetic field*, *energy* and *supercurrent* are invariant under the Gauge transformations. More precisely, the following quantities

$$|\psi|^2, \quad \operatorname{curl} \mathbf{A}, \quad |(\nabla - i\kappa H \mathbf{A})\psi|^2, \quad (1.5)$$

$$j(\psi, \mathbf{A}) = \operatorname{Re}(-i\bar{\psi}(\nabla - i\kappa H \mathbf{A})\psi), \quad (1.6)$$

are invariant under the transformation $(\psi, \mathbf{A}) \mapsto (e^{i\chi}, \mathbf{A} - \nabla\chi)$ for every given $\chi \in H^1(\Omega; \mathbb{R})$. This gauge invariance insures that all the quantities in (1.5) and (1.6) are smooth functions (cf. [23, Ch. 2]) when (ψ, \mathbf{A}) is a minimizer. The solution (ψ, \mathbf{A}) of (1.4) in the class such that $\operatorname{div} \mathbf{A} = 0$ in Ω and $\mathbf{A} \cdot \nu = 0$ on $\partial\Omega$ is indeed C^∞ .

Earlier results on the density. In this paper, we will study the asymptotics for the density in the following regime

$$H = b\kappa, \quad (1.7)$$

where $b \in (0, 1)$ is a **fixed** constant.

This corresponds to the situation of an external magnetic field with intensity strictly below the second critical field $H_{c_2}(\kappa) := \kappa$. The case where $b > 1$ in (1.7) is related to the phenomenon of surface superconductivity which is extensively studied by many authors [4, 8, 10, 22].

When (1.7) holds, Sandier-Serfaty [25] proved the following formula for the ground state energy in (1.3):

$$E_{\text{gs}}(\kappa, h_{\text{ex}}) = g(b)|\Omega|\kappa^2 + o(\kappa^2) \quad \text{as } \kappa \rightarrow +\infty, \quad (1.8)$$

where $g(b)$ is an implicitly defined quantity that depends only on b . Its precise definition will be given in (2.4). In particular, it satisfies:

$$g(0) = -\frac{1}{2}, \quad g(1) = 0 \quad \text{and} \quad g(b) < 0 \quad \text{for } b \in (0, 1).$$

The convergence in (1.8) is uniform with respect to b on every interval $[\epsilon, 1)$, $\epsilon > 0$. The uniform convergence fails on the interval $(0, 1)$. More details regarding the uniformity with respect to b are given by K. Attar in [5, 6].

Now suppose that (1.7) holds and that $(\psi, \mathbf{A})_{\kappa, H}$ is a minimizer of the functional in (1.1). The magnetic energy satisfies [5]:

$$\kappa^2 H^2 \int_{\Omega} |\operatorname{curl} \mathbf{A} - 1|^2 dx \leq C \kappa^{7/4}, \quad (1.9)$$

for $\kappa \geq \kappa_0$, where κ_0 and C are two constants that depend only on the domain Ω and the constant b in (1.7). Hence its contribution in the ground state energy is relatively small as $\kappa \rightarrow +\infty$.

Again, if $b \in [\epsilon, 1)$ for some $\epsilon > 0$, the constants κ_0 and C can be selected independently from b , but they will depend on ϵ . More details can be found in [5, 6], where it is allowed for ϵ to depend on κ , $\epsilon = \epsilon(\kappa)$, and approach 0 as $\kappa \rightarrow +\infty$.

Using the Ginzburg-Landau equation for ψ (see (1.4)), we get the following simple relation between the energy and the L^2 -norm of the density:

$$\mathcal{E}_{\text{op}}(\psi, \mathbf{A}) = -\frac{\kappa^2}{2} \int_{\Omega} |\psi(x)|^4 dx, \quad (1.10)$$

where \mathcal{E}_{op} is the energy of the order parameter introduced in (1.2). Consequently, combining the estimates in (1.8) and (1.9), we deduce the following formula regarding the L^2 -norm of the density [25]:

$$\int_{\Omega} |\psi(x)|^4 dx = -2g(b)|\Omega| + o(1) \quad \text{as } \kappa \rightarrow +\infty, \quad (1.11)$$

where the function $o(1)$ is dominated by a function $s(\kappa)$ such that $s(\kappa)$ is independent of the choice of the minimizer $(\psi, \mathbf{A})_{\kappa, H}$ and $s(\kappa) \rightarrow 0$ as $\kappa \rightarrow +\infty$. When $b \in [\epsilon, 1)$ for some $\epsilon > 0$, the function $s(\kappa)$ can be selected independently from b . More details can be found in [5, 6], where

the case $\epsilon = \epsilon(\kappa)$ tending to 0 is considered. In particular the comparison of $\epsilon(\kappa)$ with the first critical field $H_{c_1}(\kappa) \approx \frac{\ln \kappa}{\kappa}$ could play a role.

Furthermore, Sandier-Serfaty obtained the following weak-convergence of $|\psi|^4$ as $\kappa \rightarrow +\infty$ in the sense of distributions [25]:

$$|\psi|^4 \rightharpoonup -2g(b) \quad \text{in } \mathcal{D}'(\Omega). \quad (1.12)$$

Open questions. Note that for $b = 0$ in (1.7), i.e. $H = 0$, every minimizer $(\psi, \mathbf{A})_{\kappa, H}$ satisfies $|\psi| = 1$ and $\text{curl } \mathbf{A} = 1$. This is consistent with (1.11) and (1.9). Indeed, as $b \rightarrow 0_+$, we know that $g(b) \rightarrow -\frac{1}{2}$.

The regime $b \rightarrow 0_+$ (which corresponds to $H \ll \kappa$, see (1.7)) is thoroughly analyzed by Sandier-Serfaty in [26, 24]. In particular, it is proved that, for any minimizer $(\psi, \mathbf{A})_{\kappa, H}$, the density $|\psi|^2$ satisfies $|\psi|^2 \rightarrow 1$ in $L^2(\Omega)$ and it is close to 1 everywhere except in narrow regions of area $O(\kappa^{-1})$. The region where $|\psi|^2$ is not close to 1 consists of small defects accommodating isolated zeros of ψ , called *vortices*. These vortices are evenly distributed in the domain Ω along a lattice, and the distance between two vortices is $\approx H^{-1}$, much larger than κ^{-1} , the core size of the vortex.

The detailed analysis of the distribution of vortices is missing when (1.7) holds for a fixed constant $b \in (0, 1)$, even for small values of b . This is a challenging problem mainly for the following reason. For a minimizer $(\psi, \mathbf{A})_{\kappa, H}$, it is expected that ψ will have isolated zeros/vortices filling up all the domain Ω , but these zeros are separated by a distance $O(H^{-1}) = O(\kappa^{-1})$. At the same time, the core-size of every vortex is equal to $O(\kappa^{-1})$. Consequently, detecting the vortices in this regime becomes harder than when $H \ll \kappa$ (i.e. $b \rightarrow 0_+$ in (1.7)).

This problem is related to the one of the Abrikosov state near the critical field $H_{C_2} := \kappa$, where the transition to the normal state in the bulk occurs. This is visualized in the regime $b \rightarrow 1_-$ in (1.7) and is analyzed in many papers, [3, 12, 18, 19]. The same difficulty is encountered when trying to detect the vortices by the methods of Sandier-Serfaty, so that the analysis is shifted to the distribution of the density $|\psi|^2$ instead.

In this paper, we complement the results of Sandier-Serfaty by obtaining analogues of the formulas in (1.11) and (1.12) for the density $|\psi|^2$ (instead of the square of the density, $|\psi|^4$), in the regime where (1.7) holds for a fixed constant $b \in (0, 1)$. Besides that such results are new and do not follow from the analysis by Sandier-Serfaty [25], they might be helpful in the analysis of the vortices. Related to these results is the asymptotics of the supercurrent $j(\psi, \mathbf{A})$ when (1.7) holds. Even in the particular regime $H \ll \kappa$ (i.e. $b \ll 1$ in (1.7)), the analysis of the distribution of the super-current is missing. Actually, Sandier-Serfaty [23, Ch. 8, Corol. 8.1] prove only that, in the regime $\frac{\ln \kappa}{\kappa} \ll H \ll \kappa$, $\text{curl } j \rightarrow 0$ in $\mathcal{D}'(\Omega)$, as $\kappa \rightarrow +\infty$.

Main results. To state our main results, we recall some properties of g . The function g is increasing and concave (cf. [13, Thm. 2.1]). Consequently, g has at each point left- and right-sided derivatives $g'(b_-)$ and $g'(b_+)$ with

$$g'(b_+) \leq g'(b_-).$$

Therefore, we can introduce the set

$$\mathcal{R} = \{b \in (0, 1) : g'(b_-) = g'(b_+)\} \quad (1.13)$$

whose complement in the interval $(0, 1)$ is countable. Assuming that $b \in \mathcal{R}$ and (1.7) holds, we will prove that every minimizer $(\psi, \mathbf{A})_{\kappa, H}$ of the G-L functional in (1.1) satisfies (compare with (1.11))

$$\int_{\Omega} |\psi(x)|^2 dx = \left(g'(b) - 2g(b) \right) |\Omega| + o(1) \quad \text{as } \kappa \rightarrow +\infty. \quad (1.14)$$

The formula in (1.14) is consistent with the one given in [18, Eq. (1.6)] which is valid as $b \rightarrow 1_-$. We have indeed (see below (2.7)),

$$g(b) \sim E_{\text{Ab}}(b - 1)^2,$$

where $E_{Ab} \in [-\frac{1}{2}, 0)$ is a universal constant.

More precisely, our main result is:

Theorem 1.1. *Let $b \in (0, 1)$. There exist $\kappa_0 > 0$ and a function $\lambda : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\lim_{\kappa \rightarrow \infty} \lambda(\kappa) = 0$ and the following is true.*

If $(\psi, \mathbf{A})_{\kappa, H}$ is a minimizer of the functional in (1.1) for $H = b\kappa$ and $\kappa \geq \kappa_0$, then

(1)

$$g'(b_+) - \lambda(\kappa) \leq \frac{1}{\kappa^2 |\Omega|} \int_{\Omega} |(\nabla - i\kappa H)\mathbf{A}|^2 dx \leq g'(b_-) + \lambda(\kappa).$$

(2)

$$g'(b_+) - 2g(b) - \lambda(\kappa) \leq \frac{1}{|\Omega|} \int_{\Omega} |\psi(x)|^2 dx \leq g'(b_-) - 2g(b) + \lambda(\kappa) \quad \text{as } \kappa \rightarrow +\infty.$$

(3) *If $b \in \mathcal{R}$, then as $\kappa \rightarrow \infty$, the following convergence holds in the sense of distributions*

$$|\psi|^2 \rightharpoonup g'(b) - 2g(b) \quad \text{in } \mathcal{D}'(\Omega).$$

(4) *The supercurrent satisfies*

$$\frac{1}{\kappa^2 |\Omega|} \int_{\Omega} |j(\psi, \mathbf{A})|^2 dx \leq g'(b_-) + \lambda(\kappa),$$

and

$$\frac{1}{\kappa |\Omega|} \int_{\Omega} |j(\psi, \mathbf{A})| dx \leq \sqrt{g'(b_-)(g'(b_-) - 2g(b))} + \lambda(\kappa).$$

Remark 1.2. [On the leading order term]

The coefficient of the leading term in (1.14) does not vanish. Actually, $g'(b) \geq 0$ since g is increasing, and $g(b) < 0$ for $b \in (0, 1)$.

Remark 1.3. [On the L^2 -norm of $1 - |\psi|^2$]

Using (1.11) and Hölder's inequality, we get, for fixed b and as $\kappa \rightarrow +\infty$,

$$\frac{1}{|\Omega|} \int_{\Omega} |\psi(x)|^2 dx \leq |\Omega|^{-\frac{1}{2}} \left(\int_{\Omega} |\psi(x)|^4 dx \right)^{\frac{1}{2}} \leq (-2g(b))^{\frac{1}{2}} + o(1).$$

Combined with the lower bound in (1.14), we get (we use that $g'(b_+) \geq 0$)

$$-2g(b) - o(1) \leq \frac{1}{|\Omega|} \int_{\Omega} |\psi(x)|^2 dx \leq (-2g(b))^{\frac{1}{2}} + o(1).$$

Now we find the following estimate for the L^2 -norm of $1 - |\psi|^2$,

$$\frac{1}{|\Omega|} \int_{\Omega} (1 - |\psi(x)|^2)^2 dx \leq 1 + 2g(b) + o(1),$$

with the principal term on the right hand side approaching 0 as $b \rightarrow 0_+$, since

$$\lim_{b \rightarrow 0_+} g(b) = -\frac{1}{2}.$$

This is consistent with the behavior $|\psi|^2 \rightarrow 1$ in $L^2(\Omega)$ obtained in [26].

Remark 1.4. [On the potential energy]

When $b \in \mathcal{R}$ (see (1.13)), we get from Theorem 1.1 that the potential energy satisfies

$$\kappa^2 \int_{\Omega} \left(-|\psi(x)|^2 + \frac{1}{2} |\psi(x)|^4 \right) dx = \kappa^2 (g(b) - g'(b)) |\Omega| (1 + o(1)).$$

2. PRELIMINARIES

2.1. The bulk energy. Here we give the definition of the reference bulk energy $g(\cdot)$. This energy first appeared in [25] and was then extensively studied in [2, 13, 6, 7, 17].

Consider $b \in (0, +\infty)$, $r > 0$ and $Q_r = (-r/2, r/2) \times (-r/2, r/2)$. Define the functional,

$$F_{b, Q_r}(u) = \int_{Q_r} \left(b |(\nabla - i\mathbf{A}_0)u|^2 - |u|^2 + \frac{1}{2}|u|^4 \right) dx, \quad \text{for } u \in H^1(Q_r). \quad (2.1)$$

Here, \mathbf{A}_0 is the magnetic potential,

$$\mathbf{A}_0(x) = \frac{1}{2}(-x_2, x_1), \quad \text{for } x = (x_1, x_2) \in \mathbb{R}^2. \quad (2.2)$$

Define the two ground state energies,

$$\begin{aligned} e_D(b, r) &= \inf\{F_{b, Q_r}(u) : u \in H_0^1(Q_r)\}, \\ e_N(b, r) &= \inf\{F_{b, Q_r}(u) : u \in H^1(Q_r)\}. \end{aligned} \quad (2.3)$$

The function $g(\cdot)$ may be defined as follows (cf. [13, 25, 6]),

$$\forall b > 0, \quad g(b) = \lim_{r \rightarrow +\infty} \frac{e_D(b, r)}{|Q_r|} = \lim_{r \rightarrow \infty} \frac{e_N(b, r)}{|Q_r|}, \quad (2.4)$$

where $|Q_r|$ denotes the area of Q_r ($|Q_r| = r^2$). Furthermore, there exists a constant C such that, for all $r \geq 1$ and $b \in (0, 1)$,

$$g(b) \leq \frac{e_D(b, r)}{|Q_r|} \leq g(b) + C \frac{\sqrt{b}}{r} \quad \text{and} \quad e_D(b, R) - Cr\sqrt{b} \leq e_N(b, r) \leq e_D(b, r). \quad (2.5)$$

Various properties satisfied by the function $g(\cdot)$ are established in [7, 13, 20, 25]. In particular, the function $g(\cdot)$ is a non decreasing continuous and locally Lipschitz function such that

$$g(0) = -\frac{1}{2} \quad \text{and} \quad g(b) = 0 \quad \text{when } b \geq 1, \quad (2.6)$$

and

$$\lim_{b \rightarrow 1^-} \frac{g(b)}{(b-1)^2} = E_{\text{Ab}} \in \left[-\frac{1}{2}, 0\right). \quad (2.7)$$

2.2. A priori estimates and Gauge transformations. Here we collect useful estimates regarding the critical points of the Ginzburg-Landau functional (cf. [10, Prop. 10.3.1 and 11.4.4]).

Proposition 2.1. *Let $b \in (0, 1)$. There exist two constants $C > 0$ and $\kappa_0 > 0$ such that, if $\kappa \geq \kappa_0$, $H = b\kappa^2$ and $(\psi, \mathbf{A})_{\kappa, H}$ is a critical point of (1.3), then:*

$$\|\psi\|_\infty \leq 1, \quad (2.8)$$

$$\|(\nabla - i\kappa H \mathbf{A})\psi\|_{C(\bar{\Omega})} \leq C\kappa, \quad (2.9)$$

$$\|\text{curl } \mathbf{A} - 1\|_{C^1(\bar{\Omega})} \leq \frac{C}{\kappa}. \quad (2.10)$$

As a consequence of Proposition 2.1, we may pick a useful gauge transformation in every ball with small radius:

Proposition 2.2. *Let $b \in (0, 1)$. There exist two constants $C > 0$ and $\kappa_0 > 0$ such that, for any $x_0 \in \Omega$, there exists a function $\varphi_0 \in C^1(\Omega)$ such that*

$$\forall x \in \Omega, \quad \left| \mathbf{A}(x) - (\mathbf{A}_0(x - x_0) - \nabla\varphi_0(x)) \right| \leq \frac{C}{\kappa} \max(|x - x_0|, |x - x_0|^2),$$

where \mathbf{A}_0 is the vector field introduced in (2.2).

Proof. Let $B = \operatorname{curl} \mathbf{A}$. Choose a convex and open set $U \subset \mathbb{R}^2$ such that $\overline{\Omega} \subset U$. We may extend the function B to a function $B_{\text{ext}} : U \rightarrow \mathbb{R}$ such that

$$\operatorname{supp}(B_{\text{ext}}) \subset U \quad \text{and} \quad \|\nabla B_{\text{ext}}\|_{L^\infty(U)} \leq C \|\nabla B\|_{L^\infty(\Omega)}, \quad (2.11)$$

where C is a constant that depends solely on Ω and U (i.e. it is independent of B).

Define the vector field in Ω

$$\mathbf{G}(x) = 2 \left(\int_0^1 s B_{\text{ext}}(s(x - x_0) + x_0) ds \right) \mathbf{A}_0(x - x_0).$$

It is easy to check that

$$\operatorname{curl} \mathbf{G} = B_{\text{ext}} = B \quad \text{in } \Omega.$$

Consequently, since Ω is simply connected, there exists a smooth function φ_0 such that,

$$\mathbf{A}(x) = \mathbf{G}(x) - \nabla \varphi_0(x).$$

Using (2.10), (2.11) and the mean value theorem, we get further

$$|\mathbf{G}(x) - B_0(x_0) \mathbf{A}_0(x - x_0)| \leq \frac{C}{\kappa} |x - x_0|^2.$$

Again, using (2.10), we write $|(B_0(x_0) - 1) \mathbf{A}_0(x - x_0)| \leq C \kappa^{-1} |x - x_0|$. This yields the inequality

$$|\mathbf{G}(x) - \mathbf{A}_0(x - x_0)| \leq \frac{C}{\kappa} \max(|x - x_0|, |x - x_0|^2).$$

□

Remark 2.3. We will use the inequality in Proposition 2.2 for $|x - x_0| \leq \ell$ and $\ell \ll 1$, which in turn reads as follows

$$\left| \mathbf{A}(x) - (\mathbf{A}_0(x - x_0) - \nabla \varphi_0(x)) \right| \leq \frac{C}{\kappa} \ell.$$

3. ON THE LOCAL ENERGY OF MINIMIZERS

For any open set $D \subset \Omega$, we define the following local energy

$$\mathcal{E}_0(f, a; D) = \int_D \left(|\nabla - i\kappa H a| f|^2 - \kappa^2 |f|^2 + \frac{\kappa^2}{2} |f|^4 \right) dx. \quad (3.1)$$

For $x_0 \in \mathbb{R}^2$ and $\ell > 0$, $Q_\ell(x_0) = x_0 + (-\ell/2, \ell/2)^2$ denotes the square of center x_0 and side-length ℓ .

We will need the following result, essentially proved in [5] modulo a few adjustments.

Proposition 3.1. *If $b \in (0, 1)$, there exist positive constants C , R_0 , and $\kappa_0 > 0$, such that for $\kappa \geq \kappa_0$, $H = b\kappa$, $R_0 \kappa^{-1} \leq \ell \leq \kappa_0^{-1}$, $x_0 \in \Omega$, and if $\overline{Q_\ell(x_0)} \subset \Omega$, then the following inequalities hold*

$$\left| \frac{1}{|Q_\ell(x_0)|} \mathcal{E}_0 \left(e^{i\kappa H \varphi_0} \psi, \mathbf{A}_0^{x_0}; Q_\ell(x_0) \right) - \kappa^2 g(b) \right| \leq C \left(\ell + (\kappa \ell)^{-1} \right) \kappa^2,$$

and

$$\left| \frac{1}{|Q_\ell(x_0)|} \int_{Q_\ell(x_0)} |\psi(x)|^4 dx + 2g(b) \right| \leq C \left(\ell + (\kappa \ell)^{-1} \right),$$

where $\mathbf{A}_0^{x_0}(x) = \mathbf{A}_0(x - x_0)$, \mathbf{A}_0 is the vector field in (2.2), and φ_0 is the function constructed in Proposition 2.2.

Proof. In [5, Prop. 4.2 and 6.2], it is proved that

$$\left| \frac{1}{|Q_\ell(x_0)|} \mathcal{E}_0(\psi, \mathbf{A}; Q_\ell(x_0)) - \kappa^2 g(b) \right| \leq C(\ell + (\ell\kappa)^{-1})\kappa^2. \quad (3.2)$$

The estimate of the remainder term in [5] was worse because the magnetic field was assumed non-constant and a variant of the inequality in Proposition 2.2 was used (with a worse error as well). However, in our case of a constant magnetic field, we insert the inequality in Proposition 2.2 into the proof given in [5] and get the better remainder as in (3.2).

We write

$$\begin{aligned} \mathcal{E}_0(\psi, \mathbf{A}; Q_\ell(x_0)) &= \mathcal{E}_0(\psi, \mathbf{A}_0^{x_0} - \nabla\varphi_0 + (\mathbf{A} - \mathbf{A}_0^{x_0} + \nabla\varphi_0); Q_\ell(x_0)) \\ &\geq (1 - \ell) \mathcal{E}_0(\psi, \mathbf{A}_0^{x_0} - \nabla\varphi_0; Q_\ell(x_0)) \\ &\quad - \ell^{-1}\kappa^2 H^2 \int_{Q_\ell(x_0)} |\mathbf{A} - \mathbf{A}_0^{x_0} + \nabla\varphi_0|^2 |\psi|^2 dx - \ell\kappa^2 \int_{Q_\ell(x_0)} |\psi|^2 dx. \end{aligned}$$

Using the gauge invariance, the bound $|\psi| \leq 1$ and the inequality in Proposition 2.2, we get the following lower bound

$$\mathcal{E}_0(\psi, \mathbf{A}; Q_\ell(x_0)) \geq (1 - \ell) \mathcal{E}_0(e^{-\kappa H \varphi_0} \psi, \mathbf{A}_0^{x_0}; Q_\ell(x_0)) - C\kappa^2 \ell^3.$$

In a similar fashion, we prove the upper bound

$$\mathcal{E}_0(\psi, \mathbf{A}; Q_\ell(x_0)) \leq (1 + \ell) \mathcal{E}_0(e^{-\kappa H \varphi_0} \psi, \mathbf{A}_0^{x_0}; Q_\ell(x_0)) + C\kappa^2 \ell^3.$$

Inserting the foregoing lower and upper bounds into (3.2), we get the first inequality in Proposition 3.1.

Now we prove the second inequality in Proposition 3.1. We multiply the first G-L equation in (1.4) by $\bar{\psi}$ and integrate by parts in the integral over $Q_\ell(x_0)$. We get

$$-\frac{\kappa^2}{2} \int_{Q_\ell(x_0)} |\psi(x)|^4 dx = \mathcal{E}_0(\psi, \mathbf{A}; Q_\ell(x_0)) + \int_{\partial Q_\ell(x_0)} \bar{\psi} (\nu \cdot (\nabla - i\kappa H \mathbf{A})\psi) d\sigma(x).$$

Using the bounds $|\psi| \leq 1$ and $|(\nabla - i\kappa H \mathbf{A})\psi| \leq C\kappa$ in Proposition 2.1, we get that the boundary term is bounded by $\tilde{C}\kappa\ell$, where \tilde{C} is a constant.

Now, using (3.2), we get

$$\left| -\frac{\kappa^2}{2} \int_{Q_\ell(x_0)} |\psi(x)|^4 dx - g(b)\kappa^2 |Q_\ell(x_0)| \right| \leq C(\ell + (\kappa\ell)^{-1})\kappa^2 |Q_\ell(x_0)|.$$

□

4. PROOF OF THEOREM 1.1

Our proof of Theorem 1.1 has some similarities with the analysis of diamagnetism [11] and the computation of the quantum supercurrent [9].

For the proof of Theorem 1.1, it is easier to work with rescaled variables.

Definition 4.1. Let $x_0 \in \Omega$, $\ell > 0$ and $f \in H^1(\Omega)$ and suppose that $Q_\ell(x_0) \subset \Omega$. We define the new function \tilde{f} on $Q_{\ell\sqrt{\kappa H}} := Q_{\ell\sqrt{\kappa H}}(0)$ as follows:

$$\tilde{f}(y) = f\left(x_0 + \frac{y}{\sqrt{\kappa H}}\right).$$

For $H = b\kappa$ and $R = \ell\sqrt{\kappa H}$, we have the following relation:

$$\frac{1}{\kappa^2 |Q_\ell(x_0)|} \mathcal{E}_0(f, \mathbf{A}_0^{x_0}; Q_\ell(x_0)) = \frac{1}{|Q_R|} \int_{Q_R} \left(b|(\nabla - i\mathbf{A}_0)\tilde{f}|^2 - |\tilde{f}|^2 + \frac{1}{2}|\tilde{f}|^4 \right) dy. \quad (4.1)$$

Lemma 4.2. For $b \in (0, 1)$, there exist $\kappa_0, R_0 > 0$ and a positive-valued function $r(\cdot, \cdot)$ such that $\lim_{(t^{-1}, s) \rightarrow 0} r(t, s) = 0$ and the inequality

$$g'(b_+) - r(R, \ell) \leq \frac{1}{|Q_R|} \int_{Q_R} |(\nabla - i\mathbf{A}_0)\tilde{f}|^2 dy \leq g'(b_-) + r(R, \ell),$$

holds for (cf. Prop. 3.1)

$$f(x) = e^{i\kappa H \varphi_0} \psi(x),$$

$R = \ell\sqrt{\kappa H}$, $R_0\kappa^{-1} < \ell < \kappa_0^{-1}$, $\kappa \geq \kappa_0$, $H = b\kappa$ and $(\psi, \mathbf{A})_{\kappa, H}$ is a minimizer of the functional in (1.1).

Proof. Recall the definition of the function F_{b, Q_R} in (2.1). By (4.1) and Proposition 3.1,

$$F_{b, Q_R}(\tilde{f}) \leq g(b)|Q_R| + C(R + \ell R^2).$$

Let $\epsilon \in \mathbb{R} \setminus \{0\}$ such that $b + \epsilon \in (0, 1)$. Using (2.5), we get

$$F_{b+\epsilon, R}(\tilde{f}) \geq e_N(b + \epsilon, R) \geq g(b + \epsilon)|Q_R| - CR.$$

It is easy to notice that

$$\begin{aligned} \epsilon \int_{Q_R} |(\nabla - i\mathbf{A}_0)\tilde{f}|^2 dy &= F_{b+\epsilon, R}(\tilde{f}) - F_{b, R}(\tilde{f}) \\ &\geq (g(b + \epsilon) - g(b))|Q_R| - C(R + \ell R^2). \end{aligned} \tag{4.2}$$

For $\epsilon > 0$, we infer from (4.2) the lower bound

$$\int_{Q_R} |(\nabla - i\mathbf{A}_0)\tilde{f}|^2 dy \geq \frac{g(b + \epsilon) - g(b)}{\epsilon} |Q_R| - C\epsilon^{-1}(R + \ell R^2).$$

Choosing $\epsilon = \max(R^{-1/2}, \ell^{1/2})$, we get further

$$\int_{Q_R} |(\nabla - i\mathbf{A}_0)\tilde{f}|^2 dy \geq g'(b_+)|Q_R| - r_1(R, \ell)|Q_R|,$$

where

$$r_1(R, \ell) = C(R^{-1/2} + \ell^{1/2}) + \left| \frac{g(b + \epsilon) - g(b)}{\epsilon} - g'(b_+) \right| \rightarrow 0 \text{ as } (R^{-1}, \ell) \rightarrow 0.$$

In a similar fashion, we choose $\epsilon = -\max(R^{-1/2}, \ell^{1/2}) < 0$ and infer from (4.2) the upper bound

$$\int_{Q_R} |(\nabla - i\mathbf{A}_0)\tilde{f}|^2 dy \leq g'(b_-)|Q_R| + r_2(R)|Q_R|,$$

where

$$r_2(R, \ell) = C(R^{-1/2} + \ell^{1/2}) + \left| \frac{g(b + \epsilon) - g(b)}{\epsilon} - g'(b_-) \right| \rightarrow 0 \text{ as } (R^{-1}, \ell) \rightarrow 0.$$

To conclude, we choose $r(R) = \max(r_1(R, \ell), r_2(R, \ell))$.

□

Lemma 4.3. There exists a function $\tilde{r}(\cdot, \cdot)$ such that $\lim_{(t^{-1}, s) \rightarrow 0} \tilde{r}(t, s) = 0$ and, under the assumptions in Lemma 4.2, the following inequality holds

$$g'(b_+) - \tilde{r}(R, \ell) \leq \frac{1}{|Q_R|} \int_{Q_R} |\tilde{f}(y)|^2 dy \leq g'(b_-) + \tilde{r}(R, \ell).$$

Proof. By (4.1) and Proposition 3.1,

$$\left| F_{b, Q_R}(\tilde{f}) - g(b)|Q_R| \right| \leq CR^{3/2}.$$

By the formula for the L^4 -norm of ψ in Proposition 3.1 and a change of variables, we have

$$\left| \int_{Q_R} |\tilde{f}(y)|^4 dy + 2g(b)|Q_R| \right| \leq C(\ell + R^{-1})|Q_R|.$$

Combining the aforementioned formulae and the one in (4.2), we get the formula for the integral of $|\tilde{f}|^2$. \square

By rescaling, we deduce from Lemma 4.3:

Theorem 4.4. *Let $b \in (0, 1)$. There exist $C, R_0, \kappa_0 > 0$ and a positive-valued function $\lambda(\cdot)$ such that $\lim_{\kappa \rightarrow +\infty} \lambda(\kappa) = 0$ and the following is true.*

Suppose that

- $\kappa \geq \kappa_0$ and $H = b\kappa$;
- $R_0\kappa^{-1} \leq \ell \leq \kappa_0^{-1}$;
- Q_ℓ is the interior of a square of side length ℓ satisfying $\overline{Q_\ell} \subset \Omega$;
- $(\psi, \mathbf{A})_{\kappa, H}$ is a minimizer of the functional in (1.1).

Then the following inequalities hold

$$g'(b_+) - 2g(b) - \lambda(\kappa) \leq \frac{1}{|Q_\ell|} \int_{Q_\ell} |\psi(x)|^2 dx \leq g'(b_-) - 2g(b) + \lambda(\kappa).$$

Theorem 4.4 improves the results in [25], where only a non-optimal upper bound on the integral of $|\psi|^2$ is given (see [25, Eq. (1.18)]). In Theorem 4.4, we not only prove a lower bound on the integral of $|\psi|^2$, but also a matching upper bound in the case where $b \in \mathcal{R}$ (i.e. when $g'(b_+) = g'(b_-)$).

Proof of Theorem 1.1. The proof of the statements (2) and (3) regarding the estimate of the L^2 -norm of ψ and the weak convergence of $|\psi|^2$ both follow from Theorem 4.4 in a standard manner, see e.g. [5, Proof of Thm. 4.1].

Now, the proof of statement (1) regarding the L^2 -norm of the magnetic gradient is a consequence of statement (1) and the formulas in (1.10) and (1.11).

The first inequality in statement (4) regarding the supercurrent results from statement (1) and the following inequality

$$|j(\psi, \mathbf{A})| \leq |(\nabla - i\kappa H \mathbf{A})\psi|,$$

which is a consequence of the definition of the supercurrent in (1.6) and the inequality in (2.8).

The other inequality for the L^1 -norm of the supercurrent results from the inequality

$$|j(\psi, \mathbf{A})| \leq |\psi| |(\nabla - i\kappa H \mathbf{A})\psi|,$$

the Cauchy-Schwarz inequality and the conclusions in Statements (1) and (2). \square

5. NEW PROPERTIES OF THE FUNCTION g

5.1. Universal estimates of $g(b)$. As a by-product of the result in Theorem 1.1, we get new properties of the function $g(\cdot)$ introduced in (2.4).

Using the classical bound $|\psi| \leq 1$ (see (2.8)), we deduce from (1.14) that

$$\forall b \in (0, 1), \quad g'(b_+) - 2g(b) \leq 1. \quad (5.1)$$

We can obtain an upper bound on the left-derivative of g as well by expanding the square in the inequality $\int_{\Omega} (1 - |\psi(x)|^2)^2 dx \geq 0$ then using (1.11) and (1.14):

$$\forall b \in (0, 1), \quad g'(b_-) \leq \frac{1}{2} + g(b). \quad (5.2)$$

Note that (5.2) is better than (5.1) since $g'(b_+) \leq g'(b_-)$, $g(b) \geq -\frac{1}{2}$, hence $\frac{1}{2} + g(b) \leq 1 + 2g(b)$.

5.2. On the behavior of $g(b)$ as $b \rightarrow 0_+$. Taking the limit as $b \rightarrow 0_+$ in (5.1) and noticing that $g'(b_\pm) \geq 0$ and $g(0) = -\frac{1}{2}$, we get

$$\lim_{b \rightarrow 0_+} g'(b_\pm) = 0.$$

Consequently, there exists a sequence $(b_n)_{n \geq 1} \subset \mathcal{R}$ such that $b_n \rightarrow 0$ and $g'(b_n) \rightarrow 0$ (\mathcal{R} is defined in (2.5)). On the other hand, it is proved in [20] that as $b \rightarrow 0_+$,

$$g(b) = -\frac{1}{2} + \frac{b}{4} \ln \frac{1}{b} + o\left(b \ln \frac{1}{b}\right). \quad (5.3)$$

We deduce from this that:

- $g'(0_+) = +\infty$;
- the function $b \mapsto g'(b_+)$ is not continuous at 0;
- The asymptotics in (5.3) can not be differentiated, i.e. the formula

$$g'(b) \sim \frac{1}{4} \ln b - \frac{1}{4}$$

does not hold as $b \xrightarrow{b \in \mathcal{R}} 0_+$. Simply, the aforementioned sequence (b_n) violates this formula.

5.3. The radial symmetry. Next we try to extract more information about the function g by exploiting the radial symmetry. The function g may be expressed as follows

$$\forall b \in (0, 1), \quad g(b) = \lim_{R \rightarrow \infty} \frac{\epsilon_{\text{disc}}(b, R)}{\pi R^2}, \quad (5.4)$$

where

$$\epsilon_{\text{disc}}(b, R) = \inf\{F_{b, D_R}(u) : u \in H_0^1(D_R)\}, \quad (5.5)$$

$D_R = \{x \in \mathbb{R}^2 : |x| < R\}$ and F_{b, D_R} is the functional introduced in (2.1). The proof of (5.4) is standard (see [2, 13]). It follows by covering the disc $D(0, R)$ with squares $(Q_{R', j})_j$ with side-length $1 \ll R' \ll R$ and using the estimates in (2.5) (for $r = R'$). We omit the technical details.

We restrict the functional $F_{b, D_R}(u)$ on configurations of the form

$$u(r, \theta) = e^{im\theta} f(r), \quad (5.6)$$

where $f : (0, R) \rightarrow \mathbb{C}$, $m \in \mathbb{Z}$ and (r, θ) denote the polar coordinates.

Note that $u \in H_0^1(B(0, R))$ if and only if $f \in \mathcal{D}_{m, R}$, where

$$\mathcal{D}_{m, R} = \left\{ f : \sqrt{r} f', \sqrt{r} f, \frac{m}{\sqrt{r}} f \in L^2((0, R); \mathbb{R}), f(R) = 0 \right\}. \quad (5.7)$$

Furthermore,

$$F_{b, D_R}(u) = G_{m, b, R}(f),$$

where

$$G_{m, b, R}(f) = 2\pi \int_0^R \left(b|f'(r)|^2 + b\left(\frac{m}{r} - \frac{r}{2}\right)^2 |f(r)|^2 - |f(r)|^2 + \frac{1}{2}|f(r)|^4 \right) r dr. \quad (5.8)$$

Consequently, we define the following ground state energy

$$e^{1D}(m, b, R) = \inf\{G_{m, b, R}(f) : f \in \mathcal{D}_{m, R}\} \quad (5.9)$$

A minimizer $f_{m, b, R}$ exists, can be selected real-valued and non-negative (because $|f_{m, b, R}|$ is a minimizer too) and satisfies the following ODE

$$-f_{m, b, R}''(r) - \frac{1}{r} f_{m, b, R}'(r) + \left(\frac{m}{r} - \frac{r}{2}\right)^2 f_{m, b, R}(r) = \frac{1}{b} (1 - |f_{m, b, R}(r)|^2) f_{m, b, R}(r) \quad \text{in } (0, R). \quad (5.10)$$

When the magnetic field is absent (i.e. the term $\frac{r}{2}$ is dropped from (5.10)) and $R = +\infty$, (5.10) has been studied in many papers, for example [16].

Now we define

$$g_m(b) = \limsup_{R \rightarrow +\infty} \frac{\mathfrak{e}^{1D}(m, b, R)}{\pi R^2}. \quad (5.11)$$

We then have,

$$\forall b \in (0, 1), \quad \forall m \in \mathbb{Z}, \quad g(b) \leq g_m(b). \quad (5.12)$$

Remark 5.1. A natural question is then to determine if for any $b \in (0, 1)$ there exists $m \in \mathbb{Z}$ such that $g(b) = g_m(b)$ and if the discontinuity of g' corresponds to the case when two m 's satisfy this property.

6. EXTENSION TO THREE DIMENSIONAL DOMAINS

The result in Theorem 1.1 can be easily extended to the three dimensional Ginzburg-Landau model. In this section, $\Omega \subset \mathbb{R}^3$ denotes a bounded smooth open set with a smooth boundary. We introduce the Ginzburg-Landau functional in Ω as follows [10, 21],

$$\begin{aligned} \mathcal{E}^{3D}(\psi, \mathbf{A}) = \mathcal{E}_{\kappa, H}^{3D}(\psi, \mathbf{A}) = \int_{\Omega} \left[|(\nabla - i\kappa H \mathbf{A})\psi|^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2} |\psi|^4 \right] dx \\ + \kappa^2 H^2 \int_{\mathbb{R}^3} |\operatorname{curl} \mathbf{A} - \beta|^2 dx, \end{aligned} \quad (6.1)$$

where $\beta = (0, 0, 1)$.

The configuration (ψ, \mathbf{A}) belongs to the space $H^1(\Omega; \mathbb{C}) \times \dot{H}_{\operatorname{div}, \mathbf{F}}^1(\mathbb{R}^3)$ with $\dot{H}_{\operatorname{div}, \mathbf{F}}^1(\mathbb{R}^3)$ defined as follows. Let $\dot{H}^1(\mathbb{R}^3)$ be the homogeneous Sobolev space, i.e. the closure of $C_c^\infty(\mathbb{R}^3)$ under the norm $u \mapsto \|u\|_{\dot{H}^1(\mathbb{R}^3)} := \|\nabla u\|_{L^2(\mathbb{R}^3)}$. Let further $\mathbf{F}(x) = (-x_2/2, x_1/2, 0)$. Clearly $\operatorname{div} \mathbf{F} = 0$. We define the space,

$$\dot{H}_{\operatorname{div}, \mathbf{F}}^1(\mathbb{R}^3) = \{ \mathbf{A} : \operatorname{div} \mathbf{A} = 0, \text{ and } \mathbf{A} - \mathbf{F} \in \dot{H}^1(\mathbb{R}^3) \}. \quad (6.2)$$

Now we define the ground state energy,

$$E_{\text{gs}}(\kappa, h_{\text{ex}})(\kappa, H) = \inf \{ \mathcal{E}^{3D}(\psi, \mathbf{A}) : (\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times \dot{H}_{\operatorname{div}, \mathbf{F}}^1(\mathbb{R}^3) \}. \quad (6.3)$$

This energy is estimated in [13] when $H = b\kappa$, $b \in (0, 1)$ is a fixed constant and $\kappa \rightarrow \infty$. Using the methods in [13], we may easily adapt the proof of Theorems 1.1 and 4.4 to get the following result:

Theorem 6.1. *For $b \in (0, 1)$, there exist $C, R_0, \kappa_0 > 0$ and a positive-valued function $\lambda(\cdot)$ such that $\lim_{\kappa \rightarrow +\infty} \lambda(\kappa) = 0$ and the following is true.*

Suppose that

- $\kappa \geq \kappa_0$ and $H = b\kappa$;
- $R_0 \kappa^{-1} \leq \ell \leq \kappa_0^{-1}$;
- Q_ℓ is the interior of a cube of side length ℓ satisfying $\overline{Q_\ell} \subset \Omega$;
- $(\psi, \mathbf{A})_{\kappa, H}$ is a minimizer of the functional in (6.1).

Then the following inequalities hold

$$g'(b_+) - 2g(b) - \lambda(\kappa) \leq \frac{1}{|Q_\ell|} \int_{Q_\ell} |\psi|^2 dx \leq g'(b_-) - 2g(b) + \lambda(\kappa).$$

As a consequence of Theorem 6.1, we can get that the minimizer $(\psi, \mathbf{A})_{\kappa, H}$ satisfies the following weak convergence for $H = b\kappa$, $b \in \mathcal{R}$ and $\kappa \rightarrow \infty$:

$$|\psi|^2 \rightarrow g'(b) - 2g(b) \quad \text{in } \mathcal{D}'(\Omega).$$

This result is complementary to the results in [14] and [19] devoted respectively to the regimes $b > 1$ (surface superconductivity) and $b \rightarrow 1_-$ (bulk superconductivity near H_{C_2}) for three dimensional superconducting samples.

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(B. Helffer) LABORATOIRE JEAN LERAY, UNIVERSITÉ DE NANTES, 2 RUE DE LA HOUSSINIÈRE, 44322 NANTES (FRANCE) AND LABORATOIRE DE MATHÉMATIQUES, UNIV. PARIS-SUD.

E-mail address: bernard.helffer@univ-nantes.fr

(A. Kachmar) DEPARTMENT OF MATHEMATICS, LEBANESE UNIVERSITY, HADAT, LEBANON.

E-mail address: ayman.kashmar@gmail.com