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Low lying spectral gaps induced by slowly varying magnetic fields

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Horia D. Cornean[1], **Bernard Helffer**[2], **and Radu Purice**[3]

Abstract

Consider a periodic Schrödinger operator in two dimensions, perturbed by a weak magnetic field whose intensity slowly varies around a positive mean. We show in great generality that the bottom of the spectrum of the corresponding magnetic Schrödinger operator develops spectral islands separated by gaps, reminding of a Landau-level structure.

First, we construct an effective magnetic matrix which accurately describes the low lying spectrum of the full operator. The construction of this effective magnetic matrix does not require a gap in the spectrum of the non-magnetic operator, only that the first and the second Bloch eigenvalues never cross. The crossing case is more difficult and it will be considered elsewhere.

Secondly, we perform a detailed spectral analysis of the effective matrix using a gauge-covariant magnetic pseudo-differential calculus adapted for slowly varying magnetic fields.

1 Introduction

In this paper we analyze the gap structure appearing at the bottom of the spectrum of a two dimensional periodic Hamiltonian, which is perturbed by a magnetic field that is neither constant, nor vanishing at infinity, but which is supposed to have *'weak variation'*, in a sense made precise in Eq.(1.5) below. Our main purpose is to show the appearance of a structure of narrow spectral islands separated by open spectral gaps. We shall also investigate how the size of these spectral objects varies with the smallness and the weak variation of the magnetic field.

We therefore hope to contribute to the mathematical understanding of the so called Peierls substitution at weak magnetic fields [35]; this problem has been mathematically analyzed by various authors (Buslaev [6], Bellissard [3, 4], Nenciu [31, 32], Helffer-Sjöstrand [18, 38], Panati-Spohn-Teufel [34], Freund-Teufel [13], de Nittis-Lein [11], Cornean-Iftimie-Purice [24, 8]) in order to investigate the validity domain of various models developed by physicists like Kohn [27] and Luttinger [28]. An exhaustive discussion on the physical background of this problem can be found in [32].

1.1 Preliminaries

On the configuration space $\mathcal{X} := \mathbb{R}^2$ we consider a lattice $\Gamma \subset \mathcal{X}$ generated by two linearly independent vectors $\{e_1, e_2\} \subset \mathcal{X}$, and we also consider a smooth, Γ-periodic potential $V : \mathcal{X} \to \mathbb{R}$. The dual lattice of Γ is defined as

$$\Gamma_* := \{\gamma^* \in \mathcal{X}^* \mid \langle \gamma^*, \gamma \rangle / (2\pi) \in \mathbb{Z}\,,\ \forall \gamma \in \Gamma\}\,.$$

Let us fix an *elementary cell*:

$$E := \Big\{ y = \sum_{j=1}^{2} t_j e_j \in \mathbb{R}^2 \mid -1/2 \leq t_j < 1/2\,,\ \forall j \in \{1,2\} \Big\}\,.$$

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We consider the quotient group $\mathcal{X}/\Gamma$ that is canonically isomorphic to the 2-dimensional torus $\mathbb{T}$. The dual basis $\{e_1^*, e_2^*\} \subset \mathcal{X}^*$ is defined by $\langle e_j^*, e_k\rangle = (2\pi)\delta_{jk}$, and we have $\Gamma_* = \oplus_{j=1}^2 \mathbb{Z} e_j^*$. We define $\mathbb{T}_* := \mathcal{X}^*/\Gamma_*$ and E_* by

$$E_* := \Big\{\theta = \sum_{j=1}^2 t_j e_j^* \in \mathbb{R}^2 \ \mid\ -1/2 \le t_j < 1/2\,,\ \forall j \in \{1,2\}\Big\}\,.$$

Consider the differential operator $-\Delta + V$, which is essentially self-adjoint on the Schwartz set $\mathscr{S}(\mathcal{X})$. Denote by H^0 its self-adjoint extension in $\mathcal{H} := L^2(\mathcal{X})$. The map

$$\big(\mathscr{V}_\Gamma \varphi\big)(\theta, x) \ := \ |E|^{-1/2} \sum_{\gamma\in\Gamma} e^{-i<\theta, x-\gamma>} \varphi(x-\gamma)\,, \quad \forall\,(\theta, x) \in \mathcal{X}\times E_*, \forall\,\varphi \in \mathscr{S}(\mathcal{X})\,, \tag{1.1}$$

(where $|E|$ is the Lebesgue measure of the elementary cell E) induces a unitary operator $\mathscr{V}_\Gamma$ from $L^2(\mathcal{X})$ onto $L^2\big(E_*; L^2(\mathbb{T})\big)$. Its inverse $\mathscr{V}_\Gamma^{-1}$ is given by:

$$(\mathscr{V}_\Gamma^{-1}\psi)(x) = |E_*|^{-\frac{1}{2}} \int_{E_*} e^{i<\theta, x>} \psi(\theta, x)\, d\theta\,. \tag{1.2}$$

We know from the Bloch-Floquet theory (see for example [37]) the following:

1. We have a fibered structure:

$$\hat{H}^0 := \mathscr{V}_\Gamma H^0 \mathscr{V}_\Gamma^{-1} = \int_{E_*}^{\oplus} \hat{H}^0(\theta) d\theta, \quad \hat{H}^0(\theta) := \big(-i\nabla - \theta\big)^2 + V \text{ in } L^2(\mathbb{T})\,. \tag{1.3}$$

2. The map $E_* \ni \theta \mapsto \hat{H}^0(\theta)$ has an extension to $\mathcal{X}^*$ given by

$$\hat{H}^0(\theta + \gamma^*) = e^{i<\gamma^*, \cdot>} \hat{H}^0(\theta) e^{-i<\gamma^*, \cdot>},$$

and it is analytic in the norm resolvent topology.
3. There exists a family of continuous functions $E_* \ni \theta \mapsto \lambda_j(\theta) \in \mathbb{R}$ with periodic continuous extensions to $\mathcal{X}^* \supset E_*$, indexed by $j \in \mathbb{N}$ such that $\lambda_j(\theta) \le \lambda_{j+1}(\theta)$ for every $j \in \mathbb{N}$ and $\theta \in E_*$, and

$$\sigma\big(\hat{H}^0(\theta)\big) = \bigcup_{j\in\mathbb{N}} \{\lambda_j(\theta)\}.$$

4. There exists an orthonormal family of measurable eigenfunctions $E_* \ni \theta \mapsto \hat{\phi}_j(\theta, \cdot) \in L^2(\mathbb{T})$, $j \in \mathbb{N}$, such that $\|\hat{\phi}_j(\theta, \cdot)\|_{L^2(\mathbb{T})} = 1$ and $\hat{H}^0(\theta)\hat{\phi}_j(\theta, \cdot) = \lambda_j(\theta)\hat{\phi}_j(\theta, \cdot)$.

Remark 1.1. It was proved in [26] that $\lambda_0(\theta)$ is always simple in a neighborhood of $\theta = 0$ and has a nondegenerate global minimum on E_* at $\theta = 0$, minimum which we may take equal to zero (up to a shift in energy). For the convenience of the reader, we include a short proof of these facts in Appendix A.

We shall also need one of the following two conditions.

Hypothesis 1.2. *Either* $\sup(\lambda_0) < \inf(\lambda_1)$*, i.e. a non-crossing condition with a gap,*

or

Hypothesis 1.3. *The eigenvalue* $\lambda_0(\theta)$ *remains simple for all* $\theta \in \mathbb{T}_*$*, but* $\sup(\lambda_0) \ge \inf(\lambda_1)$*, i.e. a non-crossing condition with range overlapping and no gap.*

Because H^0 has a real symbol, we have $\overline{\hat{H}^0(\theta)} = \hat{H}^0(-\theta)$. Also, since $\lambda_0(\cdot)$ is simple, it must be an even function

$$\lambda_0(\theta) = \lambda_0(-\theta)\,. \tag{1.4}$$

1.2 The main result

Now let us consider the magnetic field perturbation, which is a 2-parameter family of magnetic fields

$$B_{\epsilon,\kappa}(x) \,:=\, \epsilon B_0 \,+\, \kappa\epsilon B(\epsilon x)\,, \tag{1.5}$$

indexed by $(\epsilon,\kappa)\in[0,1]\times[0,1]$.
Here $B_0>0$ is constant, while $B:\mathcal{X}\to\mathbb{R}$ is smooth and bounded together with all its derivatives. Let us choose some smooth *vector potentials* $A^0:\mathcal{X}\to\mathcal{X}$ and $A:\mathcal{X}\to\mathcal{X}$ such that:

$$B_0=\partial_1 A^0_2-\partial_2 A^0_1\,, B=\partial_1 A_2-\partial_2 A_1\,, \tag{1.6}$$

and

$$A^{\epsilon,\kappa}(x):=\epsilon A^0(x)+\kappa A(\epsilon x)\,,\; B_{\epsilon,\kappa}=\partial_1 A^{\epsilon,\kappa}_2-\partial_2 A^{\epsilon,\kappa}_1\,. \tag{1.7}$$

The vector potential A^0 is always in the *transverse gauge*, i.e.

$$A^0(x) \,=\, (B_0/2)\big(-x_2,x_1\big). \tag{1.8}$$

We consider the following magnetic Schrödinger operator, essentially self-adjoint on $\mathscr{S}(\mathcal{X})$:

$$H^{\epsilon,\kappa}:=(-i\partial_{x_1}-A^{\epsilon,\kappa}_1)^2+(-i\partial_{x_2}-A^{\epsilon,\kappa}_2)^2+V\,. \tag{1.9}$$

When $\kappa=\epsilon=0$ we recover the periodic Schrödinger Hamiltonian without magnetic field H^0.

Our main goal is to prove that for ϵ and κ small enough, the bottom of the spectrum of $H^{\epsilon,\kappa}$ develops gaps of width of order ϵ separated by spectral islands (non-empty but not necessarily connected) whose width is slightly smaller than ϵ, see below for the precise statement.

Theorem 1.4. *Consider either Hypothesis 1.2 or Hypothesis 1.3. Fix an integer $N>1$. Then there exist some constants $C_0,C_1,C_2>0$, and $\epsilon_0,\kappa_0\in(0,1)$, such that for any $\kappa\in(0,\kappa_0]$ and $\epsilon\in(0,\epsilon_0]$ there exist $a_0<b_0<a_1<\cdots<a_N<b_N$ with $a_0=\inf\{\sigma(H^{\epsilon,\kappa})\}$ so that:*

$$\sigma(H^{\epsilon,\kappa})\cap[a_0,b_N]\subset\bigcup_{k=0}^{N}[a_k,b_k]\,,\quad \dim\big(\mathrm{Ran}E_{[a_k,b_k]}(H^{\epsilon,\kappa})\big)=+\infty\,,$$

$$b_k-a_k\le C_0\,\kappa\epsilon+C_1\,\epsilon^{4/3}\,,\; 0\le k\le N\,,\quad\text{and}\quad a_{k+1}-b_k\ge\frac{1}{C_2}\epsilon\,,\; 0\le k\le N-1\,. \tag{1.10}$$

Moreover, given any compact set $K\subset\mathbb{R}$, there exists $C>0$, such that, for $(\kappa,\epsilon)\in[0,1]\times[0,1]$, we have (here dist_H means Hausdorff distance):

$$\mathrm{dist}_H\big(\sigma(H^{\epsilon,\kappa})\cap K,\sigma(H^{\epsilon,0})\cap K\big)\le C\sqrt{\kappa\epsilon}\,. \tag{1.11}$$

Remark 1.5. Exactly one of the two non-crossing conditions described in Hypothesis 1.2 and 1.3 is generically satisfied when the potential V does not have any special symmetries. The crossing case will be considered elsewhere.

Remark 1.6. A natural conjecture is that the spectrum is close (say modulo $o(\epsilon)$) to the union of the spectra of the Schrödinger operators with a constant magnetic field $\epsilon B_0+\epsilon\kappa\beta$ with β in the range of B. This leads us to the conjecture that the best C_0 in (1.10) could be the variation of B, i.e. $\sup B-\inf B$. Anyhow, from (1.10) we see that a condition for the appearance of gaps is $\kappa<1/(C_0C_2)$.

Remark 1.7. Most of this paper is dedicated to the proof of (1.10). The estimate (1.11) is a direct consequence of the results of [9], but we included it here in order to make a comparison with previous results obtained for constant magnetic fields (i.e. $\kappa=0$), where the values of the a_k's and b_k's from (1.10) are found with much better accuracy, see Theorem 2.2.

1.3 The plan of the paper

Some properties of the bottom of the spectrum of the non-magnetic periodic operator can be found in Appendix A. The magnetic quantization is summarized in Appendix B. Given a magnetic field B of class $BC^\infty(\mathcal{X})$ (i.e. bounded together with all its derivatives) and a choice of a vector potential A for it (i.e. such that $\operatorname{curl} A = B$), one can define a *twisted pseudo-differential calculus* (see [29]), that generalizes the standard Weyl calculus and associates with any symbol $F \in S^m_\rho(\Xi)$ the following operator in $\mathcal{H}$ (for all $u \in \mathscr{S}(\mathcal{X})$ and $x \in \mathcal{X}$):

$$\big(\mathfrak{Op}^A(F)u\big)(x) \;:=\; (2\pi)^{-2} \int_{\mathcal{X}} \int_{\mathcal{X}^*} e^{i\langle \xi, x-y\rangle} e^{-i\int_{[x,y]} A}\, F\left(\frac{x+y}{2}, \xi\right) u(y)\, d\xi\, dy\,, \tag{1.12}$$

where $\int_{[x,y]} A$ is the integral over the oriented segment $[x, y|$ of the 1-form associated with A. For $F(x,\xi) = \xi^2 + V(x)$, $\mathfrak{Op}^A(F)$ is the usual Schrödinger operator in (1.9).

For the convenience of the reader, we now list the main ideas appearing in the proof.

Step 1: Construction of an effective magnetic matrix, see Subsections 3.1 and 3.2. Because $\lambda_0(\theta)$ is always assumed to be isolated, one can associate with it an orthonormal projection π_0 which commutes with H^0. Note that π_0 might **not** be a spectral projection for H^0, unless there is a gap between the first band and the rest. But in both cases, the range of π_0 has a basis consisting of exponentially localized Wannier functions [7, 32, 12]. When ϵ and κ are small enough, we can construct an orthogonal system of exponentially localized *almost Wannier functions* starting from the unperturbed Wannier basis, and show that the corresponding orthogonal projection $\pi_0^{\epsilon,\kappa}$ is **almost invariant** for $H^{\epsilon,\kappa}$. Note that in the case with a gap, $\pi_0^{\epsilon,\kappa}$ can be constructed to be a true spectral projection for $H^{\epsilon,\kappa}$. Next, using a Feshbach-type argument, we prove that the low lying spectrum of $H^{\epsilon,\kappa}$ is at a Hausdorff distance of order ϵ^2 from the spectrum of the reduced operator $\pi_0^{\epsilon,\kappa} H^{\epsilon,\kappa} \pi_0^{\epsilon,\kappa}$.

In the basis of magnetic almost Wannier functions, the reduced operator $\pi_0^{\epsilon,\kappa} H^{\epsilon,\kappa} \pi_0^{\epsilon,\kappa}$ defines an *effective magnetic matrix* acting on $\ell^2(\Gamma)$. Hence if this magnetic matrix has spectral gaps of order ϵ, the same holds true for the bottom of the spectrum of $H^{\epsilon,\kappa}$.

Step 2: Replacing the magnetic matrix with a magnetic pseudodifferential operator with periodic symbol, see Subsection 3.3. Adapting the methods of [18, 8] in the case of a constant magnetic field ϵB_0, i.e. for $\kappa = 0$, one can define a *periodic magnetic Bloch band function* λ^ϵ which is a perturbation of order ϵ of λ_0. Considering this magnetic Bloch band function as a periodic symbol, we may define its *magnetic quantization* (see Subsection B.2) in the magnetic field $B_{\epsilon,\kappa}$, denoted by $\mathfrak{Op}^{A^{\epsilon,\kappa}}(\lambda^\epsilon)$. It turns out that the spectrum of $\mathfrak{Op}^{A^{\epsilon,\kappa}}(\lambda^\epsilon)$ is located at a Hausdorff distance of order $\kappa\epsilon$ from the spectrum of the effective operator $\pi_0^{\epsilon,\kappa} H^{\epsilon,\kappa} \pi_0^{\epsilon,\kappa}$. Hence if $\mathfrak{Op}^{A^{\epsilon,\kappa}}(\lambda^\epsilon)$ has gaps of order ϵ (provided that κ is smaller than some constant independent of ϵ), the same is true for the bottom of the spectrum of $H^{\epsilon,\kappa}$.

Step 3: Spectral analysis of $\mathfrak{Op}^{A^{\epsilon,\kappa}}(\lambda^\epsilon)$, see Section 4. Here we compare the spectrum of $\mathfrak{Op}^{A^{\epsilon,\kappa}}(\lambda^\epsilon)$ with the spectrum of a Landau-type quadratic symbol defined using the Hessian of λ^ϵ near its simple, isolated minimum; this is achieved by proving that the magnetic quantization of an explicitly defined symbol is in fact a *quasi-resolvent* for the magnetic quantization of λ^ϵ (see Subsection 4.3). The main technical result is Proposition 4.5.

An important technical component of the proof of Proposition 4.5 is the expansion of a magnetic Moyal calculus for symbols with *weak spatial variation* (see Appendix B.5.2), that replaces the Moyal calculus for a constant field as appearing in [5, 19]. Some often used notations and definitions are recalled in Appendix B.1.

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2 Previous results

When $\kappa = 0$ and under the gap-Hypothesis 1.2, some sharper spectral results were proved by B. Helffer and J. Sjöstrand in [18]. They showed that in this case the width of the spectral islands is of order $\mathcal{O}(\epsilon^\infty)$.

For the convenience of the reader, we recall the precise statement. Let $\lambda_0(\theta_1, \theta_2)$ denote the isolated eigenvalue and let $B_0 = 1$. Then if ϵ is small enough, $H^{\epsilon,0}$ has an isolated spectral island I_ϵ near the range of λ_0. Moreover, I_ϵ coincides with the spectrum of an *one-dimensional* ϵ-pseudo-differential operator, whose symbol λ_ϵ admits an asymptotic expansion in ϵ and whose principal symbol is λ_0 (see formula (6.13) in [18]). By ϵ-pseudo-differential operator, acting in $L^2(\mathbb{R})$, we mean the following object:

$$\big(\mathfrak{Op}^{\mathfrak{w}}_\epsilon(\lambda_\epsilon)u\big)(x) \;:=\; (2\pi\epsilon)^{-1} \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\langle \xi, x-y\rangle/\epsilon}\, \lambda_\epsilon\big(\frac{x+y}{2}, \xi\big) u(y)\, d\xi\, dy\,, \quad u \in \mathscr{S}(\mathbb{R})\,. \tag{2.1}$$

Note that the operator $\mathfrak{Op}^{\mathfrak{w}}_\epsilon(\lambda_\epsilon)$ can also be considered as the usual Weyl quantification of the symbol $(x, \xi) \mapsto \lambda_\epsilon(x, \epsilon\xi)$. It is important to note that we have quantified the symbol into an operator on $L^2(\mathbb{R})$. The spectral analysis of a Hamiltonian of the form $\mathfrak{Op}^w_\epsilon(\lambda)$ in $(1D)$ (that is standard for $\lambda(x, \xi) = x^2 + \xi^2$) has been extended to general Hamiltonians in [16] (see also [17]). Near the minimum of λ_0 one can perform a semi-classical analysis of the bottom of the spectrum under the hypothesis that the minimum is non degenerate, assumption which is satisfied in our case (see Appendix A). One can also perform a semi-classical analysis near each energy $E \in \lambda_0(\mathbb{R}^2)$ such that $\nabla\lambda_0 \neq 0$ on $\lambda_0^{-1}(E)$ (under an assumption that the connected components of $\lambda_0^{-1}(E)$ are compact) and prove a Bohr-Sommerfeld formula for the eigenvalues of one-well microlocal problems. Each time this implies as a byproduct the existence of gaps, the spectrum at the bottom being contained, due to the tunneling effect, inside the union of exponentially small (as $\epsilon \to 0$) intervals centered at a point asymptotically close to the value computed by the Bohr-Sommerfeld rule. This is indeed what is done in great detail in [17] and [19] under specific conditions on the potential and for the square lattice, which imply that $\lambda_0(\theta)$ is close in a suitable sense to $\mathbf{t}\,(\cos\theta_1 + \cos\theta_2))$, where $\mathbf{t}$ is a tunneling factor. See Section 9 in [17]. In our context, one of the main results in [18] is the following.

Theorem 2.1. *Under Hypothesis 1.2, given a positive integer $N > 1$ there exist $\epsilon_0 > 0$ and $C > 0$, such that for $\epsilon \in (0, \epsilon_0]$ there exist $a_0 < b_0 < ... < a_N < b_N$ such that $a_0 = \inf\{\sigma(H^{\epsilon,0})\}$ and:*

$$\sigma(H^{\epsilon,0}) \cap [a_0, b_N] \subset \bigcup_{k=0}^{N} [a_k, b_k]\,,$$

$$b_k - a_k = \mathcal{O}(\epsilon^\infty),\ 0 \le k \le N, \quad \text{and} \quad a_{k+1} - b_k \ge \epsilon/C,\ 0 \le k \le N-1\,.$$

Moreover,

$$\dim\big(\mathrm{Ran} E_{[a_k,b_k]}(H^{\epsilon,0})\big) = +\infty\,.$$

One can actually have the same description with $N = N(\epsilon)$ in an interval $[-\infty, E]$ under the following conditions that are always satisfied for $|E - \inf\lambda_0|$ small enough:

- $\lambda_0^{-1}((-\infty, E]) = \cup_{\gamma^* \in \Gamma^*} \tau_{\gamma^*} W_0(E)\,.$
- $W_0(E)$ is connected and contains $\theta = 0$ which is the unique critical point of λ_0 in $W_0(E)\,.$
- $\tau_{\gamma^*} W_0(E) \cap \tau_{\tilde\gamma^*} W_0(E) = \emptyset$ if $\gamma^* \neq \tilde\gamma^*\,.$

The proof in Section 2 in [17] gives also in each interval $[a_k, b_k]$ an orthonormal basis ϕ_{γ^*} (indexed by Γ^*) of the image of $E_{[a_k,b_k]}(H^{\epsilon,0})$ of functions ϕ_{γ^*} being localized near γ^*. Here we have identified this image with the image of $E_{[a_k,b_k]}(\mathfrak{Op}_\epsilon(\lambda_\epsilon))$ in $L^2(\mathbb{R})$. Finally these results imply the following statement.

Theorem 2.2. *Under Hypothesis 1.2 and the above assumptions, for any $L \in \mathbb{N}^*$, there exist $\epsilon_0 > 0$ and $C > 0$, such that for $\epsilon \in (0, \epsilon_0]$ there exist $N(\epsilon)$ and $a_0 < b_0 < ... < a_N < b_N$ such that $a_0 = \inf\{\sigma(H^{\epsilon,0})\}$ and:*

$$\sigma(H^{\epsilon,0}) \cap (-\infty, E) \subset \bigcup_{k=0}^{N} [a_k, b_k]\,,$$

$$|b_k - a_k| \leq C\,\epsilon^L \text{ for } 0 \leq k \leq N(\epsilon)\,, \quad \text{and} \quad a_{k+1} - b_k \geq \epsilon/C \text{ for } 0 \leq k \leq N(\epsilon) - 1\,.$$

Moreover a_k is determined by a Bohr-Sommerfeld rule

$$a_k = f((2k+1)\epsilon, \epsilon)\,,$$

where $t \mapsto f(t, \epsilon)$ has a complete expansion in powers of ϵ, $f(0,0) = \inf \lambda_0$, and $\partial_t f(0,0) \neq 0$ (see [16] or [17]).

Remark 2.3. We conjecture that Theorem 2.2 still holds with bands of size $\exp(-\frac{S}{\epsilon})$ for some $S > 0$. What is missing are some "microlocal" decay estimates which are only established in particular cases in [17] for Harper's like models. We conjecture that if in addition $E < \inf \lambda_1$, Theorem 2.2 is true even under the weaker Hypothesis 1.3.

Remark 2.4. The case of purely magnetic Schrödinger operators when the magnetic field has a global non-degenerate minimum has been treated in [15, 36].

3 The effective band Hamiltonian

3.1 The magnetic almost Wannier functions

We recall some results from [32, 18, 7, 8]. Under both Hypothesis 1.2 and 1.3, using the analyticity of the application $\mathcal{X}^* \ni \theta \to \hat{H}^0(\theta)$ in the norm resolvent sense and the contour integral formula for the spectral projection, it is well known (see for example Lemma 1.1 in [18]) that one can choose its first L^2-normalized eigenfunction as an analytic function $\mathcal{X}^* \ni \theta \to \hat{\phi}_0(\theta, \cdot) \in L^2(\mathbb{T})$ such that

$$\hat{\phi}_0(\theta + \gamma^*, x) = e^{i<\gamma^*, x>} \hat{\phi}_0(\theta, x)$$

and

$$\hat{H}^0(\theta)\, \hat{\phi}_0(\theta, \cdot) \;=\; \lambda_0(\theta)\, \hat{\phi}_0(\theta, \cdot)\,. \tag{3.1}$$

Then the principal *Wannier function* ϕ_0 is defined (see (1.2)) by:

$$\phi_0(x) = \left[\mathscr{V}_\Gamma^{-1} \hat{\phi}_0\right](x) \;=\; |E_*|^{-\frac{1}{2}} \int_{E_*} e^{i<\theta, x>} \hat{\phi}_0(\theta, x)\, d\theta\,. \tag{3.2}$$

ϕ_0 has rapid decay. In fact, using the analyticity property of $\theta \mapsto \hat{\phi}_0(\theta, \cdot)$ and deforming the contour of integration in (3.2) (cf (1.8) and (1.9) in [18]), we get the existence of $C > 0$ and for any $\alpha \in \mathbb{N}^2$ of $C_\alpha > 0$ such that

$$|\partial_x^\alpha \phi_0(x)| \leq C_\alpha\, \exp(-|x|/C)\,, \forall x \in \mathbb{R}^2\,.$$

We shall also consider the associated orthogonal projections

$$\hat{\pi}_0(\theta) := |\hat{\phi}_0(\theta, \cdot) >< \hat{\phi}_0(\theta, \cdot)|\,, \quad \pi_0 := \mathscr{V}_\Gamma^{-1} \left(\int_{E_*}^{\oplus} \hat{\pi}_0(\theta) d\theta \right) \mathscr{V}_\Gamma\,. \tag{3.3}$$

Then the family generated by ϕ_0 by translation with γ over the lattice Γ:

$$\phi_\gamma := \tau_{-\gamma} \phi_0$$

is an orthonormal basis for the subspace $\pi_0 \mathcal{H}$.

Remark 3.1. We note that under Hypothesis 1.2, π_0 is the spectral projection attached to the first simple band. This is no longer true under Hypothesis 1.3 where we might have $\inf(\lambda_1) < \sup(\lambda_0)$.

We now start the construction of some *magnetic almost Wannier functions*. In order to obtain the best properties for these functions adapted to the specific structure of the magnetic field (see (1.5)) we shall proceed in two steps. First we consider the constant magnetic field ϵB_0, and then we add the second perturbation $\kappa\epsilon B(\epsilon x)$.

Definition 3.2.

1. *For any $\gamma \in \Gamma$ and with A^0 defined in* (1.8) *we introduce:*

$$\overset{\circ}{\phi}{}^{\epsilon}_{\gamma}(x) \,:=\, \Lambda^{\epsilon}(x,\gamma)\phi_0(x-\gamma)\,, \quad \Lambda^{\epsilon}(x,y) \,:=\, \exp\left\{-i\,\epsilon\int_{[x,y]} A^0\right\}.$$

2. *$\widetilde{\pi}^{\epsilon}_0$ will denote the orthogonal projection on the closed linear span of $\{\overset{\circ}{\phi}{}^{\epsilon}_{\gamma}\}_{\gamma\in\Gamma}$.*

3. *$\mathbb{G}^{\epsilon}_{\alpha\beta} \,:=\, \langle \overset{\circ}{\phi}{}^{\epsilon}_{\alpha}, \overset{\circ}{\phi}{}^{\epsilon}_{\beta}\rangle_{\mathcal{H}}$ denotes the infinite Gramian matrix, indexed by $\Gamma\times\Gamma$.*

Due to the exponential decay of ϕ_0, we obtain (see Lemma 3.15 in [8]):

Proposition 3.3. *There exists $\epsilon_0 > 0$ such that, for any $\epsilon \in [0,\epsilon_0]$, the matrix $\mathbb{G}^{\epsilon}$ defines a positive bounded operator on $\ell^2(\Gamma)$. Moreover, for any $m \in \mathbb{N}$, there exists $C_m > 0$ such that*

$$\sup_{(\alpha,\beta)\in\Gamma\times\Gamma} <\alpha-\beta>^m \left|\mathbb{G}^{\epsilon}_{\alpha\beta} \,-\, \mathbb{1}\right| \,\leq\, C_m\,\epsilon\,. \tag{3.4}$$

Actually, by (4.5) in [18], we have even exponential decay but this is not needed in this paper. From (3.4) and the construction of the Wannier functions through the magnetic translations (cf Lemmas 3.1 and 3.2 in [7])), we obtain the following result:

Proposition 3.4. *For $\epsilon \in [0,\epsilon_0]$, $\mathbb{F}^{\epsilon} \,:=\, \left(\mathbb{G}^{\epsilon}\right)^{-1/2}$ has the following properties:*

1. *$\mathbb{F}^{\epsilon} \in \mathcal{L}(\ell^2(\Gamma)) \cap \mathcal{L}(\ell^{\infty}(\Gamma))$.*

2. *For any $m \in \mathbb{N}$, there exists $C_m > 0$ such that*

$$\sup_{(\alpha,\beta)\in\Gamma\times\Gamma} <\alpha-\beta>^m \left|\mathbb{F}^{\epsilon}_{\alpha\beta} \,-\, \mathbb{1}\right| \,\leq\, C_m\,\epsilon\,,\, \forall\epsilon\in[0,\epsilon_0]\,. \tag{3.5}$$

3. *There exists a rapidly decaying function $\mathbf{F}_{\epsilon} : \Gamma \to \mathbb{C}$ such that for any pair $(\alpha,\beta) \in \Gamma\times\Gamma$ we have:*

$$\mathbb{F}^{\epsilon}_{\alpha,\beta} = \Lambda^{\epsilon}(\beta,\alpha)\,\mathbf{F}_{\epsilon}(\beta-\alpha)\,.$$

Thus for all $\epsilon \in [0,\epsilon_0]$ we can define the following orthonormal basis of $\widetilde{\pi}^{\epsilon}_0\mathcal{H}$:

$$\phi^{\epsilon}_{\gamma} \,:=\, \sum_{\alpha\in\Gamma} \mathbb{F}^{\epsilon}_{\alpha\gamma}\,\overset{\circ}{\phi}{}^{\epsilon}_{\alpha}\,,\forall\gamma\in\Gamma\,. \tag{3.6}$$

Remark 3.5. Note that the operators $\{\Lambda^{\epsilon}(\cdot,\gamma)\tau_{-\gamma}\}_{\gamma\in\Gamma}$, appearing in Definition 3.2 (1), are the *magnetic translations* considered in [32] and in Section 2 (p. 147) of [18]. As proved there, these operators satisfy the following composition relation:

$$\left[\Lambda^{\epsilon}(\cdot,\alpha)\tau_{-\alpha}\right]\left[\Lambda^{\epsilon}(\cdot,\beta)\tau_{-\beta}\right] \,=\, \Lambda^{\epsilon}(\beta,\alpha)\left[\Lambda^{\epsilon}(\cdot,\alpha+\beta)\tau_{-\alpha-\beta}\right], \qquad \forall(\alpha,\beta)\in\Gamma\times\Gamma\,.$$

Remark 3.6. One can verify [18, 7] that for any $\gamma \in \Gamma$, $\Lambda^{\epsilon}(\cdot,\gamma)\tau_{-\gamma}$ commute with $H^{\epsilon,0}$ and with the *magnetic momenta* $\big(-i\partial_j - \epsilon A^0_j(x)\big)$.

Using the second point of Theorem 1.10 from [7] one shows that the second point of our Proposition 3.4 implies the next two statements.

Proposition 3.7. *With ψ_0^ϵ in $\mathscr{S}(\mathbb{R}^2)$ defined by*

$$\psi_0^\epsilon(x) = \sum_{\alpha\in\Gamma} \mathbf{F}^\epsilon(\alpha)\, \Omega^\epsilon(0,\alpha,x)\, \phi_0(x-\alpha)\,, \tag{3.7}$$

we have

$$\phi_\gamma^\epsilon \;=\; \Lambda^\epsilon(\cdot,\gamma)(\tau_{-\gamma}\psi_0^\epsilon)\,, \qquad \forall\gamma\in\Gamma\,.$$

Corollary 3.8. *There exists $\epsilon_0 > 0$ and, for any $m\in\mathbb{N}$, $\alpha\in\mathbb{N}^2$, there exists $C_{m,\alpha} > 0$ s. t.*

$$< x >^m\, |[\partial_x^\alpha(\psi_0^\epsilon - \phi_0)](x)| \,\leq\, C_{m,\alpha}\,\epsilon\,,$$

for any $\epsilon\in[0,\epsilon_0]$ and any $x\in\mathbb{R}$.

We are now ready to consider the case with $\kappa\neq 0$.

Definition 3.9. *The magnetic almost Wannier functions associated with $B_{\epsilon,\kappa}$ in* (1.5) *are defined as:*

$$\overset{\circ}{\phi}{}_\gamma^{\epsilon,\kappa} \;:=\; \Lambda^{\epsilon,\kappa}(\cdot,\gamma)\big(\tau_{-\gamma}\psi_0^\epsilon\big)\,,$$

where $\Lambda^{\epsilon,\kappa}$ is given by (B.7) *with $A_{\epsilon,\kappa}$ as defined in* (1.7)*.*

If we choose some smooth vector potential $A(y)$ such that $\operatorname{curl} A = B$ (we recall that $B_{\epsilon,\kappa} := \epsilon B_0 + \kappa\epsilon B(\epsilon x)$) and introduce

$$A_\epsilon(x) := A(\epsilon x) \;\text{ and }\; \widetilde{\Lambda}^{\epsilon,\kappa}(x,y) \;:=\; \exp\left\{-i\kappa\int_{[x,y]} A_\epsilon\right\}\,,$$

then we have the equality

$$\overset{\circ}{\phi}{}_\gamma^{\epsilon,\kappa} = \widetilde{\Lambda}^{\epsilon,\kappa}(\cdot,\gamma)\phi_\gamma^\epsilon\,.$$

The following statement is proved in Lemma 3.1 of [7]. It is based on the rapid decay of the Wannier functions and the polynomial growth of the derivatives of $\Omega^{\epsilon,\kappa}(\alpha,\beta,x)$ (see (B.9)).

Proposition 3.10. *There exists $\epsilon_0 > 0$ such that, for any $(\epsilon,\kappa)\in[0,\epsilon_0]\times[0,1]$, the Gramian matrix $\left(\mathbb{G}_{\alpha\beta}^{\epsilon,\kappa}\right)_{(\alpha,\beta)\in\Gamma^2}$ defined by*

$$\mathbb{G}_{\alpha\beta}^{\epsilon,\kappa} := \langle \overset{\circ}{\phi}{}_\alpha^{\epsilon,\kappa}, \overset{\circ}{\phi}{}_\beta^{\epsilon,\kappa}\rangle_{\mathcal{H}}$$

has the form:

$$\mathbb{G}_{\alpha\beta}^{\epsilon,\kappa} \;=\; \delta_{\alpha\beta} \;+\; \widetilde{\Lambda}^{\epsilon,\kappa}(\alpha,\beta)\mathbb{X}^{\epsilon,\kappa}(\alpha,\beta),$$

where, for all $m\in\mathbb{N}$, there exists $C_m > 0$ such that

$$|\mathbb{X}^{\epsilon,\kappa}(\alpha,\beta)| \;\leq\; C_m\,\kappa\epsilon < \alpha-\beta >^{-m},\; \forall(\alpha,\beta)\in\Gamma^2\,.$$

Definition 3.11. *If $\epsilon\in[0,\epsilon_0]$ and $\kappa\in[0,1]$ we define:*

1. $\widetilde{\phi}_\gamma^{\epsilon,\kappa} \;:=\; \sum_{\alpha\in\Gamma} \mathbb{F}_{\alpha\gamma}^{\epsilon,\kappa}\, \overset{\circ}{\phi}{}_\alpha^{\epsilon,\kappa}$ *with* $\mathbb{F}^{\epsilon,\kappa} \;:=\; \big[\mathbb{G}^{\epsilon,\kappa}\big]^{-1/2}$,

2. $\widetilde{\pi}_0^{\epsilon,\kappa}$ *to be the orthogonal projection on the closed linear span of* $\{\overset{\circ}{\phi}{}_\gamma^{\epsilon,\kappa}\}_{\gamma\in\Gamma}$.

3.2 The almost invariant magnetic subspace

We shall prove (see Proposition 3.12 below) that the orthogonal projection $\widetilde{\pi}_0^{\epsilon,\kappa}$ is almost invariant (modulo an error of order ϵ) for the Hamiltonian $H^{\epsilon,\kappa}$.

Proposition 3.12. *There exist $\epsilon_0 > 0$ and $C > 0$ such that, for any $(\epsilon,\kappa) \in [0,\epsilon_0] \times [0,1]$, the range of $\widetilde{\pi}_0^{\epsilon,\kappa}$ belongs to the domain of $H^{\epsilon,\kappa}$ and*

$$\big\| \big[H^{\epsilon,\kappa}, \widetilde{\pi}_0^{\epsilon,\kappa}\big] \big\|_{\mathcal{L}(\mathcal{H})} \leq C\,\epsilon\,.$$

Proof. All the Wannier-type functions introduced in Subsection 3.1 belong to $\mathscr{S}(\mathcal{X})$ and are in the domain of the respective Hamiltonians. We follow the ideas of Subsection 3.1 in [8] and compare the orthogonal projection $\widetilde{\pi}_0^{\epsilon,\kappa}$ with π_0. Let us denote by $p^{\epsilon,\kappa}$ (resp. p) the distribution symbol of the orthogonal projection $\widetilde{\pi}_0^{\epsilon,\kappa}$ (resp. π_0) for the corresponding quantizations, i.e.

$$\widetilde{\pi}_0^{\epsilon,\kappa} := \mathfrak{Op}^{\epsilon,\kappa}\big(p^{\epsilon,\kappa}\big), \quad \pi_0 := \mathfrak{Op}\big(p\big). \tag{3.8}$$

We have

$$\widetilde{\pi}_0^{\epsilon,\kappa} \;=\; \sum_{\gamma\in\Gamma} |\widetilde{\phi}_\gamma^{\epsilon,\kappa}\rangle\langle \widetilde{\phi}_\gamma^{\epsilon,\kappa}|\,, \qquad \pi_0 \;=\; \sum_{\gamma\in\Gamma} |\phi_\gamma\rangle\langle \phi_\gamma|\,. \tag{3.9}$$

For any $\gamma \in \Gamma$, both projections $|\widetilde{\phi}_\gamma^{\epsilon,\kappa}\rangle\langle \widetilde{\phi}_\gamma^{\epsilon,\kappa}|$ and $|\phi_\gamma\rangle\langle \phi_\gamma|$ are magnetic pseudodifferential operators with associated symbols $p_\gamma^{\epsilon,\kappa}$ and p_γ of class $\mathscr{S}(\Xi)$. We conclude then that the symbols

$$p^{\epsilon,\kappa} \;=\; \sum_{\gamma\in\Gamma} p_\gamma^{\epsilon,\kappa}, \qquad p \;=\; \sum_{\gamma\in\Gamma} p_\gamma\,, \tag{3.10}$$

where the series converge for the weak distribution topology, are in fact of class $S^{-\infty}(\Xi)$ (see Definition B.1 in Appendix B.1). Then

$$\big[H^{\epsilon,\kappa}, \widetilde{\pi}_0^{\epsilon,\kappa}\big] \;=\; \mathfrak{Op}^{\epsilon,\kappa}\big(h \,\sharp^{\epsilon,\kappa}\, p^{\epsilon,\kappa} - p^{\epsilon,\kappa} \,\sharp^{\epsilon,\kappa}\, h\big)\,, \tag{3.11}$$

and Proposition B.14 in Appendix B.4 shows that

$$h \,\sharp^{\epsilon,\kappa} p^{\epsilon,\kappa} = h \,\sharp\, p^{\epsilon,\kappa} + \kappa\epsilon\, r^{\epsilon,\kappa}(h, p^{\epsilon,\kappa})\,,$$

with $r^{\epsilon,\kappa}(h,p^{\epsilon,\kappa}) \in S^{-\infty}(\Xi)$ uniformly in $(\epsilon,\kappa) \in [0,\epsilon_0] \times [0,1]$ and similarly for $p^{\epsilon,\kappa} \,\sharp^{\epsilon,\kappa} h$.

By construction, we have the commutation relation

$$\big[H^0, \pi_0\big] \;=\; 0\,,$$

so that

$$h \,\sharp\, p - p \,\sharp\, h = 0\,.$$

Hence:

$$\big[H^{\epsilon,\kappa}, \widetilde{\pi}_0^{\epsilon,\kappa}\big] = \mathfrak{Op}^{\epsilon,\kappa}\big(h \,\sharp\, (p^{\epsilon,\kappa} - p) - (p^{\epsilon,\kappa} - p) \,\sharp\, h\big) \;+\; \kappa\epsilon\, \mathfrak{Op}^{\epsilon,\kappa}\big(r^{\epsilon,\kappa}(h,p^{\epsilon,\kappa}) - r^{\epsilon,\kappa}(p^{\epsilon,\kappa},h)\big)\,.$$

Finally let us compare the two regularizing symbols $p^{\epsilon,\kappa}$ and p. We notice that for any function $\psi \in \mathscr{S}(\mathcal{X})$ having L^2-norm equal to one, its associated 1-dimensional orthogonal projection $|\psi\rangle\langle\psi|$ has the integral kernel $K_\psi(x,y) := \psi(x)\,\overline{\psi(y)}$ and thus (see (B.14)) its magnetic symbol p_ψ^A is given by:

$$p_\psi^A(x,\xi) = (2\pi)^{-1/2} \int_{\mathcal{X}} e^{-i<\xi,z>} \psi\big(x + \frac{z}{2}\big)\, \overline{\psi\big(x - \frac{z}{2}\big)}\, \Lambda^A\big(x - \frac{z}{2}, x + \frac{z}{2}\big)\, dz\,.$$

Let us consider the difference $p_\gamma^{\epsilon,\kappa} - p_\gamma$ for some $\gamma \in \Gamma$ and compute

$$p_\gamma^{\epsilon,\kappa}(x,\xi) - p_\gamma(x,\xi) = (2\pi)^{-1/2}\int_{\mathcal{X}} e^{-i<\xi,z>}\times$$
$$\times\Big[\Lambda^{\epsilon,\kappa}\big(x-\frac{z}{2},x+\frac{z}{2}\big)\,\widetilde{\phi}_\gamma^{\epsilon,\kappa}\big(x+\frac{z}{2}\big)\,\overline{\widetilde{\phi}_\gamma^{\epsilon,\kappa}\big(x-\frac{z}{2}\big)}$$
$$-\phi_\gamma\big(x+\frac{z}{2}\big)\,\overline{\phi_\gamma\big(x-\frac{z}{2}\big)}\Big]\,dz\,.$$

In order to estimate the above difference we compare both terms with a third symbol $q_\gamma^{\epsilon,\kappa}$ associated with $\overset{\circ}{\phi}{}_\gamma^{\epsilon,\kappa}$ (see Definition 3.9):

$$q_\gamma^{\epsilon,\kappa}(x,\xi) \;:=\; (2\pi)^{-1/2}\int_{\mathcal{X}} e^{-i<\xi,z>}\Lambda^{\epsilon,\kappa}\big(x-\frac{z}{2},x+\frac{z}{2}\big)\,\overset{\circ}{\phi}{}_\gamma^{\epsilon,\kappa}\big(x+\frac{z}{2}\big)\,\overline{\overset{\circ}{\phi}{}_\gamma^{\epsilon,\kappa}\big(x-\frac{z}{2}\big)}\,dz$$

and estimate the difference

$$p_\gamma^{\epsilon,\kappa}(x,\xi) \;-\; q_\gamma^{\epsilon,\kappa}(x,\xi)$$
$$= (2\pi)^{-1/2}\sum_{(\alpha,\beta)\in\Gamma\times\Gamma}\widetilde{\Lambda}^{\epsilon,\kappa}(\gamma,\alpha)\mathbb{X}^{\epsilon,\kappa}(\gamma,\alpha)\widetilde{\Lambda}^{\epsilon,\kappa}(\beta,\gamma)\overline{\mathbb{X}^{\epsilon,\kappa}(\gamma,\beta)}\,s_{\alpha,\beta}^{\epsilon,\kappa}(x,\xi)\,,$$

with

$$s_{\alpha,\beta}^{\epsilon,\kappa}(x,\xi) := \int_{\mathcal{X}} e^{-i<\xi,z>}\Lambda^{\epsilon,\kappa}\big(x-\frac{z}{2},x+\frac{z}{2}\big)\;\overset{\circ}{\phi}{}_\alpha^{\epsilon,\kappa}\big(x+\frac{z}{2}\big)\,\overline{\overset{\circ}{\phi}{}_\beta^{\epsilon,\kappa}\big(x-\frac{z}{2}\big)}\,dz\,.$$

Let us consider the above integrals after using the Stokes Theorem as in (B.8):

$$s_{\alpha,\beta}^{\epsilon,\kappa}(x,\xi) \;=\; \Lambda^{\epsilon,\kappa}(\beta,\alpha)\int_{\mathcal{X}} e^{-i<\xi,z>}\,\Omega^{\epsilon,\kappa}\big(x-\tfrac{z}{2},\alpha,\beta\big)\Omega^{\epsilon,\kappa}\big(x-\tfrac{z}{2},x+\tfrac{z}{2},\alpha\big)\times$$
$$\times\,\psi_0^\epsilon\big(x+\tfrac{z}{2}-\alpha\big)\,\psi_0^\epsilon\big(x-\tfrac{z}{2}-\beta\big)\,dz\,.$$

Having in mind the estimate (B.9) we conclude that all the functions $s_{\alpha,\beta}^{\epsilon,\kappa}$ are symbols of class $S^{-\infty}(\Xi)$, uniformly for $(\epsilon,\kappa)\in[0,\epsilon_0]\times[0,1]$ and that, for any seminorm ν defining the topology of $S^{-\infty}(\Xi)$, there exist exponents $(a(\nu),b(\nu))\in\mathbb{N}^2$ and, for any pair $(N,M)\in\mathbb{N}^2$, a constant $C_{N,M}$ such that

$$\nu\big(s_{\alpha,\beta}^{\epsilon,\kappa}\big) \;\le\; C_{N,M}\,|\beta-\alpha|^{a(\nu)}\int_{\mathcal{X}}|z|^{b(\nu)}|x-\frac{z}{2}-\beta|^{-N}|x+\frac{z}{2}-\alpha|^{-M}\,dz\,.$$

Thus, choosing M and N large enough, we finally obtain that there exist $C_\nu>0$ and $p(\nu)\in\mathbb{N}$ such that:

$$\nu\big(s_{\alpha,\beta}^{\epsilon,\kappa}\big) \;\le\; C_\nu\,|\beta-\alpha|^{p(\nu)}, \qquad \forall(\epsilon,\kappa)\in[0,\epsilon_0]\times[0,1]\,,\,\forall(\alpha,\beta)\in\Gamma^2\,.$$

In order to control the convergence of the series in $\gamma\in\Gamma$ in the definition of $p^{\epsilon,\kappa}$, we need to consider some weights of the form $\rho_{n,\gamma}(x) :=< x-\gamma >^n$ for $(n,\gamma)\in\mathbb{N}\times\Gamma$ and notice that there exist $C_{\nu,N,M,n}>0$ and $p(\nu,n)\in\mathbb{N}$ such that:

$$\nu\big(\rho_{n,\gamma}s_{\alpha,\beta}^{\epsilon,\kappa}\big) \;\le\; C_{\nu,N,M,n}\,|\alpha-\gamma|^{p(\nu,n)}|\beta-\gamma|^{p(\nu,n)}, \qquad \forall(\epsilon,\kappa)\in[0,\epsilon_0]\times[0,1]\,,\,\forall(\alpha,\beta,\gamma)\in\Gamma^3\,.$$

From Proposition 3.10 we now conclude that $p_\gamma^{\epsilon,\kappa}-q_\gamma^{\epsilon,\kappa}$ are symbols of class $S^{-\infty}(\Xi)$ uniformly for $(\epsilon,\kappa)\in[0,\epsilon_0]\times[0,1]$ and for any seminorm ν defining the topology of this space we can find $C(\nu,n,N)$ and $\widehat{C}(\nu,n,N)$ such that:

$$\nu\big(\rho_{n,\gamma}(p_\gamma^{\epsilon,\kappa}-q_\gamma^{\epsilon,\kappa})\big) \le\; C(\nu,n,N)\,\kappa\epsilon\sum_{(\alpha,\beta)\in\Gamma\times\Gamma}<\alpha-\gamma>^{-N}<\beta-\gamma>^{-N}\le\;\widehat{C}(\nu,n,N)\,\kappa\epsilon\,,\;\forall\gamma\in\Gamma\,.$$

Finally, if we define $q^{\epsilon,\kappa} := \sum_{\gamma\in\Gamma} q_\gamma^{\epsilon,\kappa}$ and consider the weights $\rho_{n,\gamma}$ with $n\in\mathbb{N}$ large enough, we easily conclude that the series defining $p^{\epsilon,\kappa}-q^{\epsilon,\kappa}$ converges in the weak distribution topology to a limit that is a symbol in $S^{-\infty}(\Xi)$ having the defining seminorms of order $\kappa\epsilon$.

We still have to estimate the difference $q^{\epsilon,\kappa} - p$ in $S^{-\infty}(\Xi)$. We have, with $\psi^\epsilon_\gamma := \tau_{-\gamma}\psi^\epsilon_0$,

$$\begin{aligned} q^{\epsilon,\kappa}_\gamma(x,\xi) - \ p_\gamma(x,\xi) \ &= (2\pi)^{-1/2}\int_{\mathcal{X}} e^{-i<\xi,z>}\Lambda^{\epsilon,\kappa}\big(x-\tfrac{z}{2}, x+\tfrac{z}{2}\big)\,\overset{\circ}{\phi}{}^{\epsilon,\kappa}_\gamma\big(x+\tfrac{z}{2}\big)\overline{\overset{\circ}{\phi}{}^{\epsilon,\kappa}_\gamma\big(x-\tfrac{z}{2}\big)}\,dz \\ &\quad -(2\pi)^{-1/2}\int_{\mathcal{X}} e^{-i<\xi,z>}\phi_\gamma\big(x+\tfrac{z}{2}\big)\,\overline{\phi_\gamma\big(x-\tfrac{z}{2}\big)}\,dz \\ &= (2\pi)^{-1/2}\int_{\mathcal{X}} e^{-i<\xi,z>}\Omega^{\epsilon,\kappa}\big(x-\tfrac{z}{2}, x+\tfrac{z}{2},\gamma\big)\,\psi^\epsilon_\gamma\big(x+\tfrac{z}{2}\big)\,\overline{\psi^\epsilon_\gamma\big(x-\tfrac{z}{2}\big)}\,dz \\ &\quad -(2\pi)^{-1/2}\int_{\mathcal{X}} e^{-i<\xi,z>}\phi_\gamma\big(x+\tfrac{z}{2}\big)\,\overline{\phi_\gamma\big(x-\tfrac{z}{2}\big)}\,dz\,. \end{aligned}$$

Hence

$$\begin{aligned} &q^{\epsilon,\kappa}_\gamma(x,\xi) - \ p_\gamma(x,\xi) \\ &\quad = (2\pi)^{-1/2}\int_{\mathcal{X}} e^{-i<\xi,z>}\left[\Omega^{\epsilon,\kappa}\big(x-\tfrac{z}{2}, x+\tfrac{z}{2},\gamma\big) \ - \ 1\right]\psi^\epsilon_\gamma\big(x+\tfrac{z}{2}\big)\,\overline{\psi^\epsilon_\gamma\big(x-\tfrac{z}{2}\big)}\,dz \\ &\quad +(2\pi)^{-1/2}\int_{\mathcal{X}} e^{-i<\xi,z>}\left[\psi^\epsilon_\gamma\big(x+\tfrac{z}{2}\big)\,\overline{\psi^\epsilon_\gamma\big(x-\tfrac{z}{2}\big)} \ - \ \phi_\gamma\big(x+\tfrac{z}{2}\big)\,\overline{\phi_\gamma\big(x-\tfrac{z}{2}\big)}\right]dz\,. \end{aligned}$$

Let us first consider the second integrand and use (3.7) and (3.5) in order to get the estimate

$$< x >^n \,|\psi^\epsilon_0\big(x+\frac{z}{2}\big)\,\overline{\psi^\epsilon_0\big(x-\frac{z}{2}\big)} \ - \ \phi_0\big(x+\frac{z}{2}\big)\,\overline{\phi_0\big(x-\frac{z}{2}\big)}\,| \ \leq \ C_n\,\epsilon\,.$$

Using once again (B.9) and the above estimate, arguments very similar to the above ones allow us to prove that $q^{\epsilon,\kappa} - p$ is in $S^{-\infty}(\Xi)$ uniformly for $(\epsilon,\kappa) \in [0,\epsilon_0]\times[0,1]$ and for any seminorm ν defining the topology of $S^{-\infty}(\Xi)$ there is a constant $C(\nu)$ such that:

$$\nu\big(q^{\epsilon,\kappa} - p\big) \ \leq \ C(\nu)\,\epsilon\,. \tag{3.12}$$

Summarizing we have proved that $p^{\epsilon,\kappa} - p$ is in $S^{-\infty}(\Xi)$ uniformly for $(\epsilon,\kappa) \in [0,\epsilon_0]\times[0,1]$ and that, for any seminorm ν defining the topology of $S^{-\infty}(\Xi)$, there exists $C(\nu)$ such that:

$$\nu\big(p^{\epsilon,\kappa} - p\big) \ \leq \ C(\nu)\,\epsilon\,. \tag{3.13}$$

In order to finish the proof we still have to control the operator norms of $\mathfrak{Op}^{\epsilon,\kappa}\big(h\,\sharp\,(p^{\epsilon,\kappa} - p)\big)$ and $\mathfrak{Op}^{\epsilon,\kappa}\big((p^{\epsilon,\kappa} - p)\,\sharp\,h\big)$. But the above results and the usual theorem on Moyal compositions of Hörmander type symbols imply that $h\,\sharp\,(p^{\epsilon,\kappa} - p)$ and $(p^{\epsilon,\kappa} - p)\,\sharp\,h$ are symbols of type $S^{-\infty}(\Xi)$ uniformly for $(\epsilon,\kappa) \in [0,\epsilon_0]\times[0,1]$ and all their seminorms (defining the topology of $S^{-\infty}(\Xi)$) are of order ϵ. Finally, by Theorem 3.1 and Remark 3.2 in [22] we know that the operator norm is bounded by some symbol seminorm and thus will be of order ϵ. □

Definition 3.13. *We call quasi-band magnetic Hamiltonian, the operator $\widetilde{\pi}^{\epsilon,\kappa}_0 H^{\epsilon,\kappa}\widetilde{\pi}^{\epsilon,\kappa}_0$ and quasi-band magnetic matrix, its expression in the orthonormal basis $\{\widetilde{\phi}^{\epsilon,\kappa}_\gamma\}_{\gamma\in\Gamma}$.*

In order to apply a Feshbach type argument we need to control the invertibility on the orthogonal complement of $\widetilde{\pi}^{\epsilon,\kappa}_0\mathcal{H}$. Let us introduce

$$\widetilde{\pi}^{\epsilon,\kappa}_\perp \ := \ \mathbb{1} \ - \ \widetilde{\pi}^{\epsilon,\kappa}_0, \quad m_1 := \inf_{\theta\in\mathbb{T}_*}\lambda_1(\theta)\,, \tag{3.14}$$

where λ_1 is the second Bloch eigenvalue. Define:

$$K^{\epsilon,\kappa} \ := \ H^{\epsilon,\kappa} \ + \ m_1\,\widetilde{\pi}^{\epsilon,\kappa}_0\,. \tag{3.15}$$

We have:

Proposition 3.14. *There exist ϵ_0 and $C > 0$ such that, for $\epsilon \in [0,\epsilon_0]$,*

$$K^{\epsilon,\kappa} \geq m_1 - C\epsilon > 0\,.$$

Proof. Using (3.13), the conclusion just above it and the notation introduced in the proof of Proposition 3.12 we can write

$$K^{\epsilon,\kappa} \,=\, \mathfrak{Op}^{\epsilon,\kappa}\big(h+m_1p^{\epsilon,\kappa}\big) \,=\, \mathfrak{Op}^{\epsilon,\kappa}\big(h+m_1p\big) \,+\epsilon R_{\epsilon,\kappa}\,,$$

with $\|R_{\epsilon,\kappa}\|_{\mathcal{L}(\mathcal{H})}$ bounded uniformly in $(\epsilon,\kappa)\in[0,\epsilon_0]\times[0,1]$.
By Corollary 1.6 in [10] we have

$$\big|\inf\sigma\big(\mathfrak{Op}^{\epsilon,\kappa}\big(h+m_1p\big)\big)-\inf\sigma\big(\mathfrak{Op}\big(h+m_1p\big)\big)\big| \,\leq\, C\,\epsilon\,.$$

Since $\mathfrak{Op}(h)$ commutes with $\mathfrak{Op}(p)$, we have

$$\inf\sigma\big(\mathfrak{Op}\big(h+m_1p\big)\big)=\inf\sigma\big(H^0+m_1\pi_0\big)=m_1\,,$$

and we are done. □

An immediate consequence is the existence of $\epsilon_0>0$ such that, if $\epsilon\in[0,\epsilon_0]$ and $\Re\mathtt{z}\leq\frac{2}{3}m_1$, the operator $K^{\epsilon,\kappa}-\mathtt{z}$ is invertible on $\mathcal{H}$ with a uniformly bounded inverse $R^{\epsilon,\kappa}_{\mathtt{z}}$ in $\mathcal{L}(\mathcal{H})$.

Proposition 3.15. *There exists $\epsilon_0>0$ such that for $\epsilon\in[0,\epsilon_0]$, the Hausdorff distance between the spectra of $H^{\epsilon,\kappa}$ and $\widetilde{\pi}^{\epsilon,\kappa}_0H^{\epsilon,\kappa}\widetilde{\pi}^{\epsilon,\kappa}_0$, both restricted to the interval $[0,\frac{m_1}{2}]$, is of order ϵ^2.*

Proof.
Step 1.
We first establish the invertibility of $\widetilde{\pi}^{\epsilon,\kappa}_\perp\big(H^{\epsilon,\kappa}-\mathtt{z}\big)\widetilde{\pi}^{\epsilon,\kappa}_\perp$ on the range of $\widetilde{\pi}^{\epsilon,\kappa}_\perp$. We have

$$\widetilde{\pi}^{\epsilon,\kappa}_\perp\big(K^{\epsilon,\kappa}-\mathtt{z}\big)\widetilde{\pi}^{\epsilon,\kappa}_\perp \,=\, \widetilde{\pi}^{\epsilon,\kappa}_\perp\big(H^{\epsilon,\kappa}-\mathtt{z}\big)\widetilde{\pi}^{\epsilon,\kappa}_\perp.$$

We also observe that:

$$\big(\widetilde{\pi}^{\epsilon,\kappa}_\perp R^{\epsilon,\kappa}_{\mathtt{z}}\widetilde{\pi}^{\epsilon,\kappa}_\perp\big)\,\big(\widetilde{\pi}^{\epsilon,\kappa}_\perp\big(K^{\epsilon,\kappa}-\mathtt{z}\big)\widetilde{\pi}^{\epsilon,\kappa}_\perp\big)=\widetilde{\pi}^{\epsilon,\kappa}_\perp\big(\mathbb{1}-R^{\epsilon,\kappa}_{\mathtt{z}}[H^{\epsilon,\kappa},\widetilde{\pi}^{\epsilon,\kappa}_0]\big)\widetilde{\pi}^{\epsilon,\kappa}_\perp\,.$$

A similar formula holds for $\big(\widetilde{\pi}^{\epsilon,\kappa}_\perp\big(K^{\epsilon,\kappa}-\mathtt{z}\big)\widetilde{\pi}^{\epsilon,\kappa}_\perp\big)\,\big(\widetilde{\pi}^{\epsilon,\kappa}_\perp R^{\epsilon,\kappa}_{\mathtt{z}}\widetilde{\pi}^{\epsilon,\kappa}_\perp\big)$.
Using Proposition 3.12 we conclude that $\widetilde{\pi}^{\epsilon,\kappa}_\perp\big(H^{\epsilon,\kappa}-\mathtt{z}\big)\widetilde{\pi}^{\epsilon,\kappa}_\perp$ is invertible in the subspace $\widetilde{\pi}^{\epsilon,\kappa}_\perp\mathcal{H}$ and its inverse $R^{\epsilon,\kappa}_{\mathtt{z},\perp}$ verifies the estimate

$$\Big\|R^{\epsilon,\kappa}_{\mathtt{z},\perp}-\widetilde{\pi}^{\epsilon,\kappa}_\perp R^{\epsilon,\kappa}_{\mathtt{z}}\widetilde{\pi}^{\epsilon,\kappa}_\perp\Big\|_{\mathcal{L}(\widetilde{\pi}^{\epsilon,\kappa}_\perp\mathcal{H})}\,\leq\,C\,\epsilon\,.$$

Step 2.
The Feshbach inversion formula implies that if $\Re\mathtt{z}\leq\frac{m_1}{2}$, the operator $H^{\epsilon,\kappa}-\mathtt{z}$ is invertible in $\mathcal{H}$ if and only if the operator

$$T^{\epsilon,\kappa}(\mathtt{z}):=\widetilde{\pi}^{\epsilon,\kappa}_0H^{\epsilon,\kappa}\widetilde{\pi}^{\epsilon,\kappa}_0-\mathtt{z}\widetilde{\pi}^{\epsilon,\kappa}_0-\big(\widetilde{\pi}^{\epsilon,\kappa}_0H^{\epsilon,\kappa}\,\widetilde{\pi}^{\epsilon,\kappa}_\perp\big)\;R^{\epsilon,\kappa}_{\mathtt{z},\perp}\;\big(\widetilde{\pi}^{\epsilon,\kappa}_\perp\,H^{\epsilon,\kappa}\,\widetilde{\pi}^{\epsilon,\kappa}_0\big) \tag{3.16}$$

is invertible in $\widetilde{\pi}^{\epsilon,\kappa}_0\mathcal{H}$. Since $\widetilde{\pi}^{\epsilon,\kappa}_0H^{\epsilon,\kappa}\widetilde{\pi}^{\epsilon,\kappa}_\perp=\widetilde{\pi}^{\epsilon,\kappa}_0\,[\widetilde{\pi}^{\epsilon,\kappa}_0,H^{\epsilon,\kappa}]\,\widetilde{\pi}^{\epsilon,\kappa}_\perp$ and using once again Proposition 3.12, we get that the last term in (3.16) has a norm which is bounded by $C'\epsilon^2$, with C' denoting a generic constant, uniformly in $\mathtt{z}\in[0,m_1/2]$.

First, we assume that $\mathtt{z}\in[0,m_1/2]\setminus\sigma(\widetilde{\pi}^{\epsilon,\kappa}_0H^{\epsilon,\kappa}\widetilde{\pi}^{\epsilon,\kappa}_0)$. A Neumann series argument implies that if the distance between $\mathtt{z}$ and $\sigma(\widetilde{\pi}^{\epsilon,\kappa}_0H^{\epsilon,\kappa}\widetilde{\pi}^{\epsilon,\kappa}_0)$ is larger than $C'\epsilon^2$, then $T^{\epsilon,\kappa}(\mathtt{z})$ is invertible, hence $\mathtt{z}$ is also in the resolvent set of $H^{\epsilon,\kappa}$.

Secondly, we assume that $\mathtt{z}\in[0,m_1/2]\setminus\sigma(H^{\epsilon,\kappa})$ and moreover, the distance between $\mathtt{z}$ and $\sigma(H^{\epsilon,\kappa})$ is larger than $C'\epsilon^2$. From the Feshbach formula we get that $T^{\epsilon,\kappa}(\mathtt{z})^{-1}$ exists and

$$T^{\epsilon,\kappa}(\mathtt{z})^{-1}=\widetilde{\pi}^{\epsilon,\kappa}_0(H^{\epsilon,\kappa}-\mathtt{z})^{-1}\widetilde{\pi}^{\epsilon,\kappa}_0,\quad ||T^{\epsilon,\kappa}(\mathtt{z})^{-1}||<C'^{-1}\epsilon^{-2}.$$

Then (3.16) implies

$$\widetilde{\pi}^{\epsilon,\kappa}_0H^{\epsilon,\kappa}\widetilde{\pi}^{\epsilon,\kappa}_0-\mathtt{z}\widetilde{\pi}^{\epsilon,\kappa}_0=T^{\epsilon,\kappa}(\mathtt{z})+\big(\widetilde{\pi}^{\epsilon,\kappa}_0H^{\epsilon,\kappa}\,\widetilde{\pi}^{\epsilon,\kappa}_\perp\big)\;R^{\epsilon,\kappa}_{\mathtt{z},\perp}\;\big(\widetilde{\pi}^{\epsilon,\kappa}_\perp\,H^{\epsilon,\kappa}\,\widetilde{\pi}^{\epsilon,\kappa}_0\big)\,,$$

and a Neumann series argument shows that $\mathtt{z}$ also belongs to the resolvent set of the quasi-band Hamiltonian.

Thus we have shown that, the Hausdorff distance between the two spectra restricted to the interval $[0,m_1/2]$ must be of order ϵ^2. □

3.3 The magnetic quasi-Bloch function λ^ϵ

In this subsection, we study the spectrum of the operator $\widetilde{\pi}_0^{\epsilon,\kappa} H^{\epsilon,\kappa}\, \widetilde{\pi}_0^{\epsilon,\kappa}$ acting in $\widetilde{\pi}_0^{\epsilon,\kappa}\mathcal{H}$ by looking at its associated magnetic matrix (see also Definition 3.11) in the basis $(\widetilde{\phi}_\alpha^{\epsilon,\kappa})_{\alpha\in\Gamma}$:

$$\begin{aligned}
&\Big\langle \widetilde{\phi}_\alpha^{\epsilon,\kappa}\,,\, H^{\epsilon,\kappa}\widetilde{\phi}_\beta^{\epsilon,\kappa}\Big\rangle_{\mathcal{H}} \\
&\quad= \sum_{(\tilde\alpha,\tilde\beta)\in\Gamma\times\Gamma} \overline{\mathbb{F}_{\tilde\alpha\alpha}^{\epsilon,\kappa}}\,\mathbb{F}_{\tilde\beta\beta}^{\epsilon,\kappa}\,\Big\langle \widetilde{\Lambda}^{\epsilon,\kappa}(\cdot,\tilde\alpha)\,\phi_{\tilde\alpha}^\epsilon\,,\, H^{\epsilon,\kappa}\widetilde{\Lambda}^{\epsilon,\kappa}(\cdot,\tilde\beta)\,\phi_{\tilde\beta}^\epsilon\Big\rangle_{\mathcal{H}} \\
&\quad= \sum_{(\tilde\alpha,\tilde\beta)\in\Gamma\times\Gamma} \overline{\mathbb{F}_{\tilde\alpha\alpha}^{\epsilon,\kappa}}\,\mathbb{F}_{\tilde\beta\beta}^{\epsilon,\kappa}\,\Big\langle \phi_{\tilde\alpha}^\epsilon\,,\, \widetilde{\Lambda}^{\epsilon,\kappa}(\cdot,\tilde\alpha)^{-1}\big((-i\nabla-\epsilon A^0(\cdot)-\kappa A(\epsilon\cdot))^2+V\big)\widetilde{\Lambda}^{\epsilon,\kappa}(\cdot,\tilde\beta)\,\phi_{\tilde\beta}^\epsilon\Big\rangle_{\mathcal{H}}\,.
\end{aligned}$$

Introducing

$$a_\epsilon(x,\gamma)_j \;=\; \sum_k (x-\gamma)_k \int_0^1 \epsilon B_{jk}\big(\epsilon\gamma + s\epsilon(x-\gamma)\big)s\,ds \;\text{ for } j=1,2\,,$$

and using the intertwining formula (B.16), we find:

$$(-i\nabla-\epsilon A^0(x)-\kappa A(\epsilon x))\widetilde{\Lambda}^{\epsilon,\kappa}(x,\tilde\beta) = \widetilde{\Lambda}^{\epsilon,\kappa}(x,\tilde\beta)\Big\{(-i\nabla-\epsilon A^0(x)) \,+\, \kappa a_\epsilon(x,\tilde\beta)\Big\}\,, \tag{3.17}$$

and

$$(-i\nabla-\epsilon A^0(x)-\kappa A(\epsilon x))^2\widetilde{\Lambda}^{\epsilon,\kappa}(x,\tilde\beta) = \widetilde{\Lambda}^{\epsilon,\kappa}(x,\tilde\beta)\Big\{(-i\nabla-\epsilon A^0(x)) \,+\, \kappa a_\epsilon(x,\tilde\beta)\Big\}^2\,. \tag{3.18}$$

Let us notice that

$$|a_\epsilon(x,\gamma)| \le C\epsilon < x-\gamma > \;. \tag{3.19}$$

By the Stokes Formula we have

$$\widetilde{\Lambda}^{\epsilon,\kappa}(x,\tilde\alpha)^{-1}\widetilde{\Lambda}^{\epsilon,\kappa}(x,\tilde\beta) \;=\; \widetilde{\Lambda}^{\epsilon,\kappa}(\tilde\alpha,\tilde\beta)\,\Omega^{\kappa,\epsilon}(\tilde\alpha,x,\tilde\beta)\,, \tag{3.20}$$

and we know that

$$|\Omega^{\kappa,\epsilon}(\tilde\alpha,x,\tilde\beta)-\mathbb{1}| \;\le\; \kappa\epsilon\,|x-\tilde\alpha|\,|x-\tilde\beta|\,.$$

Moreover, using Remark 3.6, one easily proves the following estimates (remember the notation $H^\epsilon = H^{\epsilon,0}$):

$$\Big|\Big\langle \big(\Omega^{\kappa,\epsilon}(\tilde\alpha,\cdot,\tilde\beta)-\mathbb{1}\big)\phi_{\tilde\alpha}^\epsilon\,,\, H^\epsilon\phi_{\tilde\beta}^\epsilon\Big\rangle_{\mathcal{H}}\Big| \le C_m\kappa\epsilon < \tilde\alpha-\tilde\beta >^{-m}, \qquad \forall m\in\mathbb{N}\,. \tag{3.21}$$

Lemma 3.16. *For any $m\in\mathbb{N}$, there exists C_m such that if ψ equals either $\big(\Omega^{\kappa,\epsilon}(\tilde\alpha,\cdot,\tilde\beta)-\mathbb{1}\big)\phi_{\tilde\alpha}^\epsilon$ or $\psi=\phi_{\tilde\alpha}^\epsilon$, we have:*

$$\Big|\Big\langle \psi\,,\,\big[\big((-i\nabla-\epsilon A^0)+\kappa a_\epsilon(\cdot,\tilde\beta)\big)^2+V\big]\phi_{\tilde\beta}^\epsilon\Big\rangle_{\mathcal{H}} - \Big\langle \psi\,,\, H^\epsilon\phi_{\tilde\beta}^\epsilon\Big\rangle_{\mathcal{H}}\Big| \le C_m\,\kappa\epsilon < \tilde\alpha-\tilde\beta >^{-m}\;.$$

Proof.
The difference of the two scalar products is equal to

$$\kappa^2\Big\langle \psi\,,\, a_\epsilon(\cdot,\tilde\beta)^2\phi_{\tilde\beta}^\epsilon\Big\rangle_{\mathcal{H}} + \kappa\Big\langle \psi\,,\,(-i\nabla-\epsilon A^0)\cdot a_\epsilon(\cdot,\tilde\beta)\phi_{\tilde\beta}^\epsilon\Big\rangle_{\mathcal{H}} + \kappa\Big\langle \psi\,,\, a_\epsilon(\cdot,\tilde\beta)\cdot(-i\nabla-\epsilon A^0(x))\phi_{\tilde\beta}^\epsilon\Big\rangle_{\mathcal{H}}\,.$$

Remark 3.6, the estimate (3.19) and some arguments similar to those leading to (3.21) finish the proof. □

Proposition 3.17. *For any $m\in\mathbb{N}$, there exists C_m such that, $\forall\,(\alpha,\beta)\in\Gamma\times\Gamma$,*

$$\Big|\Big\langle \widetilde{\phi}_\alpha^{\epsilon,\kappa}\,,\, H^{\epsilon,\kappa}\widetilde{\phi}_\beta^{\epsilon,\kappa}\Big\rangle_{\mathcal{H}} \,-\, \widetilde{\Lambda}^{\epsilon,\kappa}(\alpha,\beta)\,\big\langle \phi_\alpha^\epsilon\,,\, H^\epsilon\phi_\beta^\epsilon\big\rangle_{\mathcal{H}}\Big| \le C_m\,\kappa\epsilon < \alpha-\beta >^{-m}\;.$$

Proof. Putting together all the previous estimates and using Proposition 3.10 we obtain:

$$\begin{aligned}\Big|\Big\langle \widetilde{\phi}^{\epsilon,\kappa}_\alpha\,,\, H^{\epsilon,\kappa}\widetilde{\phi}^{\epsilon,\kappa}_\beta\Big\rangle_{\mathcal{H}} - \widetilde{\Lambda}^{\epsilon,\kappa}(\alpha,\beta)\,\Big\langle \phi^\epsilon_\alpha\,,\, H^\epsilon\,\phi^\epsilon_\beta\Big\rangle_{\mathcal{H}}\Big| \\ \leq \Big|\Big\langle H^\epsilon\,\big(\Omega^{\kappa,\epsilon}(\alpha,x,\beta)-\mathbb{1}\big)\phi^\epsilon_\alpha\,,\,\phi^\epsilon_\beta\Big\rangle_{\mathcal{H}}\Big| \\ +C(m_1,m_2,m_3)\,\kappa\epsilon\left(\sum_{(\tilde\alpha,\tilde\beta)\in(\Gamma\setminus\{\alpha\})\times(\Gamma\setminus\{\beta\})} <\alpha-\tilde\alpha>^{-m_1}<\beta-\tilde\beta>^{-m_2}<\tilde\alpha-\tilde\beta>^{-m_3}\right),\end{aligned}$$

for any triple $(m_1,m_2,m_3)\in\mathbb{N}^3$ and the Proposition follows from (3.21). □

Remarks 3.6 and 3.5 imply:

$$\begin{aligned}\big\langle \phi^\epsilon_\alpha\,,\, H^\epsilon\phi^\epsilon_\beta\big\rangle_{\mathcal{H}} &= \big\langle \psi^\epsilon_0\,,\,(\Lambda^\epsilon(\cdot,\alpha))^{-1}\tau_{\alpha-\beta}\Lambda^\epsilon(\cdot,\beta)H^\epsilon\psi^\epsilon_0\big\rangle_{\mathcal{H}} \\ &= \Lambda^\epsilon(\alpha,\beta)\,\big\langle \psi^\epsilon_0\,,\,\Lambda^\epsilon(\cdot,\beta-\alpha)\tau_{-(\beta-\alpha)}H^\epsilon\psi^\epsilon_0\big\rangle_{\mathcal{H}}\,.\end{aligned}$$

Definition 3.18. *We define* $\mathfrak{h}^\epsilon\in\ell^2(\Gamma)$ *by:*

$$\mathfrak{h}^\epsilon(\gamma)\,:=\,\langle\psi^\epsilon_0\,,\,\Lambda^\epsilon(x,\gamma)\tau_{-\gamma}H^\epsilon\psi^\epsilon_0\rangle_{\mathcal{H}}\,=\,\big\langle\phi^\epsilon_0\,,\,H^\epsilon\phi^\epsilon_\gamma\big\rangle_{\mathcal{H}}\ \textit{for}\ \gamma\in\Gamma\,, \tag{3.22}$$

and the magnetic quasi Bloch function λ^ϵ *as its discrete Fourier transform:*

$$\lambda^\epsilon:\mathbb{T}_*\to\mathbb{R},\qquad \lambda^\epsilon(\theta):=\sum_{\gamma\in\Gamma}\mathfrak{h}^\epsilon(\gamma)e^{-i<\theta,\gamma>}\,. \tag{3.23}$$

The conclusion of this subsection is contained in:

Proposition 3.19. *There exists* $\epsilon_0>0$ *such that, for* $\epsilon\in[0,\epsilon_0]$ *and* $\kappa\in[0,1]$*, the Hausdorff distance between the spectra of the magnetic quasi-band Hamiltonian* $\widetilde{\pi}^{\epsilon,\kappa}_0H^{\epsilon,\kappa}\widetilde{\pi}^{\epsilon,\kappa}_0$ *and* $\mathfrak{Op}^{\epsilon,\kappa}(\lambda^\epsilon)$ *is of order* $\kappa\epsilon$*.*

Proof. Proposition 3.17 and a Schur type estimate imply that the spectrum of our magnetic quasi-band Hamiltonian is at a Hausdorff distance of order $\kappa\epsilon$ from the spectrum of the matrix with elements

$$\mathcal{E}^{\epsilon,\kappa}(\alpha,\beta):=\widetilde{\Lambda}^{\epsilon,\kappa}(\alpha,\beta)\Lambda^\epsilon(\alpha,\beta)\,\mathfrak{h}^\epsilon(\alpha-\beta)=\Lambda^{\epsilon,\kappa}(\alpha,\beta)\,\mathfrak{h}^\epsilon(\alpha-\beta) \tag{3.24}$$

acting on $\ell^2(\Gamma)$.

We may extend the function λ^ϵ from (3.23) as a Γ_*-periodic function defined on all $\mathcal{X}^*$ and thus as a symbol defined on Ξ, constant with respect to the variables in $\mathcal{X}$. Using the natural unitary isomorphism $L^2(\mathcal{X})\to\ell^2(\Gamma)\otimes L^2(E)$ we may compute the integral kernel of $\mathfrak{Op}^{\epsilon,\kappa}(\lambda^\epsilon)$ as in [10] and obtain

$$\Lambda^{\epsilon,\kappa}(\alpha+\underline{x},\beta+\underline{x})\,\mathfrak{h}^\epsilon(\alpha-\beta)\,\delta(\underline{x}-\underline{x}'),\quad \forall\underline{x}\,,\underline{x}'\in E\,,\quad\forall\alpha\,,\beta\in\Gamma\,.$$

Following [10], it turns out that one can perform an (ϵ,κ)-dependent unitary gauge transform in $\ell^2(\Gamma)\otimes L^2(E)$ such that after conjugating $\mathfrak{Op}^{\epsilon,\kappa}(\lambda^\epsilon)$ with it, the rotated operator will be (up to an error of order $\kappa\epsilon$ in the norm topology) given by an operator whose integral kernel is given by $\mathcal{E}^{\epsilon,\kappa}(\alpha,\beta)\,\delta(\underline{x}-\underline{x}')$. This operator is nothing but $\mathcal{E}^{\epsilon,\kappa}\otimes\mathbb{1}$, and it is isospectral with the matrix $\mathcal{E}^{\epsilon,\kappa}$. □

4 The magnetic quantization of the magnetic Bloch function.

4.1 Study of the magnetic Bloch function λ^ϵ.

Let us recall from (1.4) and (A.1) that $\lambda_0\in C^\infty(\mathbb{T}_*;\mathbb{R})$ is even and has a non-degenerate absolute minimum in $0\in\mathbb{T}_*$. Thus in a neighborhood of $0\in\mathbb{T}_*$ we have the Taylor expansion

$$\lambda_0(\theta)\,=\,\sum_{1\leq j,k\leq 2}a_{jk}\theta_j\theta_k\,+\,\mathscr{O}(|\theta|^4)\,,\qquad a_{jk}:=\big(\partial^2_{jk}\lambda_0\big)(0)\,. \tag{4.1}$$

Proposition 4.1. *For λ^ϵ defined in* (3.23)*, there exists $\epsilon_0 > 0$ such that*

$$\lambda^\epsilon(\theta) \,=\, \lambda_0(\theta) \,+\, \epsilon\rho^\epsilon(\theta)\,,$$

with $\rho^\epsilon \in BC^\infty(\mathbb{T}_*)$ uniformly in $\epsilon \in [0,\epsilon_0]$ and such that $\rho^\epsilon - \rho^0 = \mathscr{O}(\epsilon)$.

Proof. From the definition of λ^ϵ in Definition 3.18 we have

$$\lambda^\epsilon(\theta) = \sum_{\gamma\in\Gamma} \langle \psi_0^\epsilon\,,\, \Lambda^\epsilon(\cdot,\gamma)\tau_{-\gamma}H^\epsilon\psi_0^\epsilon\rangle_{\mathcal{H}}\, e^{-i<\theta,\gamma>}\,.$$

Due to Corollary 3.8, there exists $f^\epsilon \in \mathscr{S}(\mathcal{X})$ such that $\psi_0^\epsilon = \phi_0 + \epsilon f^\epsilon$, and for any $m \in \mathbb{N}$ there exists $C_m > 0$ such that, for any $\epsilon \in [0,\epsilon_0]$,

$$\sup_{x\in\mathcal{X}} <x>^m |f^\epsilon(x)| \,\leq\, C_m\,.$$

With $A^0(x) = (1/2)\big(-B_0x_2, B_0x_1\big)$, we have

$$H^\epsilon = (-i\nabla_x - \epsilon A^0(x))^2 + V \,=\, H^0 + 2i\epsilon A^0\cdot\nabla_x + \epsilon^2(A^0)^2\,.$$

Thus

$$\begin{aligned}
&\langle \psi_0^\epsilon\,,\, \Lambda^\epsilon(\cdot,\gamma)\tau_{-\gamma}H^\epsilon\psi_0^\epsilon\rangle_{\mathcal{H}} \\
&\qquad = \big\langle \tau_\gamma\Lambda^\epsilon(\gamma,\cdot)\psi_0^\epsilon\,,\, H^0\psi_0^\epsilon\big\rangle_{\mathcal{H}} + \big\langle \tau_\gamma\Lambda^\epsilon(\gamma,\cdot)\psi_0^\epsilon\,,\, \big(2i\epsilon A^0\cdot\nabla_x + \epsilon^2(A^0)^2\big)\,\psi_0^\epsilon\big\rangle_{\mathcal{H}} \\
&\qquad = \big\langle \tau_\gamma\Lambda^\epsilon(\gamma,\cdot)\phi_0\,,\, H^0\phi_0\big\rangle_{\mathcal{H}} + C_m\epsilon <\gamma>^{-m} \\
&\qquad = \big\langle \tau_\gamma\phi_0\,,\, H^0\phi_0\big\rangle_{\mathcal{H}} + C'_m\,\epsilon <\gamma>^{-m}\,.
\end{aligned}$$

In the last two estimates we have used the rapid decay of the functions ψ_0^ϵ and ϕ_0 and the fact that with our choice of gauge

$$|\Lambda^\epsilon(\gamma,x) - 1| \leq \epsilon\,|\gamma|\,|x|\,.$$

Hence

$$\lambda^\epsilon(\theta) = \sum_{\gamma\in\Gamma} \big\langle \tau_\gamma\phi_0\,,\, H^0\phi_0\big\rangle_{\mathcal{H}}\, e^{-i<\theta,\gamma>} + \mathcal{O}(\,\epsilon) = \lambda_0(\theta) + \mathcal{O}(\,\epsilon)\,,$$

in $C^m(\mathbb{T}_*)$ for any m.
Moreover, for any $m \in \mathbb{N}$, one can differentiate m-times the Fourier series of $\lambda^\epsilon(\theta)$, term by term, because of the exponential decay of its Fourier coefficients. Then the Fourier series of the derivative has an asymptotic expansion in ϵ which starts with $\lambda_0^{(m)}(\theta)$, just as in the case with $m = 0$. In other words, differentiation in θ commutes with taking the asymptotic expansion in powers of ϵ. □

Considering the function λ_ϵ as a Γ_*-periodic function on $\mathcal{X}^*$, a consequence of Proposition 4.1 is that the modified Bloch eigenvalue $\lambda^\epsilon \in C^\infty(\mathcal{X}^*)$ will also have an isolated non-degenerate minimum at some point $\theta^\epsilon \in \mathcal{X}^*$ close to $0 \in \mathcal{X}^*$. More precisely, if we denote by $\mathbf{a}$ the 2×2 positive definite matrix $(a_{jk})_{1\leq j,k\leq 2}$ introduced in (4.1) and by $\mathbf{a}^{-1}$ its inverse, we have

$$\theta_j^\epsilon \,=\, \epsilon\,\hat\theta_j \,+\, \mathcal{O}(\epsilon^2)\,,$$

with

$$\hat\theta_j := -\sum_{1\leq k\leq 2} \big(\mathbf{a}^{-1}\big)_{jk}\big(\partial_k\rho_0\big)(0)\,.$$

Then we can write the Taylor expansion at θ^ϵ

$$\begin{aligned}
\lambda^\epsilon(\theta) \quad &= \lambda^\epsilon(\theta^\epsilon) + \sum_{1\leq j,k\leq 2}\big(\partial_j\partial_k\lambda^\epsilon\big)(\theta^\epsilon)(\theta_j-\theta_j^\epsilon)(\theta_k-\theta_k^\epsilon) \\
&\quad + \sum_{1\leq j,k,\ell\leq 2}\big(\partial_j\partial_k\partial_\ell\lambda^\epsilon\big)(\theta^\epsilon)(\theta_j-\theta_j^\epsilon)(\theta_k-\theta_k^\epsilon)(\theta_\ell-\theta_\ell^\epsilon) \,+\, \mathcal{O}(|\theta-\theta^\epsilon|^4)\,.
\end{aligned} \tag{4.2}$$

Using the evenness of λ_0 we get:

$$\lambda^\epsilon(\theta) - \lambda^\epsilon(\theta^\epsilon) = \sum_{1\le j,k\le 2} a^\epsilon_{jk}(\theta_j - \theta^\epsilon_j)(\theta_k - \theta^\epsilon_k) + \epsilon\mathscr{O}(|\theta-\theta^\epsilon|^3) + \mathcal{O}(|\theta-\theta^\epsilon|^4)\,, \tag{4.3}$$

where

$$\lambda^\epsilon(\theta^\epsilon) = \epsilon\rho^0(0) + \mathcal{O}(\epsilon^2)\,,$$

and

$$a^\epsilon_{jk} \;=\; a_{jk} \;+\; \epsilon\big(\partial_j\partial_k\rho^0\big)(0) + \mathcal{O}(\epsilon^2)\,.$$

Hence, after a shift of energy and a change of variable $\theta \mapsto (\theta - \theta_\epsilon)$, the new λ^ϵ has the same structure as λ_0 except that we have lost the symmetry (1.4).
There exists $\epsilon_0 > 0$ such that, for $\epsilon \in [0,\epsilon_0]$, we can choose a local coordinate system on a neighborhood of $\theta^\epsilon \in \mathcal{X}^*$ that diagonalizes the symmetric positive definite matrix a^ϵ and we denote by $0 < m^\epsilon_1 \le m^\epsilon_2$ its eigenvalues. We denote by $0 < m_1 \le m_2$ the two eigenvalues of the matrix a_{jk} and notice that

$$m^\epsilon_j \;=\; m_j \;+\; \epsilon\mu_j \;+\; \mathscr{O}(\epsilon^2) \text{ for } j = 1,2\,,$$

with μ_j explicitly computable in terms of the matrix a_{jk} and $\big(\partial_j\partial_k\rho^0\big)(0)$.

4.2 Spectral analysis of the model operator

4.2.1 Quadratic approximation

In studying the bottom of the spectrum of the operator $\mathfrak{Op}^{\epsilon,\kappa}(\lambda^\epsilon)$ we shall start with the quadratic term given by the Hessian close to the minimum (see (4.2)). We introduce

$$m^\epsilon \;:=\; \sqrt{m^\epsilon_1 m^\epsilon_2} \;=\; m + \epsilon\,\mu^*(\epsilon)\,, \tag{4.4}$$

where $m := \sqrt{m_1 m_2}$ is the square root of the determinant of the matrix a_{jk} and $\mu^*(\epsilon)$ is uniformly bounded for $\epsilon \in [0,\epsilon_0]$. Our goal is to obtain spectral information concerning the Hamiltonian $\mathfrak{Op}^{\epsilon,\kappa}(\lambda^\epsilon)$ starting from the spectral information about $\mathfrak{Op}^{\epsilon,\kappa}(h_{m^\epsilon})$ with

$$h_{m^\epsilon}(\xi) \;:=\; m^\epsilon_1\xi_1^2 \;+\; m^\epsilon_2\xi_2^2\,, \tag{4.5}$$

defining an elliptic symbol of class $S^2_1(\Xi)^\circ$ (i.e. that does not depend on the configuration space variable, see (B.33) in Appendix B.5.1).

4.2.2 A perturbation result

Now we compare the bottom of the spectrum of the magnetic Hamiltonians $\mathfrak{Op}^{\epsilon,\kappa}(h_{m^\epsilon})$ with the one of the constant field magnetic Landau operator $\mathfrak{Op}^{\epsilon,0}(h_{m^\epsilon})$.

First, we have to perform a dilation. Starting from the initial model (with the magnetic field in (1.5)), we make the change of variable $y = \sqrt{\epsilon}x$ and factorize an ϵ. This leads to a scaled Landau operator with magnetic field $B_0 + \kappa B(\sqrt{\epsilon}x)$. Then, using also (4.4), we can prove the following statement:

Proposition 4.2. *For any compact set M in $\mathbb{R}$, there exist $\epsilon_K > 0$, $C > 0$ and $\kappa_K \in (0,1]$, such that for any $(\epsilon,\kappa) \in [0,\epsilon_K]\times[0,\kappa_K]$, the spectrum of the operator $\mathfrak{Op}^{\epsilon,\kappa}(h_{m^\epsilon})$ in ϵM is contained in bands of width $C\kappa\epsilon$ centered at $\{(2n+1)\,\epsilon\, m^\epsilon\, B_0\}_{n\in\mathbb{N}}$.*

The above result follows from the following slightly more general proposition.

Proposition 4.3. *Assume that $B(x) = B_0 + b(x)$ where $B_0 > 0$ and $b \in BC^1(\mathcal{X})$ and consider*

$$L_b := \big(- i\partial_1 - a_1(x)\big)^2 + \big(- i\partial_2 - B_0 x_1 - a_2(x)\big)^2 ,$$

where $a(x) = a(x,0)$ with

$$a(x,x') = (a_1(x,x'), a_2(x,x')) = \left(\int_0^1 ds\ s\ b(x' + s(x-x')) \right) \big(- x_2 + x_2', x_1 - x_1' \big) .$$

With $\beta := ||b||_{C^1(\mathcal{X})}$, for any $N \in \mathbb{N}$, there exist $C > 0$ and $\beta_0 > 0$ such that, for any $0 \le \beta \le \beta_0$, we have:

$$\sigma(L_b) \cap [0, 2(N+1)\, B_0] \subset \bigcup_{j=0}^{N} \big[(2j+1)B_0 - C\beta, (2j+1)B_0 + C\beta \big] . \tag{4.6}$$

Proof. Define $\phi_b(x,x')$ to be the magnetic flux generated by the magnetic field b through the triangle with vertices at 0, x and x'. Let $\widehat{A}_0(x) = (0, B_0 x_1)$. We have the following variant of (3.18), for the composition of L_b and the multiplication operator by $e^{i\phi_b(\cdot,x')}$, with $x' \in \mathcal{X}$ arbitrary:

$$L_b\, e^{i\phi_b(x,x')} = e^{i\phi_b(x,x')} \big(L_0 + a^2(x,x') - 2a(x,x') \cdot (-i\nabla_x - \widehat{A}_0(x)) + i\nabla_x a(x,x') \big) , \tag{4.7}$$

Let $z \notin \sigma(L_0)$. Denote by $K_0(x,x';z)$ the integral kernel of $(L_0 - z)^{-1}$. Let $S_b(z)$ be the operator whose integral kernel is given by

$$S_b(x,x';z) := e^{i\phi_b(x,x')} K_0(x,x';z) .$$

Using (4.7) and the fact that $\phi_b(x,x) = 0$, one can prove that:

$$[(L_b - z) S_b(\cdot, x'; z)](x) = \delta(x - x') + T_b(x,x';z) , \tag{4.8}$$

where the kernel $T_b(x,x';z)$ generates an operator T_b such that, for any compact set K in the resolvent set of L_0, there exists $C_K > 0$ s. t.

$$\sup_{z \in K} ||T_b(z)|| \le C_K \beta .$$

From (4.8) it follows that there exists $\beta_0 > 0$, such that if $\beta \in [0, \beta_0]$ then K is in the resolvent set of L_b and we have:

$$(L_b - z)^{-1} = S_b(z)(1 + T_b(z))^{-1} .$$

In particular, there exists $C_K > 0$ s. t.

$$\sup_{z \in K} \|(L_b - z)^{-1} - S_b(z)\| \le C_K \beta . \tag{4.9}$$

By a Riesz integral on a contour Γ_j encircling the eigenvalue $(2j+1)B_0$ of L_0, we can define the band operator

$$L_{b,j} := \frac{i}{2\pi} \int_{\Gamma_j} z\, (L_b - z)^{-1}\, dz$$

living in the range of the projector $P_{j,b} = \frac{i}{2\pi} \int_{\Gamma_j} (L_b - z)^{-1} dz$.
From (4.9) we get:

$$\big(L_{b,j} - (2j+1)B_0 \big) P_{j,b} = \mathcal{O}(\beta) , \tag{4.10}$$

which shows that the spectrum of L_b in $[0, 2(N+1)\, B_0]$ is contained in intervals of width of order β centered around the Landau levels $(2j+1)\, B_0$. □

Remark 4.4. Let us introduce $r^{\epsilon,\kappa}(\mathsf{z})$, resp. $r^{\epsilon}(\mathsf{z})$, in $S_1^{-2}(\Xi)$ such that

$$(\mathfrak{Op}^{\epsilon,\kappa}(h_{m^\epsilon})-\mathsf{z})^{-1} =: \mathfrak{Op}^{\epsilon,\kappa}\big(r^{\epsilon}(\mathsf{z})\big), \quad (\mathfrak{Op}^{\epsilon}(h_{m^\epsilon})-\mathsf{z})^{-1} =: \mathfrak{Op}^{\epsilon}\big(r^{\epsilon}(\mathsf{z})\big) \tag{4.11}$$

for z in the resolvent set of $\mathfrak{Op}^{\epsilon,\kappa}(h_{m^\epsilon})$, resp. $\mathfrak{Op}^{\epsilon}(h_{m^\epsilon})$.
Coming back to Proposition 4.2, after a scaling, the estimate (4.9) combined with (4.11) leads, for $j \in \mathbb{N}$, to the existence of C_j such that

$$\sup_{|\mathsf{z}-(2j+1)\epsilon m^\epsilon B_0|=\epsilon m^\epsilon B_0} \|\mathfrak{Op}^{\epsilon,\kappa}\big(r^{\epsilon,\kappa}(\mathsf{z})-r^{\epsilon}(\mathsf{z})\big)\| \leq C_j\, \kappa\epsilon^{-1}\,. \tag{4.12}$$

4.3 The resolvent of $\mathfrak{Op}^{\epsilon,\kappa}(\lambda^\epsilon)$

Using the results of Section 3, we need for finishing the proof of Theorem 1.4 a spectral analysis of $\mathfrak{Op}^{\epsilon,\kappa}(\lambda^\epsilon)$. In the rest of this section we use definitions, notation and results from the Appendix B dedicated to the magnetic Moyal calculus.

Cut-off functions near the minimum. We introduce a cut-off around $\theta = 0$. We choose an even function χ in $C_0^\infty(\mathbb{R})$ with $0 \leq \chi \leq 1$, with support in $(-2,+2)$ and such that $\chi(t) = 1$ on $[-1,+1]$. We note that the following relation holds on $[0,+\infty)$

$$(1-t)\big(\chi(t)-1\big) \geq 0\,. \tag{4.13}$$

For $\delta > 0$ we define

$$g_{1/\delta}(\xi) := \chi\big(h_{m^\epsilon}(\delta^{-1}\xi)\big), \quad \xi \in \mathcal{X}^*\,, \tag{4.14}$$

where h_{m^ϵ} is defined in (4.5).
Thus $0 \leq g_{1/\delta} \leq 1$ for any $\delta > 0$ and

$$g_{1/\delta}(\xi) \;=\; \begin{cases} 1 & \text{if} \quad |\xi|^2 \leq (2m_2)^{-1}\delta^2\,, \\ 0 & \text{if} \quad |\xi|^2 \geq m_1^{-1}\delta^2\,. \end{cases} \tag{4.15}$$

We choose δ^0 such that

$$D(0,\sqrt{2m_1^{-1}}\delta^0) \subset \mathring{E}_*\,, \tag{4.16}$$

where $D(0,\rho)$ denotes the disk centered at 0 of radius ρ and $\mathring{E}_*$ denotes the interior of E_*.

For any $\delta \in (0,\delta_0]$, $g_{1/\delta}\, C_0^\infty(E_*)$ and we shall consider it as an element of $C_0^\infty(\mathcal{X}^*)$ by extending it by 0. We introduce

$$\widetilde{g}_{1/\delta}(\xi) := \sum_{\gamma\in\Gamma^*} g_{1/\delta}(\xi-\gamma)\,, \tag{4.17}$$

the Γ_*-periodic continuation of $g_{1/\delta}$ to $\mathcal{X}^*$. The multiplication with these functions defines bounded linear maps in $L^2(\mathbb{T}_*)$ or in $L^2(\mathcal{X}^*)$ with operator norm bounded uniformly for $\delta \searrow 0$.
For any $\delta \in (0,\delta_0]$ we introduce:

$$\delta^\circ := \sqrt{m_1/2m_2}\;\delta\,, \tag{4.18}$$

so we have

$$g_{1/\delta^\circ} = g_{1/\delta}\; g_{1/\delta^\circ}\,.$$

Behavior near the minimum. We use now the expansion of λ^ϵ near its minimum (4.3) and the coordinates that diagonalize the Hessian (see the discussion after (4.3)) in order to get

$$\lambda^\epsilon(\xi) \;=\; \big(m_1^\epsilon \xi_1^2 + m_2^\epsilon \xi_2^2\big) \;+\; \epsilon\, \mathcal{O}(|\xi|^3) \;+\; \mathcal{O}(|\xi|^4)\,. \tag{4.19}$$

In order to localize near the minimum $\xi = 0$, we use the cut-off function $g_{1/\delta}$ and write

$$\lambda^\epsilon \;= \lambda^\epsilon_\circ \;+\; \widetilde{\lambda}^\epsilon\,.$$

with

$$\lambda^\epsilon_\circ := g_{1/\delta}\, \lambda^\epsilon \;\text{ and }\; \widetilde{\lambda}^\epsilon := (1 - g_{1/\delta})\, \lambda^\epsilon\,.$$

We introduce a second scaling:

$$\epsilon = \delta^\mu\,,\;\text{ for some } \mu \in (2,4)\,. \tag{4.20}$$

Due to (4.19) and the inequality $\mu + 3 > 4$, we can find a bounded family $\{\mathfrak{f}^\delta\}_{\delta\in(0,\delta_0]}$ in $C_0^\infty(\mathcal{X}^*)$ such that

$$\lambda^\epsilon_\circ(\xi) = g_{1/\delta}(\xi)\, h_{m^\epsilon}(\xi) + \delta^4\, \mathfrak{f}^\delta(\xi)\,. \tag{4.21}$$

The shifted operator outside the minima.
For the region outside the minima, we need the operator $\mathfrak{Op}^{\epsilon,\kappa}\big(\lambda^\epsilon \;+\; (\delta^\circ)^2 \widetilde{g}_{1/\delta^\circ}\big)$ with δ° as in (4.18) and $\widetilde{g}_{1/\delta^\circ}$ as in (4.17). The inequality (4.13) implies that there exists $C > 0$ such that

$$\lambda^\epsilon(\xi) + (\delta^\circ)^2\, \widetilde{g}_{1/\delta^\circ}(\xi) \;\geq\; C\,(\delta^\circ)^2 = C\,\frac{m_1}{2m_2}\,\delta^2\,.$$

Hence there exists $C' > 0$ such that for any $\delta \in (0, \delta_0)$

$$\mathfrak{Op}^0\big(\lambda^\epsilon + (\delta^\circ)^2\, \widetilde{g}_{1/\delta^\circ}\big) \geq C'\,\delta^2\, \mathbb{1}\,. \tag{4.22}$$

As the magnetic field is slowly variable and the symbol $\lambda^\epsilon + (\delta^\circ)^2 \widetilde{g}_{1/\delta^\circ}$ is Γ^*-periodic in $S^0_0(\Xi)$, by Corollary 1.6 in [10] there exists $\epsilon_0 > 0$ and for $(\epsilon, \kappa, \delta) \in [0, \epsilon_0] \times [0,1] \times (0, \delta_0]$, some constant $C'(\epsilon, \delta) > 0$ such that:

$$\mathfrak{Op}^{\epsilon,\kappa}\big(\lambda^\epsilon + (\delta^\circ)^2\, \widetilde{g}_{1/\delta^\circ}\big) \geq \big(C'\,\delta^2 - C'(\epsilon,\delta)\,\epsilon\,\big)\; \mathbb{1}\,. \tag{4.23}$$

The proof of Corollary 1.6 in [10] gives a control of $C'(\epsilon, \delta)$ by:

$$C'(\epsilon,\delta) \;\;\leq\; C_1 \Big[\max_{|\alpha|\leq 2} \Big\|\mathcal{F}^-_{\mathbb{T}_*}\big(\partial_\xi^\alpha \lambda^\epsilon\big)\Big\|_{\ell^1(\Gamma)} + (\delta^\circ)^2 \max_{|\alpha|\leq 2} \Big\|\mathcal{F}^-_{\mathbb{T}_*}\big[\partial_\xi^\alpha\big(g_{1/\delta^\circ}\big)\big]\Big\|_{\ell^1(\Gamma)}\Big]\,.$$

The differentiation of g_{1/δ° with respect to ξ produces $|\alpha| \leq 2$ negative powers of δ. Denote by $g \equiv g_1$. Then we change the integration variable $\xi \mapsto \delta^\circ \xi$ under the Fourier transform (this gives us an extra factor of $(\delta^\circ)^2$) and we obtain:

$$C'(\epsilon,\delta) \leq\; C_2 \Big[1 + \max_{|\alpha|\leq 2} (\delta^\circ)^2 \sum_{\gamma\in\Gamma} \Big| \int_{\mathbb{T}_*} e^{i\langle \delta^\circ \gamma, \xi\rangle} (\partial_\xi^\alpha g) d\xi \Big|\Big].$$

We notice that $\partial^\alpha g \in C_0^\infty(\mathcal{X}^*)$ has its support strictly included in the dual elementary cell, so that the above series is just a Riemann sum for the integral representing the norm $\big\|\mathcal{F}^-_{\mathcal{X}^*} \partial^\alpha g\big\|_{L^1(\mathcal{X})}$. Hence:

$$C'(\epsilon,\delta) \leq\; C_3 \Big[1 + \max_{|\alpha|\leq 2} \big\|\mathcal{F}^-_{\mathcal{X}^*}\big(\partial_\xi^\alpha g\big)\big\|_{L^1(\mathcal{X})}\Big]\,.$$

Thus we have shown the existence of $C_4 > 0$ such that $C'(\epsilon, \delta) \leq C_4$, uniformly in ϵ. Hence any $\mathsf{z} \in (-\infty, C'\delta^2 - C_4\epsilon)$ is in the resolvent set of $\big(\mathfrak{Op}^{\epsilon,\kappa}\big(\lambda_\epsilon + (\delta^\circ)^2\, \widetilde{g}_{1/\delta^\circ}\big)\big)$ and we denote by $r_{\delta,\epsilon,\kappa}(\mathsf{z})$ its magnetic symbol.

Due to the choice of ϵ made in (4.20), it follows that $C'\delta^2 - C_4\epsilon$ is of order $\epsilon^{2/\mu}$, i.e. much larger than ϵ as $\epsilon \to 0$. Since we are interested in inverting our operators for $\mathtt{z}$ in an interval of the form $[0, c\epsilon]$ for some $c > 0$, we conclude that the distance between $\mathtt{z}$ and the bottom of the spectrum of $\mathfrak{Op}^{\epsilon,\kappa}\big(\lambda_\epsilon + (\delta^\circ)^2\, \widetilde{g}_{1/\delta^\circ}\big)$ is of order $\delta^2 = \epsilon^{2/\mu}$. Thus given any $C' > 0$, there exists ϵ_0 and $C > 0$ such that for every $\epsilon < \epsilon_0$ and $\kappa < 1$ we have

$$\sup_{\mathtt{z}\in[0,C'\epsilon]} \|r_{\delta,\epsilon,\kappa}(\mathtt{z})\|_{B_{\epsilon,\kappa}} \leq C\,\epsilon^{-2/\mu} = C\delta^{-2}\,. \tag{4.24}$$

Definition of the *quasi-inverse.* Let us fix $\mu \in (2,4)$ and some compact set $K \subset \mathbb{C}$ such that:

$$K \subset \mathbb{C} \setminus \{(2n+1)m\,B_0\}_{n\in\mathbb{N}}\,. \tag{4.25}$$

We deduce from Proposition 4.2 that there exist $\epsilon_K > 0$ and $\kappa_K \in [0,1]$ such that for $(\epsilon,\kappa) \in [0,\epsilon_K] \times [0,\kappa_K]$ and for $a \in K$, the point $\epsilon a \in \mathbb{C}$ belongs to the resolvent set of $\mathfrak{Op}^{\epsilon,\kappa}(h_{m^\epsilon})$. We denote by $r^{\epsilon,\kappa}(\epsilon a)$ the magnetic symbol of the resolvent of $\mathfrak{Op}^{\epsilon,\kappa}(h_{m^\epsilon}) - \epsilon a$, i.e.

$$\big(\mathfrak{Op}^{\epsilon,\kappa}(h_{m^\epsilon}) - \epsilon a\big)^{-1} \;:=\; \mathfrak{Op}^{\epsilon,\kappa}(r^{\epsilon,\kappa}(\epsilon a))\,. \tag{4.26}$$

For $a \in K$ we want to define the following symbol in $\mathscr{S}'(\mathcal{X}^*)$ as the sum of the series on the right hand side:

$$\widetilde{r}_\lambda(\epsilon a) := \sum_{\gamma^*\in\Gamma_*} \tau_{\gamma^*}\big(g_{1/\delta}\,\sharp^{\epsilon,\kappa}\, r^{\epsilon,\kappa}(\epsilon a)\big) + \big(1 - \widetilde{g}_{1/\delta}\big)\,\sharp^{\epsilon,\kappa}\, r_{\delta,\epsilon,\kappa}(\epsilon a)\,, \quad \delta = \epsilon^{1/\mu}. \tag{4.27}$$

Then the proof of Theorem 1.4 will be a consequence of the following key result:

Proposition 4.5. *Let $\mu = 3$ in (4.27). For any compact set K satisfying (4.25), there exist $C > 0$, $\kappa_0 \in (0,1]$ and $\epsilon_0 > 0$ such that for $(\kappa,\epsilon,a) \in [0,\kappa_0] \times (0,\epsilon_0] \times K$, we have*

$$\|\mathfrak{Op}^{\epsilon,\kappa}(\widetilde{r}_\lambda(\epsilon a))\| \leq C\epsilon^{-1}\,,$$

and

$$\big(\lambda_\epsilon - \epsilon a\big)\,\sharp^{\epsilon,\kappa}\,\widetilde{r}_\lambda(\epsilon a) \;=\; 1 + \mathfrak{r}_{\delta,a}, \quad \textit{with} \quad \|\mathfrak{Op}^{\epsilon,\kappa}(\mathfrak{r}_{\delta,a})\| \;\leq\; C\,\epsilon^{1/3}\,. \tag{4.28}$$

For $N > 0$, there exist C, ϵ_0 and κ_0 such that the spectrum of $\mathfrak{Op}^{\epsilon,\kappa}(\lambda_\epsilon)$ in $[0,(2N+2)mB_0\epsilon]$ consists of spectral islands centered at $(2n+1)mB_0\epsilon$, $0 \leq n \leq N$, with a width bounded by $C\,(\epsilon\kappa + \epsilon^{4/3})$.

4.4 Proof of Proposition 4.5

For the time being, we allow μ to belong to the interval $(2,4)$ and we will explicitly mention when we make the particular choice $\mu = 3$. We begin with proving the convergence of the first series in (4.27) by analyzing the properties of the symbol $r^{\epsilon,\kappa}(\epsilon a)$. Then assuming that (4.28) is true and using the properties of $r^{\epsilon,\kappa}(\epsilon a)$ we will show how this implies the second part of Proposition 4.5, namely the localization of the spectral islands (see Subsection 4.4.3). In the last part of this section we shall prove (4.28).

4.4.1 Properties of $r^{\epsilon,\kappa}(\epsilon a)$.

From its definition in (4.26) and from Proposition 6.5 in [23] we know that $r^{\epsilon,\kappa}(\epsilon a)$ defines a symbol of class $S_1^{-2}(\Xi)$. Moreover, from Proposition 4.2 we have:

$$\|\mathfrak{Op}^{\epsilon,\kappa}(r^{\epsilon,\kappa}(\epsilon a))\| \leq C(K)\,\epsilon^{-1}. \tag{4.29}$$

In order to study the convergence of the series in (4.27) we need a more involved expression for $r^{\epsilon,\kappa}(\epsilon a)$ obtained by using the resolvent equation with respect to some point far from the

spectrum. All the operators $\mathfrak{Op}^{\epsilon,\kappa}(h_{m^\epsilon})$ (indexed by $(\epsilon,\kappa)\in[0,\epsilon_0]\times[0,\kappa_0]$) are non-negative due to the diamagnetic inequality, hence the point $\mathbf{z}=-1$ belongs to their resolvent sets and:

$$\left\|\left(\mathfrak{Op}^{\epsilon,\kappa}(h_{m^\epsilon})+\mathbb{1}\right)^{-1}\right\|\leq 1\,.$$

We denote by $r_1^{\epsilon,\kappa}$ the magnetic symbols of these resolvents, so

$$\left(\mathfrak{Op}^{\epsilon,\kappa}(h_{m^\epsilon})+\mathbb{1}\right)^{-1}\;=\;\mathfrak{Op}^{\epsilon,\kappa}(r_1^{\epsilon,\kappa})\,.$$

We shall consider all these symbols as *slowly varying symbols on* Ξ (see Appendix B.5.2).

Lemma 4.6. *The family* $\{r_1^{\epsilon,\kappa}\}_{(\epsilon,\kappa)\in[0,\epsilon_0]\times[0,\kappa_0]}$ *is bounded in* $S_1^{-2}(\Xi)$ *and the family* $\{r_1^{\epsilon,\kappa}\}_{(\epsilon,\kappa)\in[0,\epsilon_0]\times[0,\kappa_0]}$ *is bounded in* $S_1^{-2}(\Xi)^\bullet$ *(see Definition B.19).*

Proof. We have $r_1^{\epsilon,\kappa}=\left(h_{m^\epsilon}+1\right)^-_{\epsilon,\kappa}$ with $h_{m^\epsilon}\in S_1^2(\Xi)^\circ$. A straightforward verification using the arguments in the proof of Proposition 6.7 in [23] gives the first conclusion; then Proposition B.23 allows to obtain the second one. □

Lemma 4.7. *For any* $N\in\mathbb{N}^*$, *there exist two bounded families* $\{\phi_N[r_1^{\epsilon,\kappa}]\}_{(\epsilon,\kappa)\in[0,\epsilon_0]\times[0,\kappa_0]}$ *in* $S_1^{-2}(\Xi)^\bullet$ *and* $\{\psi_N[r_1^{\epsilon,\kappa}]\}_{(\epsilon,\kappa)\in[0,\epsilon_0]\times[0,\kappa_0]}$ *in* $S_1^{-2N}(\Xi)^\bullet$, *such that*

$$r^{\epsilon,\kappa}(\epsilon a)\;=\;\phi_N[r_1^{\epsilon,\kappa}]+r^{\epsilon,\kappa}(\epsilon a)\;\sharp^{\epsilon,\kappa}\,\psi_N[r_1^{\epsilon,\kappa}]=\phi_N[r_1^{\epsilon,\kappa}]+\psi_N[r_1^{\epsilon,\kappa}]\,\sharp^{\epsilon,\kappa}\,r^{\epsilon,\kappa}(\epsilon a)\,.$$

Proof. From the resolvent equation we deduce that, for any $N\in\mathbb{N}^*$,

$$\begin{aligned}r^{\epsilon,\kappa}(\epsilon a)\;&=\;\sum_{1\leq n\leq N}(1+\epsilon a)^{n-1}\left[r_1^{\epsilon,\kappa}\right]^{\sharp^{\epsilon,\kappa}\,n}\;+\;(1+\epsilon a)^N r^{\epsilon,\kappa}(\epsilon a)\;\sharp^{\epsilon,\kappa}\;\left[r_1^{\epsilon,\kappa}\right]^{\sharp^{\epsilon,\kappa}\,N}\\&=\;\sum_{1\leq n\leq N}(1+\epsilon a)^{n-1}\left[r_1^{\epsilon,\kappa}\right]^{\sharp^{\epsilon,\kappa}\,n}\;+\;(1+\epsilon a)^N\;\left[r_1^{\epsilon,\kappa}\right]^{\sharp^{\epsilon,\kappa}\,N}\sharp^{\epsilon,\kappa}\;r^{\epsilon,\kappa}(\epsilon a)\,.\end{aligned}\tag{4.30}$$

□

Proposition 4.8. *For any* $N\in\mathbb{N}^*$, *there exist two bounded families* $\{\phi_N[r_1^{\epsilon,\kappa}]\}_{(\epsilon,\kappa)\in[0,\epsilon_0]\times[0,\kappa_0]}$ *in* $S_1^{-2}(\Xi)^\bullet$ *and* $\{\psi_N[r_1^{\epsilon,\kappa}]\}_{(\epsilon,\kappa)\in[0,\epsilon_0]\times[0,\kappa_0]}$ *in* $S_1^{-2N}(\Xi)^\bullet$, *such that*

$$g_{1/\delta}\;\sharp^{\epsilon,\kappa}\;r^{\epsilon,\kappa}(\epsilon a)=g_{1/\delta}\;\sharp^{\epsilon,\kappa}\;\phi_N[r_1^{\epsilon,\kappa}]+g_{1/\delta}\;\sharp^{\epsilon,\kappa}\,r^{\epsilon,\kappa}(\epsilon a)\;\sharp^{\epsilon,\kappa}\;\psi_N[r_1^{\epsilon,\kappa}]\,.$$

Proof. We use Lemma 4.6, noticing that the constants appearing in these formulas are all uniformly bounded for $(\epsilon,\kappa)\in[0,\epsilon_0]\times[0,\kappa_0]$ and Proposition B.21. □

For the first term in the formula given by Proposition 4.8 we can use Proposition B.26 for any $\delta\in(0,\delta_0]$ but we have to control the behavior when $\delta\searrow 0$.

Proposition 4.9. *If* $f^\bullet\in S_1^{-m}(\Xi)^\bullet$ *for some* $m>0$ *there exists a bounded family* $\{F^\epsilon\}_{\epsilon\in[0,\epsilon_0]}\subset S^{-\infty}(\Xi)$ *such that for any* $\epsilon\in[0,\epsilon_0]$ *and* $\delta\in(0,\delta_0]$ *satisfying* (4.20) *we have the relation*

$$g_{1/\delta}\;\sharp^{\epsilon,\kappa}\,f^\epsilon\;=\;F^\epsilon_{(\epsilon,\delta^{-1})}\,,$$

and if we use the notation from Remark B.20 (i.e. $f^\epsilon=\widetilde{f}^\epsilon_{(\epsilon,1)}$*) the map*

$$S_1^{-m}(\Xi)\ni\widetilde{f}^\epsilon\mapsto F^\epsilon\in S^{-\infty}(\Xi)$$

is continuous.

Proof. For any $\varphi \in S_1^{-m}(\Xi)$ with $m > 0$ let us proceed as in the proof of Proposition B.26 after a change of variables $(\eta, z) \mapsto (\delta^{-1}\eta, \delta z)$:

$$\begin{aligned}
&\big(g_{1/\delta} \sharp^{\epsilon,\kappa} \varphi_{(\epsilon,1)}\big)(x,\xi) \\
&\quad = \pi^{-4} \int_{\Xi\times\Xi} e^{-2i\sigma(Y,Z)} \omega^{B_{\epsilon,\kappa}}(x,y,z) g\big(\delta^{-1}\xi - \delta^{-1}\eta\big)\varphi\big(\epsilon x - \epsilon z, \xi - \zeta\big)\, dY dZ \\
&\quad = \pi^{-4} \int_{\Xi\times\Xi} e^{-2i\sigma(Y,Z)} \omega^{B_{\epsilon,\kappa}}(x,y,\delta^{-1}z) g\big(\delta^{-1}\xi - \eta\big)\varphi\big(\epsilon x - \epsilon\delta^{-1} z, \xi - \zeta\big)\, dY dZ \\
&\quad = \pi^{-4} \int_{\Xi\times\Xi} e^{-2i\sigma(Y,Z)} e^{i\,\epsilon\Phi_\kappa(\epsilon x, y, \delta^{-1}z)} g\big(\delta^{-1}\xi - \eta\big)\varphi\big(\epsilon x - \epsilon\delta^{-1} z, \xi - \zeta\big) \times \\
&\qquad\qquad \times \Big(1 \,+\, i\,\kappa\epsilon^2 \Psi^\epsilon(\epsilon x, y, \delta^{-1}z) \textstyle\int_0^1 \Theta_\tau^{\epsilon,\kappa}(\epsilon x, y, \delta^{-1}z) d\tau\Big)\, dY dZ\,.
\end{aligned}$$

Let us note that due to (4.20) $\tau := \epsilon\delta^{-1} = \delta^{\mu-1}$ is bounded for $\delta \in (0, \delta_0]$ and we can write

$$\epsilon\Phi_\kappa(x,y,\delta^{-1}z) = \epsilon\delta^{-1}\sum_{j,k} y_j z_k \left(B_0 + \kappa B(x)\right) = \tau\Phi_\kappa(x,y,z)\,,$$

and

$$\begin{aligned}
&\epsilon^2\Psi^\epsilon(x,y,\delta^{-1}z) \\
&\quad = -8\epsilon^2\delta^{-2}\sum_{j,k} y_j z_k \sum_{\ell=1,2} \big[\delta y_\ell R_1\big(\partial_\ell B_{jk}\big)(x,\epsilon y,\epsilon\delta^{-1}z) + z_\ell R_2\big(\partial_\ell B_{jk}\big)(x,\epsilon y,\epsilon\delta^{-1}z)\big] \\
&\quad = -8\,\tau^2\sum_{j,k} y_j z_k \sum_{\ell=1,2} \big[\delta y_\ell R_1\big(\partial_\ell B_{jk}\big)(x,\delta^\mu y,\tau z) \,+\, z_\ell R_2\big(\partial_\ell B_{jk}\big)(x,\delta^\mu y,\tau z)\big]\,.
\end{aligned}$$

These equalities imply that

$$(x,y,z) \mapsto \epsilon\Phi_\kappa(x,y,\delta^{-1}z), \quad (x,y,z) \mapsto \kappa\epsilon^2\Psi^\epsilon(x,y,\delta^{-1}z) \text{ and } (x,y,z) \mapsto \Theta_t^{\epsilon,\kappa}(\epsilon x, y, \delta^{-1}z)$$

define three families of functions on $\mathcal{X}^3$ indexed by $(\kappa,\delta,\tau,t) \in [0,\kappa_0] \times (0,\delta_0] \times [0, \epsilon_0^{1-1/\mu}] \times [0,1]$ that are bounded in $BC^\infty\big(\mathcal{X}; C^\infty_{\mathsf{pol}}(\mathcal{X}\times\mathcal{X})\big)$. In conclusion we can apply Proposition B.9 with $\mu_1 = \mu_2 = \mu_4 = 1$ and $\mu_3 = \tau$. □

We can now rephrase Proposition 4.8 as

Proposition 4.10. *For any $N \in \mathbb{N}^*$, under Hypothesis* (4.20)*, there exist two bounded families $\{\mathfrak{g}_N^{\epsilon,\kappa}\}_{(\epsilon,\kappa)\in[0,\epsilon_0]\times[0,\kappa_0]}$ in $S^{-\infty}(\Xi)$ and $\{\psi_N[r_1^{\epsilon,\kappa}]\}_{(\epsilon,\kappa)\in[0,\epsilon_0]\times[0,\kappa_0]}$ in $S_1^{-2N}(\Xi)^\bullet$, such that*

$$\begin{aligned}
g_{1/\delta} \sharp^{\epsilon,\kappa} r^{\epsilon,\kappa}(\epsilon a) \quad &= \big(\mathfrak{g}_N^{\epsilon,\kappa}\big)_{(\epsilon,\delta^{-1})} \,+\, g_{1/\delta} \sharp^{\epsilon,\kappa} r^{\epsilon,\kappa}(\epsilon a) \sharp^{\epsilon,\kappa} \psi_N[r_1^{\epsilon,\kappa}] \\
&= \big(\mathfrak{g}_N^{\epsilon,\kappa}\big)_{(\epsilon,\delta^{-1})} \,+\, \psi_N[r_1^{\epsilon,\kappa}] \sharp^{\epsilon,\kappa} r^{\epsilon,\kappa}(\epsilon a) \sharp^{\epsilon,\kappa} g_{1/\delta}\,.
\end{aligned} \tag{4.31}$$

4.4.2 The first term of the quasi-inverse in (4.27).

Using Proposition 4.10 we conclude that in order to study the convergence of the series

$$\sum_{\gamma^*\in\Gamma_*} \tau_{\gamma^*}\big(g_{1/\delta} \sharp^{\epsilon,\kappa} r^{\epsilon,\kappa}(\epsilon a)\big)\,,$$

we may according to (4.31) separately study the convergence of the following two series

$$\sum_{\gamma^*\in\Gamma_*} \tau_{\gamma^*}\big(\big(\mathfrak{g}_N^{\epsilon,\kappa}\big)_{(\epsilon,\delta^{-1})}\big), \tag{4.32}$$

$$\sum_{\gamma^*\in\Gamma_*} \tau_{\gamma^*}\big(g_{1/\delta} \sharp^{\epsilon,\kappa} r^{\epsilon,\kappa}(\epsilon a) \sharp^{\epsilon,\kappa} \psi_N[r_1^{\epsilon,\kappa}]\big)\,, \tag{4.33}$$

that are both of the form discussed in Lemma B.32 and Proposition B.33, but we stil have to control the behavior when $\delta \searrow 0$.

The series (4.32)**.** Let us consider for some $\varphi \in S^{-\infty}(\Xi)$, the series

$$\Phi^{\epsilon,\delta} \ := \sum_{\gamma^* \in \Gamma_*} \tau_{\gamma^*}\big(\varphi_{(\epsilon,\delta^{-1})}\big)\,,$$

and the operator $\mathfrak{Op}^{\epsilon,\kappa}\big(\Phi^{\epsilon,\delta}\big)$ that are well defined as shown in Lemma B.32 and Proposition B.33.

Proposition 4.11. *There exist positive constants C, ϵ_0 and δ_0 such that*

$$\big\|\mathfrak{Op}^{\epsilon,\kappa}\big(\Phi^{\epsilon,\delta}\big)\big\|_{\mathcal{L}(\mathcal{H})} \leq C\,,$$

for any $(\epsilon,\delta,\kappa) \in [0,\epsilon_0] \times (0,\delta_0] \times [0,1]$ verifying (4.20)*. Moreover the application*

$$S^{-\infty}(\Xi) \ni \varphi \ \mapsto \ \Phi^{\epsilon,\delta} \in \big(S^0_0(\Xi), \|\cdot\|_{B_{\epsilon,\kappa}}\big)$$

is continuous uniformly for $(\epsilon,\delta,\kappa) \in [0,\epsilon_0] \times [0,\delta_0] \times [0,\kappa_0]$ verifying (4.20)*.*

Proof. All we have to do is to control the behavior of the norm on the right hand side of (B.49) when $\delta \searrow 0$. We note that $\epsilon\delta^{-2} = \delta^{\mu-2} \leq \delta_0^{\mu-2}$ with $\mu - 2 > 0$ and due to (4.20) we can use Proposition B.26 with $\tau = \delta^{-1}$, in order to obtain that, for any $\alpha^* \in \Gamma_*$,

$$\begin{aligned}
&\Big\|\overline{\widetilde{\varphi}^{\epsilon}}_{(\epsilon,\delta^{-1})} \ \sharp^{\epsilon,\kappa} \ \big(\tau_{\alpha^*}\widetilde{\varphi}^{\epsilon}_{(\epsilon,\delta^{-1})}\big)\Big\|_{B_{\epsilon,\kappa}} \\
&\quad \leq \Big\|\big(\overline{\widetilde{\varphi}^{\epsilon}} \natural^{\epsilon} \big(\tau_{\alpha^*})\widetilde{\varphi}^{\epsilon}\big)_{(\epsilon,\delta^{-1})}\big)\Big\|_{B_{\epsilon,\kappa}} \\
&\qquad + (\epsilon\delta^{-1}/2)\sum_{\ell\leq 2}\Big(\Big\|\big(\partial_{x_\ell}\overline{\widetilde{\varphi}^{\epsilon}} \natural^{\epsilon} \big(\tau_{\alpha^*}\partial_{\xi_\ell}\widetilde{\varphi}^{\epsilon}\big)\big)_{(\epsilon,\delta^{-1})}\Big\|_{B_{\epsilon,\kappa}} + \Big\|\big(\partial_{\xi_\ell}\overline{\widetilde{\varphi}^{\epsilon}} \natural^{\epsilon} \big(\tau_{\alpha^*}\partial_{x_\ell}\widetilde{\varphi}^{\epsilon}\big)\big)_{(\epsilon,\delta^{-1})}\Big\|_{B_{\epsilon,\kappa}}\Big) \\
&\qquad + (\epsilon\delta^{-1})^2\Big\|\mathscr{R}_{\epsilon,\kappa,\delta^{-1}}\big(\overline{\widetilde{\varphi}^{\epsilon}}, \big(\tau_{\alpha^*}\widetilde{\varphi}^{\epsilon}\big)_{(\epsilon,\delta^{-1})}\Big\|_{B_{\epsilon,\kappa}}\,.
\end{aligned}$$

We use Proposition B.8 and obtain that, for any $p > 2$ there exists $C_p > 0$ such that:

$$\begin{aligned}
&< \alpha^* >^N \Big\|\overline{\widetilde{\varphi}^{\epsilon}}_{(\epsilon,\delta^{-1})} \sharp^{\epsilon,\kappa} \big(\tau_{\alpha^*}\widetilde{\varphi}^{\epsilon}_{(\epsilon,\delta^{-1})}\big)\Big\|_{B_{\epsilon,\kappa}} \\
&\qquad \leq C_p \ < \alpha^* >^N \Big[\nu^{-p,0}_{(0,p)}\Big(\overline{\widetilde{\varphi}^{\epsilon}} \natural^{\epsilon}\big(\tau_{\alpha^*}\widetilde{\varphi}^{\epsilon}\big)\Big) \\
&\qquad\quad + (\epsilon\delta^{-1}/2)\sum_{\ell=1,2}\Big(\nu^{-p,0}_{(0,p)}\Big(\partial_{x_\ell}\overline{\widetilde{\varphi}^{\epsilon}} \natural^{\epsilon} \big(\tau_{\alpha^*}\partial_{\xi_\ell}\widetilde{\varphi}^{\epsilon}\big)\Big) + \nu^{-p,0}_{(0,p)}\Big(\partial_{\xi_\ell}\overline{\widetilde{\varphi}^{\epsilon}} \natural^{\epsilon} \big(\tau_{\alpha^*}\partial_{x_\ell}\widetilde{\varphi}^{\epsilon}\big)\Big)\Big) \\
&\qquad\quad + \ (\epsilon\delta^{-1})^2\nu^{-p,0}_{(0,p)}\Big(\mathscr{R}_{\epsilon,\kappa,\delta^{-1}}\big(\overline{\widetilde{\varphi}^{\epsilon}}, \big(\tau_{\alpha^*}\widetilde{\varphi}^{\epsilon}\big)\Big)\Big]\,.
\end{aligned}$$

The four terms appearing above can now be dealt with by similar arguments to those in the proof of Proposition B.33 by using Proposition B.9 and choosing $N > 2$ in order to control the convergence of the series indexed by $\alpha^* \in \Gamma_*$ appearing in the Cotlar-Stein criterion. □

The series (4.33)**.** Let us consider now some $f^\bullet \in S^{-m}_1(\Xi)^\bullet$ for some $m > 0$, the associated family of symbols $\{\widetilde{f}^{\epsilon}\}_{\epsilon\in[0,\epsilon_0]}$ given by Remark B.20 and the series

$$\Psi^{\epsilon,\kappa,\delta} := \sum_{\gamma^* \in \Gamma_*} \tau_{\gamma^*}\big(g_{\delta^{-1}} \ \sharp^{\epsilon,\kappa} \ r^{\epsilon,\kappa}(\epsilon a) \ \sharp^{\epsilon,\kappa} \ f^{\epsilon}\big)\,,$$

with its associated operator $\mathfrak{Op}^{\epsilon,\kappa}\big(\Psi^{\epsilon,\kappa,\delta}\big)$ obtained by using Lemma B.32 and Proposition B.33 after noticing that the magnetic composition property (Theorem 2.2 in [22]) gives:

$$g_{\delta^{-1}} \ \sharp^{\epsilon,\kappa} \ r^{\epsilon,\kappa}(\epsilon a)\sharp^{\epsilon,\kappa} f^{\epsilon} \ \in \ S^{-\infty}(\Xi)\,,$$

for any $(\epsilon,\kappa,\delta) \in [0,\epsilon_0] \times [0,\kappa_0] \times (0,\delta_0]$.

Proposition 4.12. *There exist positive constants C, ϵ_0, δ_0 and κ_0 such that*

$$\left\|\mathfrak{Op}^{\epsilon,\kappa}\big(\Psi^{\epsilon,\kappa,\delta}\big)\right\|_{\mathcal{L}(\mathcal{H})} \leq C\epsilon^{-1},$$

for any $(\epsilon,\delta,\kappa) \in [0,\epsilon_0] \times (0,\delta_0] \times [0,\kappa_0]$ verifying (4.20). Moreover the application

$$S_1^{-m}(\Xi) \ni f^\epsilon \mapsto \epsilon\Psi^{\epsilon,\kappa,\delta} \in \big(S_0^0(\Xi), \|\cdot\|_{B_{\epsilon,\kappa}}\big)$$

is continuous uniformly for $(\epsilon,\delta,\kappa) \in [0,\epsilon_0] \times [0,\delta_0] \times [0,\kappa_0]$ verifying (4.20).

Proof. The main ingredient is to note that

$$\begin{aligned}\big(\overline{g_{\delta^{-1}} \sharp^{\epsilon,\kappa} r^{\epsilon,\kappa}(\epsilon a) \sharp^{\epsilon,\kappa} f^\epsilon}\big) \sharp^{\epsilon,\kappa} \tau_{\alpha^*}\big(g_{\delta^{-1}} \sharp^{\epsilon,\kappa} r^{\epsilon,\kappa}(\epsilon a) \sharp^{\epsilon,\kappa} f^\epsilon\big)& \\ = \big(\overline{f^\epsilon} \sharp^{\epsilon,\kappa} r^{\epsilon,\kappa}(\epsilon a)\big) \sharp^{\epsilon,\kappa} \big(g_{\delta^{-1}} \sharp^{\epsilon,\kappa} \tau_{\alpha^*}(g_{\delta^{-1}})\big) \sharp^{\epsilon,\kappa} \tau_{\alpha^*}\big(r^{\epsilon,\kappa}(\epsilon a) \sharp^{\epsilon,\kappa} f^\epsilon\big),&\end{aligned}$$

and

$$\begin{aligned}&\big(g_{\delta^{-1}} \sharp^{\epsilon,\kappa} r^{\epsilon,\kappa}(\epsilon a) \sharp^{\epsilon,\kappa} f^\epsilon\big) \sharp^{\epsilon,\kappa} \tau_{\alpha^*}\big(\overline{g_{\delta^{-1}} \sharp^{\epsilon,\kappa} r^{\epsilon,\kappa}(\epsilon a) \sharp^{\epsilon,\kappa} f^\epsilon}\big) \\ &= \big(g_{\delta^{-1}} \sharp^{\epsilon,\kappa} r^{\epsilon,\kappa}(\epsilon a)\big) \sharp^{\epsilon,\kappa} \big(\widetilde{f^\epsilon}_{(\epsilon,1)} \sharp^{\epsilon,\kappa} \tau_{\alpha^*}(\overline{\widetilde{f^\epsilon}}_{(\epsilon,1)})\big) \sharp^{\epsilon,\kappa} \tau_{\alpha^*}\big(r^{\epsilon,\kappa}(\epsilon a) \sharp^{\epsilon,\kappa} g_{\delta^{-1}}\big).\end{aligned}$$

Then we fix some $m > N+2 > 4$ and we repeat the arguments in the proof of Proposition 4.11. □

4.4.3 Locating the spectral islands

In this subsection we fix $\mu = 3$ and assume moreover (4.28), in order to prove the second claim of Proposition 4.5 concerning the location and size of the spectral islands.

At the operator level, (4.28) can be rewritten as

$$\Big(\mathfrak{Op}^{\epsilon,\kappa}(\lambda^\epsilon) - \epsilon a\Big)\, \mathfrak{Op}^{\epsilon,\kappa}(\widetilde{r}_\lambda(\epsilon a)) = \mathbb{1} + \mathfrak{Op}^{\epsilon,\kappa}(\mathfrak{r}_{\delta,a}), \quad \|\mathfrak{Op}^{\epsilon,\kappa}(\mathfrak{r}_{\delta,a})\| \leq C\,\epsilon^{1/3}. \tag{4.34}$$

Let us choose $a \in \mathbb{C}$ on the positively oriented circle $|a - (2n+1)mB_0| = mB_0$ centered at some Landau level $(2n+1)mB_0$ such that the distance between a and all the Landau levels is bounded from below by mB_0 and take $\mathtt{z} = \epsilon a$. The estimate (4.34) implies that

$$\left\|\Big(\mathfrak{Op}^{\epsilon,\kappa}(\lambda^\epsilon) - \mathtt{z}\Big)^{-1}\right\| \leq C\,\epsilon^{-1}.$$

By iterating we get:

$$\Big(\mathfrak{Op}^{\epsilon,\kappa}(\lambda^\epsilon) - \mathtt{z}\Big)^{-1} = \mathfrak{Op}^{\epsilon,\kappa}(\widetilde{r}_\lambda(\mathtt{z})) - \Big(\mathfrak{Op}^{\epsilon,\kappa}(\lambda^\epsilon) - \mathtt{z}\Big)^{-1} \mathfrak{Op}^{\epsilon,\kappa}(\mathfrak{r}_{\delta,a}). \tag{4.35}$$

In order to identify the band operator corresponding to the spectrum near $(2n+1)m\epsilon B_0$ we compute the Riesz integral:

$$h_n := \frac{i}{2\pi}\int_{|\mathtt{z}-(2n+1)m\epsilon B_0| = m\epsilon B_0} \big(\mathtt{z} - (2n+1)m\epsilon B_0\big)\Big(\mathfrak{Op}^{\epsilon,\kappa}(\lambda^\epsilon) - \mathtt{z}\Big)^{-1} d\mathtt{z}.$$

We want to show that

$$\|h_n\| \leq C(\epsilon\kappa + \epsilon^{4/3}).$$

Inserting (4.35) in the Riesz integral we obtain two contributions. The second contribution containing the term $\mathfrak{Op}^{\epsilon,\kappa}(\mathfrak{r}_{\delta,a})$ is easy, and leads to something of order $\epsilon^{4/3}$. We still need to estimate the term

$$\frac{i}{2\pi}\int_{|\mathtt{z}-(2n+1)m\epsilon B_0| = m\epsilon B_0} \big(\mathtt{z} - (2n+1)m\epsilon B_0\big)\mathfrak{Op}^{\epsilon,\kappa}(\widetilde{r}_\lambda(\mathtt{z}))\, d\mathtt{z}.$$

Since $m_\epsilon - m = \mathcal{O}(\epsilon)$, replacing m with m_ϵ in the expression of h_n produces an error of order ϵ^2, hence the relevant object becomes:

$$h'_n := \frac{i}{2\pi}\int_{|\mathbf{z}-(2n+1)m_\epsilon\epsilon B_0|=m_\epsilon\epsilon B_0} \big(\mathbf{z}-(2n+1)m_\epsilon\epsilon B_0\big)\mathfrak{Op}^{\epsilon,\kappa}(\widetilde{r}_\lambda(\mathbf{z}))\,d\mathbf{z}\,.$$

From the expression of $\widetilde{r}_\lambda(\mathbf{z})$ in (4.27) we see that the second term is analytic inside the circle of integration, thus only

$$\sum_{\gamma^*\in\Gamma_*}\tau_{\gamma^*}\big(g_{1/\delta}\,\sharp^{\epsilon,\kappa}\,r^{\epsilon,\kappa}(\mathbf{z})\big)$$

will contribute to h'_n. Let us introduce the shorthand notation $E_n^\epsilon := (2n+1)m\epsilon B_0$ and notice that $h'_n = \mathfrak{Op}^{\epsilon,\kappa}(\mathfrak{h}_n)$ with $\mathfrak{h}_n := \sum_{\gamma^*\in\Gamma_*}\tau_{\gamma^*}\left(g_{1/\delta}\,\sharp^{\epsilon,\kappa}\,\mathfrak{g}_n\right)$, where

$$\mathfrak{g}_n := \frac{i}{2\pi}\int_{|\mathbf{z}-E_n^\epsilon|=m_\epsilon\epsilon B_0}\big(\mathbf{z}-E_n^\epsilon\big)\,r^{\epsilon,\kappa}(\mathbf{z})\,d\mathbf{z}\,. \tag{4.36}$$

This is a slowly varying symbol of class $S_1^{-2}(\Xi)^\bullet$ (see Definition B.19) having an operator norm of order ϵ. Using the expansion in (4.30) we notice that the first sum of N terms is a polynomial in $\mathbf{z}$ and thus vanishes when integrated along the circle $|\mathbf{z}-E_n^\epsilon| = m_\epsilon\epsilon B_0$ and we get:

$$\mathfrak{g}_n = \frac{i}{2\pi}\int_{|\mathbf{z}-E_n^\epsilon|=m_\epsilon\epsilon B_0}\big(\mathbf{z}-E_n^\epsilon\big)(1+\mathbf{z})^N r^{\epsilon,\kappa}(\mathbf{z})\,\sharp^{\epsilon,\kappa}\,\left[r_1^{\epsilon,\kappa}\right]^{\sharp^{\epsilon,\kappa}\,N}d\mathbf{z}\,,$$

and

$$\mathfrak{h}'_n = \sum_{\gamma^*\in\Gamma_*}\tau_{\gamma^*}\left(g_{1/\delta}\,\sharp^{\epsilon,\kappa}\,\widetilde{\mathfrak{g}}_n\,\sharp^{\epsilon,\kappa}\,\left[r_1^{\epsilon,\kappa}\right]^{\sharp^{\epsilon,\kappa}\,N}\right)\,,$$

where

$$\widetilde{\mathfrak{g}}_n := \frac{i}{2\pi}\int_{|\mathbf{z}-E_n^\epsilon|=m_\epsilon\epsilon B_0}\big(\mathbf{z}-E_n^\epsilon\big)(1+\mathbf{z})^N r^{\epsilon,\kappa}(\mathbf{z})\,d\mathbf{z}\,.$$

The arguments in the above subsection may be applied in order to conclude that $\mathfrak{h}'_n \in S_0^0(\Xi)$ and a-priori:

$$\|\mathfrak{Op}^{\epsilon,\kappa}(\mathfrak{h}'_n)\| \le C\epsilon\,,$$

but this is not enough in order to conclude the existence of gaps.

Let us also consider $r^\epsilon(\epsilon a) \in S_1^{-2}(\Xi)$ such that

$$(\mathfrak{Op}^\epsilon(h_{m^\epsilon}) - \epsilon a)^{-1} = \mathfrak{Op}^\epsilon\big(r^\epsilon(\epsilon a)\big)\,.$$

For a on the circle $|a-E_n^1| = m_\epsilon B_0$, the above inverse is well defined and its symbol also. Let us notice that

$$\frac{i}{2\pi}\int_{|\mathbf{z}-E_n^\epsilon|\,=\,m_\epsilon\epsilon B_0}\big(\mathbf{z}-E_n^\epsilon\big)r^\epsilon(\mathbf{z})\,d\mathbf{z} = 0\,,$$

and

$$\frac{i}{2\pi}\int_{|\mathbf{z}-E_n^\epsilon|\,=\,m_\epsilon\epsilon B_0}\big(\mathbf{z}-E_n^\epsilon\big)(1+\mathbf{z})^N r^\epsilon(\mathbf{z})\,d\mathbf{z} = 0\,,$$

both integrands being analytic inside the circle $|\mathbf{z}-E_n^\epsilon| = m_\epsilon\epsilon B_0$. Hence we can write:

$$\widetilde{\mathfrak{g}}_n = \frac{i}{2\pi}\int_{|\mathbf{z}-E_n^\epsilon|=m_\epsilon\epsilon B_0}\big(\mathbf{z}-E_n^\epsilon\big)(1+\mathbf{z})^N\big(r^{\epsilon,\kappa}(\mathbf{z}) - r^\epsilon(\mathbf{z})\big)\,d\mathbf{z}\,.$$

Using now the estimate (4.12), Proposition 4.9 and Proposition 4.11 we obtain:

$$\|\mathfrak{Op}^{\epsilon,\kappa}(h'_n)\| \;\le\; C\,\kappa\epsilon^{-1}\epsilon^2 \;=\; C\,\kappa\epsilon\,.$$

which concludes the proof of the size of the spectral islands. From now on we concentrate on proving (4.28).

4.4.4 Estimating the product $\big(\lambda^\epsilon - \epsilon a\big) \sharp^{\epsilon,\kappa} \widetilde{r}_\lambda(\epsilon a)$.

Here we no longer fix $\mu = 3$. Instead, we again allow μ to vary in the interval $(2,4)$ and actually prove that this is the maximal interval for which the operator defined in (4.27) is a quasi-resolvent.

Given the periodic lattice $\Gamma_* \subset \mathcal{X}^*$ and the function $g_{1/\delta}$ defined in (4.14), for $\delta \in (0,\delta_0]$, there exists a function $\widetilde{\chi} \in C_0^\infty(\mathcal{X}^*)$ such that:

1. $0 \le \widetilde{\chi} \le 1$,
2. $\widetilde{\chi}\, g_{1/\delta} = g_{1/\delta}$, for any $\delta \in (0,\delta_0]$,
3. $\sum\limits_{\gamma^* \in \Gamma_*} \tau_{\gamma^*} \widetilde{\chi} = 1$.

For any $\gamma^* \in \Gamma_*$ let us introduce

$$\widetilde{\lambda}^{\epsilon,\kappa}_{\gamma^*} \;:=\; \big(\lambda^\epsilon - \epsilon a\big) \sharp^{\epsilon,\kappa} \tau_{\gamma^*} \widetilde{\chi}\,. \tag{4.37}$$

Proposition 4.13. *The families* $\{\widetilde{\lambda}^{\epsilon,\kappa}_{\gamma^*}\}_{(\epsilon,\kappa)\in[0,\epsilon_0]\times[0,1]}$ *are bounded in* $S^{-\infty}(\Xi)^\bullet$.

Proof. We notice that $\lambda^\epsilon - \epsilon a \in S^{-\infty}(\Xi)^\circ$ for any $\epsilon \in [0,\epsilon_0]$ and $\widetilde{\chi}$ also, so that we can use Proposition B.21. □

We have the following properties:

1. $\widetilde{\lambda}^{\epsilon,\kappa}_{\gamma^*} = \tau_{\gamma^*} \widetilde{\lambda}^{\epsilon}_0$,
2. $\big(\lambda^\epsilon - \epsilon a\big) = \sum\limits_{\gamma^* \in \Gamma_*} \widetilde{\lambda}^{\epsilon,\kappa}_{\gamma^*}$,
   the convergence following from Lemma B.32 and Proposition B.33.

We shall use the following decomposition:

$$\begin{aligned}\big(\lambda^\epsilon - \epsilon a\big) \sharp^{\epsilon,\kappa} \widetilde{r}_\lambda(\epsilon a) = &\sum_{\gamma^* \in \Gamma_*} \widetilde{\lambda}^{\epsilon,\kappa}_{\gamma^*} \sharp^{\epsilon,\kappa} \sum_{\alpha^* \in \Gamma_*} \tau_{\alpha^*}\big(g_{1/\delta} \sharp^{\epsilon,\kappa} r^{\epsilon,\kappa}(\epsilon a)\big) \\ &+ \sum_{\gamma^* \in \Gamma_*} \widetilde{\lambda}^{\epsilon,\kappa}_{\gamma^*} \sharp^{\epsilon,\kappa} \big(1 - \widetilde{g}_{1/\delta}\big) \sharp^{\epsilon,\kappa} r_{\delta,\epsilon,\kappa}(\epsilon a)\,.\end{aligned} \tag{4.38}$$

The main contribution to the series (4.38)**.** In this paragraph we shall consider the term in the second line of (4.38) with $\gamma^* = \alpha^* = 0$:

$$\widetilde{\lambda}^{\epsilon}_0 \sharp^{\epsilon,\kappa} \big(g_{1/\delta} \sharp^{\epsilon,\kappa} r^{\epsilon,\kappa}(\epsilon a)\big) \;=\; \big(\lambda^\epsilon - \epsilon a\big) \sharp^{\epsilon,\kappa} \big(\widetilde{\chi} \sharp^{\epsilon,\kappa} g_{1/\delta}\big) \sharp^{\epsilon,\kappa} r^{\epsilon,\kappa}(\epsilon a)\,. \tag{4.39}$$

Lemma 4.14. *Given* $\epsilon_0 > 0$ *and* $\delta_0 > 0$ *satisfying* (4.16)*, for any* $(\epsilon,\delta,\kappa) \in (0,\epsilon_0] \times (0,\delta_0] \times [0,1]$ *satisfying* (4.20)*, the following relation holds*

$$\widetilde{\chi} \sharp^{\epsilon,\kappa} g_{1/\delta} \;=\; \varphi^{\epsilon,\delta,\kappa}_{(\epsilon,\delta^{-1})}\,,$$

where the family of symbols

$$\big\{\varphi^{\epsilon,\delta,\kappa} \;\big|\; (\epsilon,\delta,\kappa) \in (0,\epsilon_0] \times (0,\delta_0] \times [0,1],\, \epsilon = \delta^\mu\big\}$$

is bounded in $S^{-\infty}(\Xi)$.

Proof. We use the arguments in the proof of Proposition B.26 (formula (B.46)) and Proposition B.9. A change of variables in the definition of

$$\begin{aligned}\big(\widetilde{\chi} \sharp^{\epsilon,\kappa} g_{1/\delta}\big)(x,\xi) \;&=\; \int_{\Xi\times\Xi} dY dZ \,\Big[e^{-2i\sigma(Y,Z)} e^{i\,\epsilon\Phi_\kappa(\epsilon x,y,z)} \widetilde{\chi}(\xi-\eta) g(\delta^{-1}(\xi-\zeta)) \times \\ &\qquad \times \Big(1 + i\,\kappa\epsilon^2 \Psi^\epsilon(\epsilon x,y,z) \textstyle\int_0^1 \Theta^{\epsilon,\kappa}_\tau(\epsilon x,y,z) d\tau\Big)\Big]\end{aligned} \tag{4.40}$$

allows to write

$$\big(\widetilde{\chi}\,\sharp^{\epsilon,\kappa}\,g_{1/\delta}\big)(x,\xi)\;=\;\varphi^{\epsilon,\delta,\kappa}(\epsilon x,\delta^{-1}\xi)\,,$$

where

$$\varphi^{\epsilon,\delta,\kappa}(x,\xi):=\int\limits_{\Xi\times\Xi}dY\,dZ\,e^{-2i\sigma(Y,Z)}\widetilde{\chi}_\delta(\xi-\eta)g(\xi-\zeta)e^{i\,\epsilon\Phi_\kappa(x,\delta^{-1}y,\delta^{-1}z)}\times$$
$$\times\;\left(1\,+\,i\,\kappa\epsilon^2\Psi^\epsilon(x,\delta^{-1}y,\delta^{-1}z)\int_0^1\Theta_\tau^{\epsilon,\kappa}(x,\delta^{-1}y,\delta^{-1}z)d\tau\right).$$

We notice first that $\{\widetilde{\chi}_\delta\}_{\delta\in(0,\delta_0]}$ is a bounded subset of $S^{-\infty}(\Xi)^\circ$ and that the symbol $\varphi^{\epsilon,\delta,\kappa}$ is defined by an oscillatory integral of the form $\mathcal{L}^{1,1}_{(1,1,1,1)}\big(\widetilde{\chi}_\delta,g\big)$ (see Proposition B.9 for this notation) with the integral kernel

$$L(x,y,z):=e^{i\,\epsilon\Phi_\kappa(x,\delta^{-1}y,\delta^{-1}z)}\left(1\,+\,i\,\kappa\epsilon^2\Psi^\epsilon(x,\delta^{-1}y,\delta^{-1}z)\int_0^1\Theta_\tau^{\epsilon,\kappa}(x,\delta^{-1}y,\delta^{-1}z)d\tau\right).\qquad(4.41)$$

We notice that

1. $\epsilon\Phi_\kappa(x,\delta^{-1}y,\delta^{-1}z)=\epsilon\delta^{-2}\big(B_0+\kappa B(x)\big)(y_2z_3-y_3z_2)=\epsilon\delta^{-2}\Phi_\kappa(x,y,z)\,,$
   and by (4.20) $\epsilon\delta^{-2}=\delta^{\mu-2}\leq\delta_0^{\mu-2}$ goes to 0 when $\delta\searrow 0\,.$
2. $\kappa\epsilon^2\Psi^\epsilon(x,\delta^{-1}y,\delta^{-1}z)=\kappa\epsilon^2\delta^{-3}\Psi^1(x,\epsilon\delta^{-1}y,\epsilon\delta^{-1}z)\,,$
   and by (4.20) we have $\epsilon^2\delta^{-3}=\delta^{2\mu-3}\leq\delta_0^{2\mu-3}$ and $\epsilon\delta^{-1}=\delta^{\mu-1}\leq\delta_0^{\mu-1}$ and both go to 0 when $\delta\searrow 0\,.$
3. $\Theta_\tau^{\epsilon,\kappa}(x,\delta^{-1}y,\delta^{-1}z)=\exp\big\{i\tau\kappa\epsilon^2\Psi^\epsilon(x,\delta^{-1}y,\delta^{-1}z)\big\}=\exp\big\{i\tau\kappa\epsilon^2\delta^{-3}\Psi^1(x,\epsilon\delta^{-1}y,\epsilon\delta^{-1}z)\big\}\,.$

We conclude that for $(\tau,\kappa)\in[0,1]\times[0,1]$ and $(\epsilon,\delta)\in[0,\epsilon_0]\times(0,\delta_0]$ verifying (4.20), the corresponding family of integral kernels L defined in (4.41) is bounded in $BC^\infty\big(\mathcal{X};C^\infty_{\mathsf{pol}}(\mathcal{X}\times\mathcal{X})\big)$ and we can apply Corollary B.11 in order to finish the proof. □

Lemma 4.15. *Given $\epsilon_0>0$ and $\delta_0>0$ satisfying (4.16), there exists $C>0$ such that, for any $(\epsilon,\delta,\kappa)\in(0,\epsilon_0]\times(0,\delta_0]\times[0,1]$ satisfying (4.20), we have*

$$\widetilde{\chi}\,\sharp^{\epsilon,\kappa}\,g_{1/\delta}\;=\;g_{1/\delta}\,+\,\mathfrak{r}^{\epsilon,\delta,\kappa}_{(\epsilon,\delta^{-1})}\,,$$

where the family

$$\{\mathfrak{r}^{\epsilon,\delta,\kappa}\;|\;(\epsilon,\delta,\kappa)\in[0,\epsilon_0]\times(0,\delta_0]\times[0,1]\,,\,\epsilon=\delta^\mu\}$$

is a bounded subset in $S^{-\infty}(\Xi)$ and satisfies

$$\|\mathfrak{r}^{\epsilon,\delta,\kappa}_{(\epsilon,\delta^{-1})}\|_{B_{\epsilon,\kappa}}\leq C\,\delta^{2(\mu-1)}\,.$$

Proof. Let us come back to the formula (4.40) in the proof of Lemma 4.14 and notice that

$$\int\limits_{\Xi\times\Xi}dY\,dZ\,e^{-2i\sigma(Y,Z)}e^{i\,\epsilon\Phi_\kappa(\epsilon x,y,z)}\widetilde{\chi}(\xi-\eta)g(\delta^{-1}(\xi-\zeta))=\big(\widetilde{\chi}\,\natural^\epsilon\,g_{\delta^{-1}}\big)(x,\xi)\,,$$

so that

$$\widetilde{\chi}\,\sharp^{\epsilon,\kappa}\,g_{1/\delta}=\widetilde{\chi}\,\natural^\epsilon\,g_{\delta^{-1}}\,+\,\psi^{\epsilon,\delta,\kappa}_{(\epsilon,\delta^{-1})}\,,$$

with

$$\psi^{\epsilon,\delta,\kappa}(x,\xi)$$
$$:=\kappa\epsilon^2\int_0^1d\tau\int\limits_{\Xi\times\Xi}dY\,dZ\,e^{-2i\sigma(Y,Z)}e^{i\,\epsilon\Phi_\kappa(x,y,z)}\Psi^\epsilon(x,y,z)\Theta_\tau^{\epsilon,\kappa}(x,y,z)\widetilde{\chi}(\delta\xi-\eta)g(\xi-\delta^{-1}\zeta)\,,$$

being similar with the symbols $\{\varphi^{\epsilon,\delta,\kappa}\}$ in Lemma 4.14.
By the same arguments as in the previous proof we have:

$$\psi^{\epsilon,\delta,\kappa}(x,\xi) = \delta^{2(\mu-1)}\kappa(\epsilon\delta^{-\mu})^2 \int_0^1 d\tau \int\limits_{\Xi\times\Xi} dY\,dZ\, e^{-2i\sigma(Y,Z)}\widetilde{\chi}(\delta\xi-\eta)g(\xi-\zeta)\times$$
$$\times\ e^{i\,\epsilon\delta^{-1}\Phi_\kappa(x,y,z)} i\widetilde{\Psi}_\delta(x,\epsilon\delta^{-1}y,\epsilon z)\exp i\left\{\tau\kappa\epsilon^2\delta^{-2}\widetilde{\Psi}_\delta(x,\epsilon\delta^{-1}y,\epsilon z)\right\}$$

with

$$\widetilde{\Psi}_\delta(x,y,z)\ :=\ 8\sum_{1\le j,k\le 2} y_j z_k \sum_{1\le \ell\le 2}\left[y_\ell R_1\big(\partial_\ell B_{jk}\big)(x,\epsilon y,\epsilon z)\ +\ \delta z_\ell R_2\big(\partial_\ell B_{jk}\big)(x,\epsilon y,\epsilon z)\right]\ .$$

Moreover the family

$$\{\widetilde{\psi}^{\epsilon,\delta,\kappa} := \delta^{2(1-\mu)}\psi^{\epsilon,\delta,\kappa}\ \mid\ (\epsilon,\delta,\kappa)\in[0,\epsilon_0]\times(0,\delta_0]\times[0,1],\ \epsilon=\delta^\mu\}$$

is a bounded subset in $S^{-\infty}(\Xi)$.

We now use Proposition B.31 with $N=2$ in order to obtain that

$$\begin{aligned}\big(\widetilde{\chi}\,\natural^\epsilon\, g_{\delta^{-1}}\big)(x,\xi) &= \widetilde{\chi}(\xi)g_{\delta^{-1}}(\xi)\ +\ \epsilon\big(B_0+\kappa B(\epsilon x)\big)\sum_{j=1,2}\big(\partial_{\xi_j}\widetilde{\chi}\big)(\xi)\,\big(\partial_{\xi_j^\perp}g_{\delta^{-1}}\big)(\xi)\\ &\quad +\ \epsilon^2\big(B_0+\kappa B(\epsilon x)\big)^2\sum_{|\alpha|=2}\mathscr{M}_2^{\epsilon,\kappa}\big(\partial_\xi^\alpha\widetilde{\chi},\partial_{\xi^\perp}^\alpha g_{\delta^{-1}}\big)(x,\xi)\\ &=\ g_{\delta^{-1}}(\xi)\ +\ \epsilon^2\delta^{-2}\big(B_0+\kappa B(\epsilon x)\big)^2\sum_{|\alpha|=2}\mathscr{M}_2^{\epsilon,\kappa}\big(\partial_\xi^\alpha\widetilde{\chi},(\partial_{\xi^\perp}^\alpha g)_{\delta^{-1}}\big)(x,\xi)\,,\end{aligned}\tag{4.42}$$

because $g_{\delta^{-1}}=0$ on the support of $\nabla_\xi\widetilde{\chi}$.
Taking into account formula (B.47) in the proof of Proposition B.31 we see that

$$\epsilon^2\delta^{-2}\big(B_0+\kappa B_\epsilon\big)^2\sum_{|\alpha|=2}\mathscr{M}_2^{\epsilon,\kappa}\big(\partial_\xi^\alpha\widetilde{\chi},(\partial_{\xi^\perp}^\alpha g)_{\delta^{-1}}\big)\ =\ \delta^{2(\mu-1)}\theta^{\epsilon,\delta,\kappa}_{(\epsilon,\delta^{-1})}\,,\tag{4.43}$$

for some family $\{\theta^{\epsilon,\delta,\kappa}\ \mid\ (\epsilon,\delta\,,\kappa)\in[0,\epsilon_0]\times(0,\delta_0]\times[0,1],\ \epsilon=\delta^\mu\}$ that is bounded in $S^{-\infty}(\Xi)$.

We define $\mathfrak{r}^{\epsilon,\delta,\kappa} := \delta^{2(\mu-1)}\big(\widetilde{\psi}^{\epsilon,\delta,\kappa}+\theta^{\epsilon,\delta,\kappa}\big)$ and the above results imply the conclusion of the lemma by using Proposition B.8 and the continuity criterion for the magnetic pseudodifferential calculus (Theorem 3.1 in [22]). □

Thus we can write our *main contribution term* in (4.39) as:

$$\begin{aligned}&\big(\lambda^\epsilon-\epsilon a\big)\ \sharp^{\epsilon,\kappa}\ \big(\widetilde{\chi}\ \sharp^{\epsilon,\kappa}\ g_{1/\delta}\big)\ \sharp^{\epsilon,\kappa}\ r^{\epsilon,\kappa}(\epsilon a)\\ &\qquad=\big(\lambda^\epsilon-\epsilon a\big)\ \sharp^{\epsilon,\kappa}\ g_{1/\delta}\ \sharp^{\epsilon,\kappa}\ r^{\epsilon,\kappa}(\epsilon a)\ +\ \big(\lambda^\epsilon-\epsilon a\big)\,\sharp^{\epsilon,\kappa}\,\mathfrak{r}^{\epsilon,\delta,\kappa}_{(\epsilon,\delta^{-1})}\,\sharp^{\epsilon,\kappa}r^{\epsilon,\kappa}(\epsilon a)\end{aligned}\tag{4.44}$$

and notice that

$$\begin{aligned}\Big\|\big(\lambda^\epsilon-\epsilon a\big)\,\sharp^{\epsilon,\kappa}\,\mathfrak{r}^{\epsilon,\delta,\kappa}_{(\epsilon,\delta^{-1})}\,\sharp^{\epsilon,\kappa}r^{\epsilon,\kappa}(\epsilon a)\Big\|_{B_{\epsilon,\kappa}} &\le C\delta^{2(\mu-1)}\epsilon^{-1}\,\|\lambda^\epsilon-\epsilon a\|_{B_{\epsilon,\kappa}}\,\|\epsilon r^{\epsilon,\kappa}(\epsilon a)\|_{B_{\epsilon,\kappa}}\\ &\le C\delta^{\mu-2}.\end{aligned}\tag{4.45}$$

Proposition 4.16. *Given $\epsilon_0>0$ and $\delta_0>0$ satisfying* (4.16)*, there exist $C>0$ such that for any $(\epsilon,\delta,\kappa)\in(0,\epsilon_0]\times(0,\delta_0]\times[0,1]$ satisfying* (4.20) *we have:*

$$\big(\lambda^\epsilon-\epsilon a\big)\ \sharp^{\epsilon,\kappa}\ g_{1/\delta}\ \sharp^{\epsilon,\kappa}\ r^{\epsilon,\kappa}(a)\ =\ g_{1/\delta}\ +\ \delta^{\mu-2}\mathfrak{z}^{\epsilon,\delta,\kappa}\,,$$

with $\mathfrak{z}^{\epsilon,\delta,\kappa}\in S^{-\infty}(\Xi)$ and $\|\mathfrak{z}^{\epsilon,\delta,\kappa}\|_{B_{\epsilon,\kappa}}\le C$.

Proof. We repeat the arguments in the proof of Lemma 4.15 and write that

$$\big(\lambda^\epsilon - \epsilon a\big) \sharp^{\epsilon,\kappa} g_{1/\delta} \;=\; \big(\lambda^\epsilon - \epsilon a\big) \natural^\epsilon g_{1/\delta} \;+\; \delta^{2(\mu-1)} \psi^{\epsilon,\delta,\kappa}_{(\epsilon,\delta^{-1})}\,, \tag{4.46}$$

where the subset $\{\psi^{\epsilon,\delta,\kappa} \mid (\epsilon,\delta,\kappa) \in [0,\epsilon_0] \times (0,\delta_0] \times [0,1],\, \epsilon = \delta^\mu\} \subset S^{-\infty}(\Xi)$ is bounded in $S^{-\infty}(\Xi)$.

For the first term in (4.46) we use Proposition B.31 with $N = 2$ and write that

$$\begin{aligned}\big[\big(\lambda^\epsilon - \epsilon a\big) \natural^\epsilon g_{1/\delta}\big](x,\xi) \\ = \;& \big(\lambda^\epsilon - \epsilon a\big)(\xi)\, g_{1/\delta}(\xi) + \epsilon\big(B_0 + \kappa B(\epsilon x)\big) \sum_{j=1,2} \big(\partial_{\xi_j}\lambda^\epsilon\big)(\xi)\, \big(\partial_{\xi_j^\perp} g_{\delta^{-1}}\big)(\xi) \\ & + \epsilon^2\big(B_0 + \kappa B(\epsilon x)\big)^2 \sum_{|\alpha|=2} \mathscr{M}_2^{\epsilon,\kappa}\big(\partial_\xi^\alpha \lambda^\epsilon, \partial_{\xi^\perp}^\alpha g_{\delta^{-1}}\big)(x,\xi)\,.\end{aligned}$$

For the first term in (4.47) we use (4.21) and notice that $\delta^4\epsilon^{-1} = \delta^{4-\mu}$ with $4-\mu > 0$ by (4.20) and thus

$$\delta^4 \mathfrak{f}^{\epsilon,\delta}_{(\delta^{-1})} \sharp^{\epsilon,\kappa} r^{\epsilon,\kappa}(a) \;=\; (\delta^4\epsilon^{-1})\big[\mathfrak{f}^{\epsilon,\delta}_{(\delta^{-1})} \sharp^{\epsilon,\kappa} \big(\epsilon r^{\epsilon,\kappa}(a)\big)\big]\,,$$

where $\Big\{\mathfrak{f}^{\epsilon,\delta}_{(\delta^{-1})} \mid (\epsilon,\delta) \in [0,\epsilon_0] \times (0,\delta_0],\, \epsilon = \delta^\mu\Big\}$ is a bounded set in $S^{-\infty}(\Xi)^\circ$.
Thus

$$\|\delta^4 \mathfrak{f}^{\epsilon,\delta}_{(\delta^{-1})} \sharp^{\epsilon,\kappa} r^{\epsilon,\kappa}(a)\|_{B_{\epsilon,\kappa}} \le (\delta^4\epsilon^{-1}) \Big\|\mathfrak{f}^{\epsilon,\delta}_{(\delta^{-1})}\Big\|_{B_{\epsilon,\kappa}} \|\epsilon r^{\epsilon,\kappa}(a)\|_{B_{\epsilon,\kappa}} \le C\,\delta\,.$$

The analysis of the second term is more difficult and makes use of the special form of the cut-off function g in (4.14). We denote by $\nabla_{\xi_j^\perp} := (-\partial_{\xi_2}, \partial_{\xi_1})$ the orthogonal gradient operator. The support of all derivatives of g_{1/δ° is contained in the support of g_{1/δ°, thus:

$$\big(\partial_{\xi_j}\lambda^\epsilon\big)(\xi)\, \big(\partial_{\xi_j^\perp} g_{1/\delta^\circ}\big)(\xi) \;=\; \big(\partial_{\xi_j}(\lambda^\epsilon\, g_{1/\delta})\big)(\xi)\, \big(\partial_{\xi_j^\perp} g_{1/\delta^\circ}\big)(\xi)\,.$$

Moreover, if we denote by χ' the derivative of χ having support in the annulus $1 \le |t| \le 2$, then due to the choice (4.14) we have the following equality:

$$\begin{aligned}\big(\nabla_\xi(\lambda^\epsilon\, g_{1/\delta})\big)(\xi) \cdot \big(\nabla_{\xi^\perp} g_{1/\delta^\circ}\big)(\xi) \\ &= 4\, m_1^\epsilon m_2^\epsilon\, (\delta^\circ)^{-1} \lambda^\epsilon(\xi) \chi'\big(h_{m^\epsilon}((\delta^\circ)^{-1}\xi)\big)\big(-\xi_1\xi_2 + \xi_2\xi_1\big) \;+\; \delta^3 \textstyle\sum_{j=1}^2 \mathfrak{f}^{j,\epsilon,\delta}(\delta^{-1}\xi) \\ &= \delta^3 \textstyle\sum_{j=1}^2 \mathfrak{f}^{j,\epsilon,\delta}(\delta^{-1}\xi)\,.\end{aligned}$$

In conclusion, the second term in (4.47) is of the form $\epsilon\delta^3 F^{\epsilon,\delta,\kappa}_{(1,\delta^{-1})}$ for a bounded family of symbols in $S^{-\infty}(\Xi)$

$$\big\{F^{\epsilon,\delta,\kappa} \mid (\epsilon,\delta,\kappa) \in [0,\epsilon_0] \times (0,\delta_0] \times [0,1]\,,\, \epsilon = \delta^\mu\big\}$$

and we have the estimate:

$$\epsilon\delta^3 \|F^{\epsilon,\delta,\kappa}_{(1,\delta^{-1})} \sharp^{\epsilon,\kappa} r^{\epsilon,\kappa}(a)\|_{B_{\epsilon,\kappa}} \le \delta^3 \Big\|F^{\epsilon,\delta,\kappa}_{(1,\delta^{-1})}\Big\|_{B_{\epsilon,\kappa}} \|\epsilon r^{\epsilon,\kappa}(a)\|_{B_{\epsilon,\kappa}} \le C\,\delta^3\,.$$

For the term in the last line of (4.47) the same procedure as in the previous proof (see (4.43)) allows us to conclude that it defines a bounded magnetic pseudodifferential operator that is small in norm of order $\delta^{\mu-2}$.

This allows us to conclude, using also (4.21), that there exists a constant $C > 0$ such that the following relation is true:

$$\big(\lambda^\epsilon - \epsilon a\big) \sharp^{\epsilon,\kappa} g_{1/\delta} \;\sharp^{\epsilon,\kappa}\; r^{\epsilon,\kappa}(a) \;=\; \big[g_{1/\delta}\big(h_{m^\epsilon} - \epsilon a\big)\big] \sharp^{\epsilon,\kappa} r^{\epsilon,\kappa}(a) \;+\; \delta^{\mu-2}\mathfrak{r}^{\epsilon,\delta,\kappa}\,,$$

with $\mathfrak{r}^{\epsilon,\delta,\kappa} \in S^{-\infty}(\Xi)$ and

$$\|\mathfrak{r}^{\epsilon,\delta,\kappa}\|_{B_{\epsilon,\kappa}} \le C\,,$$

for all $(\epsilon,\delta,\kappa) \in [0,\epsilon_0] \times (0,\delta_0] \times [0,\kappa_0]$ s. t. $\epsilon = \delta^\mu$.

A similar procedure allows us to transform $g_{1/\delta}\big(h_{m^\epsilon}-\epsilon a\big)$ into $g_{1/\delta}\sharp^{\epsilon,\kappa}\big(h_{m^\epsilon}-\epsilon a\big)$ and use the equality

$$\big(h_{m^\epsilon}-\epsilon a\big)\sharp^{\epsilon,\kappa}r^{\epsilon,\kappa}(a)=1\,.$$

This time the formula similar to (4.47) will contain factors of the form $\xi_j\big(\partial_{\xi_j^\perp}g_{\delta^{-1}}\big)(\xi)$ that are still symbols of class $S^{-\infty}(\Xi)$. Finally we conclude the existence of $C>0$ such that:

$$\begin{aligned}\big[g_{1/\delta}\big(h_{m^\epsilon}-\epsilon a\big)\big]\sharp^{\epsilon,\kappa}r^{\epsilon,\kappa}(a) &= g_{1/\delta}\ \sharp^{\epsilon,\kappa}\ \big(\lambda^\epsilon-\epsilon a\big)\ \sharp^{\epsilon,\kappa}\ r^{\epsilon,\kappa}(a)\ +\ \delta^{\mu-2}\mathfrak{z}^{\epsilon,\delta,\kappa}\\ &= g_{1/\delta}\ +\ \delta^{\mu-2}\mathfrak{z}^{\epsilon,\delta,\kappa}\,,\end{aligned}$$

with

$$\mathfrak{z}^{\epsilon,\delta,\kappa}\in S^{-\infty}(\Xi)\text{ and }\|\mathfrak{z}^{\epsilon,\delta,\kappa}\|_{B_{\epsilon,\kappa}}\le C\,,$$

for all $(\epsilon,\delta,\kappa)\in[0,\epsilon_0]\times(0,\delta_0]\times[0,\kappa_0]$ s.t. $\epsilon=\delta^\mu$.

□

The series in (4.38) **continued.** We shall rewrite the expression in the last line of (4.38) as:

$$\begin{aligned}&\sum_{\gamma^*\in\Gamma_*}\widetilde{\lambda}^{\epsilon,\kappa}_{\gamma^*}\ \sharp^{\epsilon,\kappa}\ \big(1-\widetilde{g}_{1/\delta}\big)\,\sharp^{\epsilon,\kappa}r_{\delta,\epsilon,\kappa}(\epsilon a)\\ &\quad=\big(\lambda^\epsilon-\epsilon a\big)\ \sharp^{\epsilon,\kappa}\ \big(1-\widetilde{g}_{1/\delta}\big)\,\sharp^{\epsilon,\kappa}r_{\delta,\epsilon,\kappa}(\epsilon a)\\ &\quad=\ \big(1-\widetilde{g}_{1/\delta}\big)\ -\ \Big[\lambda^\epsilon,\widetilde{g}_{1/\delta}\Big]_{\epsilon,\kappa}\sharp^{\epsilon,\kappa}r_{\delta,\epsilon,\kappa}(\epsilon a)\ -\ (\delta^\circ)^2\,\big(1-\widetilde{g}_{1/\delta}\big)\ \sharp^{\epsilon,\kappa}\ \widetilde{g}_{1/\delta^\circ}\ \sharp^{\epsilon,\kappa}\ r_{\delta,\epsilon,\kappa}(\epsilon a)\\ &\quad=\ 1-\sum_{\gamma^*\in\Gamma_*}\tau_{\gamma^*}\big(g_{1/\delta}\big)-\Big[\lambda^\epsilon,\widetilde{g}_{1/\delta}\Big]_{\epsilon,\kappa}\sharp^{\epsilon,\kappa}r_{\delta,\epsilon,\kappa}(\epsilon a)-(\delta^\circ)^2\,\big(1-\widetilde{g}_{1/\delta}\big)\ \sharp^{\epsilon,\kappa}\ \widetilde{g}_{1/\delta^\circ}\ \sharp^{\epsilon,\kappa}\ r_{\delta,\epsilon,\kappa}(\epsilon a)\,,\end{aligned}$$

where

$$[f,g]_{\epsilon,\kappa}:=f\ \sharp^{\epsilon,\kappa}\ g\ -\ g\sharp^{\epsilon,\kappa}f\,.$$

We start with the second term on the right hand side of the second line of (4.47).

Lemma 4.17. *There exist positive constants C, ϵ_0, κ_0 and δ_0 such that, for $(\epsilon,\kappa,\delta)\in[0,\epsilon_0]\times[0,\kappa_0]\times(0,\delta_0]$ verifying* (4.20)*,*

$$\left\|\Big[\lambda^\epsilon,\widetilde{g}_{1/\delta}\Big]_{\epsilon,\kappa}\sharp^{\epsilon,\kappa}r_{\delta,\epsilon,\kappa}(\epsilon a)\right\|_{B_{\epsilon,\kappa}}\le\ C\,\big(\delta+\delta^{2(\mu-2)}\big)\,.$$

Proof. We note that both symbols λ^ϵ and $\widetilde{g}_{1/\delta}$ are of class $S^0_0(\Xi)^\circ$ and thus we can apply Proposition B.29 in order to obtain the existence of positive ϵ_0 and κ_0 such that, for $(\epsilon,\kappa)\in[0,\epsilon_0]\times[0,\kappa_0]$,

$$\begin{aligned}&\Big[\lambda^\epsilon,\widetilde{g}_{1/\delta}\Big]_{\epsilon,\kappa}(x,\xi)\\ &\qquad=\ -4i\epsilon B_\kappa(\epsilon x)\,\big[\big(\partial_{\xi_1}\lambda^\epsilon\big)(\xi)\big(\partial_{\xi_2}\widetilde{g}_{1/\delta}\big)(\xi)-\big(\partial_{\xi_2}\lambda^\epsilon\big)(\xi)\big(\partial_{\xi_1}\widetilde{g}_{1/\delta}\big)(\xi)\big]\ +\ \epsilon^2\widetilde{\mathscr{R}}_{\epsilon,\kappa}(\lambda^\epsilon,\widetilde{g}_{1/\delta})_{(\epsilon,1)}\,,\end{aligned}$$

where $\{\widetilde{\mathscr{R}}_{\epsilon,\kappa}\big(\lambda^\epsilon,\widetilde{g}_{1/\delta}\big)\}_{\epsilon\in[0,\epsilon_0],\kappa\in[0,\kappa_0]}$ is a bounded family in $S^{-\infty}(\Xi)$.
For the first two terms we apply once again (4.47) and obtain

$$\big(\partial_{\xi_1}\lambda^\epsilon\big)(\xi)\big(\partial_{\xi_2}\widetilde{g}_{1/\delta}\big)-\big(\partial_{\xi_2}\lambda^\epsilon\big)\big(\partial_{\xi_1}\widetilde{g}_{1/\delta}\big)\ =\ \delta^3\mathfrak{f}^{\epsilon,\delta}_{1/\delta}\,,$$

where $\big\{\mathfrak{f}^{\epsilon,\delta}\ \mid\ (\epsilon,\delta)\in[0,\epsilon_0]\times(0,\delta_0],\ \epsilon=\delta^\mu\big\}$ is a bounded subset in $C^\infty_0(\mathcal{X}^*)$.
For the third term we have

$$\begin{aligned}\widetilde{\mathscr{R}}_{\epsilon,\kappa}\big(\lambda^\epsilon,\widetilde{g}_{1/\delta}\big)(x,\xi)=-\frac{B^2_\kappa(x)}{\pi^2}\sum_{|\alpha|=2}(\alpha!)^{-1}\int_{\mathcal{X}^*\times\mathcal{X}^*}e^{i(\eta\wedge\zeta)}\int_0^1 s\,ds\,\mathfrak{T}_{\epsilon,\kappa,\delta}(X,s\eta,\zeta)d^2\eta\,d^2\zeta\\ +\ \kappa\int_{\Xi\times\Xi}e^{-2i\sigma(Y,Z)}\left(\int_0^1\Theta^{\epsilon,\kappa}_\tau(x,y,z)d\tau\right)\ \mathfrak{R}_{\epsilon,\kappa,\delta}(X,Y,Z)\,dY\,dZ\,,\end{aligned}$$

where

$$\mathfrak{T}_{\epsilon,\kappa,\delta}(X, s\eta, \zeta) := \big(\partial_\xi^\alpha \lambda^\epsilon\big)(\xi - s\epsilon^{1/2} b_\kappa(x)\eta)\big(\partial_{\xi_\perp}^\alpha \widetilde{g}_{1/\delta}\big)(\xi - \epsilon^{1/2} b_\kappa(x)\zeta)\,,$$

and

$$\begin{aligned}\mathfrak{R}_{\epsilon,\kappa,\delta}(X, Y, Z) := \sum_{j,k,\ell} \Big(& R_1\big(\partial_\ell B_{jk}\big)(x, \epsilon y, \epsilon z)\big(\partial_{\xi_k}\lambda^\epsilon\big)(\xi - \eta)\big(\partial^2_{\xi_j\xi_\ell}\widetilde{g}_{1/\delta}\big)(\xi - \zeta) \\ & + R_2\big(\partial_\ell B_{jk}\big)(x, \epsilon y, \epsilon z)\big(\partial^2_{\xi_k\xi_\ell}\lambda^\epsilon\big)(\xi - \eta)\big(\partial_{\xi_j}\widetilde{g}_{1/\delta}\big)(\xi - \zeta)\Big) \\ = \sum_{j,k,\ell} \Big(& R_1\big(\partial_\ell B_{jk}\big)(x, \epsilon\delta^{-1} y, \epsilon z)\big(\partial_{\xi_k}\lambda^\epsilon\big)(\xi - \eta)\big(\partial^2_{\xi_j\xi_\ell}\widetilde{g}\big)(\delta^{-1}\xi - \zeta) \\ & + R_2\big(\partial_\ell B_{jk}\big)(x, \epsilon\delta^{-1} y, \epsilon z)\big(\partial^2_{\xi_k\xi_\ell}\lambda^\epsilon\big)(\xi - \eta)\big(\partial_{\xi_j}\widetilde{g}\big)(\delta^{-1}\xi - \zeta)\Big)\,,\end{aligned}$$

with $R_1(\cdot)$ and $R_2(\cdot)$ defined by (B.40) and (B.41).
Taking into account (4.20) and using Proposition B.9 we obtain:

$$\Big[\lambda^\epsilon, \widetilde{g}_{1/\delta}\Big]_{\epsilon,\kappa} \;=\; \delta^3\, \mathfrak{f}^{\epsilon,\delta}_{1/\delta} \;+\; \epsilon^2\, \delta^{-2}\, \widetilde{\mathscr{R}'}_{\epsilon,\kappa,\delta}(\lambda^\epsilon, \widetilde{g})_{(\epsilon,\delta^{-1})}$$

where, for some $\epsilon_0 > 0$ and $\kappa_0 > 0$ small enough, the following families

$$\Big\{\widetilde{\mathscr{R}'}_{\epsilon,\kappa,\delta}\big(\lambda^\epsilon, \widetilde{g}\big) \;\mid\; (\epsilon, \kappa, \delta) \in [0, \epsilon_0] \times [0, \kappa_0] \times (0, \delta_0],\, \epsilon\delta^\mu = 1\Big\} \;\subset\; S^{-\infty}(\Xi)$$

and

$$\big\{\mathfrak{f}^{\epsilon,\delta} \;\mid\; (\epsilon, \delta) \in [0, \epsilon_0] \times (0, \delta_0],\, \epsilon = \delta^\mu\big\} \subset C_0^\infty(\mathcal{X}^*)$$

are bounded. □

For the third term in the right hand side of (4.47) we proceed similarly using Proposition B.30 and obtain

Lemma 4.18. *There exist $\epsilon_0 > 0$, $\kappa_0 > 0$, $\delta_0 > 0$ and $C > 0$ such that:*

$$\big\|(\delta^\circ)^2\,\big(1 - \widetilde{g}_{1/\delta}\big)\;\sharp^{\epsilon,\kappa}\;\widetilde{g}_{1/\delta^\circ}\;\sharp^{\epsilon,\kappa}\;r_{\delta,\epsilon,\kappa}(\epsilon a)\big\|_{B_{\epsilon,\kappa}} \le\; C\,\delta^{2(\mu-2)}\,,$$

for all $(\epsilon, \kappa, \delta) \in [0, \epsilon_0] \times [0, \kappa_0] \times (0, \delta_0]$ verifying (4.20).

The rest of the double series in the first term of (4.38). We still have to study the following double series:

$$\begin{aligned}&\sum_{\gamma^*\in\Gamma_*} \widetilde{\lambda}^{\epsilon,\kappa}_{\gamma^*}\;\sharp^{\epsilon,\kappa} \sum_{\alpha^*\in\Gamma_*\setminus\{\gamma^*\}} \tau_{\alpha^*}\big(g_{1/\delta}\;\sharp^{\epsilon,\kappa}\;r^{\epsilon,\kappa}(a)\big) \\ &\quad = \;\big(\lambda^\epsilon \;-\; \epsilon a\big)\;\sharp^{\epsilon,\kappa} \sum_{\gamma^*\in\Gamma_*} \tau_{\gamma^*}\widetilde{\chi}\sharp^{\epsilon,\kappa} \sum_{\alpha^*\in\Gamma_*\setminus\{\gamma^*\}} \tau_{\alpha^*}\big(g_{1/\delta}\;\sharp^{\epsilon,\kappa}\;r^{\epsilon,\kappa}(a)\big) \\ &\quad = \;\big(\lambda^\epsilon \;-\; \epsilon a\big)\;\sharp^{\epsilon,\kappa}\;\left[\sum_{\gamma^*\in\Gamma_*} \tau_{\gamma^*}\left(\sum_{\alpha^*\in\Gamma_*\setminus\{0\}} \big(\widetilde{\chi}\;\sharp^{\epsilon,\kappa}\;\tau_{\alpha^*}(g_{1/\delta})\big)\;\sharp^{\epsilon,\kappa}\;\tau_{\alpha^*}(r^{\epsilon,\kappa}(a))\right)\right]\,.\end{aligned} \tag{4.47}$$

For the symbol $\widetilde{\chi}\;\sharp^{\epsilon,\kappa}\;\tau_{\alpha^*}(g_{1/\delta})$ we proceed as in the proof of Proposition 4.15 noticing that for $\alpha^* \neq 0$ the supports of $\widetilde{\chi}$ and $\tau_{\alpha^*}(g_{1/\delta})$ are disjoint and we can use Proposition B.30 in order to obtain the following statement.

Lemma 4.19. *Given $\epsilon_0 > 0$ and $\delta_0 > 0$ satisfying (4.16), for any N, there exists $C_N > 0$ such that, for any $\alpha^* \in \Gamma_*$,*

$$<\alpha^*>^N\;\widetilde{\chi}\;\sharp^{\epsilon,\kappa}\;\tau_{\alpha^*}(g_{1/\delta}) \;=\; \mathfrak{r}^{\epsilon,\delta,\kappa}_{(\epsilon,\delta^{-1})}\,,$$

where the family

$$\{\mathfrak{r}^{\epsilon,\delta,\kappa} \;\mid\; (\epsilon, \delta, \kappa) \in [0, \epsilon_0] \times (0, \delta_0] \times [0, 1]\,,\, \epsilon = \delta^\mu\}\,,$$

is bounded in $S^{-\infty}(\Xi)$ and

$$\|\mathfrak{r}^{\epsilon,\delta,\kappa}_{(\epsilon,\delta^{-1})}\|_{B_{\epsilon,\kappa}} \le C_N\,\delta^{2(\mu-1)}\,.$$

Thus for any $N \in \mathbb{N}$ and for any $\alpha^* \neq 0$,

$$\left\| \left(\widetilde{\chi} \sharp^{\epsilon,\kappa} \tau_{\alpha^*}(g_{1/\delta}) \right) \sharp^{\epsilon,\kappa} \tau_{\alpha^*}(r^{\epsilon,\kappa}(a)) \right\|_{B_{\epsilon,\kappa}} \leq C_N < \alpha^* >^{-N} \delta^{\mu-2} .$$

We now use once again the formulas in (4.30) and proceed similarly with the proofs of Propositions 4.11 and 4.12 in order to obtain the following final result.

Proposition 4.20. *There exist $\epsilon_0 > 0$, $\kappa_0 > 0$, $\delta_0 > 0$ and $C > 0$ such that:*

$$\left\| \sum_{\gamma^* \in \Gamma_*} \widetilde{\lambda}^{\epsilon,\kappa}_{\gamma^*} \sharp^{\epsilon,\kappa} \sum_{\alpha^* \in \Gamma_* \setminus \{\gamma^*\}} \tau_{\alpha^*}\left(g_{1/\delta} \sharp^{\epsilon,\kappa} r^{\epsilon,\kappa}(a) \right) \right\|_{B_{\epsilon,\kappa}} \leq C \, \delta^{\mu-2} .$$

for all $(\epsilon, \kappa, \delta) \in [0, \epsilon_0] \times [0, \kappa_0] \times (0, \delta_0]$ verifying (4.20).

Appendices

A Study of $\lambda_0(\theta)$

Being isolated from the rest of the spectrum, the first Bloch eigenvalue λ_0 is analytic [25, 37]. In order to prove the statement in Remark 1.1 (see also [26]) it is enough to prove the following

Lemma A.1. *There exists $C > 0$ such that*

$$\lambda_0(\theta) - \lambda_0(0) \geq C \, |\theta|^2 , \, \forall \theta \in E_* . \tag{A.1}$$

Proof. Up to a Perron-Frobenius type argument [37], the first L^2-normalized eigenfunction of the elliptic operator $-\Delta + V$ on the compact manifold $\mathbb{T}$ is non-degenerate and can be chosen positive; let us denote it by u_0. We also have $0 < \min(u_0) \leq u_0(x) \leq \max(u_0)$. For $u \in C^\infty(\mathbb{T})$, if we denote by $u = u_0 v$ we can write $(-i\nabla - \theta)u = -i[u_0(\nabla - i\theta)v + v\nabla u_0]$. This implies

$$|(-i\nabla - \theta)u|^2 = u_0^2 \, |(\nabla - i\theta)v|^2 + |\nabla u_0|^2 \, |v|^2 + u_0(\nabla |v|^2) \cdot \nabla u_0 .$$

Integrating on $\mathbb{T}$ leads to

$$\int_{\mathbb{T}} |(-i\nabla - \theta)u|^2 \, dx = \int_{\mathbb{T}} u_0^2 \, |(\nabla - i\theta)v|^2 \, dx + \int_{\mathbb{T}} |\nabla u_0|^2 \, |v|^2 \, dx - \int_{\mathbb{T}} |v|^2 \, \mathrm{div}(u_0 \nabla u_0) \, dx$$

and

$$\int_{\mathbb{T}} |(-i\nabla - \theta)u|^2 \, dx + \int_{\mathbb{T}} (V - \lambda_0(0)) \, |u|^2 \, dx = \int_{\mathbb{T}} u_0^2 \, |(\nabla - i\theta)v|^2 \, dx .$$

If $u = u_\theta$ is an eigenfunction of $(-i\nabla - \theta)^2 + V$ associated with $\lambda(\theta)$, we get, with $v_\theta = u_\theta / u_0$,

$$\lambda(\theta) - \lambda(0) = \frac{\int_{\mathbb{T}} u_0^2 \, |(\nabla - i\theta)v_\theta|^2 \, dx}{\int_{\mathbb{T}} |v_\theta|^2 \, u_0^2 \, dx} \geq \frac{\min(u_0)^2}{\max(u_0)^2} \inf_{v \in C^\infty(\mathbb{T})} \frac{\int_{\mathbb{T}} (|(\nabla - i\theta)v|^2) dx}{\int_{\mathbb{T}} v^2 \, dx} .$$

On the right hand side we recognize the variational characterization of the ground state energy of the free Laplacian on the torus, which equals $|\theta|^2$ and it is achieved for a constant v. Hence (A.1) holds, with $C = \frac{\min(u_0)^2}{\max(u_0)^2}$. □

B The magnetic Moyal calculus with slowly varying symbols

This appendix is devoted to a brief reminder of the main definitions, notation and results concerning the magnetic pseudodifferential calculus [22, 9, 10] and to prove some special properties in the case of *slowly varying symbols and magnetic fields* (see Subsection B.5).

B.1 Hörmander classes of symbols

Let us recall the notation $\mathcal{X} := \mathbb{R}^2$ and let us denote by $\mathcal{X}^*$ the dual of $\mathcal{X}$ (*the momentum space*) with $\langle \cdot, \cdot \rangle : \mathcal{X}^* \times \mathcal{X} \to \mathbb{R}$ denoting the duality map. Let $\Xi := \mathcal{X} \times \mathcal{X}^*$ be *the phase space* with the canonical symplectic form

$$\sigma(X, Y) := \langle \xi, y \rangle - \langle \eta, x \rangle \,, \tag{B.1}$$

for $X := (x, \xi) \in \Xi$ and $Y := (y, \eta) \in \Xi^*$.
We consider the spaces $BC(\mathcal{V})$ of bounded continuous functions on any finite dimensional real vector space $\mathcal{V}$ with the $\| \cdot \|_\infty$ norm. We shall denote by $C^\infty(\mathcal{V})$ the space of smooth functions on $\mathcal{V}$ and by $C^\infty_{\mathsf{pol}}(\mathcal{V})$ (resp. by $BC^\infty(\mathcal{V})$) its subspace of smooth functions that are polynomially bounded together with all their derivatives, (resp. smooth and bounded together with all their derivatives), endowed with the usual locally convex topologies. We denote by τ_v the translation with $v \in \mathcal{V}$. For any $v \in \mathcal{V}$ we write $< v >:= \sqrt{1 + |v|^2}$. $\mathscr{S}(\mathcal{V})$ denotes the space of Schwartz functions on $\mathcal{V}$ endowed with the Fréchet topology defined by the following family $\{\nu_{m,n}\}_{(m,n)\in\mathbb{N}^2}$ of seminorms:

$$\mathscr{S}(\mathcal{V}) \ni \phi \mapsto \nu_{n,m}(\phi) := \sup_{v\in\mathcal{V}} < v >^n \sum_{|\alpha|\leq m} \big|\big(\partial^\alpha \phi\big)(v)\big| \,.$$

We will use the following class of Hörmander type symbols.

Definition B.1. *For any $s \in \mathbb{R}$ and any $\rho \in [0, 1]$, we denote by*

$$S^s_\rho(\Xi) := \{F \in C^\infty(\Xi) \mid \nu^{s,\rho}_{n,m}(F) < +\infty \,, \forall (n, m) \in \mathbb{N} \times \mathbb{N}\} \,,$$

where

$$\nu^{s,\rho}_{n,m}(f) := \sup_{(x,\xi)\in\Xi} \sum_{|\alpha|\leq n} \sum_{|\beta|\leq m} \Big| \langle \xi \rangle^{-s+\rho m} \big(\partial^\alpha_x \partial^\beta_\xi f\big)(x, \xi)\Big| \,,$$

and

$$S^\infty_\rho(\Xi) := \bigcup_{s\in\mathbb{R}} S^s_\rho(\Xi) \text{ and } S^{-\infty}(\Xi) := \bigcap_{s\in\mathbb{R}} S^s_\rho(\Xi) \,.$$

Definition B.2. *A symbol F in $S^s_\rho(\Xi)$ is called elliptic if there exist two positive constants R and C such that*

$$|F(x, \xi)| \geq C \langle \xi \rangle^s \,,$$

for any $(x, \xi) \in \Xi$ with $|\xi| \geq R$.

Definition B.3. *For $h \in S^m_1(\Xi)$ the Weyl quantization associates the operator $\mathfrak{Op}^{\mathfrak{w}}(h)$ defined, for $u \in \mathscr{S}(\mathcal{X})$, by*

$$\big(\mathfrak{Op}^{\mathfrak{w}}(h)u\big)(x) \;:=\; (2\pi)^{-2} \int_{\mathcal{X}} \int_{\mathcal{X}^*} e^{i\langle \xi, x-y\rangle}\, h\big(\frac{x+y}{2}, \xi\big) u(y)\, d\xi\, dy \,. \tag{B.2}$$

This operator is continuous on $\mathscr{S}(\mathcal{X})$ and has a natural extension by duality to $\mathscr{S}'(\mathcal{X})$. Moreover it is just the special case $\epsilon = 1$ of the operator in (2.1) (but in dimension 2).

B.2 Magnetic pseudodifferential calculus

We consider a vector potential A with components of class $C^\infty_{\mathsf{pol}}(\mathcal{X})$, such that the magnetic field $B = \operatorname{curl} A$ belongs to $BC^\infty(\mathcal{X})$. We recall from [22] that the functional calculus (see (1.12))

$$u \mapsto \big(\mathfrak{Op}^A(F)u\big)(x) \;:=\; (2\pi)^{-2} \int_{\mathcal{X}} \int_{\mathcal{X}^*} e^{i\langle \xi, x-y\rangle} e^{-i\int_{[x,y]} A}\, F\left(\frac{x+y}{2}, \xi\right) u(y)\, d\xi\, dy \,, \tag{B.3}$$

generalizes the usual Weyl calculus and extends to the Hörmander classes of symbols. Moreover the usual *composition of symbols theorem* is still valid (Theorem 2.2 in [22]). In fact for our special classes of Hörmander symbols ($S^m_\rho(\Xi) \equiv S^m_{\rho,0}(\Xi)$) the result concerning the composition of symbols is a corollary of Proposition B.9.

We recall (see for example Proposition 3.4 in [29]) that two vector potentials that are gauge equivalent define two unitarily equivalent functional calculi and that (Proposition 3.5 in [29]) the application $\mathfrak{Op}^A$ extends to a linear and topological isomorphism between $\mathscr{S}'(\Xi)$ and $\mathcal{L}\big(\mathscr{S}(\mathcal{X});\mathscr{S}'(\mathcal{X})\big)$ (considered with the strong topologies).

In the same spirit as the Calderon-Vaillancourt theorem for classical pseudo-differential operators, Theorem 3.1 in [22] states that any symbol $F \in S^0_0(\Xi)$ defines a bounded operator $\mathfrak{Op}^A(F)$ in $L^2(\mathcal{X})$ with an upper bound of the operator norm by some symbol seminorm of F. We denote by $\|F\|_B$ the operator norm of $\mathfrak{Op}^A(F)$ in $\mathcal{L}(L^2(\mathcal{X}))$:

$$\|F\|_B := ||\mathfrak{Op}^A(F)||_{\mathcal{L}(L^2(\mathcal{X}))}\,. \tag{B.4}$$

This norm only depends on the magnetic field B and not on the choice of the vector potential (different choices being unitary equivalent).

We also recall from [29] that the operator composition of the operators $\mathfrak{Op}^A(F)$ and $\mathfrak{Op}^A(G)$ induces a *twisted Moyal product*, also called magnetic Moyal product, such that

$$\mathfrak{Op}^A(F)\,\mathfrak{Op}^A(G) = \mathfrak{Op}^A(F \sharp^B G)\,.$$

This product depends only on the magnetic field B and is given by the following oscillating integral:

$$\begin{aligned}\big(F \sharp^B G\big)(X) &:= \pi^{-4}\int_\Xi dY \int_\Xi dZ\, e^{-2i\sigma(Y,Z)} e^{-i\int_{T(x,y,z)}{}^B} F(X-Y)\,G(X-Z) \\ &= \pi^{-4}\int_\Xi dY \int_\Xi dZ\, e^{-2i\sigma(X-Y,X-Z)} e^{-i\int_{\widetilde{T}(x,y,z)}{}^B} F(Y)\,G(Z)\,,\end{aligned} \tag{B.5}$$

where $T(x,y,z)$ denotes the triangle in $\mathcal{X}$ of vertices $x-y-z\,,\ x+y-z\,,\ x-y+z$ and $\widetilde{T}(x,y,z)$ the triangle in $\mathcal{X}$ of vertices $x-y+z\,,\ y-z+x\,,\ z-x+y\,.$

For any symbol F we denote by F^-_B its inverse with respect to the magnetic Moyal product, if it exists. It is shown in Subsection 2.1 of [30] that, for any $m>0$ and for $a>0$ large enough (depending on m) the symbol $\mathfrak{s}_m(x,\xi) :=< \xi >^m +a$, has an inverse for the magnetic Moyal product. We shall use the shorthand notation $\mathfrak{s}^B_{-m}$ instead of $\big(\mathfrak{s}_m\big)^-_B$ and extend it to any $m\in\mathbb{R}$ (thus for $m>0$ we have $\mathfrak{s}^B_m \equiv \mathfrak{s}_m$).
The following results have been established in [23] (Propositions 6.2 and 6.3):

Proposition B.4.

1. *If $F \in S^0_\rho(\Xi)$ is invertible for the magnetic Moyal product, then the inverse F^-_B also belongs to $S^0_\rho(\Xi)$.*
2. *For $m<0$, if $f \in S^m_\rho(\Xi)$ is such that $1+f$ is invertible for the magnetic Moyal product, then $(1+f)^-_B - 1 \in S^m_\rho(\Xi)$.*
3. *Let $m>0$ and $\rho\in[0,1]$. If $G \in S^m_\rho(\Xi)$ is invertible for the magnetic Moyal product, with $\mathfrak{Op}^A\big(\mathfrak{s}_m \sharp^B G^-_B\big) \in \mathcal{L}\big(L^2(\mathcal{X})\big)$, then $G^-_B \in S^{-m}_\rho(\Xi)$.*

We recall from [23] that one can associate with $X\in\Xi$, a linear symbol by:

$$\mathfrak{l}_X(Y) := \sigma(X,Y)\,,\ \forall\, Y\in\Xi\,,$$

and, for $\epsilon\geq 0$, an operator $\mathfrak{ad}^\epsilon_X$ on $\mathcal{S}'(\Xi)$

$$\mathfrak{ad}^\epsilon_X[\psi] := \mathfrak{l}_X \sharp^\epsilon \psi - \psi \sharp^\epsilon \mathfrak{l}_X\,, \qquad \forall\,\psi\in\mathscr{S}'(\Xi)\,. \tag{B.6}$$

Proposition B.5. *Let us consider a lattice $\Gamma_*\subset\mathcal{X}^*$. If f and g are Γ_*-periodic symbols, then their magnetic Moyal product is also Γ_*-periodic.*

The proof is evident by (B.5).

Remark B.6. By Theorem 4.1 in [22], for any real elliptic symbol $h \in S_1^m(\Xi)_\Gamma$ (with $m > 0$) and for any A in $C^\infty_{\mathsf{pol}}(\mathcal{X}, \mathbb{R}^2)$, the operator $\mathfrak{Op}^A(h)$ has a closure H^A in $L^2(\mathcal{X})$ that is self-adjoint on a domain $\mathcal{H}^m_A$ (a *magnetic Sobolev space*) and lower semibounded. Thus we can define its resolvent $(H^A - \mathsf{z})^{-1}$ for any $\mathsf{z} \notin \sigma(H^A)$ and Theorem 6.5 in [23] states that it exists a symbol $r^B_{\mathsf{z}}(h) \in S_1^{-m}(\Xi)$ such that

$$(H^A - \mathsf{z})^{-1} \;=\; \mathfrak{Op}^A(r^B_{\mathsf{z}}(h))\,.$$

Remark B.7. For symbols of class $S^0_\rho(\Xi)$ with $\rho \in [0,1]$, we have seen that the associated magnetic pseudodifferential operator is bounded in $\mathcal{H}$ and is self-adjoint if and only if its symbol is real. In that case we can also define its resolvent and the results in [23], cited above, show that it is also defined by a symbol of class $S^0_\rho(\Xi)$.

Using the notation

$$\Lambda^A(x,y) := e^{-i\int_{[x,y]} A}\,, \tag{B.7}$$

we notice that

$$\Lambda^A(x,z)\Lambda^A(z,y)\Lambda^A(y,x) \;=\; \exp\left\{-i\int_{<x,y,z>} B\right\} \;:=\; \Omega^B(x,y,z)\,, \tag{B.8}$$

and there exists $C > 0$ such that, for any magnetic field B of class $BC^\infty(\mathcal{X})$,

$$\big|\Omega^B(x,y,z) - 1\big| \le C\,\|B\|_\infty |(y-x)\wedge(z-x)|\,. \tag{B.9}$$

The above integral of the 2-form $B = dA$ is taken on the positively oriented triangle $< x\,,\, y\,,\, z >$.

As we are working in a two-dimensional framework, we use the following notation for the *vector product* of two vectors u and v in $\mathbb{R}^2$:

$$u \wedge v \;:=\; u_1 v_2 \,-\, u_2 v_1\,. \tag{B.10}$$

and the $-\pi/2$ rotation of v:

$$v^\perp := \big(v_2, -v_1\big)\,. \tag{B.11}$$

We can make the connection with the '*twisted integral kernels*' formalism in [33], where for any integral kernel $K \in \mathscr{S}'(\mathcal{X}\times\mathcal{X})$ one associates a twisted integral kernel

$$K^A(x,y) \;:=\; \Lambda^A(x,y)\,K(x,y)\,. \tag{B.12}$$

For any integral kernel $K \in \mathscr{S}'(\mathcal{X}\times\mathcal{X})$ we denote by $\mathcal{I}\mathsf{nt}\,K$ its corresponding linear operator from $\mathscr{S}(\mathcal{X})$ into $\mathscr{S}'(\mathcal{X})$:

$$\big((\mathcal{I}\mathsf{nt}\;K)u\big)(\phi) = \int_{\mathcal{X}\times\mathcal{X}} K(x,y)\phi(x)u(y)\,dxdy\,,\ \forall\phi\in\mathcal{S}(\mathcal{X})\,.$$

Let us recall that there exists a linear bijection $\mathfrak{W} : \mathscr{S}'(\Xi) \to \mathscr{S}'(\mathcal{X}\times\mathcal{X})$ defined by

$$\big(\mathfrak{W}F\big)(x,y) \;:=\; (2\pi)^{-2}\int_{\mathcal{X}^*} e^{i<\xi,x-y>} F\big(\frac{x+y}{2},\xi\big)\,d\xi\,, \tag{B.13}$$

such that

$$\mathfrak{Op}^w(F) = \mathcal{I}\mathsf{nt}(\mathfrak{W}F)\,.$$

In the magnetic calculus, we have the equality

$$\mathfrak{Op}^A(F) \;=\; \mathcal{I}\mathsf{nt}(\Lambda^A\,\mathfrak{W}F)\,. \tag{B.14}$$

As our main operator H^0 is a differential operator, we shall have to use instead of formula (B.12) the following commutation relation (valid for any $z \in \mathcal{X}$)

$$\Lambda^A(x,z)^{-1}\big(-i\partial_{x_j}\big)\Lambda^A(x,z) \;=\; A_j(x) + a_j(x,z)\,, \tag{B.15}$$

with

$$a_j(x,z) \;:=\; \sum_k (x_k - z_k)\int_0^1 B_{jk}\big(z + s(x-z)\big)s\,ds\,. \tag{B.16}$$

B.3 The scaling of symbols

For any symbol $F \in S^{\infty}_{\rho}(\Xi)$ and for any pair $(\epsilon, \tau) \in [0, +\infty) \times [0, +\infty)$ we define the scaled symbol:

$$F_{(\epsilon,\tau)}(x, \xi) := F(\epsilon x, \tau\xi)\,. \tag{B.17}$$

Proposition B.8. *Suppose given a symbol $F \in S^{m}_{\rho}(\Xi)$, with either $m < 0$ and $\rho \in (0, 1]$ or $m < -2$ and $\rho = 0$ and a magnetic field $B \in BC^{\infty}(\mathcal{X})$. Then, there exists $p_0 > 0$ such that for any $p > p_0$ there exists $C_p > 0$ such that*

$$\|F_{(\epsilon,\tau)}\|_B \;\le\; C_p\, \nu^{m,\rho}_{(0,p)}\big(F\big)\,,$$

for any $(\epsilon, \tau) \in [0, +\infty) \times [0, +\infty)$.
In fact $p_0 = (2 - m)\rho^{-1}$ for the first case and $p_0 = 2$ for the second case.

Proof. We use the Schur-Holmgren criterion for the integral kernel associated with the magnetic quantization of the given symbol, introducing the following notation for the Schur-Holmgren norm:

$$\|K\|_{SH} := \max\left\{\sup_{x\in\mathcal{X}} \int_{\mathcal{X}} |K(x, y)|\, dy\,, \quad \sup_{y\in\mathcal{X}} \int_{\mathcal{X}} |K(x, y)|\, dx\right\}\,.$$

Then we have the Schur estimate:

$$\big\|\mathscr{I}\mathsf{nt}\, K\big\|_{\mathcal{L}(\mathcal{H})} \le \|K\|_{SH}.$$

If A is a vector potential for the given magnetic field B we recall from (B.14):

$$\mathfrak{Op}^{A}(F) = \mathscr{I}\mathsf{nt}\big(\Lambda^{A}\mathfrak{W}F\big)\,.$$

Thus we obtain the estimate:

$$\big\|\mathfrak{Op}^{A}(F)\big\|_{\mathcal{L}(\mathcal{H})} = \big\|\mathscr{I}\mathsf{nt}\big(\Lambda^{A}\mathfrak{W}F\big)\big\|_{\mathcal{L}(\mathcal{H})} \le \big\|\big(\Lambda^{A}\mathfrak{W}F\big)\big\|_{SH} = \big\|\mathfrak{W}F\big\|_{SH}\,.$$

Now using (B.13) we can write:

$$\begin{aligned}
\big(\mathfrak{W}F_{(\epsilon,\tau)}\big)(x, y) &= (2\pi)^{-2} \int_{\mathcal{X}^*} e^{i<\xi,x-y>} F_{(\epsilon,\tau)}\big(\frac{x + y}{2}, \xi\big)\, d\xi \\
&= (2\pi)^{-2} \int_{\mathcal{X}^*} e^{i<\xi,x-y>} F\big(\epsilon\frac{x + y}{2}, \tau\xi\big)\, d\xi \\
&= (2\pi)^{-2}\tau^{-1} \int_{\mathcal{X}^*} e^{i<\xi,\tau^{-1}(x-y)>} F\big(\epsilon\frac{x + y}{2}, \xi\big)\, d\xi\,,
\end{aligned}$$

and thus

$$\begin{aligned}
\big\|\mathfrak{W}F_{(\epsilon,\tau)}\big\|_{SH} &= \sup_{x\in\mathcal{X}} \textstyle\int_{\mathcal{X}} |\mathfrak{W}F_{(\epsilon,\tau)}(x, y)|\, dy \\
&= (2\pi)^{-2}\tau^{-1} \sup_{x\in\mathcal{X}} \textstyle\int_{\mathcal{X}} \Big|\int_{\mathcal{X}^*} e^{i<\xi,\tau^{-1}(x-y)>} F\big(\epsilon\frac{x+y}{2}, \xi\big) d\xi\Big|\, dy \\
&= (2\pi)^{-2}\tau^{-1} \sup_{x\in\mathcal{X}} \textstyle\int_{\mathcal{X}} \big|\int_{\mathcal{X}^*} e^{i<\xi,(2/\tau)v>} F\big(\epsilon(x - v), \xi\big) d\xi\big|\, dv \\
&= (4\pi)^{-2} \sup_{x\in\mathcal{X}} \textstyle\int_{\mathcal{X}} \big|\int_{\mathcal{X}^*} e^{i<\xi,u>} F\big(\epsilon(x - (\tau/2)u), \xi\big) d\xi\big|\, du \\
&= (4\pi)^{-2} \sup_{x\in\mathcal{X}} \textstyle\int_{\mathcal{X}} < u >^{-p} \big|\int_{\mathcal{X}^*} e^{i<\xi,u>} \big[(\mathbb{1} - (1/2)\Delta_{\xi})^{p/2} F\big] \big(\epsilon(x - (\tau/2)u), \xi\big) d\xi\big|\, du\,.
\end{aligned}$$

Now suppose that $m < -2$ so that for any $x \in \mathcal{X}$ and any $(\alpha, \beta) \in \mathbb{N}^2 \times \mathbb{N}^2$, $\big(\partial^{\alpha}_{x}\partial^{\beta}_{\xi} F\big)(x, \cdot)$ is in $L^1(\mathcal{X}^*)$ and

$$\sup_{x\in\mathcal{X}} \big\|\big(\partial^{\alpha}_{x}\partial^{\beta}_{\xi} F\big)(x, \cdot)\big\|_{L^1(\mathcal{X}^*)} \le C_{\alpha,\beta}\, \nu^{m,0}_{|\alpha|,|\beta|}(F)\,.$$

Then from (B.18) we deduce that

$$\begin{aligned}\big\|\mathfrak{W}F_{(\epsilon,\tau)}\big\|_{SH} &\leq (4\pi)^{-2}\sup_{z\in\mathcal{X}}\big\|(\mathbb{1}-(1/2)\Delta_\xi)^{p/2}F\big(z,\cdot\big)\big\|_{L^1(\mathcal{X}^*)}\int_{\mathcal{X}} <u>^{-p}\,du\\ &\leq C_p\,\nu^{m,0}_{0,p}(F),\quad \forall p>2\,.\end{aligned}$$

If we have $m<0$ but $\rho\in(0,1]$ we can use Lemma A.4 in [30] (based on Propositions 1.3.3 and 1.3.6 in [1]) in order to obtain from (B.18) the following estimate:

$$\big\|\mathfrak{W}F_{(\epsilon,\tau)}\big\|_{SH}\;\leq\;(2\pi)^{-1}\sup_{z\in\mathcal{X}}\big\|\mathcal{F}F\big(z,\cdot\big)\big\|_{L^1(\mathcal{X})}\;\leq\;C_p\,\nu^{m,\rho}_{0,p}(F)\,,$$

for some $p>(2-m)\rho^{-1}$. □

Proposition B.9. *Let $F\in S^m_\rho(\Xi)$ and $G\in S^p_\rho(\Xi)$, with $m\in\mathbb{R}$, $p\in\mathbb{R}$ and $\rho\in[0,1]$ and let L some integral kernel in $BC^\infty\big(\mathcal{X};C^\infty_{\mathsf{pol}}(\mathcal{X}\times\mathcal{X})\big)$. Let $D\subset(0,\infty)\times(0,\infty)$ and K some compact interval in $[0,+\infty)$. For any $(\epsilon,\tau)\in D$, $(\mu_1,\cdots,\mu_4)\in K^4$, let us consider the oscillatory integral:*

$$\begin{aligned}&\mathcal{L}^{\epsilon,\tau}_{(\mu_1,\mu_2,\mu_3,\mu_4)}\big(F,G\big)(X)\\ &\quad:=\textstyle\int_{\Xi\times\Xi}e^{-2i\sigma(Y,Z)}L(\epsilon x,y,z))F\big(\epsilon x-\mu_1y,\tau\xi-\mu_2\eta\big)G\big(\epsilon x-\mu_3z,\tau\xi-\mu_4\zeta\big)\,dYdZ\,.\end{aligned}$$

Then there exists a symbol $\boldsymbol{T}:=\boldsymbol{T}^{(\mu_1,\mu_2,\mu_3,\mu_4)}$ in $S^{m+p}_\rho(\mathcal{X})$, such that:

$$\mathcal{L}^{\epsilon,\tau}_{(\mu_1,\mu_2,\mu_3,\mu_4)}\big(F,G\big)\;=\;\boldsymbol{T}_{(\epsilon,\tau)}\,.$$

Moreover, for any (m,p,n_1,n_2) (with $(n_1,n_2)\in\mathbb{N}\times\mathbb{N}$), there exist (q_1,p_1,q_2,p_2), $\nu:BC^\infty\big(\mathcal{X};C^\infty_{\mathsf{pol}}(\mathcal{X}\times\mathcal{X})\big)\to\mathbb{R}_+$, and positive constants $C_K(n_1,n_2,q_1,q_2,p_1,p_2)$, such that:

$$\nu^{(m+p),\rho}_{(n_1,n_2)}\Big(\mathcal{L}^{\epsilon,\tau}_{(\mu_1,\mu_2,\mu_3,\mu_4)}\big(F,G\big)\Big)\;\leq\;C_K(n_1,n_2,q_1,q_2,p_1,p_2)\,\nu\big(L\big)\,\nu^{m,\rho}_{(q_1,p_1)}\big(F\big)\,\nu^{p,\rho}_{(q_2,p_2)}\big(G\big),$$

for any $(\epsilon,\tau)\in D$, $(\mu_1,\cdots,\mu_4)\in K^4$.

Proof. If we define:

$$\mathbf{T}^{(\mu_1,\mu_2,\mu_3,\mu_4)}(X):=\int\limits_{\Xi\times\Xi}e^{-2i\sigma(Y,Z)}L(x,y,z))\,F\big(x-\mu_1y,\xi-\mu_2\eta\big)\,G\big(x-\mu_3z,\xi-\mu_4\zeta\big)\,dYdZ\,,\tag{B.18}$$

as an oscillatory integral we have

$$\mathcal{L}^{\epsilon,\tau}_{(\mu_1,\mu_2,\mu_3,\mu_4)}\big(F,G\big)\;=\;\mathbf{T}^{(\mu_1,\mu_2,\mu_3,\mu_4)}_{(\epsilon,\tau)}.$$

Let us estimate the behavior of the expressions $<\xi>^{-(m+p)+\rho|\beta|}\,\big(\partial^\alpha_x\partial^\beta_\xi T\big)(x,\xi)$ appearing in the norms on $S^{m+p}_\rho(\Xi)$ (we take $|\alpha|=n_1$, $|\beta|=n_2$). A simple calculus shows that we have to estimate integrals of the form:

$$\begin{aligned}&<\xi>^{-(m+p)+\rho|\beta|}\textstyle\int_{\Xi\times\Xi}e^{-2i\sigma(Y,Z)}\big(\partial^{\alpha_3}_xL\big)(x,y,z)\\ &\qquad\times\big(\partial^{\alpha_1}_x\partial^{\beta_1}_\xi F\big)\big(x-\mu_1y,\xi-\mu_2\big)\big(\partial^{\alpha_2}_x\partial^{\beta_2}_\xi G\big)\big(x-\mu_3z,\xi-\mu_4\zeta\big)\,dYdZ\\ &=\textstyle\int_{\Xi\times\Xi}e^{-2i\sigma(Y,Z)}\big(\partial^{\alpha_3}_xL\big)(x,y,z)<\mu_2\eta>^{m-\rho|\beta_1|}<\mu_4\zeta>^{p-\rho|\beta_2|}\\ &\qquad\times\;<\xi-\mu_2\eta>^{-m+\rho|\beta_1|}\big(\partial^{\alpha_1}_x\partial^{\beta_1}_\xi F\big)\big(x-\mu_1y,\xi-\mu_2\eta\big)\\ &\qquad\qquad\times<\xi-\mu_4\zeta>^{-p+\rho|\beta_2|}\big(\partial^{\alpha_2}_x\partial^{\beta_2}_\xi G\big)\big(x-\mu_3z,\xi-\mu_4\zeta\big)\,dYdZ\,,\end{aligned}$$

where $|\alpha_1+\alpha_2+\alpha_3|=n_1$ and $|\beta_1+\beta_2|=n_2$.
By the usual integration by parts using the exponential $e^{-2i\sigma(Y,Z)}$, we are reduced to integrals of

the form:

$$\int_{\Xi\times\Xi} <\eta>^{-s_2/2}<\mu_2\eta>^{m-\rho|\beta_1|}<\zeta>^{-s_3/2}<\mu_4\zeta>^{p-\rho|\beta_2|}\ \big(<y><z>\big)^{-(s_1-r(n_1,s_2,s_3))/2}$$
$$\times\ e^{-2i\sigma(Y,Z)}\mu_1^{|\theta_1|}\big((1-(\mu_2^2/2)\Delta_\xi)^{s_1/2}\partial_x^{\alpha_1+\theta_1}\partial_\xi^{\beta_1}F\big)\big(x-\mu_1 y,\xi-\mu_2\eta\big)$$
$$\times\ \mu_3^{\theta_2|}\big((1-(\mu_4^2/2)\Delta_\xi)^{s_1/2}\partial_x^{\alpha_2+\theta_2}\partial_\xi^{\beta_2}G\big)\big(x-\mu_2 z\,,\xi-\mu_4\zeta\big)$$
$$\times\ \big(\partial_x^{\alpha_3}\partial_y^{\rho_1}\partial_z^{\rho_2}L\big)(x,y,z)\,dY dZ\,,$$

where $s_2>d+[m-\rho|\beta_1|]_+$, $s_3>d+[p-\rho|\beta_2|]_+$, $|\theta_1+\rho_1|\le s_3$, and $|\theta_2+\rho_2|\le s_2$.

Then by the hypothesis on L , for $|\alpha_3|\le n_1$ there exists $r(n_1,s_2,s_3)\in\mathbb{N}$ such that

$$\nu(L)\ :=\ \sup_{(x,y,z)\in\mathcal{X}^3}\max_{|\alpha_3|\le n_1}\max_{|\rho_1|\le s_3}\max_{|\rho_2|\le s_2}(<y><z>)^{r(n_1,s_2,s_3)}\,\big|\big(\partial_x^{\alpha_3}\partial_y^{\rho_1}\partial_z^{\rho_2}L\big)(x,y,z)\big|\ <\ +\infty\,.$$

With this last choice, we take $s_1>d+r(n_1,s_2,s_3)$ and the proof is finished. □

Remark B.10. The following relations hold true:

$$\partial_{\xi_j}\mathcal{L}^{\epsilon,\tau}_{(\mu_1,\mu_2,\mu_3,\mu_4)}\big(F,G\big)=\tau\,\Big(\mathcal{L}^{\epsilon,\tau}_{(\mu_1,\mu_2,\mu_3,\mu_4)}\big(\partial_{\xi_j}F,G\big)+\mathcal{L}^{\epsilon,\tau}_{(\mu_1,\mu_2,\mu_3,\mu_4)}\big(F,\partial_{\xi_j}G\big)\Big)\ ,$$

$$\partial_{x_j}\mathcal{L}^{\epsilon,\tau}_{(\mu_1,\mu_2,\mu_3,\mu_4)}\big(F,G\big)$$
$$=\epsilon\,\Big(\mathcal{L}^{\epsilon,\tau}_{(\mu_1,\mu_2,\mu_3,\mu_4)}\big(\partial_{x_j}F,G\big)+\mathcal{L}^{\epsilon,\tau}_{(\mu_1,\mu_2,\mu_3,\mu_4)}\big(F,\partial_{x_j}G\big)+\mathcal{L}^{\epsilon,\tau,(j)}_{(\mu_1,\mu_2,\mu_3,\mu_4)}\big(F,G\big)\Big)\ ,$$

where

$$\mathcal{L}^{\epsilon,\tau,(j)}_{(\mu_1,\mu_2,\mu_3,\mu_4)}\big(F,G\big)(X) \tag{B.19}$$
$$:=\int_{\Xi\times\Xi}e^{-2i\sigma(Y,Z)}(\partial_{x_j}L)(\epsilon x,y,z))F\big(\epsilon x-\mu_1 y,\tau\xi-\mu_2\eta\big)G\big(\epsilon x-\mu_3 z,\tau\xi-\mu_4\zeta\big)\,dY dZ\,.$$

Corollary B.11. *For any bounded subsets*

$$\mathcal{B}_m\subset S^m_\rho(\Xi),\ \mathcal{B}_p\subset S^p_\rho(\Xi),\ \mathcal{B}_L\subset BC^\infty\big(\mathcal{X};C^\infty_{\mathsf{pol}}(\mathcal{X}\times\mathcal{X})\big)$$

and for any compact set $K\subset[0,+\infty)$, the set

$$\big\{\mathcal{L}^{1,1}_{(\mu_1,\mu_2,\mu_3,\mu_4)}\big(F,G\big)\ \mid\ F\in\mathcal{B}_m,\ G\in\mathcal{B}_p,\ L\in\mathcal{B}_L,\ \mu_j\in K, j=1,2,3,4\big\}$$

is bounded in $S^{m+p}_\rho(\mathcal{X})$.

The following statement (a simplified version of Proposition 8.1 in [23]) is a simple corollary of Proposition B.9.

Proposition B.12. *Let $\phi\in S^m_\rho(\Xi)$, $\psi\in S^p_\rho(\Xi)$ and $\theta\in BC^\infty(\mathcal{X};C^\infty_{\mathsf{pol}}(\mathcal{X}^2))$. Then*

$$X\mapsto\mathfrak{L}(\theta;\phi,\psi)(X):=\int_\Xi dY\int_\Xi dZ\,e^{-2i\sigma(Y,Z)}\,\theta(x,y,z)\,\phi(X-Y)\,\psi(X-Z)$$

defines a symbol of class $S^{m+p}_\rho(\Xi)$ and the map

$$S^m_\rho(\Xi)\times S^p_\rho(\Xi)\ni(\phi,\psi)\mapsto\mathfrak{L}(\theta;\phi,\psi)\in S^{m+p}_\rho(\Xi)$$

is continuous. If ϕ or ψ belongs to $\mathscr{S}(\Xi)$ then $\mathfrak{L}(\theta;\phi,\psi)$ also belongs to $\mathscr{S}(\Xi)$ and the map $(\phi,\psi)\mapsto\mathfrak{L}(\theta;\phi,\psi)$ considered in the corresponding spaces is jointly continuous with respect to the associated Fréchet topologies.

Proof. While the first conclusion is a particular case of Proposition B.9 , the second one follows as in the proof of Proposition B.9 , noticing that for any $p>0$ we have the inequality

$$<x>^p\le C_p<x-y>^p<y>^p\ .$$

□

B.4 Weak magnetic fields

We are interested in *weak* magnetic fields, controlled by a small parameter $\epsilon \in [0, \epsilon_0]$ for some $\epsilon_0 > 0$, that we intend to treat as a small perturbation of the situation without magnetic field. In this subsection we work under the following hypothesis, which is more general than the situation considered in (1.5), noticing that we can write $B_{\epsilon,\kappa} = \epsilon B^0_\epsilon$ for $B^0_\epsilon(x) := B_0 + \kappa B(\epsilon x)$.

Hypothesis B.13. The family of magnetic fields $\{B_\epsilon\}_{\epsilon\in[0,\epsilon_0]}$ has the form $B_\epsilon := \epsilon\, B^0_\epsilon$, with $B^0_\epsilon \in BC^\infty\big(\mathcal{X}\big)$ uniformly with respect to $\epsilon \in [0, \epsilon_0]$.

To simplify the notation, when dealing with weak magnetic fields, the indexes (or the exponents) A_ϵ or B_ϵ shall be replaced by ϵ and we shall use the notation $\|\cdot\|_\epsilon$ instead of $\|\cdot\|_{B_\epsilon}$.

Let us first consider the difference between the magnetic and the usual Moyal products, for a weak magnetic field.

Proposition B.14. *For $\epsilon \in [0, \epsilon_0]$ there exists a continuous application $r_\epsilon : S^m_\rho(\Xi) \times S^{m'}_\rho(\Xi) \to S^{m+m'-2\rho}_\rho(\Xi)$ such that:*

$$a \,\sharp^\epsilon\, b \,=\, a \,\sharp^0\, b \,+\, \epsilon\, r_\epsilon(a,b)\,, \qquad \forall (a,b) \in S^m_\rho(\Xi) \times S^{m'}_\rho(\Xi)\,. \tag{B.20}$$

Proof. Let us introduce the notation:

$$F_\epsilon(x,y,z) := \epsilon\,(y \wedge z)\, F^\circ_\epsilon(x,y,z)\,,$$

with

$$F^\circ_\epsilon(x,y,z) := \left(\int_{-1/2}^{1/2} ds \int_{-1/2}^{s} dt\, B^0_\epsilon\big(x + sy + tz\big) \right)\,,$$

and notice that F°_ϵ belongs to $BC^\infty\big(\mathbb{R}^6\big)$ uniformly with respect to $\epsilon \in [0, \epsilon_0]$.

Then, using the Taylor expansion at first order for the exponential $\exp\{-4iF_\epsilon\}$, we can write:

$$\begin{aligned} &\frac{1}{\epsilon}(a \,\sharp^\epsilon\, b \,-\, a \,\sharp^0\, b)(X) \\ &\quad = -\frac{4i}{(2\pi)^4\epsilon} \int_{\Xi\times\Xi} e^{-2i\sigma(Y,Z)} \left(\int_0^1 e^{-4itF_\epsilon(x,y,z)} dt \right) F_\epsilon(x,y,z)\, a(X-Y)\, b(X-Z)\, dY dZ\,. \end{aligned} \tag{B.21}$$

Integrating by parts in the variables η, ζ, we obtain:

$$\begin{aligned} \big[r_\epsilon(a,b)\big](X) \,=\, -\frac{4i}{\pi^4} \int_{\Xi\times\Xi} dY dZ\, e^{-2i\sigma(Y,Z)} \left(\int_0^1 e^{-4itF_\epsilon(x,y,z)} dt \right) \times \\ \times F^\circ_\epsilon(x,y,z) \Big(\partial_\xi a(X-Y) \wedge \partial_\xi b(X-Z) \Big)\,. \end{aligned} \tag{B.22}$$

The proof can be completed by using Proposition B.12 with

$$\theta(x,y,z) = \left(\int_0^1 e^{-4itF_\epsilon(x,y,z)} dt \right) F^\circ_\epsilon(x,y,z)\,.$$

□

Remark B.15. Using the N'th order Taylor expansion of the exponential and similar arguments, we obtain that, for any $N \in \mathbb{N}^*$,

$$a \,\sharp^\epsilon\, b = a \,\sharp^0\, b + \sum_{1\le k\le N-1} \epsilon^k c^{(k)}_\epsilon(a,b) + \epsilon^N \rho^{(N)}_\epsilon(a,b)\,, \tag{B.23}$$

with $c^{(k)}_\epsilon(a,b) \in S^{m+m'-2k\rho}_1(\Xi)$ and $\rho^{(N)}_\epsilon(a,b) \in S^{m+m'-2N\rho}_1(\Xi)$ uniformly for $\epsilon \in [0, \epsilon_0]$.

With Remark B.6 in mind, let us consider, for an elliptic real symbol $h \in S_1^m(\Xi)$ with $m > 0$, the self-adjoint extension of $\mathfrak{Op}^\epsilon(h)$ denoted by H^ϵ, whose domain is given by the magnetic Sobolev space of order $m > 0$.

Proposition B.16. *For $\mathtt{z} \in \rho(H^\epsilon)$, let $r^\epsilon_{\mathtt{z}}(h) \in S_1^{-m}(\Xi)$ denote the symbol of $(H^\epsilon - \mathtt{z})^{-1}$. For any compact subset K of $\mathbb{C} \setminus \sigma(H)$, there exists $\epsilon_0 > 0$ such that:*

1. *$K \subset \mathbb{C} \setminus \sigma(H^\epsilon)$, for $\epsilon \in [0, \epsilon_0]$.*
2. *The following expansion is convergent in $\mathcal{L}(\mathcal{H})$ uniformly with respect to $(\epsilon, \mathtt{z}) \in [0, \epsilon_0] \times K$:*

$$r^\epsilon_{\mathtt{z}}(h) \;=\; \sum_{n\in\mathbb{N}} \epsilon^n r_n(h;\epsilon,\mathtt{z})\,, \quad r_0(h;\epsilon,\mathtt{z}) = r^0_{\mathtt{z}}(h)\,, \quad r_n(h;\epsilon,\mathtt{z}) \in S_1^{-(m+2n)}(\Xi)\,.$$

3. *The map $K \ni \mathtt{z} \mapsto r^\epsilon_{\mathtt{z}}(h) \in S_1^{-m}(\Xi)$ is a $S_1^{-m}(\Xi)$-valued analytic function, uniformly in $\epsilon \in [0, \epsilon_0]$.*

Proof. The first point follows from the spectral stability (see Corollary 1.2 in [24] or Theorem 1.4 in [2] or Theorem 3.1 in [9] for a more precise result).

For the last two statements we start from the analyticity in norm of the application

$$\mathbb{C} \setminus \sigma(H^\epsilon) \ni \mathtt{z} \mapsto (H^\epsilon - \mathtt{z})^{-1} = \mathfrak{Op}^\epsilon\big(r^\epsilon_{\mathtt{z}}(h)\big) \in \mathcal{L}(\mathcal{H}).$$

In order to obtain a control for the topology of $S_1^{-m}(\Xi)$ we recall Theorem 5.2 in [23]. Using (B.6) one shows that for any $\epsilon \in [0, \epsilon_0]$ and any $\mathtt{z} \notin \sigma(H^\epsilon)$

$$\mathfrak{ad}^\epsilon_X[r^\epsilon_{\mathtt{z}}(h)] = -r^\epsilon_{\mathtt{z}}(h) \,\sharp^\epsilon\, \mathfrak{ad}^\epsilon_X[h] \,\sharp^\epsilon\, r^\epsilon_{\mathtt{z}}(h)\,. \tag{B.24}$$

Using the resolvent equation:

$$r^\epsilon_{\mathtt{z}}(h) = r^\epsilon_i(h) + (i - \mathtt{z}) r^\epsilon_i(h) r^\epsilon_{\mathtt{z}}(h)\,,$$

and Propositions 3.6 and 3.7 from [23] we easily prove that for any pair of natural numbers $(p, q) \in \mathbb{N}\times\mathbb{N}$ and any families of points $\{u_1, \dots, u_p\} \subset \mathcal{X}$ and $\{\mu_1, \dots, \mu_q\} \subset \mathcal{X}^*$, the applications:

$$K \ni \mathtt{z} \mapsto \mathfrak{s}^\epsilon_{m+q} \,\sharp^\epsilon\, \big(\mathfrak{ad}^\epsilon_{u_1} \cdots \mathfrak{ad}^\epsilon_{u_p} \mathfrak{ad}^\epsilon_{\mu_1} \cdots \mathfrak{ad}^\epsilon_{\mu_q}[r^\epsilon_{\mathtt{z}}(h)]\big) \in S_0^0(\Xi) \tag{B.25}$$

are well defined, bounded and uniformly continuous for the norm $\|\cdot\|_\epsilon$ for any $\epsilon \in [0, \epsilon_0]$.

The second point follows by noticing that Proposition B.14 implies the equality

$$1 \;=\; (h - \mathtt{z}) \,\sharp^0\, r^0_{\mathtt{z}}(h) \;=\; (h - \mathtt{z}) \,\sharp^\epsilon\, r^0_{\mathtt{z}}(h) - \epsilon\, r_\epsilon\big(h, r^0_{\mathtt{z}}(h)\big)\,, \tag{B.26}$$

where the family $\{r_\epsilon\big(h, r^0_{\mathtt{z}}(h)\big)\}_{\epsilon\in[0,\epsilon_0]}$ is a bounded subset in $S_1^{-m}(\Xi)$.

We conclude that for some $\epsilon_0 > 0$, $1 + \epsilon r_\epsilon(h, r^0_{\mathtt{z}}(h))$ defines an invertible magnetic operator for any $\epsilon \in [0, \epsilon_0]$ and its inverse has a symbol $s^\epsilon(\mathtt{z})$ given as the limit of the following norm convergent series:

$$s^\epsilon(\mathtt{z}) := \sum_{n\in\mathbb{N}} \big(-\epsilon r_\epsilon(h, r^0_{\mathtt{z}}(h))\big)^{\sharp^\epsilon n} \;\in\; S_1^0(\Xi)\,. \tag{B.27}$$

This clearly gives us the expansion in point (2) of the theorem with

$$r_n(h;\epsilon,\mathtt{z}) := (-1)^n\, r^0_{\mathtt{z}}(h) \,\sharp^\epsilon\, \big(r_\epsilon(h, r^0_{\mathtt{z}}(h))\big)^{\sharp^\epsilon n} \in S_1^{-(m+2n)}(\Xi)\,.$$

In order to control the uniform continuity with respect to $\epsilon \in [0, \epsilon_0]$ of the application in (B.25) let us notice that

$$r^\epsilon_{\mathtt{z}}(h) - r^\epsilon_{\mathtt{z}'}(h) = (\mathtt{z}' - \mathtt{z})\, r^\epsilon_{\mathtt{z}}(h) \,\sharp^\epsilon\, r^\epsilon_{\mathtt{z}'}(h)\,,$$

and that for any $\mathtt{z} \in K$ the family of symbols $\{r^\epsilon_{\mathtt{z}}(h)\}_{\epsilon\in[0,\epsilon_0]}$ is a bounded set in $S_1^{-m}(\Xi)$ due to the uniform convergence of the series expansion obtained at point (2). □

Associated with the series expansion of the symbol $r^{\epsilon}_{\mathbf{z}}(h)$ given in Proposition B.16 , we shall also use the notation

$$\widetilde{r^{\epsilon}}_{\mathbf{z},n}(h) := \sum_{n+1\leq k} \epsilon^k r_k(h;\epsilon,\mathbf{z}) \in S_1^{-(m+2n+2)}(\Xi)\,. \tag{B.28}$$

Remark B.17. Having in mind (B.20) we conclude that there exist $\epsilon_0 > 0$ and $C > 0$ such that, for $\epsilon \in [0,\epsilon_0]$, the remainder $\widetilde{r^{\epsilon}}_{\mathbf{z},n}$ has the following properties:

1. $\widetilde{r^{\epsilon}}_{\mathbf{z},n} \,=\, \epsilon^{n+1}\, \widetilde{\widetilde{r^{\epsilon}}}_{\mathbf{z},n}$, where $\widetilde{\widetilde{r^{\epsilon}}}_{\mathbf{z},n} \in S_1^{-m}(\Xi)_\Gamma$ uniformly for $\epsilon \in [0,\epsilon_0)$.
2. $\|h \,\sharp^{\epsilon}\, \widetilde{r^{\epsilon}}_{\mathbf{z},n}\|_{\epsilon} \leq C\,\epsilon^{n+1}$.

B.5 The slowly varying magnetic fields

B.5.1 Some properties of the pseudodifferential calculus in a constant magnetic field.

In this case, for symbols independent of x, some interesting particularities have to be pointed out. One can also mention in this context the pseudo-calculus developed for other purposes in [5] which is applied in the magnetic context in Helffer-Sjöstrand [20].

Considering a constant magnetic field B_0 , we notice that $\omega^{B^0}(x,y,z) = \exp\{-2iB^0(y\wedge z)\}$ and thus for ϕ and ψ in $\mathcal{S}$, we can write

$$\begin{aligned}&\big(\phi\,\sharp^{B_0}\,\psi\big)(x,\xi) = \\ &= \pi^{-4}\int_{\mathcal{X}*}\int_{\mathcal{X}*}\left(\int_{\mathcal{X}}\int_{\mathcal{X}} e^{-2i<\eta,z>}e^{2i<\zeta,y>}e^{-2iB^0(y\wedge z)}\,d^2y\,d^2z\right)\phi(\xi-\eta)\psi(\xi-\zeta)\,d^2\eta\,d^2\zeta\,.\end{aligned} \tag{B.29}$$

We may compute the Fourier transform of $e^{-2iB^0(y\wedge z)}$ in $\mathcal{S}'$ using Theorem 7.6.1 in [21] and get that, in the sense of tempered distributions

$$\int_{\mathcal{X}}\int_{\mathcal{X}} e^{-2i<\eta,z>}e^{2i<\zeta,y>}e^{-2iB^0(y\wedge z)}\,d^2y\,d^2z = \left(\frac{\pi}{2B^0}\right)^2 e^{(i/B^0)(\eta\wedge\zeta)}.$$

Thus we conclude that

$$\begin{aligned}\big(\phi\,\sharp^{B_0}\,\psi\big)(x,\xi) &= \left(\tfrac{1}{2B^0\frac{\pi}{2}}\right)^2\int_{\mathcal{X}^*}\int_{\mathcal{X}^*} e^{(i/B^0)(\eta\wedge\zeta)}\phi(\xi-\eta)\psi(\xi-\zeta)\,d^2\eta\,d^2\zeta \\ &= \left(\tfrac{1}{2\pi}\right)^2\int_{\mathcal{X}^*}\int_{\mathcal{X}^*} e^{i(\eta\wedge\zeta))}\phi(\xi-\sqrt{B^0}\eta)\psi(\xi-\sqrt{B^0}\zeta)\,d^2\eta\,d^2\zeta\,.\end{aligned} \tag{B.30}$$

Using now a Taylor expansion of some order $N\in\mathbb{N}^*$ in (B.29), we obtain

$$\begin{aligned}\big(\phi\,\sharp^{B_0}\,\psi\big)(x,\xi) &= \phi(\xi)\psi(\xi) \\ &\quad + \sum_{1\leq p\leq N-1}(-2i)^p(B^0)^p\sum_{|\alpha|=p}(\alpha!)^{-1}\big(\partial^\alpha_\xi\phi\big)(\xi)\big(\partial^\alpha_{\xi^\perp}\psi\big)(\xi) \\ &\quad + (2\pi)^{-2}(-2iB^0)^N r_N(\phi,\psi,B_0)(\xi)\,,\end{aligned} \tag{B.31}$$

with

$$\begin{aligned}&r_N(\phi,\psi,B_0)(\xi) \\ &:= \sum_{|\alpha|=N}(\alpha!)^{-1}\int_{\mathcal{X}^*\times\mathcal{X}^*} e^{i\eta\wedge\zeta}\left(\int_0^1(1-s)^{N-1}\,\big(\partial^\alpha_\xi\phi\big)(\xi-\sqrt{sB^0}\eta)\big(\partial^\alpha_{\xi^\perp}\psi\big)(\xi-\sqrt{sB^0}\zeta)\,ds\right)d^2\eta\,d^2\zeta\,.\end{aligned} \tag{B.32}$$

In writing the remainder r_N we have used (B.30) with B^0 replaced by sB^0 (with $s\in[0,1]$) .

From these formulas and the results in [23] concerning the functional calculus for a magnetic pseudodifferential self-adjoint operator in $L^2(\mathcal{X})$, we directly deduce the following statement.

Proposition B.18. *For a constant magnetic field, the subspace generated by tempered distributions which are constant with respect to $\mathcal{X}$ is stable for the magnetic Moyal product.*
Moreover, if a self-adjoint magnetic pseudodifferential operator H in $L^2(\mathcal{X})$ has a constant symbol with respect to $\mathcal{X}$, then all the operators $f(H)$ obtained by functional calculus have the same property.

Motivated by the above results we shall introduce the following classes of symbols which are constant along the directions in $\mathcal{X}$:

$$S^m_\rho(\Xi)^\circ \;:=\; S^m_\rho(\Xi)\cap(\mathbb{C}\otimes\mathscr{S}'(\mathcal{X}^*))\;. \tag{B.33}$$

B.5.2 The class of slowly varying symbols.

The interest in working with slowly varying magnetic fields comes from the fact that their derivatives of order $k\in\mathbb{N}$ produce a factor ϵ^k. In order to systematically keep track of this property it will be useful to consider the following class of ϵ-indexed families of symbols, replacing the classes $S^m_\rho(\Xi)^\circ$ introduced in (B.33). Let us recall the definition of *a scaled symbol* in (B.17); here we shall consider the situation $\tau=1$ and $\epsilon\searrow 0$.

Definition B.19. *For any $(m,\rho)\in\mathbb{R}\times[0,1]$ and for some $\epsilon_0>0$, we denote by $S^m_\rho(\Xi)^\bullet$ the families of symbols $\{F^\epsilon\}_{\epsilon\in[0,\epsilon_0]}$ satisfying the following properties:*

1. $F^\epsilon\in S^m_\rho(\Xi)\,,\forall\epsilon\in[0,\epsilon_0]\,;$
2. $\exists\lim\limits_{\epsilon\searrow 0}F^\epsilon:=F^0\in S^m_\rho(\Xi)^\circ$ *in the topology of* $S^m_\rho(\Xi)\,;$
3. $\forall(\alpha,\beta)\in\mathbb{N}^2\times\mathbb{N}^2,\ \exists C_{\alpha\beta}>0$ *such that*

$$\sup_{\epsilon\in(0,\epsilon_0]}\epsilon^{-|\alpha|}\left\|\partial_x^\alpha\partial_\xi^\beta F^\epsilon\right\|_\infty\le C_{\alpha\beta}\,. \tag{B.34}$$

$S^m_\rho(\Xi)^\bullet$ is endowed with the topology defined by the seminorms indexed by $(p,q)\in\mathbb{N}^2$,

$$F^\bullet\mapsto\tilde\nu^{m,\rho}_{p,q}(F^\bullet)\;:=\;\sup_{\epsilon\in[0,\epsilon_0]}\epsilon^{-p}\sum_{|\alpha|=p}\sum_{|\beta|=q}\sup_{(x,\xi)\in\Xi}<\xi>^{-(m-q\rho)}\left|\big(\partial_x^\alpha\partial_\xi^\beta F^\epsilon\big)(x,\xi)\right|\,.$$

Remark B.20. Defining $\widetilde F^\epsilon(x,\xi):=F^\epsilon(\epsilon^{-1}x,\xi)$, it is easy to see that $\{F^\epsilon\}_{\epsilon\in[0,\epsilon_0]}\subset S^m_\rho(\Xi)$ belongs to $S^m_\rho(\Xi)^\bullet$ if and only if it is of the form $F^\epsilon(x,\xi)=\widetilde F^\epsilon(\epsilon x,\xi)=\widetilde F^\epsilon_{(\epsilon,1)}(x,\xi)$ for some bounded family $\{\widetilde F^\epsilon\}_{\epsilon\in[0,\epsilon_0]}\subset S^m_\rho(\Xi)$ verifying the condition (for the topology on $S^m_\rho(\Xi)^\circ$)

$$\exists\lim_{\epsilon\searrow 0}\widetilde F^\epsilon(0,\cdot):=F^0\in S^m_\rho(\Xi)^\circ.$$

Let us notice that for a slowly varying magnetic field of the form (1.5) we have:

$$\omega^{\kappa\epsilon B_\epsilon}(x,y,z)\;=\;\exp\left\{-4i\kappa\epsilon(y\wedge z)\int_{-1/2}^{1/2}ds\int_{-1/2}^{s}dt\,B\big(\epsilon x+\epsilon(2sy+2tz)\big)\right\}\,. \tag{B.35}$$

Proposition B.21. *Let $B_{\epsilon,\kappa}(x)$ be a magnetic field of the form (1.5). If $f^\bullet\in S^m_\rho(\Xi)^\bullet$ and $g^\bullet\in S^p_\rho(\Xi)^\bullet$, then $\{f^\epsilon\,\sharp^{B_{\epsilon,\kappa}}g^\epsilon\}_{\epsilon\in[0,\epsilon_0]}$ belongs to $S^{m+p}_\rho(\Xi)^\bullet$ uniformly with respect to $\kappa\in[0,1]$.*

Proof. Using Remark B.20, (1.5) and (B.35), we have, $\forall\epsilon\in[0,\epsilon_0]$,

$$f^\epsilon\,\sharp^{B_{\epsilon,\kappa}}g^\epsilon\;=\;\widetilde f^\epsilon_{(\epsilon,1)}\,\sharp^{\kappa\epsilon B_\epsilon}\,\widetilde g^\epsilon_{(\epsilon,1)}\,,$$

and

$$\Big(\widetilde f^\epsilon_{(\epsilon,1)}\,\sharp^{\kappa\epsilon B_\epsilon}\,\widetilde g^\epsilon_{(\epsilon,1)}\Big)(x,\xi)$$

$$= \pi^{-4} \int\limits_{\Xi\times\Xi} e^{-2i\sigma(Y,Z)} \omega^{\kappa\epsilon B_\epsilon(x,y,z)} \widetilde{f}^\epsilon\big(\epsilon(x-y),\xi-\eta\big)\widetilde{g}^\epsilon\big(\epsilon(x-z),\xi-\zeta\big)\, dY dZ$$

$$= \pi^{-4} \int\limits_{\Xi\times\Xi} e^{-2i\sigma(Y,Z)} \exp\left\{-4i\epsilon(y\wedge z)\left(B_0+\kappa\int\limits_{-1/2}^{1/2} ds \int\limits_{-1/2}^{s} dt\, B\big(\epsilon x+\epsilon(2sy+2tz)\big)\right)\right\}$$
$$\times\ \widetilde{f}^\epsilon\big(\epsilon x-\epsilon y,\xi-\eta\big)\,\widetilde{g}^\epsilon\big(\epsilon x-\epsilon z,\xi-\zeta\big)\, dY dZ\,.$$

Using Proposition B.9 with

$$L^{\epsilon,\kappa}(x,y,z) \ := \ \exp\left\{-4i\epsilon(y\wedge z)\left(B_0+\kappa\int\limits_{-1/2}^{1/2} ds \int\limits_{-1/2}^{s} dt\, B\big(x+\epsilon(2sy+2tz)\big)\right)\right\}\,,$$

we obtain:

$$f^\epsilon \sharp^{B_{\epsilon,\kappa}} g^\epsilon \ = \ \widetilde{f}^\epsilon_{(\epsilon,1)} \sharp^{\kappa\epsilon B_\epsilon} \widetilde{g}^\epsilon_{(\epsilon,1)} \ = \ \Big(\mathfrak{L}^{\epsilon,\kappa}_{(\epsilon,1,\epsilon,1)}\big(\widetilde{f}^\epsilon,\widetilde{g}^\epsilon\big)\Big)_{(\epsilon,1)}\,,$$

where $\big\{\mathfrak{L}^{\epsilon,\kappa}_{(\epsilon,1,\epsilon,1)}\big(\widetilde{f}^\epsilon,\widetilde{g}^\epsilon\big)\big\}_{(\epsilon,\kappa)\in[0,\epsilon_0]\times[0,1]}$ is a bounded subset of $S^{m+p}_\rho(\Xi)$.

Using Remark B.20 we only have to compute $\lim\limits_{\epsilon\searrow 0} \mathfrak{L}^{\epsilon,\kappa}_{(\epsilon,1,\epsilon,1)}\big(\widetilde{f}^\epsilon,\widetilde{g}^\epsilon\big)(x,\xi)$. By hypothesis we know that $\exists\lim\limits_{\epsilon\searrow 0}\widetilde{f}^\epsilon(0,\xi) = \widetilde{f}_0(\xi)$ and $\exists\lim\limits_{\epsilon\searrow 0}\widetilde{g}^\epsilon(0,\xi) = \widetilde{g}_0(\xi)$ with respect to the topology of each associated symbol class. It is also easy to notice that, in $BC^\infty\big(\mathcal{X};C^\infty_{\mathsf{pol}}(\mathcal{X}\times\mathcal{X})\big)$,

$$\exists\lim_{\epsilon\searrow 0} L^{\epsilon,\kappa}(0,y,z) = 1\,.$$

Thus taking into account Proposition B.9 we obtain:

$$\exists\lim_{\epsilon\searrow 0} \mathfrak{L}^{\epsilon,\kappa}_{(\epsilon,1,\epsilon,1)}\big(\widetilde{f}^\epsilon,\widetilde{g}^\epsilon\big)(x,\xi) \ = \ \widetilde{f}_0(\xi)\,\widetilde{g}_0(\xi)\,.$$

□

The following proposition will be useful in working with inverses of slowly varying symbols.

Proposition B.22. *For given $\epsilon_0>0$, $(m,\rho)\in\mathbb{R}_+\times[0,1]$, let $\{F_\epsilon\}_{\epsilon\in[0,\epsilon_0]}$ be a family in $S^m_\rho(\Xi)$ which is continuous at 0. Let us consider a family of magnetic fields $B_\epsilon=\epsilon B^0_\epsilon$ satisfying Hypothesis B.13 and suppose that each symbol F_ϵ, for $\epsilon\in[0,\epsilon_0]$ has an inverse $F^-_\epsilon := \big(F_\epsilon\big)^-_{B_\epsilon}$ with respect to the magnetic Moyal product $\sharp^\epsilon$. If moreover $\mathfrak{s}_m\sharp^\epsilon F^-_\epsilon$ defines a bounded operator in $\mathcal{H}$ and the family $\{F^-_\epsilon\}_{\epsilon\in[0,\epsilon_0]}$ is bounded in $S^{-m}_\rho(\Xi)$, then F^-_ϵ is continuous at 0.*

Proof. By Propositions 6.1 and 6.2 in [23] each F^-_ϵ belongs to $S^{-m}_\rho(\Xi)$. Due to the continuity at $\epsilon=0$ of the family $\{F_\epsilon\}_{\epsilon\in[0,\epsilon_0]}$ we conclude that for $\epsilon>0$ small enough, the inverse $\big(F_0\big)^-_{B_\epsilon}$ still exists. For any seminorm $\nu: S^{-m}_\rho(\Xi)\to\mathbb{R}_+$ defining its topology, we can write:

$$\nu\big[\big(F_\epsilon\big)^-_{B_\epsilon}-\big(F_0\big)^-_0\big] \le \nu\big[\big(F_\epsilon\big)^-_{B_\epsilon}-\big(F_0\big)^-_{B_\epsilon}\big] \ +\nu\big[\big(F_0\big)^-_{B_\epsilon}-\big(F_0\big)^-_0\big]\,. \tag{B.36}$$

For the first seminorm a simple usual computation shows that for any bounded smooth magnetic field B

$$\big(F_\epsilon\big)^-_B-\big(F_0\big)^-_B \ = \ -\big(F_\epsilon\big)^-_B\sharp^B\big[F_\epsilon-F_0\big]\sharp^B\big(F_0\big)^-_B\,. \tag{B.37}$$

Theorem 5.2 in [23] (Beals like criterion) states that the topology on space $S^{-m}_1(\Xi)$ (for any $m\in\mathbb{R}$) may be also defined by the following equivalent family of seminorms:

$$S^{-m}_\rho(\Xi)\ni\psi\mapsto\Big\|\mathfrak{s}^\epsilon_{m+q\rho}\sharp^\epsilon\big(\mathfrak{ad}^\epsilon_{u_1}\cdots\mathfrak{ad}^\epsilon_{u_p}\mathfrak{ad}^\epsilon_{\mu_1}\cdots\mathfrak{ad}^\epsilon_{\mu_q}[\psi]\big)\Big\|_\epsilon\,, \tag{B.38}$$

indexed by a pair of natural numbers $(p,q)\in\mathbb{N}\times\mathbb{N}$ and by two families of points $\{u_1,\dots,u_p\}\subset\mathcal{X}$ and $\{\mu_1,\dots,\mu_q\}\subset\mathcal{X}^*$. A simple computation using (B.37) shows that for any magnetic field B

$$\begin{aligned}
\mathfrak{ad}_X^B\Big[\big(F_\epsilon\big)_B^- - \big(F_0\big)_B^-\Big] &= -\big(F_\epsilon\big)_B^-\,\sharp^B\,\mathfrak{ad}_X^B\big[F_\epsilon - F_0\big]\,\sharp^B\,\big(F_0\big)_B^- \\
&\quad -\mathfrak{ad}_X^B\Big[\big(F_\epsilon\big)_B^-\Big]\,\sharp^B\,\big[F_\epsilon - F_0\big]\,\sharp^B\,\big(F_0\big)_B^- \\
&\quad -\big(F_\epsilon\big)_B^-\,\sharp^B\,\big[F_\epsilon - F_0\big]\,\sharp^B\,\mathfrak{ad}_X^B\Big[\big(F_0\big)_B^-\Big] \\
&= -\big(F_\epsilon\big)_B^-\,\sharp^B\,\mathfrak{ad}_X^B\big[F_\epsilon - F_0\big]\,\sharp^B\,\big(F_0\big)_B^- \\
&\quad +\big(F_\epsilon\big)_B^-\,\sharp^B\,\mathfrak{ad}_X^B\big[F_\epsilon\big]\,\sharp^B\,\big(F_\epsilon\big)_B^-\,\sharp^B\,\big[F_\epsilon - F_0\big]\,\sharp^B\,\big(F_0\big)_B^- \\
&\quad +\big(F_\epsilon\big)_B^-\,\sharp^B\,\big[F_\epsilon - F_0\big]\,\sharp^B\,\big(F_0\big)_B^-\,\sharp^B\,\mathfrak{ad}_X^B\big[F_0\big]\,\sharp^B\,\big(F_0\big)_B^-\,.
\end{aligned}$$

Iterating the above computation and using Theorem 5.2 in [23] once again, we prove that

$$\lim_{\epsilon\searrow 0}\nu\big[\big(F_\epsilon\big)_{B_\epsilon}^- - \big(F_0\big)_{B_\epsilon}^-\big] \;=\; 0\,.$$

For the second seminorm in (B.36) we use Proposition B.14 above with $a=F_0$ and $b=\big(F_0\big)_0^-$ in order to obtain

$$F_0\,\sharp^{B_\epsilon}\big(F_0\big)_0^- \;=\; 1\,+\,\epsilon\,r_\epsilon\big(F_0,\big(F_0\big)_0^-\big)\,,$$

and finally

$$\lim_{\epsilon\searrow 0}\nu\big[\big(F_0\big)_{B_\epsilon}^- - \big(F_0\big)_0^-\big] \;=\; 0\,.$$

□

Proposition B.23. *Let $B_{\epsilon,\kappa}(x)$ be a magnetic field of the form* (1.5)*. If $f^\bullet\in S_\rho^m(\Xi)^\bullet$ and if the inverse $(f^\epsilon)^-\equiv(f^\epsilon)^-_{B_{\epsilon,\kappa}}\in S_\rho^{-m}(\Xi)$ exists for every $\epsilon\in[0,\epsilon_0]$, then $\{(f^\epsilon)^-\}_{\epsilon\in[0,\epsilon_0]}\in S_\rho^{-m}(\Xi)^\bullet$.*

Proof. The first condition in Definition B.19 is verified by hypothesis and the second follows from Proposition B.22. In order to prove the third condition in Definition B.19 we recall from Subsection 3.3 in [23] the subspace $\mathfrak{A}(\Xi)$ of symbols in $BC^\infty\big(\mathcal{X};L^1(\mathcal{X}^*)\big)$ having rapid decay in $\xi\in\mathcal{X}^*$ together with all their derivatives with respect to $x\in\mathcal{X}$, and the fact that using the operators (B.6) we can write

$$\begin{aligned}
\partial_{x_j}f^\epsilon &= i\mathfrak{ad}_{e_j}^{B_{\epsilon,\kappa}}[f^\epsilon] + i\delta^{B_{\epsilon,\kappa}}f \\
&= i\mathfrak{ad}_{e_j}^{B_{\epsilon,\kappa}}[f^\epsilon] + i\sum_{1\le|\alpha|\le 5}c_{j,\alpha}^{B_{\epsilon,\kappa}}\star\big(\partial_\xi^\alpha f^\epsilon\big) \\
&= i\mathfrak{ad}_{e_j}^{B_{\epsilon,\kappa}}[f^\epsilon] + i\sum_{1\le|\alpha|\le 5}c_{j,\alpha}^{B_{\epsilon,\kappa}}\star\big((\mathfrak{ad}_{e^*}^{B_{\epsilon,\kappa}})^\alpha f^\epsilon\big)\,,
\end{aligned}$$

where $c_{j,\alpha}^{B_{\epsilon,\kappa}}\in\mathfrak{A}(\Xi)$ for any $(j,\alpha)\in\{1,2\}\times\mathbb{N}^2$ and, for ϕ and ψ in $BC^\infty\big(\mathcal{X};L^1(\mathcal{X}^*)\big)$,

$$\big(\phi\star\psi\big)(x,\xi)\;:=\;\int_{\mathcal{X}^*}\phi(x,\xi-\eta)\,\psi(x,\eta)\,d\eta\,.$$

Then, the explicit description of the coefficients $c_{j,\alpha}^{B_{\epsilon,\kappa}}\in\mathfrak{A}(\Xi)$ given in Propositions 3.6 and 3.7 in [23] easily implies that

$$\partial_x^\beta c_{j,\gamma}^{B_{\epsilon,\kappa}} \;=\; c_{j,\gamma}^{\partial^\beta B_{\epsilon,\kappa}} \;=\; \epsilon^{|\beta|+1}\,\widetilde{c}_{j,\gamma,\beta}^{\epsilon,\kappa}\,,$$

where $\{\widetilde{c}_{j,\gamma,\beta}^{\epsilon,\kappa}\}_{(\epsilon,\kappa)\in[0,\epsilon_0]\times[0,1]}$ is a bounded set in $\mathfrak{A}(\Xi)$.

By Theorem 5.2 in [23] we know that the topology on any space $S_\rho^m(\Xi)$ may be also defined by the family of seminorms:

$$S_\rho^m(\Xi)\ni\psi\mapsto\Big\|\mathfrak{s}_{-(m-q\rho)}^{B_{\epsilon,\kappa}}\,\sharp^{\epsilon,\kappa}\,\big(\mathfrak{ad}_{u_1}^{B_{\epsilon,\kappa}}\cdots\mathfrak{ad}_{u_p}^{B_{\epsilon,\kappa}}\mathfrak{ad}_{\mu_1}^{B_{\epsilon,\kappa}}\cdots\mathfrak{ad}_{\mu_q}^{B_{\epsilon,\kappa}}[\psi]\big)\Big\|_{B_{\epsilon,\kappa}}\,, \tag{B.39}$$

which is indexed by $(p,q)\in\mathbb{N}\times\mathbb{N}$, $\{u_1,\dots,u_p\}\subset\mathcal{X}$ and $\{\mu_1,\dots,\mu_q\}\subset\mathcal{X}^*$.
Moreover, we can repeat in this situation the argument in the proof of Proposition 6.1 in [23] and note that, with

$$\mathfrak{D}^{\epsilon,\kappa}_{\{j_1,\dots,j_r\}} := \mathfrak{ad}^{\epsilon,\kappa}_{X_{j_1}}\dots\mathfrak{ad}^{\epsilon,\kappa}_{X_{j_r}}\,,$$

and for coefficients $C_{J_1,\dots,J_r}$ taking only the values ±1, we have:

$$\begin{aligned}
&\mathfrak{s}^{B_{\epsilon,\kappa}}_{-(m-q\rho)}\ \sharp^{\epsilon,\kappa}\ \big(\mathfrak{ad}^{B_{\epsilon,\kappa}}_{u_1}\cdots\mathfrak{ad}^{B_{\epsilon,\kappa}}_{u_p}\mathfrak{ad}^{B_{\epsilon,\kappa}}_{\mu_1}\cdots\mathfrak{ad}^{B_{\epsilon,\kappa}}_{\mu_q}\big[(f^\epsilon)^-\big]\big)\\
&\qquad := \mathfrak{s}^{B_{\epsilon,\kappa}}_{-(m-q\rho)}\ \sharp^{\epsilon,\kappa}\ \mathfrak{D}^{\epsilon,\kappa}_{\{1,\dots,p+q\}}\big[(f^\epsilon)^-\big]\\
&\qquad = \sum_{1\le r\le p+q}\ \sum_{J_1\sqcup\dots\sqcup J_r} C_{J_1,\dots,J_r}\,\mathfrak{s}^{B_{\epsilon,\kappa}}_{-(m-q\rho)}\ \sharp^{\epsilon,\kappa}\ (f^\epsilon)^-\ \sharp^{\epsilon,\kappa}\ \mathfrak{D}^{\epsilon,\kappa}_{J_1}\big[f^\epsilon\big]\ \sharp^{\epsilon,\kappa}\ (f^\epsilon)^-\ \sharp^{\epsilon,\kappa}\ \dots\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \dots\sharp^{\epsilon,\kappa}\ (f^\epsilon)^-\ \sharp^{\epsilon,\kappa}\ \mathfrak{D}^{\epsilon,\kappa}_{J_r}\big[f^\epsilon\big]\,,
\end{aligned}$$

the sum being over all partitions $\{1,\dots,p+q\}=\sqcup_{i=1}^p J_i$ where, for example, the partition (J_1,J_2) is considered different from (J_2,J_1).
These remarks allow us to replace the condition (B.34) in Definition B.19 with the following condition:

$$\sup_{\epsilon\in(0,\epsilon_0]}\epsilon^{-q}\left\|\mathfrak{s}^{B_{\epsilon,\kappa}}_{-(m-q\rho)}\ \sharp^{\epsilon,\kappa}\ \big(\mathfrak{ad}^{B_{\epsilon,\kappa}}_{u_1}\cdots\mathfrak{ad}^{B_{\epsilon,\kappa}}_{u_p}\mathfrak{ad}^{B_{\epsilon,\kappa}}_{\mu_1}\cdots\mathfrak{ad}^{B_{\epsilon,\kappa}}_{\mu_q}[\psi]\big)\right\|_{B_{\epsilon,\kappa}}<+\infty\,,$$

for any pair of natural numbers $(p,q)\in\mathbb{N}\times\mathbb{N}$ and any two families of points $\{u_1,\dots,u_p\}\subset\mathcal{X}$ and $\{\mu_1,\dots,\mu_q\}\subset\mathcal{X}^*$.

Combining these remarks and the identity (B.24) achieves the proof. □

A 'localized' approximant for the magnetic Moyal product with slowly varying magnetic field. Starting from (B.35) and using the Taylor formula with integral remainder one gets:

$$\begin{aligned}
&\textstyle\int_{-1/2}^{1/2}ds\int_{-1/2}^{s}dt\,B\big(\epsilon x+\epsilon(2sy+2tz)\big)\\
&\qquad = \tfrac12 B(\epsilon x)\ +2\epsilon\sum_{1\le\ell\le2}\big[y_\ell R_1\big(\partial_\ell B\big)(\epsilon x,\epsilon y,\epsilon z)\ +\ z_\ell R_2\big(\partial_\ell B\big)(\epsilon x,\epsilon y,\epsilon z)\big]\ ,
\end{aligned}$$

with

$$R_1\big(F\big)(x,y,z)\ :=\ \int_{-1/2}^{1/2}s\,ds\int_{-1/2}^{s}dt\int_0^1du\,F\big(x+u(2sy+2tz)\big)\,, \tag{B.40}$$

$$R_2\big(F\big)(x,y,z)\ :=\ \int_{-1/2}^{1/2}ds\int_{-1/2}^{s}t\,dt\int_0^1du\,F\big(x+u(2sy+2tz)\big)\,. \tag{B.41}$$

If we introduce

$$\Psi^\epsilon(x,y,z)\ :=-8(y\wedge z)\sum_{1\le\ell\le2}\big[y_\ell R_1\big(\partial_\ell B\big)(x,\epsilon y,\epsilon z)\ +\ z_\ell R_2\big(\partial_\ell B\big)(x,\epsilon y,\epsilon z)\big]\,, \tag{B.42}$$

we notice that Ψ^ϵ is real and we can write:

$$\omega^{\kappa\epsilon B_\epsilon}(x,y,z)=\exp\left\{-2i\kappa\epsilon B(\epsilon x)(y\wedge z)\right\}\left(1\ +\ i\,\kappa\epsilon^2\Psi^\epsilon(\epsilon x,y,z)\int_0^1\exp\left\{i\,\tau\kappa\epsilon^2\Psi^\epsilon(\epsilon x,y,z)\right\}d\tau\right)\,.$$

Remark B.24. Having in mind Proposition B.9, we notice that the function

$$\Theta^{\epsilon,\kappa}_\tau(x,y,z):=\exp\left\{i\,\tau\kappa\epsilon^2\Psi^\epsilon(x,y,z)\right\}$$

has modulus one and belongs to the space $BC^\infty\big(\mathbb{R}^2;C^\infty_{\mathsf{pol}}(\mathbb{R}^2\times\mathbb{R}^2)\big)$ uniformly for $(\kappa,\epsilon)\in[0,1]\times[0,\epsilon_0]$ and the functions $R_j\big(\partial_\ell B_{jk}\big)(x,y,z)$ (for $j=1,2$) are also of class $BC^\infty\big(\mathbb{R}^2;C^\infty_{\mathsf{pol}}(\mathbb{R}^2\times\mathbb{R}^2)\big)$ uniformly in $\epsilon\in[0,\epsilon_0]$.

It is clear that the space of symbols $\bigcup_{m\in\mathbb{R}} S^m_\rho(\Xi)^\circ$ is no longer closed for the magnetic Moyal composition for a magnetic field that is no longer constant. In order to treat this difficulty we consider the formula (B.30) that is a well defined composition in $\bigcup_{m\in\mathbb{R}} S^m_\rho(\Xi)^\circ$ and extend it to the following frozen magnetic product of ϕ and ψ in $\mathscr{S}(\Xi)$ (obtained by fixing the value of the magnetic field under the integrals at the point $\in \mathcal{X}$ where the product is computed):

$$\begin{aligned}&\big(\phi\,\natural^\epsilon\,\psi\big)(x,\xi) \\ &\quad:=\left(\frac{1}{2\pi}\right)^2\int_{\mathcal{X}^*}\int_{\mathcal{X}^*} e^{-i(\eta\wedge\zeta))}\phi(x,\xi-\epsilon^{1/2}b_\kappa(x)\,\eta)\psi(x,\xi-\epsilon^{1/2}b_\kappa(x)\,\zeta)\,d^2\eta\,d^2\zeta\,.\end{aligned} \tag{B.43}$$

Remark B.25. The proof of Proposition B.31 remains true also for $N=0$ and for symbols depending also on the configuration variable $x\in\mathcal{X}$ and shows that the truncated magnetic product $\natural^\epsilon$ is a continuous map $S^m_\rho(\Xi)\times S^p_\rho(\Xi)\to S^{m+p}_\rho(\Xi)$, equicontinuous for $\epsilon\in[0,\epsilon_0]$.

Using now the Taylor expansions up to some order $N\in\mathbb{N}^*$ we obtain the following formulas similar to (B.31):

$$\begin{aligned}\big(\phi\,\natural^\epsilon\psi\big)(x,\xi) &= \phi(x,\xi)\psi(x,\xi)\\ &\quad+\sum_{1\le p\le N-1}(-2i\epsilon)^pB_\kappa(x)^p\sum_{|\alpha|=p}(\alpha!)^{-1}\big(\partial^\alpha_\xi\phi\big)(x,\xi)\,\big(\partial^\alpha_\xi\psi\big)(x,\xi)\\ &\quad+(2\pi)^{-2}(-2i\epsilon)^NB_\kappa(x)^Nr_N(\phi,\psi,B_\kappa(x),\epsilon)(x,\xi)\,,\end{aligned} \tag{B.44}$$

where

$$\begin{aligned}r_N(\phi,\psi,B,\epsilon)(x,\xi) &:= \sum_{|\alpha|=N}(\alpha!)^{-1}\int_{\mathcal{X}^*}\int_{\mathcal{X}^*}e^{i(\eta\wedge\zeta)}\int_0^1 s^{N-1}ds\,\times\\ &\qquad\times\big(\partial^\alpha_\xi\phi\,\big)(x,\xi-s\epsilon^{1/2}B^{\frac12}\eta)\,\big(\partial^\alpha_\xi\psi\big)(x,\xi-s\epsilon^{1/2}B^{\frac12}\zeta)\,d^2\eta\,d^2\zeta\,.\end{aligned} \tag{B.45}$$

Some basic estimates.

Proposition B.26. *For a magnetic field $B_{\epsilon,\kappa}(x)$ given in (1.5), there exists $\epsilon_0>0$ and $M>0$ such that for any $(\epsilon,\kappa,\tau)\in[0,\epsilon_0]\times[0,1]\times[1,\infty)$ with $\epsilon\tau^2\in[0,M]$ and for any $\phi\in S^m_\rho(\Xi)$ and $\psi\in S^p_\rho(\Xi)$,*

$$\begin{aligned}\phi_{(\epsilon,\tau)}\,\sharp^{B_{\epsilon,\kappa}}\,\psi_{(\epsilon,\tau)} &= \big(\phi\,\natural^{\epsilon\tau^2}\,\psi\big)_{(\epsilon,\tau)}\\ &\quad+\epsilon\tau\,\frac{i}{2}\sum_{1\le\ell\le2}\Big[\Big(\big(\partial_{x_\ell}\phi\big)\,\natural^{\epsilon\tau^2}\,\big(\partial_{\xi_\ell}\psi\big)\Big)_{(\epsilon,\tau)}-\Big(\big(\partial_{\xi_\ell}\phi\big)\,\natural^{\epsilon\tau^2}\,\big(\partial_{x_\ell}\psi\big)\Big)_{(\epsilon,\tau)}\Big]\\ &\quad+(\epsilon\tau)^2\mathscr{R}_{\epsilon,\kappa,\tau}(\phi,\psi)_{(\epsilon,\tau)}\,,\end{aligned}$$

where the family $\{\mathscr{R}_{\epsilon,\kappa,\tau}(\phi,\psi)\}$ is in a bounded subset of $S^{m+p-3\rho}_\rho(\Xi)$.
Moreover, the application: $\mathscr{R}_{\epsilon,\kappa,\tau}:\ S^m_\rho(\Xi)\times S^p_\rho(\Xi)\to S^{m+p-3\rho}_\rho(\Xi)$ is continuous.

Proof. With $B_{\epsilon,\kappa}$ and ϵ as in the statement of the proposition, using the notation in (B.42) and Remark B.24, we compute

$$\begin{aligned}&\big(\phi_{(\epsilon,\tau)}\,\sharp^{B_{\epsilon,\kappa}}\,\psi_{(\epsilon,\tau)}\big)(X)\\ &\quad=\int_{\Xi\times\Xi}dY\,dZ\,e^{-2i\sigma(Y,Z)}e^{i\,\epsilon\Phi_\kappa(\epsilon x,y,z)}\phi_{(\epsilon,\tau)}(X-Y)\psi_{(\epsilon,\tau)}(X-Z)\times\\ &\qquad\times\Big(1\,+\,i\,\kappa\epsilon^2\Psi^\epsilon(\epsilon x,y,z)\int_0^1\Theta^{\epsilon,\kappa}_t(\epsilon x,y,z)dt\Big)\,.\end{aligned}$$

We notice that $\phi_{(\epsilon,\tau)}(X-Y)\psi_{(\epsilon,\tau)}(X-Z) = \phi\big(\epsilon(x-y),\tau(\xi-\eta)\big)\psi\big(\epsilon(x-z),\tau(\xi-\zeta)\big)$ and use a Taylor expansion up to second order with respect to the variables ϵy and ϵz in order to obtain:

$$
\begin{aligned}
&\phi_{(\epsilon,\tau)}(X-Y)\psi_{(\epsilon,\tau)}(X-Z)\\
&\quad= \phi\big(\epsilon x,\tau(\xi-\eta)\big)\psi\big(\epsilon x,\tau(\xi-\zeta)\big)\\
&\qquad- \epsilon\int_0^1 dt\,\phi\big(\epsilon x,\tau(\xi-\eta)\big)\Big(z\cdot\big(\partial_x\psi\big)\big(\epsilon x-t\epsilon z,\tau(\xi-\zeta)\big)\Big)\\
&\qquad- \epsilon\int_0^1 dt\,\Big(y\cdot\big(\partial_x\phi\big)\big(\epsilon x-t\epsilon y,\tau(\xi-\eta)\big)\Big)\psi\big(\epsilon x,\tau(\xi-\zeta)\big)\\
&\qquad+ \epsilon^2\int_0^1 dt\int_0^1 ds\Big(y\cdot\big(\partial_x\phi\big)\big(\epsilon x-t\epsilon y,\tau\xi-\tau\eta\big)\Big)\Big(z\cdot\big(\partial_x\psi\big)\big(\epsilon x-s\epsilon z,\tau\xi-\tau\zeta\big)\Big)\\
&\quad= \phi\big(\epsilon x,\tau\xi-\tau\eta\big)\psi\big(\epsilon x,\tau\xi-\tau\zeta\big)\\
&\qquad-\epsilon\Big[\phi\big(\epsilon x,\tau\xi-\tau\eta\big)\Big(z\cdot\big(\partial_x\psi\big)\big(\epsilon x,\tau\xi-\tau\zeta\big)\Big)+\Big(y\cdot\big(\partial_x\phi\big)\big(\epsilon x,\tau\xi-\tau\eta\big)\Big)\psi\big(\epsilon x,\tau\xi-\tau\zeta\big)\Big]\\
&\qquad+\epsilon^2\Big\{\Big(y\cdot\big(\partial_x\phi\big)\big(\epsilon x,\tau\xi-\tau\eta\big)\Big)\Big(z\cdot\big(\partial_x\psi\big)\big(\epsilon x,\tau\xi-\tau\zeta\big)\Big)\\
&\qquad\qquad+\int_0^1 dt\Big[\phi\big(\epsilon x,\tau\xi-\tau\eta\big)\big\langle z,\big(D_x^2\psi\big)\big(\epsilon(x-tz),\tau\xi-\tau\zeta\big)\,z\big\rangle\\
&\qquad\qquad\qquad+\big\langle y,\big(D_x^2\phi\big)\big(\epsilon(x-ty),\tau\xi-\tau\eta\big)\,y\big\rangle)\psi\big(\epsilon x,\tau\xi-\tau\zeta\big)\Big]\\
&\qquad\qquad-\epsilon\int_0^1 dt\,\Big[\Big(y\cdot\big(\partial_x\phi\big)\big(\epsilon x,\tau\xi-\tau\eta\big)\Big)\big\langle z,\big(D_x^2\psi\big)\big(\epsilon(x-tz),\tau\xi-\tau\zeta\big)\,z\big\rangle\\
&\qquad\qquad\qquad+\big\langle y,\big(D_x^2\phi\big)\big(\epsilon(x-ty),\tau\xi-\tau\eta\big)\,y\big\rangle\Big(z\cdot\big(\partial_x\psi\big)\big(\epsilon x,\tau\xi-\tau\zeta\big)\Big)\Big]\\
&\qquad\qquad+\epsilon^2\int_0^1 dt\int_0^1 ds\,\big\langle y,\big(D_x^2\phi\big)\big(\epsilon(x-ty),\tau\xi-\tau\eta\big)\,y\big\rangle\big\langle z,\big(D_x^2\psi\big)\big(\epsilon(x-sz),\tau\xi-\tau\zeta\big)\,z\big\rangle\Big\}.
\end{aligned}
$$

We make now the usual integration by parts using the exponential factor $e^{-2i\sigma(Y,Z)}$ and use formula (B.30). We note that a change of variables allows to write

$$
\begin{aligned}
&\int_{\Xi\times\Xi} dY\,dZ\,e^{-2i\sigma(Y,Z)}e^{i\,\epsilon\Phi_\kappa(\epsilon x,y,z)}\phi\big(\epsilon x,\tau\xi-\tau\eta\big)\psi\big(\epsilon x,\tau\xi-\tau\zeta\big)\\
&\qquad\qquad= \int_{\Xi\times\Xi} dY\,dZ\,e^{-2i\sigma(Y,Z)}e^{i\,\epsilon\tau^2\Phi_\kappa(\epsilon x,y,z)}\phi\big(\epsilon x,\tau\xi-\eta\big)\psi\big(\epsilon x,\tau\xi-\zeta\big)\\
&\qquad\qquad= \big(\phi\,\natural^{\epsilon\tau^2}\,\psi\big)_{(\epsilon,\tau)}(x,\xi)\,,
\end{aligned}
$$

and the terms of order 0 and 1 in ϵ give us the first three terms of the formula stated in the proposition. The remaining terms of order 2 in ϵ may be also integrated by parts using the exponential factor $e^{-2i\sigma(Y,Z)}$ in order to obtain:

$$
\begin{aligned}
&\int_{\Xi\times\Xi} dY\,dZ\,e^{-2i\sigma(Y,Z)}e^{i\,\epsilon\Phi_\kappa(\epsilon x,y,z)}\phi_{(\epsilon,\tau)}(X-Y)\psi_{(\epsilon,\tau)}(X-Z)\\
&\qquad= \big(\phi\,\natural^{\epsilon\tau^2}\,\psi\big)_{(\epsilon,\tau)}\\
&\qquad\quad+\ \epsilon\tau\,\frac{i}{2}\sum_{1\le\ell\le 2}\Big[\Big(\big(\partial_{x_\ell}\phi\big)\,\natural^{\epsilon\tau^2}\,\big(\partial_{\xi_\ell}\psi\big)\Big)_{(\epsilon,\tau)}-\Big(\big(\partial_{\xi_\ell}\phi\big)\,\natural^{\epsilon\tau^2}\,\big(\partial_{x_\ell}\psi\big)\Big)_{(\epsilon,\tau)}\Big]\\
&\qquad\quad+\ (\epsilon\tau)^2\int_{\Xi\times\Xi} dY\,dZ\,e^{-2i\sigma(Y,Z)}e^{i\,\epsilon\Phi_\kappa(\epsilon x,y,z)}\,\mathcal{R}^{(1)}_{\epsilon,\kappa,\tau}(X,Y,Z,\phi,\psi)\,,
\end{aligned}
$$

with

$$
\begin{aligned}
&\mathcal{R}^{(1)}_{\epsilon,\kappa,\tau}(X,Y,Z,\phi,\psi)\\
&\quad:=\tfrac{1}{4}\sum_{j,k}\big(\partial_{x_j}\partial_{\xi_k}\phi\big)(x,\xi-\eta)\big(\partial_{x_k}\partial_{\xi_j}\psi\big)(x,\xi-\zeta)\\
&\qquad-\tfrac{1}{4}\int_0^1 dt\Big(\sum_{k,\ell}\big(\partial^2_{\xi_k\xi_\ell}\phi\big)(x,\xi-\eta)\big(\partial^2_{x_kx_\ell}\psi\big)(x-t\epsilon\tau z,\xi-\zeta)\\
&\qquad\qquad\qquad-\sum_{j,k}\big(\partial^2_{x_jx_k}\phi\big)(x-t\epsilon\tau y,\xi-\eta)\big(\partial^2_{\xi_j\xi_k}\psi\big)(x,\xi-\zeta)\Big)\\
&\qquad+\tfrac{i\epsilon\tau}{8}\int_0^1 dt\Big(\sum_{j,k,\ell}\big(\partial_{x_j}\partial^2_{\xi_k\xi_\ell}\phi\big)(x,\xi-\eta)\big(\partial^2_{x_kx_\ell}\partial_{\xi_j}\psi\big)(x-t\epsilon\tau z,\xi-\zeta)\\
&\qquad\qquad\qquad-\sum_{j,k,\ell}\big(\partial^2_{x_jx_k}\partial_{\xi_\ell}\phi\big)(x-t\epsilon\tau y,\xi-\eta)\big(\partial_{x_\ell}\partial^2_{\xi_j\xi_k}\psi\big)(x,\xi-\zeta)\Big)\\
&\qquad+\tfrac{\epsilon^2\tau^2}{16}\int_0^1 dt\int_0^1 ds\sum_{j,k,\ell,m}\big(\partial^2_{x_jx_k}\partial^2_{\xi_\ell\xi_m}\phi\big)(x-t\epsilon\tau y,\xi-\eta)\big(\partial^2_{x_\ell x_m}\partial^2_{\xi_j\xi_k}\psi\big)(x-s\epsilon\tau z,\xi-\zeta)
\end{aligned}
$$

Thus the remainder in the statement of the proposition is explicitely given by:

$$
\mathscr{R}_{\epsilon,\kappa,\tau}(\phi,\psi)(X)=\int\limits_{\Xi\times\Xi}dY\,dZ\,e^{-2i\sigma(Y,Z)}e^{i\,\epsilon\tau^2\Phi_\kappa(x,y,z)}\,\widehat{\mathcal{R}}_{\epsilon,\kappa,\tau}(X,Y,Z,\phi,\psi)\,,\tag{B.46}
$$

with

$$
\begin{aligned}
\widehat{\mathcal{R}}_{\epsilon,\kappa,\tau}(X,Y,Z,\phi,\psi)&=\mathcal{R}^{(1)}_{\epsilon,\kappa,\tau}(X,Y,Z,\phi,\psi)-\kappa\left(\textstyle\int_0^1\Theta^{\epsilon,\kappa}_\tau(x,\tau y,\tau z)dt\right)\times\\
&\times\Big(\sum_\ell R_1\big(\partial_\ell B\big)(x,\epsilon y,\epsilon z)\big[\big(\nabla_\xi\phi\big)(x-\epsilon\tau y,\xi-\eta)\big]\wedge\big[\big(\nabla_\xi\partial_{\xi_\ell}\psi\big)(x-\epsilon\tau z,\xi-\zeta)\big]\\
&\qquad+\sum_\ell R_2\big(\partial_\ell B\big)(x,\epsilon\tau y,\epsilon\tau z)\big[\big(\nabla_\xi\partial_{\xi_\ell}\phi\big)(x-\epsilon\tau y,\xi-\eta)\big]\wedge\big[\big(\nabla_\xi\psi\big)(x-\epsilon\tau z,\xi-\zeta)\big]\Big)\,,
\end{aligned}
$$

with $R_1(\cdot)$ and $R_2(\cdot)$ defined by (B.40) and (B.41). All these terms are of the form considered in Proposition B.9. □

Remark B.27. Starting above with a Taylor expansion of order $N\in\mathbb{N}$, one can prove that for any $N\in\mathbb{N}^*$, there exist $\epsilon_0>0$ and constants $C_j>0$ for $1\le j\le N-1$, such that, for any ϕ, ψ,

$$
\begin{aligned}
\phi_{(\epsilon,1)}\,\sharp^{B_{\epsilon,\kappa}}\psi_{(\epsilon,1)}\;&=\;\big(\phi\,\natural^\epsilon\psi\big)_{(\epsilon,1)}\\
&+\;\sum_{1\le j\le N-1}C_j(i\epsilon)^j\sum_{|\gamma|=j}\left[\Big(\big(\partial_x^\gamma\phi\big)\,\natural^\epsilon\big(\partial_\xi^\gamma\psi\big)\Big)_{(\epsilon,1)}+(-1)^j\Big(\big(\partial_\xi^\gamma\phi\big)\,\natural^\epsilon\big(\partial_x^\gamma\psi\big)\Big)_{(\epsilon,1)}\right]\\
&+\;\epsilon^N\mathscr{R}^N_{\epsilon,\kappa}(\phi,\psi)\,,
\end{aligned}
$$

where $\mathscr{R}^N_{\epsilon,\kappa}(\phi,\psi)$ is bounded uniformly in $\mathscr{S}(\Xi)$ for $(\epsilon,\kappa)\in[0,\epsilon_0]\times[0,1]$.

Corollary B.28. *If $B_{\epsilon,\kappa}(x)$ satisfies* (1.5)*, there exists $\epsilon_0>0$ such that, for any $\epsilon\in[0,\epsilon_0]$, for any $F^\bullet\in S^m_\rho(\Xi)^\bullet$ and $G^\bullet\in S^p_\rho(\Xi)^\bullet$, we have, with the notation from Remark B.20,*

$$
\begin{aligned}
&F^\epsilon\,\sharp^{B_{\epsilon,\kappa}}G^\epsilon\\
&=\;\big(\widetilde{F}^\epsilon\,\natural^\epsilon\widetilde{G}^\epsilon\big)_{(\epsilon,1)}\;+\;\frac{i\epsilon}{2}\sum_{1\le\ell\le2}\Big(\big(\partial_{x_\ell}\widetilde{F}^\epsilon\,\natural^\epsilon\,\partial_{\xi_\ell}\widetilde{G}^\epsilon\big)_{(\epsilon,1)}-\big(\partial_{\xi_\ell}\widetilde{F}^\epsilon\,\natural^\epsilon\,\partial_{x_\ell}\widetilde{G}^\epsilon\big)_{(\epsilon,1)}\Big)\;+\;\epsilon^2\mathscr{R}_{\epsilon,\kappa}(\widetilde{F}^\epsilon,\widetilde{G}^\epsilon)_{(\epsilon,1)}\,,
\end{aligned}
$$

where the family $\{\mathscr{R}_{\epsilon,\kappa}(\widetilde{F}^\epsilon,\widetilde{G}^\epsilon)\}_{\epsilon\in[0,\epsilon_0],\kappa\in[0,1]}$ is a bounded set in $S^{m+p-3\rho}_\rho(\Xi)$.

Proposition B.29. *If $B_{\epsilon,\kappa}(x)$ satisfies* (1.5)*, there exists $\epsilon_0>0$ such that for any $\epsilon\in[0,\epsilon_0]$, any $F\in S^m_\rho(\Xi)^\circ$ and $G\in S^p_\rho(\Xi)^\circ$, if we define*

$$
\big[F,G\big]_{B_{\epsilon,\kappa}}:=F\,\sharp^{B_{\epsilon,\kappa}}G\;-\;G\,\sharp^{B_{\epsilon,\kappa}}F\,,
$$

we have the expansion

$$\big[F,G\big]_{B_{\epsilon,\kappa}}(x,\xi) = -4i\epsilon B_\kappa(\epsilon x)\,\big[\big(\partial_{\xi_1}F\big)(\xi)\big(\partial_{\xi_2}G\big)(\xi) - \big(\partial_{\xi_2}F\big)(\xi)\big(\partial_{\xi_1}G\big)(\xi)\big] + \epsilon^2\widetilde{\mathscr{R}}_{\epsilon,\kappa}(F,G)_{(\epsilon,1)}(x,\xi)\,,$$

where the family $\{\widetilde{\mathscr{R}}_{\epsilon,\kappa}(F,G)\}_{\epsilon\in[0,\epsilon_0],\kappa\in[0,1]}$ *is a bounded set in* $S^{m+p-3\rho}_\rho(\Xi)$*.*
More explicitly we have

$$\begin{aligned}
&\widetilde{\mathscr{R}}_{\epsilon,\kappa}(F,G)(x,\xi)\\
&\quad= -\tfrac{1}{\pi^2}B_\kappa^2(x)\sum_{|\alpha|=2}(\alpha!)^{-1}\times\\
&\qquad\times\int_{\mathcal{X}^*}\int_{\mathcal{X}^*}e^{i(\eta\cdot\zeta^\perp)}\int_0^1 s\,ds\big(\partial_\xi^\alpha F\big)(\xi-s\epsilon^{1/2}b_\kappa(x)\eta)\big(\partial_{\xi^\perp}^\alpha G\big)(\xi-\epsilon^{1/2}b_\kappa(x)\zeta)\,d^2\eta\,d^2\zeta\\
&\quad-\kappa\int_{\Xi\times\Xi}dY\,dZ\,e^{-2i\sigma(Y,Z)}\Big(\int_0^1\Theta_\tau^{\epsilon,\kappa}(x,y,z)\,d\tau\Big)\times\\
&\qquad\times\Big(\sum_\ell R_1\big(\partial_\ell B\big)(x,\epsilon y,\epsilon z)\big[\big(\nabla_\xi F\big)(\xi-\eta)\big]\wedge\big[\big(\nabla_\xi\partial_{\xi_\ell}G\big)(\xi-\zeta)\big]\\
&\qquad\qquad+\sum_\ell R_2\big(\partial_\ell B\big)(x,\epsilon y,\epsilon z)\big[\big(\nabla_\xi\partial_{\xi_\ell}F\big)(\xi-\eta)\big]\wedge\big[\big(\nabla_\xi G\big)(\xi-\zeta)\big]\Big)\,,
\end{aligned}$$

with $R_1(\cdot)$ *and* $R_2(\cdot)$ *defined by* (B.40) *and* (B.41)*.*

Proof. Using Corollary B.28, and taking into account that both symbols do not depend on the configuration space variable $x\in\mathcal{X}$, we obtain:

$$F\,\sharp^{B_{\epsilon,\kappa}}G \;=\; F\,\natural^\epsilon G + \epsilon^2\,\widetilde{\mathscr{R}}_{\epsilon,\kappa}(F,G)_{(\epsilon,1)}\,.$$

Using one of the two formulas for r_N in (B.44) with $N=2$ we obtain the desired result. □

Proposition B.30. *If* $B_{\epsilon,\kappa}(x)$ *satisfies* (1.5)*, there exists* $\epsilon_0>0$ *such that for any* $\epsilon\in[0,\epsilon_0]$ *and any* $F\in S^m_\rho(\Xi)^\circ$ *and* $G\in S^p_\rho(\Xi)^\circ$ *with disjoint supports,*

$$F\,\sharp^{B_{\epsilon,\kappa}}G \;=\; \epsilon^2\,\widetilde{\mathscr{R}}_{\epsilon,\kappa}(F,G)_{(\epsilon,1)}\,,$$

where $\{\widetilde{\mathscr{R}}_{\epsilon,\kappa}(F,G)\}_{\epsilon\in[0,\epsilon_0]}$ *belongs to* $S^{m+p-3\rho}_\rho(\Xi)$ *uniformly with respect to* $(\epsilon,\kappa)\in[0,\epsilon_0]\times[0,1]$*.*
Explicitly:

$$\begin{aligned}
&\widetilde{\mathscr{R}}_{\epsilon,\kappa}(F,G)(x,\xi)\\
&\quad= -\tfrac{1}{\pi^2}B_\kappa^2(x)\sum_{|\alpha|=2}(\alpha!)^{-1}\times\\
&\qquad\times\int_{\mathcal{X}^*}\int_{\mathcal{X}^*}e^{i(\eta\cdot\zeta^\perp)}\int_0^1 s\,ds\big(\partial_\xi^\alpha F\big)(\xi-s\epsilon^{1/2}b_\kappa(x)\eta)\big(\partial_{\xi^\perp}^\alpha G\big)(\xi-\epsilon^{1/2}b_\kappa(x)\zeta)\,d^2\eta\,d^2\zeta\\
&\quad-\kappa\int_{\Xi\times\Xi}dY\,dZ\,e^{-2i\sigma(Y,Z)}\Big(\int_0^1\Theta_\tau^{\epsilon,\kappa}(x,y,z)d\tau\Big)\times\\
&\qquad\times\Big(\sum_\ell R_1\big(\partial_\ell B\big)(x,\epsilon y,\epsilon z)\big[\big(\nabla_\xi F\big)(\xi-\eta)\big]\wedge\big[\big(\nabla_\xi\partial_{\xi_\ell}G\big)(\xi-\zeta)\big]\\
&\qquad\qquad+\sum_\ell R_2\big(\partial_\ell B\big)(x,\epsilon y,\epsilon z)\big[\big(\nabla_\xi\partial_{\xi_\ell}F\big)(\xi-\eta)\big]\wedge\big[\big(\nabla_\xi G\big)(\xi-\zeta)\big]\Big)\,,
\end{aligned}$$

with $R_1(\cdot)$ *and* $R_2(\cdot)$ *defined by* (B.40) *and* (B.41)*.*

The proof is quite similar to the one of Proposition B.29.

Proposition B.31. *Given a magnetic field of the form* (1.5)*, for any* $N\in\mathbb{N}^*$ *and any* $\epsilon\in[0,\epsilon_0]$ *there exists a family of continuous bilinear maps*

$$\mathscr{M}_N^{\epsilon,\kappa}: S^m_\rho(\Xi)^\circ\times S^p_\rho(\Xi)^\circ\to S^{m+p}_\rho(\Xi)$$

for any $m \in \mathbb{R}$, $p \in \mathbb{R}$ and $\rho \in [0,1]$, that are uniformly bounded for $\epsilon \in [0,\epsilon_0]$ and such that the frozen magnetic product (B.43) *satisfies the following relation:*

$$\big(\phi \natural^\epsilon \psi\big)(x,\xi) = \phi(\xi)\psi(\xi) + \sum_{1\le p\le N-1} \big(-2i\epsilon)^p B^\kappa(x)\big)^p \sum_{|\alpha|=p} (\alpha!)^{-1}\big(\partial_\xi^\alpha \phi\big)(\xi)\big(\partial_{\xi^\perp}^\alpha \psi\big)(\xi)$$
$$+\ \epsilon^N B^\kappa(x)\big)^N \sum_{|\alpha|=N} (\alpha!)^{-1} \mathscr{M}_N^{\epsilon,\kappa}\big(\partial_\xi^\alpha \phi, \partial_{\xi^\perp}^\alpha \psi\big)\big)\,.$$

Proof. From formula (B.43), after a Taylor expansion up to order $N \in \mathbb{N}^*$ and the usual integration by parts argument (using the exponential factor $\exp\{-2i\sigma(Y,Z)\}$) we obtain:

$$\big(\phi \natural^\epsilon \psi\big)(x,\xi) = \phi(\xi)\psi(\xi) + \sum_{1\le p\le N-1} (-2i\epsilon)^p B^\kappa(x)^p \sum_{|\alpha|=p} (\alpha!)^{-1}\big(\partial_\xi^\alpha \phi\big)(\xi)\big(\partial_{\xi^\perp}^\alpha \psi\big)(\xi)$$
$$+ \left(\frac{1}{2\pi}\right)^2 (-2i\epsilon)^N B^\kappa(x)^N \sum_{|\alpha|=N} (\alpha!)^{-1} \int_0^1 s^{N-1}\, ds \int_{\mathcal{X}^*}\int_{\mathcal{X}^*} e^{i(\eta\cdot\zeta^\perp)}$$
$$\times \big(\partial_\xi^\alpha \phi\big)(\xi - s\epsilon^{1/2} b_\kappa(x)\eta)\big(\partial_{\xi^\perp}^\alpha \psi\big)(\xi - \epsilon^{1/2} b_\kappa(\epsilon x)\zeta)\, d^2\eta\, d^2\zeta\,.$$

Thus, if we define

$$\mathscr{M}_N^{\epsilon,\kappa}\big(f,g\big)(x,\xi) := \frac{(-2i)^N}{(2\pi)^2} \int_0^1 s^{N-1}\, ds \ \times$$
$$\times \int_{\mathcal{X}^*}\int_{\mathcal{X}^*} e^{i(\eta\cdot\zeta^\perp)} f(\xi - s\epsilon^{1/2} b_\kappa(x)\eta)\ g(\xi - \epsilon^{1/2} b_\kappa(x)\zeta)\, d^2\eta\, d^2\zeta\,, \tag{B.47}$$

we notice that $\big(\partial_x^\alpha \partial_\xi^\beta \mathscr{M}_N^{\epsilon,\kappa}(f,g)\big)(x,\xi)$ is a finite sum of terms of the form

$$F_{\gamma_1}^B(x) F_{\gamma_2}^B(x) \int_0^1 s^{N-1+|\gamma_1|}\, ds \int_{\mathcal{X}^*}\int_{\mathcal{X}^*} e^{i(\eta\cdot\zeta^\perp)} \eta^{\gamma_1}\zeta^{\gamma_2}\, d^2\eta\, d^2\zeta$$
$$\times \big(\partial_\xi^{\gamma_1+\beta_1} f\big)(\xi - s\epsilon^{1/2} b_\kappa(x)\eta)\ \big(\partial_\xi^{\gamma_2+\beta_2} g\big)(\xi - \epsilon^{1/2} b_\kappa(x)\zeta)$$
$$= (-i)^{|\gamma+\gamma_2|}\epsilon^{|\gamma_1+\gamma_2|/2} \widetilde{F}_{\gamma_1,\gamma_2}^B(x) \int_0^1 s^{N-1+|\gamma_1|}\, ds \int_{\mathcal{X}^*}\int_{\mathcal{X}^*} e^{i(\eta\cdot\zeta^\perp)}\, d^2\eta\, d^2\zeta$$
$$\times \big(\partial_{\xi^\perp}^{\gamma_2} \partial_\xi^{\gamma_1+\beta_1} f\big)(\xi - s\epsilon^{1/2} b_\kappa(x)\eta)\ \big(\partial_{\xi^\perp}^{\gamma_1} \partial_\xi^{\gamma_2+\beta_2} g\big)(\xi - \epsilon^{1/2} b_\kappa(x)\zeta)\,,$$

with $|\gamma_1+\gamma_2| \le |\alpha|$ and $\beta_1+\beta_2 = \beta$ and the functions F_γ^B and $\widetilde{F}_{\gamma_1,\gamma_2}^B$ of class $BC^\infty(\mathcal{X})$.

Then we have the following equality:

$$\int_{\mathcal{X}^*}\int_{\mathcal{X}^*} e^{i(\eta\cdot\zeta^\perp)}\big(\partial_\xi^\alpha \phi\big)(\xi - s\epsilon^{1/2} b_\kappa(x)\eta)\big(\partial_\xi^\beta \psi\big)(\xi - \epsilon^{1/2} b_\kappa(x)\zeta)\, d^2\eta\, d^2\zeta$$
$$= \int_{\mathcal{X}^*} <\eta>^{-N_1} \sum_{|a|\le N_2} \mathcal{P}_a(\eta)\Big[\partial_\eta^a \big(\partial_\xi^\alpha \phi\big)(\xi - s\epsilon^{1/2} b_\kappa(x)\eta)\Big]\, d^2\eta \times$$
$$\times \int_{\mathcal{X}^*} e^{i(\eta\cdot\zeta^\perp)} <\zeta>^{-N_2} \big(1+\Delta_\zeta\big)^{N_1/2}\Big[\big(\partial_\xi^\beta \psi\big)(\xi - \epsilon^{1/2} b_\kappa(x)\zeta)\Big]\, d^2\zeta\,,$$

where the $\mathcal{P}_a(\cdot)$ are bounded functions on $\mathcal{X}^*$. We notice that:

$$\int_{\mathcal{X}^*}\int_{\mathcal{X}^*} e^{i(\eta\cdot\zeta^\perp)} \phi(\xi-\tau\eta)\psi(\xi-\tau\zeta)\, d^2\eta\, d^2\zeta$$
$$= \int_{\mathcal{X}^*} \big(1+\Delta_\eta\big)^{N_2/2}\big(<\eta>^{-N_1}\ \phi(\xi-\tau\eta)\big)\, d^2\eta$$
$$\times \int_{\mathcal{X}^*} e^{i(\eta\cdot\zeta^\perp)} <\zeta>^{-N_2} \big(\big(1+\Delta_\zeta\big)^{N_1/2}\psi\big)(\xi-\tau\zeta)\, d^2\zeta \tag{B.48}$$
$$= \int_{\mathcal{X}^*} <\eta>^{-N_1} \sum_{|a|\le N_2} \mathcal{P}_a(\eta)\big(\partial_\eta^a \phi\big)(\xi-\tau\eta)\, d^2\eta$$
$$\times \int_{\mathcal{X}^*} e^{i(\eta\cdot\zeta^\perp)} <\zeta>^{-N_2} \big(\big(1+\Delta_\zeta\big)^{N_1/2}\psi\big)(\xi-\tau\zeta)\, d^2\zeta\,,$$

where the $\mathcal{P}_a(\cdot)$ are bounded functions on $\mathcal{X}^*$ and $N_1 > 2$, $N_2 > 2$.

Finally these arguments allow to prove estimates of the following type:

$$\nu_{n,m}^{m+p,\rho}\big(\mathscr{M}_N^{\epsilon,\kappa}(f,g)\big) \;\le\; C\,\nu_{0,q}^{m,\rho}(f)\,\nu_{0,q}^{p,\rho}(g)\,,$$

with $q > 2+n+m$ and some constant $C>0$ that may depend on B and on the three seminorms but not on $\epsilon \in [0,\epsilon_0]$. □

B.6 Control of some Γ_*-indexed series of symbols

Consider a symbol $\varphi \in S^{-\infty}(\Xi)$ and let us study the convergence of the series

$$\Phi := \sum_{\gamma^* \in \Gamma_*} \tau_{\gamma^*}\big(\varphi\big)\,.$$

For any $N \in \mathbb{N}$ let us introduce

$$\Gamma_*^N := \{\gamma^* \in \Gamma_* \mid |\gamma^*| \le N\} \;\text{ and }\; \Phi_N := \sum_{\gamma^* \in \Gamma_*^N} \tau_{\gamma^*}\big(\varphi\big)\,.$$

Lemma B.32. *With the above notation, for any symbol $\varphi \in S^{-\infty}(\Xi)$ the sequence $\{\Phi_N\}_{N\in\mathbb{N}}$ in $S^{-\infty}(\Xi)$ is weakly convergent in $\mathscr{S}'(\Xi)$ and the limit Φ is a Γ_*-periodic C^∞-function on Ξ. The sequence also converges for the norms $\nu_{p,q}^{-m,0}$ with any $m>2$ and $(p,q)\in\mathbb{N}^2$ and for any $(p,q)\in\mathbb{N}^2$ there exists $C>0$ such that, for all $\varphi \in S^{-\infty}(\Xi)$,*

$$\nu_{p,q}^{0,0}(\Phi_N) \le C\,\nu_{p,q}^{0,0}(\varphi)\,,\; \forall N \in \mathbb{N}\,.$$

Proof. We clearly have, for any $m \in \mathbb{R}_+$,

$$\sup_{(x,\xi)\in\Xi} <\xi>^m \big|\big(\partial_x^a\partial_\xi^b\varphi\big)(x,\xi)\big| \le \nu_{|a|,|b|}^{-m,0}(\varphi)\,,$$

so that, for any $N_2 \ge N_1$ in $\mathbb{N}$,

$$\begin{aligned}
|\big(\partial_x^a\partial_\xi^b\Phi_{N_1}\big)(x,\xi) - \big(\partial_x^a\partial_\xi^b\Phi_{N_2}\big)(x,\xi)| \;&\le\; \sum_{\gamma^*\in\Gamma_*^{N_2}\setminus\Gamma_*^{N_1}} \Big|\tau_{\gamma^*}\big[\big(\partial_x^a\partial_\xi^b\varphi\big)\big](x,\xi)\Big| \\
&\le\; \nu_{|a|,|b|}^{-m,0}(\varphi)\Big(\sup_{\xi\in\mathcal{X}^*}\sum_{\gamma^*\in\Gamma_*^{N_2}\setminus\Gamma_*^{N_1}} <\xi+\gamma^*>^{-m}\Big) \\
&\le\; C\nu_{|a|,|b|}^{-m,0}(\varphi) <N_1>^{-s},
\end{aligned}$$

for any $m > 2+s$ with $s>0$.
Thus the weak convergence in $\mathscr{S}'(\mathcal{V})$ follows easily and also the other conclusions of the lemma by some standard arguments. □

Proposition B.33. *With the notation from Lemma B.32, suppose that we have a family of symbols $\{\varphi^\epsilon\}_{\epsilon\in[0,\epsilon_0]} \in S^{-\infty}(\Xi)^\bullet$ and a magnetic field given by (1.5). Then $\{\Phi^\epsilon\}_{\epsilon\in[0,\epsilon_0]} \in S_0^0(\Xi)^\bullet$ and the sequence $\{\mathfrak{Op}^{\epsilon,\kappa}(\Phi_N^\epsilon)\}_{N\in\mathbb{N}}$ converges strongly to $\mathfrak{Op}^{\epsilon,\kappa}(\Phi^\epsilon)$ in $\mathcal{L}\big(L^2(\mathcal{X})\big)$ uniformly for $(\epsilon,\kappa)\in[0,\epsilon_0]\times[0,\kappa_0]$. Moreover, the application*

$$S^{-\infty}(\Xi) \ni \varphi^\epsilon \;\mapsto\; \Phi^\epsilon \in \big(S_0^0(\Xi), \|\cdot\|_{B_{\epsilon,\kappa}}\big)$$

is continuous uniformly for $(\epsilon,\kappa)\in[0,\epsilon_0]\times[0,\kappa_0]$.

Proof. We write

$$\mathfrak{Op}^{\epsilon,\kappa}(\Phi_N) \;=\; \sum_{|\gamma^*|\le N} \mathfrak{Op}^{\epsilon,\kappa}\big(\tau_{\gamma^*}[\varphi^\epsilon]\big)\,,$$

and we introduce the following simpler notation :

$$X_{\gamma^*} := \mathfrak{Op}^{\epsilon,\kappa}\big(\tau_{\gamma^*}[\varphi^\epsilon]\big)\,,\quad \widetilde{X}_\infty := \mathfrak{Op}^{\epsilon,\kappa}(\Phi),\quad \widetilde{X}_N := \mathfrak{Op}^{\epsilon,\kappa}(\Phi_N)\,,$$

so that $\widetilde{X}_N = \sum_{\gamma\in\Gamma_*^N} X_{\gamma^*}$ and we use the Cotlar-Stein Lemma. For this we need to verify some estimates. Let us first consider the products (using also Remark B.20)

$$\|X^*_{\beta^*}X_{\gamma^*}\|_{\mathcal{L}(\mathcal{H})} = \left\|\overline{\varphi^\epsilon}\,\sharp^{B_{\epsilon,\kappa}}\left(\tau_{(\gamma^*-\beta^*)}\varphi^\epsilon\right)\right\|_{B_{\epsilon,\kappa}} = \left\|\overline{\widetilde{\varphi}^\epsilon_{(\epsilon,1)}}\,\sharp^{B_{\epsilon,\kappa}}\left(\tau_{(\gamma^*-\beta^*)}\widetilde{\varphi}^\epsilon_{(\epsilon,1)}\right)\right\|_{B_{\epsilon,\kappa}}, \tag{B.49}$$

and use Proposition B.26 and (B.43) in order to obtain that

$$\begin{aligned}
&\overline{\widetilde{\varphi}^\epsilon}_{(\epsilon,1)}\sharp^{B_{\epsilon,\kappa}}\left(\tau_{(\gamma^*-\beta^*)}\widetilde{\varphi}^\epsilon\right)_{(\epsilon,1)}\\
&\quad= \overline{\widetilde{\varphi}^\epsilon}_{(\epsilon,1)}\,\natural^\epsilon\left(\tau_{(\gamma^*-\beta^*)}\widetilde{\varphi}^\epsilon\right)_{(\epsilon,1)}\\
&\qquad+\epsilon\,\tfrac{i}{2}\sum_{1\le l\le 2}\Big(\left(\partial_{x_l}\overline{\widetilde{\varphi}^\epsilon}\right)_{(\epsilon,1)}\natural^\epsilon\left(\tau_{(\gamma^*-\beta^*)}\left(\partial_{\xi_l}\widetilde{\varphi}^\epsilon\right)\right)_{(\epsilon,1)} - \left(\partial_{\xi_l}\overline{\widetilde{\varphi}^\epsilon}\right)_{(\epsilon,1)}\natural^\epsilon\left(\tau_{(\gamma^*-\beta^*)}\left(\partial_{x_l}\widetilde{\varphi}^\epsilon\right)\right)_{(\epsilon,1)}\Big)\\
&\qquad+\epsilon^2\mathscr{R}_{\epsilon,\kappa}\left(\overline{\widetilde{\varphi}^\epsilon},\left(\tau_{(\gamma^*-\beta^*)}\widetilde{\varphi}^\epsilon\right)\right)_{(\epsilon,1)}.
\end{aligned}$$

For ϕ and ψ in $S^{-\infty}(\Xi)$ we can repeat the arguments in the proof of Proposition B.31 and obtain:

$$\begin{aligned}
\left(\phi_{(\epsilon,1)}\natural^\epsilon\left(\tau_{\alpha^*}\psi\right)_{(\epsilon,1)}\right)(x,\xi) &= \left(\frac{1}{2\pi}\right)^2\int_{\mathcal{X}^*}\int_{\mathcal{X}^*} e^{-i(\eta^\perp\cdot\zeta))}\phi(\epsilon x,\xi-\epsilon^{1/2}b_\kappa(\epsilon x)\,\eta)\times\\
&\qquad\times\psi(\epsilon x,\xi+\alpha^*-\epsilon^{1/2}b_\kappa(\epsilon x)\,\zeta)\,d^2\eta\,d^2\zeta\\
&= \int_{\mathcal{X}^*} <\eta>^{-N_1}\sum_{|a|\le N_2}\mathcal{P}_a(\eta)\partial_\eta^a\phi\Big[(\epsilon x,\xi-s\epsilon^{1/2}b_\kappa(\epsilon x)\eta)\Big]\,d^2\eta\,\times\\
&\qquad\times\int_{\mathcal{X}^*} e^{i(\eta\cdot\zeta^\perp)} <\zeta>^{-N_2}\ \times\\
&\qquad\times\left(1+\Delta_\zeta\right)^{N_1/2}\Big[\psi(\epsilon x,\xi+\alpha^*-\epsilon^{1/2}b_\kappa(\epsilon x)\zeta)\Big]\,d^2\zeta\,,
\end{aligned}\tag{B.50}$$

where the $\mathcal{P}_a(\cdot)$ are bounded functions on $\mathcal{X}^*$ and $N_1 = N_2 > 2$.

Thus we obtain, for any $\alpha^*\in\Gamma_*$ and for any $N\in\mathbb{N}$, the estimate:

$$\begin{aligned}
&<\alpha^*>^N\ \nu_{0,0}^{0,0}\left(\phi_{(\epsilon,1)}\natural^\epsilon\left(\tau_{\alpha^*}\psi\right)_{(\epsilon,1)}\right)\\
&\quad= \sup_{(x,\xi)\in\Xi} <\alpha^*>^N\left|\left(\phi_{(\epsilon,1)}\natural^\epsilon\left(\tau_{\alpha^*}\psi\right)_{(\epsilon,1)}\right)(x,\xi)\right|\\
&\quad\le C_N\left|\int_{\mathcal{X}^*} <\eta>^{-N_1}\left\langle\xi-s\epsilon^{1/2}b_\kappa(\epsilon x)\eta\right\rangle^N\sum_{|a|\le N_2}\mathcal{P}_a(\eta)\partial_\eta^a\Big[\phi(\epsilon x,\xi-s\epsilon^{1/2}b_\kappa(\epsilon x)\eta)\Big]\,d^2\eta\,\times\right.\\
&\qquad\times\int_{\mathcal{X}^*} e^{i(\eta\cdot\zeta^\perp)} <\zeta>^{-N_2}\left\langle\xi+\alpha^*-s\epsilon^{1/2}b_\kappa(\epsilon x)\eta\right\rangle^N\times\\
&\qquad\left.\times\left(1+\Delta_\zeta\right)^{N_1/2}\Big[\psi(\epsilon x,\xi+\alpha^*-\epsilon^{1/2}b_\kappa(\epsilon x)\zeta)\Big]\,d^2\zeta\right|\\
&\quad\le\ C_N\,\nu_{0,0}^{-N,0}(\phi)\,\nu_{0,0}^{-N,0}(\psi)\,.
\end{aligned}$$

Noticing that $\epsilon^{-|a|}\partial_x^a\partial_\xi^b\left(\phi_{(\epsilon,1)}\natural^\epsilon\left(\tau_{\alpha^*}\psi\right)_{(\epsilon,1)}\right)$ is an oscillating integral of the same type as the expression (B.50) with ϕ and ψ replaced by some derivatives of them, we conclude that we have similar estimates for all the seminorms defining the Fréchet topology. Choosing now $N>2$ large enough, we verify the hypothesis of the Cotlar-Stein Lemma and obtain the conclusion of the proposition. □

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