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HIGH-FREQUENCY APPROXIMATION OF THE INTERIOR DIRICHLET-TO-NEUMANN MAP AND APPLICATIONS TO THE TRANSMISSION EIGENVALUES

GEORGI VODEV

Abstract. We study the high-frequency behavior of the Dirichlet-to-Neumann map for an arbitrary compact Riemannian manifold with a non-empty smooth boundary. We show that far from the real axis it can be approximated by a simpler operator. We use this fact to get new results concerning the location of the transmission eigenvalues on the complex plane. In some cases we obtain optimal transmission eigenvalue-free regions.

1. Introduction and statement of results

Let $(X, \mathcal{G})$ be a compact Riemannian manifold of dimension $n = \dim X \geq 2$ with a non-empty smooth boundary ∂X and let Δ_X denote the negative Laplace-Beltrami operator on $(X, \mathcal{G})$. Denote also by $\Delta_{\partial X}$ the negative Laplace-Beltrami operator on $(\partial X, \mathcal{G}_0)$, which is a Riemannian manifold without boundary of dimension $n-1$, where $\mathcal{G}_0$ is the Riemannian metric on ∂X induced by the metric $\mathcal{G}$. Given a function $f \in H^{m+1}(\partial X)$, let u solve the equation

$$\begin{cases} \left(\Delta_X + \lambda^2 n(x)\right) u = 0 & \text{in} \quad X, \\ u = f & \text{on} \quad \partial X, \end{cases} \tag{1.1}$$

where $\lambda \in \mathbf{C}$, $1 \ll |\operatorname{Im}\lambda| \ll \operatorname{Re}\lambda$ and $n \in C^\infty(\overline{X})$ is a strictly positve function. Then the Dirichlet-to-Neumann (DN) map

$$\mathcal{N}(\lambda; n) : H^{m+1}(\partial X) \to H^m(\partial X)$$

is defined by

$$\mathcal{N}(\lambda; n) f := \partial_\nu u|_{\partial X}$$

where ν is the unit inner normal to ∂X. One of our goals in the present paper is to approximate the operator $\mathcal{N}(\lambda; n)$ when $n(x) \equiv 1$ in X by a simpler one of the form $p(-\Delta_{\partial X})$ with a suitable complex-valued function $p(\sigma)$, $\sigma \geq 0$. More precisely, the function p is defined as follows

$$p(\sigma) = \sqrt{\sigma - \lambda^2}, \quad \operatorname{Re} p < 0.$$

Our first result is the following

Theorem 1.1. *Let $0 < \epsilon < 1$ be arbitrary. Then, for every $0 < \delta \ll 1$ there are constants $C_\delta, C_{\epsilon,\delta} > 1$ such that we have*

$$\|\mathcal{N}(\lambda; 1) - p(-\Delta_{\partial X})\|_{L^2(\partial X) \to L^2(\partial X)} \leq \delta |\lambda| \tag{1.2}$$

for $C_\delta \leq |\operatorname{Im}\lambda| \leq (\operatorname{Re}\lambda)^{1-\epsilon}$, $\operatorname{Re}\lambda \geq C_{\epsilon,\delta}$.

Note that this result has been previously proved in [11] in the case when X is a ball in $\mathbf{R}^d$ and the metric being the Euclidean one. In fact, in this case we have a better approximation of the operator $\mathcal{N}(\lambda; 1)$. In the general case when the function n is arbitrary the DN map can be approximated by $h - \Psi$DOs, where $0 < h \ll 1$ is a semi-classical parameter such that

$\mathrm{Re}\,(h\lambda)^2 = 1$. To describe this more precisely let us introduce the class of symbols $S^k_\delta(\partial X)$, $0 \le \delta < 1/2$, as being the set of all functions $a(x',\xi') \in C^\infty(T^*\partial X)$ satisfying the bounds

$$\left|\partial^\alpha_{x'}\partial^\beta_{\xi'}a(x',\xi')\right| \le C_{\alpha,\beta}h^{-\delta(|\alpha|+|\beta|)}\langle\xi'\rangle^{k-|\beta|}$$

for all multi-indices α and β with constants $C_{\alpha,\beta}$ independent of h. We let $\mathrm{OP}S^k_\delta(\partial X)$ denote the set of all $h-\Psi$DOs, $\mathrm{Op}_h(a)$, with symbol $a \in S^k_\delta(\partial X)$, defined as follows

$$(\mathrm{Op}_h(a)f)\,(x') = (2\pi h)^{-n+1}\int_{T^*\partial X} e^{-\frac{i}{h}\langle x'-y',\xi'\rangle}a(x',\xi')f(y')dy'd\xi'.$$

It is well-known that for this class of symbols we have a very nice pseudo-differential calculus (e.g. see [2]). It was proved in [15] that for $|\mathrm{Im}\,\lambda| \ge |\lambda|^{1/2+\epsilon}$, $0<\epsilon\ll 1$, the operator $h\mathcal{N}(\lambda;n)$ is an $h-\Psi$DO of class $\mathrm{OP}S^1_{1/2-\epsilon}(\partial X)$ with a principal symbol

$$\rho(x',\xi') = \sqrt{r_0(x',\xi') - (h\lambda)^2 n_0(x')}, \quad \mathrm{Re}\,\rho < 0, \quad n_0 := n|_{\partial X},$$

$r_0 \ge 0$ being the principal symbol of $-\Delta_{\partial X}$. Note that it is still possible to construct a semiclassical parametrix for the operator $h\mathcal{N}(\lambda;n)$ when $|\mathrm{Im}\,\lambda| \ge |\lambda|^\epsilon$, $0<\epsilon\ll 1$, if one supposes that the boundary ∂X is strictly concave (see [16]). This construction, however, is much more complex and one has to work with symbols belonging to much worse classes near the glancing region $\Sigma = \{(x',\xi') \in T^*\partial X : r_\sharp(x',\xi') = 1\}$, where $r_\sharp = n_0^{-1}r_0$. On the other hand, it seems that no parametrix construction near Σ is possible in the important region $1 \ll Const \le |\mathrm{Im}\,\lambda| \le |\lambda|^\epsilon$. Therefore, in the present paper we follow a different approach which consists of showing that, for arbitrary manifold X, the norm of the operator $h\mathcal{N}(\lambda;n)\mathrm{Op}_h(\chi^0_\delta)$ is $\mathcal{O}(\delta)$ for every $0<\delta\ll 1$ independent of λ, provided $|\mathrm{Im}\,\lambda|$ and $\mathrm{Re}\,\lambda$ are taken big enough (see Proposition 3.3 below). Here the function $\chi^0_\delta \in C^\infty_0(T^*\partial X)$ is supported in $\{(x',\xi') \in T^*\partial X : |r_\sharp(x',\xi')-1| \le 2\delta^2\}$ and $\chi^0_\delta = 1$ in $\{(x',\xi') \in T^*\partial X : |r_\sharp(x',\xi')-1| \le \delta^2\}$ (see Section 3 for the precise definition of χ^0_δ). Theorem 1.1 is an easy consequence of the following semi-classical version.

Theorem 1.2. *Let $0<\epsilon<1$ be arbitrary. Then, for every $0<\delta\ll 1$ there are constants $C_\delta, C_{\epsilon,\delta} > 1$ such that we have*

$$\left\|h\mathcal{N}(\lambda;n) - \mathrm{Op}_h(\rho(1-\chi^0_\delta) + hb)\right\|_{L^2(\partial X)\to H^1_h(\partial X)} \le C\delta \tag{1.3}$$

for $C_\delta \le |\mathrm{Im}\,\lambda| \le (\mathrm{Re}\,\lambda)^{1-\epsilon}$, $\mathrm{Re}\,\lambda \ge C_{\epsilon,\delta}$, where $C>0$ is a constant independent of λ and δ, and $b \in S^0_0(\partial X)$ is independent of λ and the function n.

Here $H^1_h(\partial X)$ denotes the Sobolev space equipped with the semi-classical norm (see Section 3 for the precise definition). Thus, to prove (1.3) (resp. (1.2)) it suffices to construct semi-classical parametrix outside a δ^2- neighbourhood of Σ, which turns out to be much easier and can be done for an arbitrary X. In the elliptic region $\{(x',\xi') \in T^*\partial X : r_\sharp(x',\xi') \ge 1+\delta^2\}$ we use the same parametrix construction as in [15] with slight modifications. In the hyperbolic region $\{(x',\xi') \in T^*\partial X : r_\sharp(x',\xi') \le 1-\delta^2\}$, however, we need to improve the parametrix construction of [15]. We do this in Section 4 for $1 \ll Const \le |\mathrm{Im}\,\lambda| \le |\lambda|^{1-\epsilon}$. Then we show that the difference between the operator $h\mathcal{N}(\lambda;n)$ microlocalized in the hyperbolic region and its parametrix is $\mathcal{O}\left(e^{-\beta|\mathrm{Im}\,\lambda|}\right) + \mathcal{O}_{\epsilon,M}\left(|\lambda|^{-M}\right)$, where $\beta>0$ is some constant and $M \ge 1$ is arbitrary. So, we can do it small by taking $|\mathrm{Im}\,\lambda|$ and $|\lambda|$ big enough.

This kind of approximations of the DN map are important for the study of the location of the complex eigenvalues associated to boundary-value problems with dissipative boundary conditions (e.g. see [9]). In particular, Theorem 1.2 leads to significant improvements of the eigenvalue-free regions in [9]. In the present paper we use Theorem 1.2 to study the location of the interior transmission egenvalues (see the next section). We improve most of the results

in [15] as well as those in [11], [16], and provide a simpler proof. In some cases we get optimal transmission eigenvalue-free regions (see Theorem 2.1). Note that for the applications in the anisotropic case it suffices to have an weaker analogue of the estimate (1.3) with the space H^1_h replaced by L^2, in which case the operator $\mathrm{Op}_h(hb)$ becomes negligible. In the isotropic case, however, it is essential to have in (1.3) the space H^1_h and that the function b does not depend on the refraction index n.

Note finally that Theorem 1.2 can be also used to study the location of the resonances for the exterior transmission problems considered in [1] and [3]. For example, it allows to simplify the proof of the resonance-free regions in [1] and to extend it to more general boundary conditions.

2. Applications to the transmission eigenvalues

Let $\Omega \subset \mathbf{R}^d$, $d \geq 2$, be a bounded, connected domain with a C^∞ smooth boundary $\Gamma = \partial\Omega$. A complex number $\lambda \in \mathbf{C}$, $\mathrm{Re}\,\lambda \geq 0$, will be said to be a transmission eigenvalue if the following problem has a non-trivial solution:

$$\begin{cases} \left(\nabla c_1(x)\nabla + \lambda^2 n_1(x)\right) u_1 = 0 & \text{in} \quad \Omega, \\ \left(\nabla c_2(x)\nabla + \lambda^2 n_2(x)\right) u_2 = 0 & \text{in} \quad \Omega, \\ u_1 = u_2, \;\; c_1\partial_\nu u_1 = c_2 \partial_\nu u_2 & \text{on} \quad \Gamma, \end{cases} \tag{2.1}$$

where ν denotes the Euclidean unit inner normal to Γ, $c_j, n_j \in C^\infty(\overline{\Omega})$, $j = 1, 2$ are strictly positive real-valued functions. We will consider two cases:

$$c_1(x) \equiv c_2(x) \equiv 1 \quad \text{in} \quad \Omega, \quad n_1(x) \neq n_2(x) \quad \text{on} \quad \Gamma, \quad \text{(isotropic case)} \tag{2.2}$$

$$(c_1(x) - c_2(x))(c_1(x)n_1(x) - c_2(x)n_2(x)) \neq 0 \quad \text{on} \quad \Gamma. \quad \text{(anisotropic case)} \tag{2.3}$$

In Section 6 we will prove the following

Theorem 2.1. *Assume either the condition (2.2) or the condition*

$$(c_1(x) - c_2(x))(c_1(x)n_1(x) - c_2(x)n_2(x)) < 0 \quad \textit{on} \quad \Gamma. \tag{2.4}$$

Then there exists a constant $C > 0$ such that there are no transmission eigenvalues in the region

$$\{\lambda \in \mathbf{C} : \mathrm{Re}\,\lambda > 1, \; |\mathrm{Im}\,\lambda| \geq C\}. \tag{2.5}$$

Remark. It is proven in [15] that under the condition (2.2) (as well as the condition (2.6) below) there exists a constant $\widetilde{C} > 0$ such that there are no transmission eigenvalues in the region

$$\left\{\lambda \in \mathbf{C} : 0 \leq \mathrm{Re}\,\lambda \leq 1, \; |\mathrm{Im}\,\lambda| \geq \widetilde{C}\right\}.$$

This is no longer true under the condition (2.4) in which case there exist infinitely many transmission eigenvalues very close to the imaginary axis.

Note that the eigenvalue-free region (2.5) is optimal and cannot be improved in general. Indeed, it follows from the analysis in [7] (see Section 4) that in the isotropic case when the domain Ω is a ball and the refraction indices n_1 and n_2 constant, there may exist infinitely many transmission eigenvalues whose imaginary parts are bounded from below by a positive constant. Note also that the above result has been previously proved in [11] in the case when the domain Ω is a ball and the coefficients constant. In the isotropic case the eigenvalue-free region (2.5) has been also obtained in [14] when the dimension is one. In the general case of arbitrary domains transmission eigenvalue-free regions have been previously proved in [5], [6]

and [12] (isotropic case), [15] and [16] (both cases). For example, it has been proved in [15] that, under the conditions (2.2) and (2.4), there are no transmission eigenvalues in

$$\left\{\lambda \in \mathbf{C} : \operatorname{Re}\lambda > 1,\ |\operatorname{Im}\lambda| \geq C_\varepsilon\,(\operatorname{Re}\lambda)^{\frac{1}{2}+\varepsilon}\right\}, \quad C_\varepsilon > 0,$$

for every $0 < \varepsilon \ll 1$. This eigenvalue-free region has been improved in [16] under an additional strict concavity condition on the boundary Γ to the following one

$$\{\lambda \in \mathbf{C} : \operatorname{Re}\lambda > 1,\ |\operatorname{Im}\lambda| \geq C_\varepsilon\,(\operatorname{Re}\lambda)^{\varepsilon}\}, \quad C_\varepsilon > 0,$$

for every $0 < \varepsilon \ll 1$. When the function in the left-hand side of (2.3) is strictly positive, parabolic eigenvalue-free regions have been proved in [15] for arbitrary domains, which however are worse than the eigenvalue-free regions we have under the conditions (2.2) and (2.4). In Section 7 we will prove the following

Theorem 2.2. *Assume the conditions*

$$(c_1(x) - c_2(x))(c_1(x)n_1(x) - c_2(x)n_2(x)) > 0 \quad on \quad \Gamma \tag{2.6}$$

and

$$\frac{n_1(x)}{c_1(x)} \neq \frac{n_2(x)}{c_2(x)} \quad on \quad \Gamma. \tag{2.7}$$

Then there exists a constant $C > 0$ such that there are no transmission eigenvalues in the region

$$\{\lambda \in \mathbf{C} : \operatorname{Re}\lambda > 1,\ |\operatorname{Im}\lambda| \geq C\log(\operatorname{Re}\lambda + 1)\}. \tag{2.8}$$

Note that in the case when (2.6) is fulfilled but (2.7) is not, the method developed in the present paper does not work and it is not clear if improvements are possible compared with the results in [15]. To our best knowledge, no results exist in the degenerate case when the function in the left-hand side of (2.3) vanishes without being identically zero.

It has been proved in [10] that the counting function $N(r) = \#\{\lambda - \text{trans. eig.} : |\lambda| \leq r\}$, $r > 1$, satisfies the asymptotics

$$N(r) = (\tau_1 + \tau_2)r^d + O_\varepsilon(r^{d-\kappa+\varepsilon}), \quad \forall\, 0 < \varepsilon \ll 1,$$

where $0 < \kappa \leq 1$ is such that there are no transmission eigenvalues in the region

$$\left\{\lambda \in \mathbf{C} : \operatorname{Re}\lambda > 1,\ |\operatorname{Im}\lambda| \geq C\,(\operatorname{Re}\lambda)^{1-\kappa}\right\}, \quad C > 0,$$

and

$$\tau_j = \frac{\omega_d}{(2\pi)^d}\int_\Omega \left(\frac{n_j(x)}{c_j(x)}\right)^{d/2} dx,$$

ω_d being the volume of the unit ball in $\mathbf{R}^d$. Using this we obtain from the above theorems the following

Corollary 2.3. *Under the conditions of Theorems 2.1 and 2.2, the counting function of the transmission eigenvalues satisfies the asymptotics*

$$N(r) = (\tau_1 + \tau_2)r^d + O_\varepsilon(r^{d-1+\varepsilon}), \quad \forall\, 0 < \varepsilon \ll 1. \tag{2.9}$$

This result has been previously proved in [16] under an additional strict concavity condition on the boundary Γ. In the present paper we remove this additional condition to conclude that in fact the asymptotics (2.9) holds true for an arbitrary domain. We also expect that (2.9) holds with $\varepsilon = 0$, but this remains an interesting open problem. In the isotropic case asymptotics for the counting function $N(r)$ with remainder $o(r^d)$ have been previously obtained in [4], [8], [13].

3. A priori estimates in the glancing region

Let $\lambda \in \mathbf{C}$, $\mathrm{Re}\,\lambda > 1$, $1 < |\mathrm{Im}\,\lambda| \leq \theta_0 \mathrm{Re}\,\lambda$, where $0 < \theta_0 < 1$ is a fixed constant, and set $h = \mu^{-1}$, where

$$\mu = \mathrm{Re}\,\lambda \sqrt{1 - \left(\frac{\mathrm{Im}\,\lambda}{\mathrm{Re}\,\lambda}\right)^2} \sim \mathrm{Re}\,\lambda \sim |\lambda|.$$

Clearly, we have $\mathrm{Re}\,(h\lambda)^2 = 1$ and

$$\lambda^2 = \mu^2(1 + izh), \quad z = 2\mu^{-1}\mathrm{Im}\,\lambda \mathrm{Re}\,\lambda \sim 2\mathrm{Im}\,\lambda.$$

Given an integer $m \geq 0$, denote by $H_h^m(X)$ the Sobolev space equipped with the semi-classical norm

$$\|v\|_{H_h^m(X)} = \sum_{|\alpha| \leq m} h^{|\alpha|} \left\|\partial_x^\alpha v\right\|_{L^2(X)}.$$

We define similarly the Sobolev space $H_h^m(\partial X)$. It is well-known that

$$\|v\|_{H_h^m(\partial X)} \sim \|\mathrm{Op}_h(\langle \xi' \rangle^m) v\|_{L^2(\partial X)} \sim \|v\|_{L^2(\partial X)} + \|\mathrm{Op}_h((1-\eta)|\xi'|^m) v\|_{L^2(\partial X)}$$

for any function $\eta \in C_0^\infty(T^*\partial X)$ independent of h. Hereafter, $\langle \xi' \rangle = (1 + |\xi'|^2)^{1/2}$.

Given functions $V \in L^2(X)$ and $f \in L^2(\partial X)$, we let the function u solve the equation

$$\begin{cases} \left(\Delta_X + \lambda^2 n(x)\right) u = \lambda V & \text{in} \quad X, \\ u = f & \text{on} \quad \partial X, \end{cases} \tag{3.1}$$

and set $g = h\partial_\nu u|_{\partial X}$. We will first prove the following

Lemma 3.1. *There is a constant $C > 0$ such that the following estimate holds*

$$\|u\|_{H_h^1(X)} \leq C|\mathrm{Im}\,\lambda|^{-1}\|V\|_{L^2(X)} + C|\mathrm{Im}\,\lambda|^{-1/2}\|f\|_{L^2(\partial X)}^{1/2}\|g\|_{L^2(\partial X)}^{1/2}. \tag{3.2}$$

Proof. By Green's formula we have

$$\mathrm{Im}\,(\lambda^2)\|n^{1/2}u\|_{L^2(X)}^2 = \mathrm{Im}\,\langle \lambda V, u\rangle_{L^2(X)} + \mathrm{Im}\,\langle \partial_\nu u|_{\partial X}, f\rangle_{L^2(\partial X)}$$

which implies

$$|\mathrm{Im}\,\lambda|\|u\|_{L^2(X)}^2 \lesssim \|V\|_{L^2(X)}\|u\|_{L^2(X)} + \|f\|_{L^2(\partial X)}\|g\|_{L^2(\partial X)}. \tag{3.3}$$

On the other hand, we have

$$\|\nabla_X u\|_{L^2(X)}^2 - \mathrm{Re}\,(\lambda^2)\|n^{1/2}u\|_{L^2(X)}^2 = -\mathrm{Re}\,\langle \lambda V, u\rangle_{L^2(X)} - \mathrm{Re}\,\langle \partial_\nu u|_{\partial X}, f\rangle_{L^2(\partial X)}$$

which yields

$$\|h\nabla_X u\|_{L^2(X)}^2 \lesssim \|u\|_{L^2(X)}^2 + \mathcal{O}(h^2)\|V\|_{L^2(X)}^2 + \mathcal{O}(h)\|f\|_{L^2(\partial X)}\|g\|_{L^2(\partial X)}. \tag{3.4}$$

Since $h \lesssim |\mathrm{Im}\,\lambda|^{-1}$, the estimate (3.2) follows from (3.3) and (3.4). □

We now equip X with the Riemannian metric $n\mathcal{G}$. We will write the operator $n^{-1}\Delta_X$ in the normal coordinates (x_1, x') with respect to the metric $n\mathcal{G}$ near the boundary ∂X, where $0 < x_1 \ll 1$ denotes the distance to the boundary and x' are coordinates on ∂X. Set $\Gamma(x_1) = \{x \in X : \mathrm{dist}(x, \partial X) = x_1\}$, $\Gamma(0) = \partial X$. Then $\Gamma(x_1)$ is a Riemannian manifold without boundary of dimension $n-1$ with a Riemannian metric induced by the metric $n\mathcal{G}$, which depends smoothly in x_1. It is well-known that the operator $n^{-1}\Delta_X$ writes as follows

$$n^{-1}\Delta_X = \partial_{x_1}^2 + Q(x_1) + R$$

where $Q(x_1) = \Delta_{\Gamma(x_1)}$ is the negative Laplace-Beltrami operator on $\Gamma(x_1)$ and R is a first-order differential operator. Clearly, $Q(x_1)$ is a second-order differential operator with smooth

coefficients and $Q(0) = \Delta^{(n)}_{\partial X}$ is the negative Laplace-Beltrami operator on ∂X equipped with the Riemannian metric induced by the metric $n\mathcal{G}$.

Let $\chi \in C_0^\infty(\mathbf{R})$, $0 \le \chi(t) \le 1$, $\chi(t) = 1$ for $|t| \le 1$, $\chi(t) = 0$ for $|t| \ge 2$. Given a parameter $0 < \delta_1 \ll 1$ independent of λ and an integer $k \ge 0$, set $\phi_k(x_1) = \chi(2^{-k}x_1/\delta_1)$. Given integers $0 \le s_1 \le s_2$ we define the norm $\|u\|_{s_1,s_2,k}$ by

$$\|u\|^2_{s_1,s_2,k} = \|u\|^2_{H_h^{s_1}(X)} + \sum_{\ell_1=0}^{s_1}\sum_{\ell_2=0}^{s_2-\ell_1}\int_0^\infty \|(h\partial_{x_1})^{\ell_1}(\phi_k u)(x_1,\cdot)\|^2_{H_h^{\ell_2}(\partial X)}dx_1.$$

Clearly, we have

$$\|u\|_{H_h^{s_1}(X)} \le \|u\|_{s_1,s_2,k} \lesssim \|u\|_{H_h^{s_2}(X)}.$$

Throughout this paper $\eta \in C_0^\infty(T^*\partial X)$, $0 \le \eta \le 1$, $\eta = 1$ in $|\xi'| \le A$, $\eta = 0$ in $|\xi'| \ge A+1$, will be a function independent of λ, where $A > 1$ is a parameter we may take as large as we want. We will now prove the following

Lemma 3.2. *Let u solve the equation (3.1) with $V \in H^{s-1}(X)$ and $f \in H^{2s}(\partial X)$ for some integer $s \ge 1$. Then the following estimate holds*

$$\|u\|_{1,s+1,k} \lesssim \|u\|_{H_h^1(X)} + \|V\|_{0,s-1,k+s-1} + \|\mathrm{Op}_h(1-\eta)f\|^{1/2}_{H_h^{2s}(\partial X)}\|g\|^{1/2}_{L^2(\partial X)}. \tag{3.5}$$

Proof. Note that

$$\|u\|_{1,s+1,k} \lesssim \|u\|_{H_h^1(X)} + \|u_{s,k}\|_{H_h^1(X)}$$

where the function $u_{s,k} = \mathrm{Op}_h((1-\eta)|\xi'|^s)(\phi_k u)$ satisfies the equation

$$\left(h^2\partial_{x_1}^2 + h^2Q(x_1) + 1 + ihz\right)u_{s,k} = U_{s,k}$$

with

$$U_{s,k} = \left[h^2Q(x_1), \mathrm{Op}_h((1-\eta)|\xi'|^s)\right](\phi_k u) + \mathrm{Op}_h((1-\eta)|\xi'|^s)\left[h^2\partial_{x_1}^2, \phi_k\right]\phi_{k+1}u$$
$$-h^2\mathrm{Op}_h((1-\eta)|\xi'|^s)\phi_k R\phi_{k+1}u + h^2\lambda\mathrm{Op}_h((1-\eta)|\xi'|^s)(\phi_k V).$$

We also have

$$f_s := u_{s,k}|_{x_1=0} = \mathrm{Op}_h((1-\eta)|\xi'|^s)f,$$
$$g_s := h\partial_{x_1}u_{s,k}|_{x_1=0} = \mathrm{Op}_h((1-\eta)|\xi'|^s)g_\flat,$$

where $g_\flat := h\partial_{x_1}u|_{x_1=0}$. Integrating by parts the above equation and taking the real part, we get

$$\begin{aligned}\|h\partial_{x_1}u_{s,k}\|^2_{L^2(X)} &- \left\langle (h^2Q(x_1)+1)u_{s,k}, u_{s,k}\right\rangle_{L^2(X)}\\ &\le \left|\langle U_{s,k}, u_{s,k}\rangle_{L^2(X)}\right| + h\left|\langle f_s, g_s\rangle_{L^2(\partial X)}\right|\\ &\lesssim \|u_{s,k}\|_{H_h^1(X)}\left(\|V\|_{0,s-1,k} + \|u\|_{1,s,k+1}\right)\\ &+ \left\|\mathrm{Op}_h((1-\eta)|\xi'|^s)^*\mathrm{Op}_h((1-\eta)|\xi'|^s)f\right\|_{L^2(\partial X)}\|g_\flat\|_{L^2(\partial X)}.\end{aligned} \tag{3.6}$$

The principal symbol r of the operator $-Q(x_1)$ satisfies $r(x,\xi') \ge C'|\xi'|^2$, $C' > 0$, on $\mathrm{supp}\phi_k$, provided δ_1 is taken small enough. Therefore, we can arrange by taking the parameter A big enough that $r - 1 \ge C\langle\xi'\rangle$ on $\mathrm{supp}\,(1-\eta)\phi_k$, where $C > 0$ is some constant. Hence, by Gärding's inequality we have

$$-\left\langle (h^2Q(x_1)+1)u_{s,k}, u_{s,k}\right\rangle_{L^2(X)} \ge C\|\mathrm{Op}_h(\langle\xi'\rangle)u_{s,k}\|^2_{L^2(X)} \tag{3.7}$$

with possibly a new constant $C > 0$. Since the norms of g and $g_\flat$ are equivalent, by (3.6) and (3.7) we get

$$\|u_{s,k}\|_{H_h^1(X)} \lesssim \|V\|_{0,s-1,k} + \|u\|_{H_h^1(X)} + \|u_{s-1,k+1}\|_{H_h^1(X)}$$

$$+\|\mathrm{Op}_h(1-\eta)f\|^{1/2}_{H^{2s}_h(\partial X)}\,\|g\|^{1/2}_{L^2(\partial X)}. \tag{3.8}$$

We may now apply the same argument to $u_{s-1,k+1}$. Thus, repeating this argument a finite number of times we can eliminate the term involving $u_{s-1,k+1}$ in the RHS of (3.8) and obtain the estimate (3.5). □

Let the functions $\chi_j \in C^\infty(\mathbf{R})$, $0 \le \chi_j(t) \le 1$, $j = 1,2,3$, be such that $\chi_1+\chi_2+\chi_3 \equiv 1$, $\chi_2 = \chi$, $\chi_1(t) = 1$ for $t \le -2$, $\chi_1(t) = 0$ for $t \ge -1$, $\chi_3(t) = 0$ for $t \le 1$, $\chi_3(t) = 1$ for $t \ge 2$. Given a parameter $0 < \delta \ll 1$ independent of λ, set

$$\begin{aligned}\chi^-_\delta(x',\xi') &= \chi_1((r_\sharp(x',\xi')-1)/\delta^2),\\ \chi^0_\delta(x',\xi') &= \chi_2((r_\sharp(x',\xi')-1)/\delta^2),\\ \chi^+_\delta(x',\xi') &= \chi_3((r_\sharp(x',\xi')-1)/\delta^2),\end{aligned}$$

where $r_\sharp = n_0^{-1}r_0$ is the principal symbol of the operator $-\Delta^{(n)}_{\partial X}$. Since $(r_\sharp-1)^k\chi^0_\delta = \mathcal{O}(\delta^{2k})$, we have

$$(h^2\Delta^{(n)}_{\partial X}+1)^k\mathrm{Op}_h(\chi^0_\delta) = \mathcal{O}(\delta^{2k}) : L^2(\partial X) \to L^2(\partial X) \tag{3.9}$$

for every integer $k \ge 0$. Clearly, we also have

$$\mathrm{Op}_h(\chi^0_\delta) = \mathcal{O}(1) : L^2(\partial X) \to H^m_h(\partial X), \quad \forall\, m \ge 0,$$

uniformly in δ. Using (3.9) we will prove the following

Proposition 3.3. *Let u solve (3.1) with $f \equiv 0$ and $V \in H^s(X)$ for some integer $s \ge 0$. Then the function $g = h\partial_\nu u|_{\partial X}$ satisfies the estimate*

$$\|g\|_{H^s_h(\partial X)} \le C'|\mathrm{Im}\,\lambda|^{-1/2}\|V\|_{0,s,s} \tag{3.10}$$

with a constant $C' > 0$ independent of λ.

Let u solve (3.1) with f replaced by $\mathrm{Op}_h(\chi^0_\delta)f$ and $V \in H^{s+2}(X)$ for some integer $s \ge 0$. Then the function $g = h\partial_\nu u|_{\partial X}$ satisfies the estimate

$$\|g\|_{H^s_h(\partial X)} \le C\left(\delta + |\mathrm{Im}\,\lambda|^{-1/4}\right)\|f\|_{L^2(\partial X)} + C\left(\delta^{1/2} + |\mathrm{Im}\,\lambda|^{-1/8}\right)\|V\|_{0,s+2,s+2} \tag{3.11}$$

for $1 < |\mathrm{Im}\,\lambda| \le \delta^2\mathrm{Re}\,\lambda$, $\mathrm{Re}\,\lambda \ge C_\delta \gg 1$, with a constant $C > 0$ independent of λ and δ.

Proof. Set $w = \phi_0(x_1)u$. We will first show that the estimates (3.10) and (3.11) with $s \ge 1$ follow from (3.10) and (3.11) with $s = 0$, respectively. This follows from the estimate

$$\|g\|_{H^s_h(\partial X)} \lesssim \|g\|_{L^2(\partial X)} + \|h\partial_{x_1}v_s|_{x_1=0}\|_{L^2(\partial X)} \tag{3.12}$$

where the function $v_s = \mathrm{Op}_h((1-\eta)|\xi'|^s)w$ satisfies the equation (3.1) with V replaced by

$$V_s = n\mathrm{Op}_h((1-\eta)|\xi'|^s)\phi_0 n^{-1}V + \lambda^{-1}n\left[n^{-1}\Delta_X, \mathrm{Op}_h((1-\eta)|\xi'|^s)\phi_0\right]u.$$

We can write the commutator as

$$\left[\partial^2_{x_1}+R, \phi_0(x_1)\right]\mathrm{Op}_h((1-\eta)|\xi'|^s)\phi_1(x_1) + \phi_0\left[Q(x_1)+R, \mathrm{Op}_h((1-\eta)|\xi'|^s)\right]\phi_1(x_1).$$

Therefore, if $f \equiv 0$, in view of Lemmas 3.1 and 3.2, the function V_s satisfies the bound

$$\begin{aligned}\|V_s\|_{0,0,0} &\lesssim \|V\|_{0,s,0} + \|u\|_{1,s+1,1}\\ &\lesssim \|u\|_{H^1_h(X)} + \|V\|_{0,s,s} \lesssim \|V\|_{0,s,s}.\end{aligned} \tag{3.13}$$

Clearly, the assertion concerning (3.10) follows from (3.12) and (3.13). The estimate (3.11) can be treated similarly. Indeed, in view of Lemma 3.2, the function V_s satisfies the bound

$$\|V_s\|_{0,2,2} \lesssim \|V\|_{0,s+2,0} + \|u\|_{1,s+3,1}$$

$$\lesssim \|u\|_{H^1_h(X)} + \|V\|_{0,s+2,s+2}$$
$$+\|\mathrm{Op}_h(1-\eta)\mathrm{Op}_h(\chi^0_\delta)f\|^{1/2}_{H^{2s+4}_h(\partial X)}\|g\|^{1/2}_{L^2(\partial X)}. \tag{3.14}$$

Taking the parameter A big enough we can arrange that $\mathrm{supp}\,\chi^0_\delta \cap \mathrm{supp}\,(1-\eta) = \emptyset$. Hence

$$\mathrm{Op}_h(1-\eta)\mathrm{Op}_h(\chi^0_\delta) = \mathcal{O}(h^\infty) : L^2(\partial X) \to H^m_h(\partial X), \quad \forall m \geq 0. \tag{3.15}$$

By (3.14) and (3.15) together with Lemma 3.1 we conclude

$$\|V_s\|_{0,2,2} \lesssim \|u\|_{H^1_h(X)} + \|V\|_{0,s+2,s+2} + \mathcal{O}(h^\infty)\|f\|^{1/2}_{L^2(\partial X)}\|g\|^{1/2}_{L^2(\partial X)}$$
$$\lesssim \|V\|_{0,s+2,s+2} + \mathcal{O}\left(|\mathrm{Im}\,\lambda|^{-1/2} + h^\infty\right)\|f\|^{1/2}_{L^2(\partial X)}\|g\|^{1/2}_{L^2(\partial X)}.$$

We now apply (3.11) with $s = 0$ to the function v_s and note that

$$v_s|_{x_1=0} = \mathrm{Op}_h((1-\eta)|\xi'|^s)\mathrm{Op}_h(\chi^0_\delta)f = \mathcal{O}(h^\infty)f.$$

Hence

$$\|h\partial_{x_1}v_s|_{x_1=0}\|_{L^2(\partial X)} \leq \mathcal{O}(h^\infty)\|f\|_{L^2(\partial X)} + \mathcal{O}\left(\delta^{1/2} + |\mathrm{Im}\,\lambda|^{-1/8}\right)\|V_s\|_{0,2,2}$$
$$\leq \mathcal{O}\left(\delta^{1/2} + |\mathrm{Im}\,\lambda|^{-1/8}\right)\|V\|_{0,s+2,s+2} + \mathcal{O}\left(|\mathrm{Im}\,\lambda|^{-1/2} + h^\infty\right)\|f\|^{1/2}_{L^2(\partial X)}\|g\|^{1/2}_{L^2(\partial X)}. \tag{3.16}$$

Therefore, the assertion concerning (3.11) follows from (3.12) and (3.16).

We now turn to the proof of (3.10) and (3.11) with $s = 0$. In view of Lemma 3.1, the function

$$U := h(n^{-1}\Delta_X + \lambda^2)w = h[n^{-1}\Delta_X, \phi_0(x_1)]u + h\lambda n^{-1}\phi_0 V$$

satisfies the bound

$$\|U\|_{L^2(X)} \lesssim \|u\|_{H^1_h(X)} + \|V\|_{L^2(X)}$$
$$\lesssim \|V\|_{L^2(X)} + \mathcal{O}\left(|\mathrm{Im}\,\lambda|^{-1/2}\right)\|f\|^{1/2}_{L^2(\partial X)}\|g\|^{1/2}_{L^2(\partial X)}. \tag{3.17}$$

Observe now that the derivative of the function

$$E(x_1) = \|h\partial_{x_1}w\|^2 + \left\langle \left(h^2Q(x_1) + 1\right)w, w\right\rangle,$$

$\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ being the norm and the scalar product in $L^2(\partial X)$, satisfies

$$E'(x_1) = 2\mathrm{Re}\,\left\langle \left(h^2\partial^2_{x_1} + h^2Q(x_1) + 1\right)w, \partial_{x_1}w\right\rangle + \left\langle h^2Q'(x_1)w, w\right\rangle$$
$$= 2\mathrm{Re}\,\left\langle (U - izw - hRw), h\partial_{x_1}w\right\rangle + \left\langle h^2Q'(x_1)w, w\right\rangle.$$

If we put $g_\flat := h\partial_{x_1}u|_{x_1=0}$, we have

$$\|g_\flat\|^2 + \left\langle \left(h^2\Delta^{(n)}_{\partial X} + 1\right)\mathrm{Op}_h(\chi^0_\delta)f, \mathrm{Op}_h(\chi^0_\delta)f\right\rangle = E(0) = -\int_0^\infty E'(x_1)dx_1$$
$$\lesssim \left(\|U\|_{L^2(X)} + |z|\|w\|_{L^2(X)} + \|hRw\|_{L^2(X)}\right)\|h\partial_{x_1}w\|_{L^2(X)} + \|w\|^2_{H^1_h(X)}$$
$$\leq \mathcal{O}(|z|)\|h\partial_{x_1}w\|_{L^2(X)}\|w\|_{L^2(X)} + \mathcal{O}\left(|\mathrm{Im}\,\lambda|^{-1}\right)F^2 \tag{3.18}$$

where we have used Lemma 3.1 together with (3.17) and we have put

$$F = \|f\|^{1/2}\|g\|^{1/2} + \|V\|_{L^2(X)}.$$

Clearly, (3.10) with $s = 0$ follows from (3.18) applied with $f \equiv 0$ and Lemma 3.1. To prove (3.11) with $s = 0$, observe that (3.9) and (3.18) lead to

$$\|g\| \leq \mathcal{O}(\delta)\|f\| + \mathcal{O}\left(|\mathrm{Im}\,\lambda|^{-1/2}\right)F$$
$$+\mathcal{O}(|\mathrm{Im}\,\lambda|^{1/2})\|h\partial_{x_1}w\|^{1/2}_{L^2(X)}\|w\|^{1/2}_{L^2(X)}. \tag{3.19}$$

We need now to better bound the norm $\|h\partial_{x_1} w\|_{L^2(X)}$ in the RHS of (3.19) than what the estimate (3.2) gives. To this end, observe that integrating by parts yields

$$\begin{aligned}
&\|h\partial_{x_1} w\|^2_{L^2(X)} - \left\langle \left(h^2 Q(x_1)+1\right) w, w\right\rangle_{L^2(X)} \\
&= -h\mathrm{Re}\,\langle (U-hRw), w\rangle_{L^2(X)} - h\mathrm{Re}\,\langle f, g_\flat\rangle \\
&\le \mathcal{O}(h)\|w\|^2_{H^1_h(X)} + \mathcal{O}(h)\|U\|^2_{L^2(X)} + \mathcal{O}(h)\|f\|\|g\| \le \mathcal{O}(h)F^2. \qquad (3.20)
\end{aligned}$$

By (3.19) and (3.20) together with Lemma 3.1 we get

$$\begin{aligned}
\|g\| &\le \mathcal{O}(\delta)\|f\| + \mathcal{O}(|\mathrm{Im}\,\lambda|^{1/2})\|w_1\|^{1/4}_{L^2(X)}\|w\|^{3/4}_{L^2(X)} \\
&+\mathcal{O}(h^{1/4}|\mathrm{Im}\,\lambda|^{1/2})F^{1/2}\|w\|^{1/2}_{L^2(X)} + \mathcal{O}\left(|\mathrm{Im}\,\lambda|^{-1/2}\right)F \\
&\le \mathcal{O}(\delta)\|f\| + \mathcal{O}(|\mathrm{Im}\,\lambda|^{1/8})\|w_1\|^{1/4}_{L^2(X)}F^{3/4} \\
&+\mathcal{O}\left(|\mathrm{Im}\,\lambda|^{-1/2} + h^{1/4}|\mathrm{Im}\,\lambda|^{1/4}\right)F \qquad (3.21)
\end{aligned}$$

where we have put $w_1 := \left(h^2 Q(x_1)+1\right) w$. We need now the following

Lemma 3.4. *The function w_1 satisfies the estimate*

$$\begin{aligned}
|\mathrm{Im}\,\lambda|^{1/2}\|w_1\|_{L^2(X)} &\le \mathcal{O}\left(\delta^2 + |\mathrm{Im}\,\lambda|^{-1} + h^\infty\right)\|f\|^{1/2}\|g\|^{1/2} \\
&+\mathcal{O}\left(h^{1/2}\right)\|f\| + \mathcal{O}\left(|\mathrm{Im}\,\lambda|^{-1/2}\right)\|V\|_{0,2,2}. \qquad (3.22)
\end{aligned}$$

Let us see that this lemma implies the estimate (3.11) with $s=0$. Set

$$\widetilde{F} = \|f\|^{1/2}\|g\|^{1/2} + \|V\|_{0,2,2} \ge F.$$

By (3.21) and (3.22),

$$\begin{aligned}
\|g\| &\le \mathcal{O}(\delta)\|f\| + \mathcal{O}\left(\delta^{1/2} + |\mathrm{Im}\,\lambda|^{-1/8} + h^\infty\right)\widetilde{F} \\
&+\mathcal{O}\left(h^{1/8}\right)(\|f\| + F) + \mathcal{O}\left(|\mathrm{Im}\,\lambda|^{-1/2} + h^{1/4}|\mathrm{Im}\,\lambda|^{1/4}\right)F \\
&\le \mathcal{O}\left(\delta + h^{1/8}\right)\|f\| + \mathcal{O}\left(\delta^{1/2} + |\mathrm{Im}\,\lambda|^{-1/8} + h^{1/8} + h^{1/4}|\mathrm{Im}\,\lambda|^{1/4}\right)\widetilde{F}. \qquad (3.23)
\end{aligned}$$

Since by assumption $h^{1/4}|\mathrm{Im}\,\lambda|^{1/4} = \mathcal{O}\left(\delta^{1/2}\right)$, one can easily see that (3.11) with $s=0$ follows from (3.23). □

Proof of Lemma 3.4. Observe that the function w_1 satisfies the equation

$$\left(h^2\partial^2_{x_1} + h^2 Q(x_1) + 1 + ihz\right) w_1 = hU_1$$

where

$$U_1 := \left(h^2 Q(x_1)+1\right)(U - hRw) + 2h^3 Q'(x_1)\partial_{x_1} w + h^3 Q''(x_1) w.$$

We also have

$$\begin{aligned}
f_1 &:= w_1|_{x_1=0} = (h^2 Q(0)+1)\mathrm{Op}_h(\chi^0_\delta) f, \\
g_1 &:= h\partial_{x_1} w_1|_{x_1=0} = (h^2 Q(0)+1) g_\flat + h^2 Q'(0)\mathrm{Op}_h(\chi^0_\delta) f.
\end{aligned}$$

Integrating by parts the above equation and taking the imaginary part, we get

$$\begin{aligned}
|z|\|w_1\|^2_{L^2(X)} &\le \left|\langle U_1, w_1\rangle_{L^2(X)}\right| + |\langle f_1, g_1\rangle| \\
&\le \|U_1\|_{L^2(X)}\|w_1\|_{L^2(X)} + \mathcal{O}(1)\left\|(h^2 Q(0)+1)^2\mathrm{Op}_h(\chi^0_\delta) f\right\|\|g\| \\
&+\mathcal{O}(h)\left\|\mathrm{Op}_h(\chi^0_\delta) f\right\|_{H^2_h(\partial X)}\left\|(h^2 Q(0)+1)\mathrm{Op}_h(\chi^0_\delta) f\right\|
\end{aligned}$$

$$\leq \|U_1\|_{L^2(X)}\|w_1\|_{L^2(X)} + \mathcal{O}(\delta^4)\|f\|\|g\| + \mathcal{O}(h)\|f\|^2$$

where we have used (3.9). Hence

$$|z|\|w_1\|^2_{L^2(X)} \leq \mathcal{O}\left(|z|^{-1}\right)\|U_1\|^2_{L^2(X)} + \mathcal{O}(\delta^4)\|f\|\|g\| + \mathcal{O}(h)\|f\|^2. \tag{3.24}$$

Recall that the function U is of the form $(2h\partial_{x_1} + a(x))\phi_1(x_1)u + h\lambda n^{-1}\phi_0 V$, where a is some smooth function. Hence the function U_1 satisfies the estimate

$$\|U_1\|_{L^2(X)} \lesssim \|u\|_{1,3,1} + \|V\|_{0,2,0}$$

$$\lesssim \|u\|_{H^1_h(X)} + \|V\|_{0,2,2} + \mathcal{O}(h^\infty)\|f\|^{1/2}_{L^2(\partial X)}\|g\|^{1/2}_{L^2(\partial X)} \tag{3.25}$$

where we have used Lemma 3.2 together with (3.15). By (3.24) and (3.25),

$$|z|\|w_1\|^2_{L^2(X)} \leq \mathcal{O}\left(|z|^{-1}\right)\|u\|^2_{H^1_h(X)} + \mathcal{O}\left(|z|^{-1}\right)\|V\|^2_{0,2,2}$$

$$+\mathcal{O}(\delta^4 + h^\infty)\|f\|\|g\| + \mathcal{O}(h)\|f\|^2. \tag{3.26}$$

Clearly, (3.22) follows from (3.26) and Lemma 3.1. □

4. Parametrix construction in the hyperbolic region

Let λ be as in Theorems 1.1 and 1.2, and let h, z, δ, r_0, n_0, $r_\sharp$, χ and χ^-_δ be as in the previous sections. Set $\theta = \mathrm{Im}\,(h\lambda)^2 = hz = \mathcal{O}(h^\epsilon)$, $|\theta| \gg h$, and

$$\rho(x',\xi') = \sqrt{r_0(x',\xi') - (1+i\theta)n_0(x')}, \quad \mathrm{Re}\,\rho < 0.$$

It is easy to see that $\rho\chi^-_\delta \in S^0_0(\partial X)$. In this section we will prove the following

Proposition 4.1. *There are constants $C, C_1 > 0$ depending on δ but independent of λ such that*

$$\left\|h\mathcal{N}(\lambda;n)\mathrm{Op}_h(\chi^-_\delta) - \mathrm{Op}_h(\rho\chi^-_\delta)\right\|_{L^2(\partial X)\to H^1_h(\partial X)} \leq C_1\left(h + e^{-C|\mathrm{Im}\,\lambda|}\right). \tag{4.1}$$

Proof. To prove (4.1) we will build a parametrix near the boundary of the solution to the equation (1.1) with f replaced by $\mathrm{Op}_h(\chi^-_\delta)f$. Let $x = (x_1, x')$, $x_1 > 0$, be the normal coordinates with respect to the metric $\mathcal{G}$, which of course are different from those introduced in the previous section. In these coordinates the operator Δ_X writes as follows

$$\Delta_X = \partial^2_{x_1} + \widetilde{Q} + \widetilde{R}$$

where $\widetilde{Q} \leq 0$ is a second-order differential operator with respect to the variables x' and $\widetilde{R}$ is a first-order differential operator with respect to the variables x, both with coefficients depending smoothly on x. Let $(x^0,\xi^0) \in \mathrm{supp}\,\chi^-_\delta$ and let $\mathcal{U} \subset T^*\partial X$ be a small open neighbourhood of (x^0,ξ^0) contained in $\{r_\sharp \leq 1 - \delta^2/2\}$. Take a function $\psi \in C^\infty_0(\mathcal{U})$. We will construct a parametrix $\widetilde{u}^-_\psi$ of the solution of (1.1) with $\widetilde{u}^-_\psi|_{x_1=0} = \mathrm{Op}_h(\psi)f$ in the form $\widetilde{u}^-_\psi = \phi(x_1)\mathcal{K}^-f$, where $\phi(x_1) = \chi(x_1/\delta_1)$, $0 < \delta_1 \ll 1$ being a parameter independent of λ to be fixed later on depending on δ, and

$$(\mathcal{K}^-f)(x) = (2\pi h)^{-n+1}\int\int e^{\frac{i}{h}(\langle y',\xi'\rangle + \varphi(x,\xi',\theta))}a(x,\xi',\lambda)f(y')d\xi' dy'.$$

The phase φ is complex-valued such that $\varphi|_{x_1=0} = -\langle x',\xi'\rangle$ and satisfies the eikonal equation mod $\mathcal{O}(\theta^M)$:

$$(\partial_{x_1}\varphi)^2 + \langle B(x)\nabla_{x'}\varphi, \nabla_{x'}\varphi\rangle = (1+i\theta)n(x) + \theta^M\mathcal{R}_M \tag{4.2}$$

where $M \gg 1$ is an arbitrary integer, the function $\mathcal{R}_M$ is bounded uniformly in θ, and B is a matrix-valued function such that $r(x,\xi') = \langle B(x)\xi',\xi'\rangle$, $r(x,\xi') \geq 0$ being the principal symbol

of the operator $-\widetilde{Q}$. We clearly have $r_0(x',\xi') = r(0,x',\xi')$. Let us see that for $(x',\xi') \in \mathcal{U}$, $0 \le x_1 \le 3\delta_1$, the equation (4.2) has a smooth solution safisfying

$$\partial_{x_1}\varphi|_{x_1=0} = -i\rho + \mathcal{O}(\theta^{M/2}) \tag{4.3}$$

provided δ_1 and $\mathcal{U}$ are small enough. We will be looking for φ in the form

$$\varphi = \sum_{j=0}^{M-1} (i\theta)^j \varphi_j(x,\xi')$$

where φ_j are real-valued functions depending only on the sign of θ and satisfying the equations

$$(\partial_{x_1}\varphi_0)^2 + \langle B(x)\nabla_{x'}\varphi_0, \nabla_{x'}\varphi_0\rangle = n(x), \tag{4.4}$$

$$\sum_{j=0}^{k} \partial_{x_1}\varphi_j \partial_{x_1}\varphi_{k-j} + \sum_{j=0}^{k} \langle B(x)\nabla_{x'}\varphi_j, \nabla_{x'}\varphi_{k-j}\rangle = \varepsilon_k n(x), \quad 1 \le k \le M-1, \tag{4.5}$$

$\varphi_0|_{x_1=0} = -\langle x',\xi'\rangle$, $\varphi_j|_{x_1=0} = 0$ for $j \ge 1$, where $\varepsilon_1 = 1$, $\varepsilon_k = 0$ for $k \ge 2$. It is easy to check that with this choice the function φ satisfies (4.2) with $\mathcal{R}_M$ being polynomial in θ.

Clearly, if φ_0 is a solution to (4.4), then we have $(\partial_{x_1}\varphi_0|_{x_1=0})^2 = n_0(x') - r_0(x',\xi') \ge C'$ with some constant $C' > 0$ depending on δ. It is well-known that the equation (4.4) has a local (that is, for δ_1 and $\mathcal{U}$ small enough) real-valued solution $\varphi_0^{\pm}$ such that $\partial_{x_1}\varphi_0^{\pm}|_{x_1=0} = \pm\sqrt{n_0 - r_0}$. We now define the function φ_0 by $\varphi_0 = \varphi_0^+$ if $\theta > 0$, $\varphi_0 = \varphi_0^-$ if $\theta < 0$. Hence $|\partial_{x_1}\varphi_0(x,\xi')| \ge Const > 0$ for x_1 small enough. Therefore, the equations (4.5) can be solved locally. Taking $x_1 = 0$ in the equation (4.5) with $k = 1$ we find

$$\theta\partial_{x_1}\varphi_1|_{x_1=0} = \theta n_0 \left(2\partial_{x_1}\varphi_0|_{x_1=0}\right)^{-1} = \frac{|\theta|}{2} n_0 (n_0 - r_0)^{-1/2} \ge \frac{C|\theta|}{2} \tag{4.6}$$

on $\mathcal{U}$, where $C = \min\sqrt{n_0(x')}$. Hence

$$\operatorname{Im}\partial_{x_1}\varphi|_{x_1=0} = \theta\partial_{x_1}\varphi_1|_{x_1=0} + \mathcal{O}(\theta^2) \ge \frac{C|\theta|}{3} \tag{4.7}$$

if $|\theta|$ is taken small enough. On the other hand, taking $x_1 = 0$ in the equation (4.2) we find

$$(\partial_{x_1}\varphi|_{x_1=0})^2 = (i\rho)^2 + \mathcal{O}(\theta^M) = (i\rho)^2(1 + \mathcal{O}(\theta^M)) \tag{4.8}$$

where we have used that $|\rho| \ge Const > 0$ on $\mathcal{U}$. Since $\operatorname{Re}\rho < 0$, we get (4.3) from (4.7) and (4.8). By (4.6) we also get

$$\theta\varphi_1(x_1,x',\xi') = \theta x_1 \partial_{x_1}\varphi_1(0,x',\xi') + \mathcal{O}(\theta x_1^2) \ge \frac{Cx_1|\theta|}{2} - \mathcal{O}(|\theta|x_1^2) \ge \frac{Cx_1|\theta|}{3}$$

provided x_1 is taken small enough. This implies

$$\operatorname{Im}\varphi(x,\xi',\theta) = \theta\varphi_1(x_1,x',\xi') + \mathcal{O}(\theta^2 x_1) \ge \frac{Cx_1|\theta|}{4}. \tag{4.9}$$

The amplitude a is of the form

$$a = \sum_{k=0}^{m} h^k a_k(x,\xi',\theta)$$

where $m \gg 1$ is an arbitrary integer and the functions a_k satisfy the transport equations mod $\mathcal{O}(\theta^M)$:

$$2i\partial_{x_1}\varphi\partial_{x_1}a_k + 2i\left\langle B(x)\nabla_{x'}\varphi, \nabla_{x'}a_k\right\rangle + i\left(\Delta_X\varphi\right)a_k + \Delta_X a_{k-1} = \theta^M \mathcal{Q}_M^{(k)}, \quad 0 \le k \le m, \tag{4.10}$$

$a_0|_{x_1=0}=\psi$, $a_k|_{x_1=0}=0$ for $k\geq 1$, where $a_{-1}=0$. Let us see that the transport equations have smooth solutions for $(x',\xi')\in\mathcal{U}$, $0\leq x_1\leq 3\delta_1$, provided δ_1 and $\mathcal{U}$ are taken small enough. As above, we will be looking for a_k in the form

$$a_k=\sum_{j=0}^{M-1}(i\theta)^j a_{k,j}(x,\xi').$$

We let $a_{k,j}$ satisfy the equations

$$2i\sum_{\nu=0}^{j}\partial_{x_1}\varphi_\nu\partial_{x_1}a_{k,j-\nu}+2i\sum_{\nu=0}^{j}\langle B(x)\nabla_{x'}\varphi_\nu,\nabla_{x'}a_{k,j-\nu}\rangle+i\left(\Delta_X\varphi_j\right)a_{k,j}+\Delta_X a_{k-1,j}=0, \quad (4.11)$$

$0\leq j\leq M-1$, $a_{0,0}|_{x_1=0}=\psi$, $a_{k,j}|_{x_1=0}=0$ for $k+j\geq 1$. Then the functions a_k satisfy (4.10) with $\mathcal{Q}_M^{(k)}$ being polynomial in θ. As in the case of the equations (4.5) one can solve (4.11) locally. Then we can write

$$V_-:=h^{-1}(h^2\Delta_X+(1+i\theta)n(x))\widetilde{u}_\psi^-=\mathcal{K}_1^-f+\mathcal{K}_2^-f$$

where

$$\mathcal{K}_1^-f=h[\Delta_X,\phi]\mathcal{K}^-f=h(2\phi'(x_1)\partial_{x_1}+c(x)\phi''(x_1))\mathcal{K}^-f$$
$$=(2\pi h)^{-n+1}\int\int e^{\frac{i}{h}(\langle y',\xi'\rangle+\varphi(x,\xi',\theta))}A_1^-(x,\xi',\lambda)f(y')d\xi'dy'$$

c being some smooth function,

$$A_1^-=2i\phi'a\partial_{x_1}\varphi+hc\phi''\partial_{x_1}a$$

and

$$(\mathcal{K}_2^-f)(x)=(2\pi h)^{-n+1}\int\int e^{\frac{i}{h}(\langle y',\xi'\rangle+\varphi(x,\xi',\theta))}A_2^-(x,\xi',\lambda)f(y')d\xi'dy'$$

where

$$A_2^-=\phi(x_1)\left(h^{-1}\theta^M\mathcal{R}_Ma+\theta^M\sum_{k=0}^{m}h^k\mathcal{Q}_M^{(k)}+h^{m+1}\Delta_Xa_m\right).$$

Let us see that Proposition 4.1 follows from the following

Lemma 4.2. *The function V_- satisfies the estimate*

$$\|V_-\|_{H_h^1(X)}\lesssim e^{-C|\mathrm{Im}\,\lambda|}\|f\|+\mathcal{O}_m\left(h^{m-n}\right)\|f\|+\mathcal{O}_M\left(h^{\epsilon M-n}\right)\|f\| \quad (4.12)$$

with some constant $C>0$.

Indeed, if u_ψ^- denotes the solution to the equation (1.1) with f replaced by $\mathrm{Op}_h(\psi)f$ and $\widetilde{u}_\psi^-$ is the parametrix built above, then the function $v=u_\psi^--\widetilde{u}_\psi^-$ satisfies the equation (3.1) with $f\equiv 0$. Therefore, by the estimates (3.10) and (4.12) we have

$$\left\|h\mathcal{N}(\lambda;n)\mathrm{Op}_h(\psi)-T_\psi^-\right\|_{L^2(\partial X)\to H_h^1(\partial X)}\lesssim e^{-C|\mathrm{Im}\,\lambda|}+\mathcal{O}_m\left(h^{m-n}\right)+\mathcal{O}_M\left(h^{\epsilon M-n}\right) \quad (4.13)$$

where the operator T_ψ^- is defined by

$$T_\psi^-f=h\partial_{x_1}\mathcal{K}^-f|_{x_1=0}.$$

Hence, in view of (4.3),

$$\left(T_\psi^-f\right)(x')=(2\pi h)^{-n+1}\int\int e^{\frac{i}{h}\langle y'-x',\xi'\rangle}(i\psi\partial_{x_1}\varphi(0,x',\xi',\theta)+h\partial_{x_1}a(0,x',\xi',\lambda))f(y')d\xi'dy'$$

$$= \mathrm{Op}_h(\rho\psi + \mathcal{O}(\theta^{M/2}))f + \sum_{k=0}^{m} h^{k+1}\mathrm{Op}_h(\partial_{x_1} a_k(0, x', \xi', \theta))f.$$

Since

$$\mathrm{Op}_h(\partial_{x_1} a_k(0, x', \xi', \theta)) = \mathcal{O}(1) : L^2(\partial X) \to H_h^1(\partial X)$$

uniformly in θ, it follows from (4.13) that

$$\|h\mathcal{N}(\lambda; n)\mathrm{Op}_h(\psi) - \mathrm{Op}_h(\rho\psi)\|_{L^2(\partial X)\to H_h^1(\partial X)} \lesssim e^{-C|\mathrm{Im}\,\lambda|} + \mathcal{O}(h). \tag{4.14}$$

On the other hand, using a suitable partition of the unity we can write the function χ_δ^- as $\sum_{j=1}^J \psi_j$, where each function ψ_j has the same properties as the function ψ above. In other words, we have (4.14) with ψ replaced by each ψ_j, which after summing up leads to (4.1). □

Proof of Lemma 4.2. Let α be a multi-index such that $|\alpha| \le 1$. Since

$$i|\alpha|A_2^- \partial_x^\alpha \varphi + (h\partial_x)^\alpha A_2^- = \mathcal{O}_m\left(h^{m+1}\right) + \mathcal{O}_M\left(h^{\epsilon M-1}\right)$$

and $\mathrm{Im}\,\varphi \ge 0$, the kernel of the operator $(h\partial_x)^\alpha \mathcal{K}_2^- : L^2(\partial X) \to L^2(X)$ is $\mathcal{O}_m\left(h^{m-n}\right) + \mathcal{O}_M\left(h^{\epsilon M-n}\right)$, and hence so is its norm. Since the function A_1^- is supported in the interval $[\delta_1/2, 3\delta_1]$ with respect to the variable x_1, to bound the norm of the operator $\mathcal{K}_{1,\alpha}^- := (h\partial_x)^\alpha \mathcal{K}_1^- : L^2(\partial X) \to L^2(X)$ it suffices to show that

$$\|\mathcal{K}_{1,\alpha}^-\|_{L^2(\partial X)\to L^2(\partial X)} \lesssim e^{-C|\theta|/h} + \mathcal{O}(h^\infty) \tag{4.15}$$

for every fixed $x_1 \in [\delta_1/2, 3\delta_1]$ uniformly in x_1. Since $|\theta|/h \sim |\mathrm{Im}\,\lambda|$, (4.15) will imply (4.12). We would like to consider $\mathcal{K}_{1,\alpha}^-$ as an $h-$FIO with phase $\mathrm{Re}\,\varphi$ and amplitude

$$A_\alpha = e^{-\mathrm{Im}\,\varphi/h}\left(i|\alpha|A_1^- \partial_x^\alpha \varphi + (h\partial_x)^\alpha A_1^-\right).$$

To do so, we need to have that the phase satisfies the condition

$$\left|\det\left(\frac{\partial^2 \mathrm{Re}\,\varphi}{\partial x' \partial \xi'}\right)\right| \ge \widetilde{C} > 0 \tag{4.16}$$

for $|\theta|$ small enough, where $\widetilde{C}$ is a constant independent of θ. Since $\mathrm{Re}\,\varphi = \varphi_0 + \mathcal{O}(|\theta|)$, it suffices to show (4.16) for the phase φ_0. This, however, is easy to arrange by taking x_1 small enough because $\varphi_0 = -\langle x', \xi'\rangle + \mathcal{O}(x_1)$ and (4.16) is trivially fulfilled for the phase $-\langle x', \xi'\rangle$. On the other hand, using that $\mathrm{Im}\,\varphi = \mathcal{O}(|\theta|)$ together with (4.9) we get the following bounds for the amplitude:

$$\left|\partial_{x'}^{\beta_1}\partial_{\xi'}^{\beta_2} A_\alpha\right| \le C_{\beta_1,\beta_2} \sum_{0\le k\le|\beta_1|+|\beta_2|} \left(\frac{|\theta|}{h}\right)^k e^{-\frac{C\delta_1|\theta|}{8h}} \le \widetilde{C}_{\beta_1,\beta_2} e^{-\frac{C\delta_1|\theta|}{9h}} \tag{4.17}$$

for all multi-indices β_1 and β_2. It follows from (4.16) and (4.17) that, mod $\mathcal{O}(h^\infty)$, the operator $(\mathcal{K}_{1,\alpha}^-)^*\mathcal{K}_{1,\alpha}^-$ is an $h-\Psi$DO in the class $\mathrm{OP}S_0^0(\partial X)$ uniformly in θ with a symbol which is $\mathcal{O}\left(e^{-2C|\theta|/h}\right)$ together with all derivatives, where $C > 0$ is a new constant. Therefore, its norm is also $\mathcal{O}\left(e^{-2C|\theta|/h}\right)$, which clearly implies (4.15). □

5. Parametrix construction in the elliptic region

We keep the notations from the previous sections and note that $\rho\chi_\delta^+ \in S_0^1(\partial X)$. It is easy also to see that $0 < C_1\langle\xi'\rangle \le |\rho| \le C_2\langle\xi'\rangle$ on $\mathrm{supp}\,\chi_\delta^+$, where C_1 and C_2 are constants depending on δ. In this section we will prove the following

Proposition 5.1. *There is a constant $C > 0$ depending on δ but independent of λ such that*

$$\left\|h\mathcal{N}(\lambda;n)\mathrm{Op}_h(\chi_\delta^+) - \mathrm{Op}_h(\rho\chi_\delta^+ + hb)\right\|_{L^2(\partial X)\to H_h^1(\partial X)} \le Ch \tag{5.1}$$

where $b \in S_0^0(\partial X)$ does not depend on λ and the function n.

Proof. The estimate (5.1) is a consequence of the parametrix built in [15]. In what follows we will recall this construction. We will first proceed locally and then we will use partition of the unity to get the global parametrix. Fix a point $x^0 \in \partial X$ and let $\mathcal{U}_0 \subset \partial X$ be a small open neighbourhood of x^0. Let (x_1, x'), $x_1 > 0$, $x' \in \mathcal{U}_0$, be the normal coordinates used in the previous section. Take a function $\psi^0 \in C_0^\infty(\mathcal{U}_0)$ and set $\psi = \psi^0\chi_\delta^+$. As in the previous section, we will construct a parametrix $\widetilde{u}_\psi^+$ of the solution of (1.1) with $\widetilde{u}_\psi^+|_{x_1=0} = \mathrm{Op}_h(\psi)f$ in the form $\widetilde{u}_\psi^+ = \phi(x_1)\mathcal{K}^+f$, where $\phi(x_1) = \chi(x_1/\delta_1)$, $0 < \delta_1 \ll 1$ being a parameter independent of λ to be fixed later on, and

$$(\mathcal{K}^+f)(x) = (2\pi h)^{-n+1}\int\int e^{\frac{i}{h}(\langle y',\xi'\rangle+\varphi(x,\xi',\theta))}a(x,\xi',\lambda)f(y')d\xi'dy'.$$

The phase φ is complex-valued such that $\varphi|_{x_1=0} = -\langle x',\xi'\rangle$ and satisfies the eikonal equation mod $\mathcal{O}(x_1^M)$:

$$(\partial_{x_1}\varphi)^2 + \langle B(x)\nabla_{x'}\varphi, \nabla_{x'}\varphi\rangle - (1+i\theta)n(x) = x_1^M\widetilde{\mathcal{R}}_M \tag{5.2}$$

where $M \gg 1$ is an arbitrary integer, the function $\widetilde{\mathcal{R}}_M$ is smooth up to the boundary $x_1 = 0$. It is shown in [15], Section 4, that for $(x',\xi') \in \operatorname{supp}\psi$, the equation (5.2) has a smooth solution of the form

$$\varphi = \sum_{k=0}^{M-1} x_1^k\varphi_k(x',\xi',\theta), \quad \varphi_0 = -\langle x',\xi'\rangle,$$

safisfying

$$\partial_{x_1}\varphi|_{x_1=0} = \varphi_1 = -i\rho. \tag{5.3}$$

Moreover, taking δ_1 small enough we can arrange that

$$\operatorname{Im}\varphi \ge -\frac{x_1}{2}\operatorname{Re}\rho \ge Cx_1\langle\xi'\rangle, \quad C > 0, \tag{5.4}$$

for $0 \le x_1 \le 3\delta_1$, $(x',\xi') \in \operatorname{supp}\psi$. The amplitude a is of the form

$$a = \sum_{j=0}^{m} h^j a_j(x,\xi',\theta)$$

where $m \gg 1$ is an arbitrary integer and the functions a_j satisfy the transport equations mod $\mathcal{O}(x_1^M)$:

$$2i\partial_{x_1}\varphi\partial_{x_1}a_j + 2i\left\langle B(x)\nabla_{x'}\varphi, \nabla_{x'}a_j\right\rangle + i\left(\Delta_X\varphi\right)a_j + \Delta_X a_{j-1} = x_1^M\widetilde{\mathcal{Q}}_M^{(j)}, \quad 0 \le j \le m, \tag{5.5}$$

$a_0|_{x_1=0} = \psi$, $a_j|_{x_1=0} = 0$ for $j \ge 1$, where $a_{-1} = 0$ and the functions $\widetilde{\mathcal{Q}}_M^{(j)}$ are smooth up to the boundary $x_1 = 0$. It is shown in [15], Section 4, that the equations (5.5) have unique smooth solutions of the form

$$a_j = \sum_{k=0}^{M-1} x_1^k a_{k,j}(x',\xi',\theta)$$

with functions $a_{k,j} \in S_0^{-j}(\partial X)$ uniformly in θ. We can write

$$V_+ := h^{-1}(h^2\Delta_X + (1+i\theta)n(x))\widetilde{u}_\psi^+ = \mathcal{K}_1^+f + \mathcal{K}_2^+f$$

where

$$\mathcal{K}_1^+ f = h[\Delta_X, \phi]\mathcal{K}^+ f = h(2\phi'(x_1)\partial_{x_1} + c(x)\phi''(x_1))\mathcal{K}^+ f$$
$$= (2\pi h)^{-n+1} \int\int e^{\frac{i}{h}(\langle y',\xi'\rangle + \varphi(x,\xi',\theta))} A_1^+(x,\xi',\lambda) f(y') d\xi' dy',$$
$$A_1^+ = 2i\phi' a \partial_{x_1}\varphi + hc\phi''\partial_{x_1} a$$

and

$$(\mathcal{K}_2^+ f)(x) = (2\pi h)^{-n+1} \int\int e^{\frac{i}{h}(\langle y',\xi'\rangle + \varphi(x,\xi',\theta))} A_2^+(x,\xi',\lambda) f(y') d\xi' dy'$$

where

$$A_2^+ = \phi(x_1)\left(h^{-1} x_1^M \widetilde{\mathcal{R}}_M a + x_1^M \sum_{j=0}^m h^j \widetilde{\mathcal{Q}}_M^{(j)} + h^{m+1}\Delta_X a_m \right).$$

As in the previous section, we will derive Proposition 5.1 from (5.3) and the following

Lemma 5.2. *The function V_+ satisfies the estimate*

$$\|V_+\|_{H_h^1(X)} \le \mathcal{O}_m\left(h^{m-n}\right)\|f\| + \mathcal{O}_M\left(h^{M-n}\right)\|f\|. \tag{5.6}$$

Proof. Let α be a multi-index such that $|\alpha| \le 1$. In view of (5.4) we have

$$\left| e^{i\varphi/h}\left(i|\alpha| A_1^+ \partial_x^\alpha \varphi + (h\partial_x)^\alpha A_1^+ \right) \right|$$
$$\lesssim \sup_{\delta_1/2 \le x_1 \le 3\delta_1} e^{-\operatorname{Im}\varphi/h} \lesssim e^{-C\langle\xi'\rangle/h} = \mathcal{O}_M\left((h/\langle\xi'\rangle)^M\right)$$

for every integer $M \gg 1$. Therefore, the kernel of the operator $(h\partial_x)^\alpha \mathcal{K}_1^+ : L^2(\partial X) \to L^2(X)$ is $\mathcal{O}_M\left(h^{M-n+1}\right)$, and hence so is its norm. By (5.4) we also have

$$x_1^M e^{-\operatorname{Im}\varphi/h} \le x_1^M e^{-Cx_1\langle\xi'\rangle/h} = \mathcal{O}_M\left((h/\langle\xi'\rangle)^M\right).$$

This implies that

$$e^{i\varphi/h}\left(i|\alpha| A_2^+ \partial_x^\alpha \varphi + (h\partial_x)^\alpha A_2^+ \right) = \mathcal{O}_M\left((h/\langle\xi'\rangle)^{M-1}\right) + \mathcal{O}_m\left((h/\langle\xi'\rangle)^m\right)$$

which again implies the desired bound for the norm of the operator $(h\partial_x)^\alpha \mathcal{K}_2^+$. □

By the estimates (3.10) and (5.6) we have

$$\left\| h\mathcal{N}(\lambda;n)\operatorname{Op}_h(\psi) - T_\psi^+ \right\|_{L^2(\partial X)\to H_h^1(\partial X)} \le \mathcal{O}_m\left(h^{m-n}\right) + \mathcal{O}_M\left(h^{M-n}\right) \tag{5.7}$$

where the operator T_ψ^+ is defined by

$$T_\psi^+ f = h\partial_{x_1}\mathcal{K}^+ f|_{x_1=0}.$$

In view of (5.3), we have

$$\left(T_\psi^+ f\right)(x') = (2\pi h)^{-n+1} \int\int e^{\frac{i}{h}\langle y'-x',\xi'\rangle}(i\psi\partial_{x_1}\varphi(0,x',\xi',\theta) + h\partial_{x_1} a(0,x',\xi',\lambda)) f(y') d\xi' dy'$$
$$= \operatorname{Op}_h(\rho\psi) f + \sum_{j=0}^m h^{j+1}\operatorname{Op}_h(a_{1,j}(x',\xi',\theta)) f$$

where $a_{1,j} \in S_0^{-j}(\partial X)$. Hence

$$\operatorname{Op}_h(a_{1,j}) = \mathcal{O}(1) : L^2(\partial X) \to H_h^j(\partial X).$$

Therefore it follows from (5.7) that

$$\|h\mathcal{N}(\lambda;n)\mathrm{Op}_h(\psi)-\mathrm{Op}_h(\rho\psi+ha_{1,0})\|_{L^2(\partial X)\to H^1_h(\partial X)}\le\mathcal{O}(h). \tag{5.8}$$

We need now the following

Lemma 5.3. *There exists a function $b^0\in S^0_0(\partial X)$ independent of λ and n such that*

$$a_{1,0}-b^0\in S^{-1}_0(\partial X). \tag{5.9}$$

Proof. We will calculate the function $a_{1,0}$ explicitly. Note that this lemma (resp. Proposition 5.1) is also used in [15], but the proof therein is not correct since $a_{1,0}$ is calculated incorrectly. Therefore we will give here a new proof. Clearly, it suffices to prove (5.9) with $a_{1,0}$ replaced by $(1-\eta)a_{1,0}$ with some function $\eta\in C^\infty_0(T^*\partial X)$ independent of h. Since $\rho=-\sqrt{r_0}\,\big(1+\mathcal{O}(r_0^{-1})\big)$ as $r_0\to\infty$, it is easy to see that

$$(1-\eta)\rho^{-k}-(1-\eta)(-\sqrt{r_0})^{-k}\in S_0^{-k-1}(\partial X) \tag{5.10}$$

for every integer $k\ge 0$, provided η is taken such that $\eta=1$ for $|\xi'|\le A$ with some $A>1$ big enough. We will now calculate the function φ_2 from the eikonal equation. To this end, write

$$B(x)=B_0(x')+x_1B_1(x')+\mathcal{O}(x_1^2),\quad n(x)=n_0(x')+x_1n_1(x')+\mathcal{O}(x_1^2)$$

and observe that the LHS of (5.2) is equal to

$$x_1\left(4\varphi_1\varphi_2+2\langle B_0\nabla_{x'}\varphi_0,\nabla_{x'}\varphi_1\rangle+\langle B_1\nabla_{x'}\varphi_0,\nabla_{x'}\varphi_0\rangle-(1+i\theta)n_1\right)+\mathcal{O}(x_1^2).$$

Hence, taking into account that $\varphi_0=-\langle x',\xi'\rangle$ and $\varphi_1=-i\rho$, we get

$$\varphi_2=(2\rho)^{-1}\langle B_0\xi',\nabla_{x'}\rho\rangle+(4i\rho)^{-1}\langle B_1\xi',\xi'\rangle-(1+i\theta)(4i\rho)^{-1}n_1.$$

Using the identity

$$2\rho\nabla_{x'}\rho=\nabla_{x'}r_0-(1+i\theta)\nabla_{x'}n_0$$

we can write φ_2 in the form

$$\varphi_2=(2\rho)^{-2}\langle B_0\xi',\nabla_{x'}r_0\rangle+(4i\rho)^{-1}\langle B_1\xi',\xi'\rangle$$
$$-(1+i\theta)(2\rho)^{-2}\langle B_0\xi',\nabla_{x'}n_0\rangle-(1+i\theta)(4i\rho)^{-1}n_1.$$

By (5.10) we conclude that, mod $S_0^{-1}(\partial X)$,

$$(1-\eta)\frac{\varphi_2}{\varphi_1}=-i4^{-1}(1-\eta)r_0^{-3/2}\langle B_0\xi',\nabla_{x'}r_0\rangle+(1-\eta)(4r_0)^{-1}\langle B_1\xi',\xi'\rangle. \tag{5.11}$$

Write now the operator Δ_X in the form

$$\Delta_X=\partial^2_{x_1}+\langle B_0\nabla_{x'},\nabla_{x'}\rangle+q_1(x')\partial_{x_1}+\langle q_2(x'),\nabla_{x'}\rangle+\mathcal{O}(x_1)$$

and observe that

$$\Delta_X\varphi=2\varphi_2+q_1\varphi_1-\langle q_2(x'),\xi'\rangle+\mathcal{O}(x_1).$$

We now calculate the LHS of the equation (5.5) with $j=0$ modulo $\mathcal{O}(x_1)$. Recall that $a_{0,0}=\psi$. We obtain

$$2i\varphi_1a_{1,0}+2i\langle B_0\nabla_{x'}\varphi_0,\nabla_{x'}a_{0,0}\rangle+i(\Delta_X\varphi)a_{0,0}$$
$$=2i\varphi_1a_{1,0}+2i\langle B_0\xi',\nabla_{x'}\psi\rangle+i(2\varphi_2+q_1\varphi_1-\langle q_2(x'),\xi'\rangle)\psi.$$

Since the RHS is $\mathcal{O}(x_1^M)$, the above function must be identically zero. Thus we get the following expression for the function $a_{1,0}$:

$$a_{1,0}=-\varphi_1^{-1}\langle B_0\xi',\nabla_{x'}\psi\rangle-(\varphi_1^{-1}\varphi_2+2^{-1}q_1-(2\varphi_1)^{-1}\langle q_2(x'),\xi'\rangle)\psi. \tag{5.12}$$

Taking into account that $\psi = \psi^0$ on $\operatorname{supp}(1-\eta)$, we find from (5.10), (5.11) and (5.12) that (5.9) holds with

$$b^0 = i(1-\eta)r_0^{-1/2}\langle B_0\xi', \nabla_{x'}\psi^0\rangle$$
$$-4^{-1}(1-\eta)\psi^0\left(-ir_0^{-3/2}\langle B_0\xi', \nabla_{x'}r_0\rangle + r_0^{-1}\langle B_1\xi', \xi'\rangle + 2q_1 + 2r_0^{-1/2}\langle q_2(x'), \xi'\rangle\right). \tag{5.13}$$

Clearly, $b^0 \in S_0^0(\partial X)$ is independent of λ and n, as desired. □

Lemma 5.3 implies that

$$\operatorname{Op}_h(a_{1,0} - b^0) = \mathcal{O}(1) : L^2(\partial X) \to H_h^1(\partial X). \tag{5.14}$$

Now, using a suitable partition of the unity on ∂X we can write $1 = \sum_{j=1}^J \psi_j^0$. Hence, we can write the function χ_δ^+ as $\sum_{j=1}^J \psi_j$, where $\psi_j = \psi_j^0\chi_\delta^+$. Since we have (5.8) and (5.14) with ψ replaced by each ψ_j, we get (5.1) by summing up all the estimates. □

It follows from the estimate (3.11) applied with $V \equiv 0$ that

$$h\mathcal{N}(\lambda; n)\operatorname{Op}_h(\chi_\delta^0) = \mathcal{O}(\delta) : L^2(\partial X) \to H_h^1(\partial X) \tag{5.15}$$

provided $|\operatorname{Im}\lambda| \geq \delta^{-4}$ and $\operatorname{Re}\lambda \geq C_\delta \gg 1$. Now Theorem 1.2 follows from (5.15) and Propositions 4.1 and 5.1. Let us now see that Theorem 1.1 follows from Theorem 1.2. Since the operator $-h^2\Delta_{\partial X} \geq 0$ is self-adjoint, we have the bound

$$\left\|hp(-\Delta_{\partial X})\chi_2((-h^2\Delta_{\partial X} - 1)\delta^{-2})\right\|$$
$$= \left\|\sqrt{-h^2\Delta_{\partial X} - 1 - i\theta}\,\chi((-h^2\Delta_{\partial X} - 1)\delta^{-2})\right\|$$
$$\leq \sup_{\sigma\geq 0}\left|\sqrt{\sigma - 1 - i\theta}\,\chi((\sigma-1)\delta^{-2})\right| \leq \sup_{\delta^2\leq|\sigma-1|\leq 2\delta^2}\sqrt{|\sigma-1| + |\theta|}$$
$$\leq \mathcal{O}(\delta + |\theta|^{1/2}) = \mathcal{O}(\delta + h^{\epsilon/2}). \tag{5.16}$$

On the other hand, it is well-known that the operator $hp(-\Delta_{\partial X})(1-\chi_2)((-h^2\Delta_{\partial X} - 1)\delta^{-2})$ is an $h-\Psi$DO in the class $\operatorname{OP}S_0^1(\partial X)$ with principal symbol $\rho(1-\chi_\delta^0)$. This implies the bound

$$hp(-\Delta_{\partial X})(1-\chi_2)((-h^2\Delta_{\partial X} - 1)\delta^{-2}) - \operatorname{Op}_h(\rho(1-\chi_\delta^0)) = \mathcal{O}(h) : L^2(\partial X) \to L^2(\partial X). \tag{5.17}$$

It is easy to see that Theorem 1.1 follows from (1.3) together with (5.16) and (5.17). □

6. Proof of Theorem 2.1

Define the DN maps $\mathcal{N}_j(\lambda)$, $j = 1, 2$, by

$$\mathcal{N}_j(\lambda)f = \partial_\nu u_j|_\Gamma$$

where ν is the Euclidean unit normal to Γ and u_j is the solution to the equation

$$\begin{cases} \left(\nabla c_j(x)\nabla + \lambda^2 n_j(x)\right)u_j = 0 & \text{in} \quad \Omega, \\ u_j = f & \text{on} \quad \Gamma, \end{cases} \tag{6.1}$$

and consider the operator

$$T(\lambda) = c_1\mathcal{N}_1(\lambda) - c_2\mathcal{N}_2(\lambda).$$

Clearly, λ is a transmission eigenvalue if there exists a non-trivial function f such that $T(\lambda)f = 0$. Therefore Theorem 2.1 is a consequence of the following

Theorem 6.1. *Under the conditions of Theorem 2.1, the operator $T(\lambda)$ sends $H^{\frac{1+k}{2}}(\Gamma)$ into $H^{\frac{1-k}{2}}(\Gamma)$, where $k=-1$ if (2.2) holds and $k=1$ if (2.4) holds. Moreover, there exists a constant $C>0$ such that $T(\lambda)$ is invertible for $\mathrm{Re}\,\lambda\geq 1$ and $|\mathrm{Im}\,\lambda|\geq C$ with an inverse satisfying in this region the bound*

$$\left\|T(\lambda)^{-1}\right\|_{H^{\frac{1-k}{2}}(\Gamma)\to H^{\frac{1+k}{2}}(\Gamma)}\lesssim|\lambda|^{\frac{k-1}{2}} \tag{6.2}$$

where the Sobolev spaces are equipped with the classical norms.

Proof. We may suppose that $\lambda\in\Lambda_\epsilon=\{\lambda\in\mathbf{C}:\mathrm{Re}\,\lambda\geq C_\epsilon\gg 1,\ |\mathrm{Im}\,\lambda|\leq|\lambda|^\epsilon\}$, $0<\epsilon\ll 1$, since the case when $\lambda\in\{\mathrm{Re}\,\lambda\geq 1\}\setminus\Lambda_\epsilon$ follows from the analysis in [15]. We will equip the boundary Γ with the Riemannian metric induced by the Euclidean metric g_E in Ω and will denote by r_0 the principal symbol of the Laplace-Beltrami operator $-\Delta_\Gamma$. We would like to apply Theorem 1.2 to the operators $\mathcal{N}_j(\lambda)$. However, some modifications must be done comming from the presence of the function c_j in the equation (6.1). Indeed, in the definition of the operator $\mathcal{N}(\lambda;n)$ in Section 1 the normal derivative is taken with respect to the Riemannian metric $g_j=c_j^{-1}g_E$, while in the definition of the operator $\mathcal{N}_j(\lambda)$ it is taken with respect to the metric g_E. The first observation to be done is that the glancing region corresponding to the problem (6.1) is defined by $\Sigma_j:=\{(x',\xi')\in T^*\Gamma: r_j(x',\xi')=1\}$, where $r_j:=m_j^{-1}r_0$, $m_j:=\frac{n_j}{c_j}|_\Gamma$. We define now the cut-off functions $\chi^0_{\delta,j}$ by replacing in the definition of χ^0_δ the function $r_\sharp$ by r_j. Secondly, the function ρ must be replaced by

$$\rho_j(x',\xi')=\sqrt{r_0(x',\xi')-(1+i\theta)m_j(x')},\quad \mathrm{Re}\,\rho_j<0.$$

With these changes the operator $\mathcal{N}_j(\lambda)$ satisfies the estimate (1.3). Set

$$\tau_\delta=c_1\rho_1(1-\chi^0_{\delta,1})-c_2\rho_2(1-\chi^0_{\delta,2})=\tau-c_1\rho_1\chi^0_{\delta,1}+c_2\rho_2\chi^0_{\delta,2}$$

where

$$\tau=c_1\rho_1-c_2\rho_2=\frac{\widetilde{c}(x')(c_0(x')r_0(x',\xi')-1-i\theta)}{c_1\rho_1+c_2\rho_2} \tag{6.3}$$

where $\widetilde{c}$ and c_0 are the restrictions on Γ of the functions

$$c_1n_1-c_2n_2\quad\text{and}\quad\frac{c_1^2-c_2^2}{c_1n_1-c_2n_2}$$

respectively. Clearly, under the conditions of Theorem 2.1, we have $\widetilde{c}(x')\neq 0$, $\forall x'\in\Gamma$. Moreover, (2.2) implies $c_0\equiv 0$, while (2.4) implies $c_0(x')<0$, $\forall x'\in\Gamma$. Hence,

$$0<C_1\leq|c_0r_0-1-i\theta|\leq C_2,$$

if (2.2) holds, and

$$0<C_1\langle r_0\rangle\leq|c_0r_0-1-i\theta|\leq C_2\langle r_0\rangle,$$

if (2.4) holds. Using this together with (6.3) and the fact that $\rho_j\sim-\sqrt{r_0}$ as $r_0\to\infty$, we get

$$0<C_1'\langle\xi'\rangle^k\leq C_1\langle r_0\rangle^{k/2}\leq|\tau|\leq C_2\langle r_0\rangle^{k/2}\leq C_2'\langle\xi'\rangle^k \tag{6.4}$$

where $k=-1$ if (2.2) holds, $k=1$ if (2.4) holds. Let $\eta\in C_0^\infty(T^*\Gamma)$ be such that $\eta=1$ on $|\xi'|\leq A$, $\eta=0$ on $|\xi'|\geq A+1$, where $A\gg 1$ is a big parameter independent of λ and δ. Taking A big enough we can arrange that $(1-\eta)\tau_\delta=(1-\eta)\tau$. On the other hand, we have $\eta\tau_\delta=\eta\tau+\mathcal{O}(\delta+|\theta|^{1/2})$. Therefore, taking δ and $|\theta|$ small enough we get from (6.4) that the function τ_δ satisfies the bounds

$$\widetilde{C}_1\langle\xi'\rangle^k\leq|\tau_\delta|\leq\widetilde{C}_2\langle\xi'\rangle^k \tag{6.5}$$

with positive constants $\widetilde{C}_1$ and $\widetilde{C}_2$ independent of δ and θ. Furthermore, one can easily check that $(1-\eta)\tau \in S_0^k(\Gamma)$ and $\eta\tau_\delta \in S_0^{-2}(\Gamma)$. Hence, $\tau_\delta \in S_0^k(\Gamma)$, which in turn implies that the operator $\mathrm{Op}_h(\tau_\delta)$ sends $H^{\frac{1+k}{2}}(\Gamma)$ into $H^{\frac{1-k}{2}}(\Gamma)$. Moreover, it follows from (6.5) that the operator $\mathrm{Op}_h(\tau_\delta) : H_h^{\frac{1+k}{2}}(\Gamma) \to H_h^{\frac{1-k}{2}}(\Gamma)$ is invertible with an inverse satisfying the bound

$$\left\|\mathrm{Op}_h(\tau_\delta)^{-1}\right\|_{H_h^{\frac{1-k}{2}}(\Gamma)\to H_h^{\frac{1+k}{2}}(\Gamma)} \leq \widetilde{C} \tag{6.6}$$

with a constant $\widetilde{C} > 0$ independent of λ and δ. We now apply Theorem 2.1 to the operators $\mathcal{N}_j(\lambda)$. We get, for $\lambda \in \Lambda_\epsilon$, $|\mathrm{Im}\,\lambda| \geq C_\delta \gg 1$, $\mathrm{Re}\,\lambda \geq C_{\epsilon,\delta} \gg 1$, that

$$\|hT(\lambda) - \mathrm{Op}_h(\tau_\delta)\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq C\delta \tag{6.7}$$

in the anisotropic case, and

$$\|hT(\lambda) - \mathrm{Op}_h(\tau_\delta)\|_{L^2(\Gamma)\to H_h^1(\Gamma)} \leq C\delta \tag{6.8}$$

in the isotropic case, where $C > 0$ is a constant independent of λ and δ. Introduce the operators

$$\mathcal{A}_1(\lambda) = (hT(\lambda) - \mathrm{Op}_h(\tau_\delta))\,\mathrm{Op}_h(\tau_\delta)^{-1},$$

$$\mathcal{A}_2(\lambda) = \mathrm{Op}_h(\tau_\delta)^{-1}\,(hT(\lambda) - \mathrm{Op}_h(\tau_\delta))\,.$$

It follows from (6.6), (6.7) and (6.8) that in the anisotropic case we have the bound

$$\|\mathcal{A}_1(\lambda)\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq C'\delta \tag{6.9}$$

while in the isotropic case we have the bound

$$\|\mathcal{A}_2(\lambda)\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq C'\delta \tag{6.10}$$

where $C' > 0$ is a constant independent of λ and δ. Hence, taking δ small enough we can arrange that the operators $1 + \mathcal{A}_j(\lambda)$ are invertible on $L^2(\Gamma)$ with inverses whose norms are bounded by 2. We now write the operator $hT(\lambda)$ as

$$hT(\lambda) = (1 + \mathcal{A}_1(\lambda))\mathrm{Op}_h(\tau_\delta)$$

in the anisotropic case, and as

$$hT(\lambda) = \mathrm{Op}_h(\tau_\delta)(1 + \mathcal{A}_2(\lambda))$$

in the isotropic case. Therefore, the operator $hT(\lambda)$ is invertible in the desired region and by (6.6) we get the bound

$$\left\|(hT(\lambda))^{-1}\right\|_{H_h^{\frac{1-k}{2}}(\Gamma)\to H_h^{\frac{1+k}{2}}(\Gamma)} \leq 2\widetilde{C}. \tag{6.11}$$

Passing from semi-classical to classical Sobolev norms one can easily see that (6.11) implies (6.2). □

7. Proof of Theorem 2.2

We keep the notations from the previous section. Theorem 2.2 is a consequence of the following

Theorem 7.1. *Under the conditions of Theorem 2.2, there exists a constant $C > 0$ such that the operator $T(\lambda) : H^1(\Gamma) \to L^2(\Gamma)$ is invertible for $\mathrm{Re}\,\lambda \geq 1$ and $|\mathrm{Im}\,\lambda| \geq C\log(\mathrm{Re}\,\lambda + 1)$ with an inverse satisfying in this region the bound*

$$\left\|T(\lambda)^{-1}\right\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim 1. \tag{7.1}$$

Proof. As in the previous section we may suppose that $\lambda \in \Lambda_\epsilon$. We will again make use of the identity (6.3) with the difference that under the condition (2.6) we have $c_0(x') > 0$, $\forall x' \in \Gamma$. This means that $|\tau|$ can get small near the characteristic variety $\Sigma = \{(x', \xi') \in T^*\Gamma : r(x', \xi') = 1\}$, where $r := c_0 r_0$. Clearly, the assumption (2.7) implies that $\Sigma_1 \cap \Sigma_2 = \emptyset$. This in turn implies that $\Sigma \cap \Sigma_j = \emptyset$, $j = 1, 2$. Indeed, if we suppose that there is a $\zeta^0 \in \Sigma \cap \Sigma_j$ for $j = 1$ or $j = 2$, then it is easy to see that $\zeta^0 \in \Sigma_1 \cap \Sigma_2$, which however is impossible in view of (2.7). Therefore, we can choose a cut-off function $\chi^0 \in C^\infty(T^*\Gamma)$ such that $\chi^0 = 1$ in a small neighbourhood of Σ, $\chi^0 = 0$ outside another small neighbourhood of Σ, and $\operatorname{supp} \chi^0 \cap \Sigma_j = \emptyset$, $j = 1, 2$. This means that $\operatorname{supp} \chi^0$ belongs either to the hyperbolic region $\{r_j \le 1 - \delta^2\}$ or to the elliptic region $\{r_j \ge 1 + \delta^2\}$, provided $\delta > 0$ is taken small enough. Therefore, we can use Propositions 4.1 and 5.1 to get the estimate

$$\left\| h\mathcal{N}_j(\lambda) \mathrm{Op}_h(\chi^0) - \mathrm{Op}_h(\rho_j \chi^0) \right\|_{L^2(\Gamma) \to L^2(\Gamma)} \lesssim h + e^{-C|\mathrm{Im}\,\lambda|}$$

which implies

$$\left\| hT(\lambda) \mathrm{Op}_h(\chi^0) - \mathrm{Op}_h(\tau \chi^0) \right\|_{L^2(\Gamma) \to L^2(\Gamma)} \lesssim h + e^{-C|\mathrm{Im}\,\lambda|}. \tag{7.2}$$

It follows from (6.3) that near Σ the function τ is of the form $\tau = \tau_0(r - 1 - i\theta)$ with some smooth function $\tau_0 \neq 0$. We now extend τ_0 globally on $T^*\Gamma$ to a function $\widetilde{\tau}_0 \in S^0_0(\Gamma)$ such that $\widetilde{\tau}_0 = \tau_0$ on $\operatorname{supp} \chi^0$ and $|\widetilde{\tau}_0| \ge Const > 0$ on $T^*\Gamma$. Hence, we can write the operator $\mathrm{Op}_h(\tau\chi^0)$ as follows

$$\mathrm{Op}_h(\tau\chi^0) = \mathrm{Op}_h(\chi^0)\mathrm{Op}_h(\widetilde{\tau}_0)(\mathcal{B} - i\theta) + \mathcal{O}(h)$$

where $\mathcal{B} = \frac{1}{2}\mathrm{Op}_h(r-1) + \frac{1}{2}\mathrm{Op}_h(r-1)^*$ is a self-adjoint operator. Hence

$$(\mathcal{B} - i\theta)^{-1} = \mathcal{O}(|\theta|^{-1}) : L^2(\Gamma) \to L^2(\Gamma).$$

Since $\widetilde{\tau}_0$ is globally elliptic, we also have

$$\mathrm{Op}_h(\widetilde{\tau}_0)^{-1} = \mathcal{O}(1) : L^2(\Gamma) \to L^2(\Gamma).$$

This implies

$$K_1 := \mathrm{Op}_h(\chi^0)(\mathcal{B} - i\theta)^{-1}\mathrm{Op}_h(\widetilde{\tau}_0)^{-1} = \mathcal{O}(|\theta|^{-1}) : L^2(\Gamma) \to L^2(\Gamma)$$

and (7.2) leads to the estimate

$$\begin{aligned} \left\| hT(\lambda)K_1 - \mathrm{Op}_h(\chi^0) \right\|_{L^2(\Gamma) \to L^2(\Gamma)} &\lesssim |\theta|^{-1} \left(h + e^{-C|\mathrm{Im}\,\lambda|} \right) \\ &\lesssim |\mathrm{Im}\,\lambda|^{-1} + \mathrm{Re}\,\lambda\, e^{-C|\mathrm{Im}\,\lambda|} \le \delta \end{aligned} \tag{7.3}$$

for any $0 < \delta \ll 1$, provided $|\mathrm{Im}\,\lambda| \ge C_\delta \log(\mathrm{Re}\,\lambda)$, $\mathrm{Re}\,\lambda \ge \widetilde{C}_\delta$ with some constants $C_\delta, \widetilde{C}_\delta > 0$. On the other hand, by Theorem 1.2 we have, for $\lambda \in \Lambda_\epsilon$, $|\mathrm{Im}\,\lambda| \ge C_\delta \gg 1$, $\mathrm{Re}\,\lambda \ge C_{\epsilon,\delta} \gg 1$,

$$\left\| hT(\lambda)\mathrm{Op}_h(1 - \chi^0) - \mathrm{Op}_h(\tau_\delta(1 - \chi^0)) \right\|_{L^2(\Gamma) \to L^2(\Gamma)} \le C\delta. \tag{7.4}$$

As in the proof of (6.5) one can see that the function τ_δ satisfies

$$\widetilde{C}_1 \langle \xi' \rangle \le |\tau_\delta| \le \widetilde{C}_2 \langle \xi' \rangle \quad \text{on} \quad \operatorname{supp}(1 - \chi^0) \tag{7.5}$$

with positive constants $\widetilde{C}_1$ and $\widetilde{C}_2$ independent of δ and θ. Moreover, $\tau_\delta \in S^1_0(\Gamma)$. We extend the function τ_δ on the whole $T^*\Gamma$ to a function $\widetilde{\tau}_\delta \in S^1_0(\Gamma)$ such that $\widetilde{\tau}_\delta(1 - \chi^0) = \tau_\delta(1 - \chi^0)$ and

$$\widetilde{C}'_1 \langle \xi' \rangle \le |\widetilde{\tau}_\delta| \le \widetilde{C}'_2 \langle \xi' \rangle \quad \text{on} \quad T^*\Gamma. \tag{7.6}$$

Hence

$$\left\| \mathrm{Op}_h(\widetilde{\tau}_\delta)^{-1} \right\|_{L^2(\Gamma) \to L^2(\Gamma)} \le \widetilde{C} \tag{7.7}$$

with a constant $\widetilde{C} > 0$ independent of λ and δ. By (7.4) and (7.7) we obtain

$$\left\| hT(\lambda)K_2 - \mathrm{Op}_h(1-\chi^0) \right\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq C\delta \tag{7.8}$$

with a new constant $C > 0$ independent of λ and δ, where

$$K_2 := \mathrm{Op}_h(1-\chi^0)\mathrm{Op}_h(\widetilde{\tau}_\delta)^{-1} = \mathcal{O}(1) : L^2(\Gamma) \to L^2(\Gamma).$$

By (7.3) and (7.8),

$$\|hT(\lambda)(K_1+K_2) - 1\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq (C+1)\delta. \tag{7.9}$$

It follows from (7.9) that if δ is taken small enough, the operator $hT(\lambda)$ is invertible with an inverse satisfying the bound

$$\left\| (hT(\lambda))^{-1} \right\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq 2\|K_1\|_{L^2(\Gamma)\to L^2(\Gamma)} + 2\|K_2\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim |\theta|^{-1} + 1. \tag{7.10}$$

It is easy to see that (7.10) implies (7.1) □

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