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When Does Output Feedback Enlarge the Capacity of the Interference Channel?

Victor Quintero, Samir M. Perlaza, Iñaki Esnaola, Jean-Marie Gorce

Abstract

In this paper, the impact of additive noise over the feedback links in the two-user interference channel is studied using linear deterministic approximations. More specifically, only one transmitter-receiver pair is let to use feedback. Under this condition, the exact value of the signal to noise ratio (SNR) on the feedback link beyond which the capacity region of the linear deterministic approximation is enlarged is fully described in terms of the forward SNRs and interference to noise ratios (INRs). In general, three scenarios can be observed depending on the exact values of the forward SNRs and INRs. In the first case, either individual information transmission rate can be improved; in the second case, using feedback from receiver i to transmitter i exclusively increases the individual rate of transmitter-receiver pair j, with $i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$. In the third case, implementing feedback turns out to be useless for enlarging the capacity region. The same analysis is performed for the sum-rate and the scenarios in which the sum-rate can be improved, as well as the corresponding minimum feedback SNR, are fully described.

Index Terms

Linear Deterministic Interference Channel, Noisy Channel-Output Feedback, Capacity Region.

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I. INTRODUCTION

Channel-output feedback is an interference management technique that significantly increases the number of generalized degrees of freedom (GDoF) [2] for the two-user Gaussian interference channel (IC) in most of the interference regimes [3]. Essentially, in the very strong interference regime, the GDoF gain provided by perfect-channel output feedback can be arbitrarily large when the interference to noise ratios (INRs) and signal to noise ratios (SNRs) grow to infinity at a constant rate $\alpha = \frac{\log \mathrm{INR}}{\log \mathrm{SNR}}$ greater than 2. One of the reasons why feedback provides such a surprising benefit stems from the fact that it uses interference to create alternative paths to the existing point-to-point paths. For instance, in the two-user IC, feedback creates a path from transmitter 1 (resp. transmitter 2) to receiver 1 (resp. receiver 2) in which symbols that are received at receiver 2 (resp. receiver 1) are fed back to transmitter 2 (resp. transmitter 1), which decodes the messages and retransmits them to receiver 1 (resp. receiver 2). The GDoF gain due to feedback in the IC depends on the topology of the network and the number of transmitter-receiver pairs in the network. For instance, in the symmetric K-user cyclic Z-interference channel, the GDoF gain does not increase with K [4]. In particular, in the very strong interference regime, the GDoF gain is shown to be monotonically decreasing with K. Alternatively, in the fully connected symmetric K-user IC with perfect feedback, the number of GDoF per user is shown to be identical to the one in the two-user case, with an exception in a particular singularity, and totally independent of the exact number of transmitter-receiver pairs [5]. It is important to highlight that the network topology, the number of transmitter-receiver pairs and the interference regime are not the only parameters determining the effect of feedback. Indeed, the capacity of the feedback links turns out to be another relevant factor [6]. As shown later in this paper, in the case in which one transmitter-receiver pair is in a high interference regime (the interfering signal is stronger than the intended signal) and the other is in a low interference regime (the interfering signal is weaker than the intended signal), the use of feedback in the former does not enlarge the capacity region, even in the case of perfect output feedback. Conversely, using feedback in the latter might enlarge the capacity region depending on the SNR of the feedback link. In the following the feedback links are assumed to be rate limited due to the action of additive noise. The exact values of the feedback SNRs beyond which the capacity region is enlarged depend on all the other channel parameters: two forward SNRs and two forward INRs. In [7], the capacity

region of the two-user Gaussian IC (GIC) with noisy channel output feedback is approximated to within a constant number of bits for the symmetric case. These results are generalized in [8] for non-symmetric cases. However, from the available descriptions of the capacity regions with and without feedback, identifying whether or not the existence of a feedback link with a given SNR enlarges the capacity region is not a trivial task.

An alternative for dealing with the challenges described above is to study the GIC via its linear deterministic IC (LD-IC) approximation [9], for which the capacity region is perfectly known [8], [10]. The two-user LD-IC with noisy channel output feedback (LD-IC-NOF) is fully described by six parameters: $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22}) \in \mathbb{N}^6$. There exists a mapping between the parameters describing the two-user LD-IC and the parameters describing the GIC. More specifically, there are two forward SNRs ($\overrightarrow{\mathrm{SNR}}_i \geqslant 1$); two forward INRs ($\mathrm{INR}_{ij} \geqslant 1$); and two backward SNRs ($\overleftarrow{\mathrm{SNR}}_i \geqslant 1$), with $i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$. In the LD-IC, the parameters of the GIC are mapped into the number of bit-pipes between each transmitter and its corresponding intended receiver, i.e., $\overrightarrow{n}_{ii} = \lfloor \frac{1}{2} \log_2(\overrightarrow{\mathrm{SNR}}_i) \rfloor$; between transmitter j and receiver i i.e., $n_{ij} = \lfloor \frac{1}{2} \log_2(\mathrm{INR}_{ij}) \rfloor$; and between each receiver and its corresponding transmitter, i.e., $\overleftarrow{n}_{ii} = \lfloor \frac{1}{2} \log_2(\overleftarrow{\mathrm{SNR}}_i) \rfloor$. An LD-IC without feedback corresponds to the case in which $\overleftarrow{n}_{11} = \overleftarrow{n}_{22} = 0$ and the capacity region is denoted by $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, 0)$. In the case in which feedback is available at both transmitters, $\overleftarrow{n}_{11} > 0$ and $\overleftarrow{n}_{22} > 0$, the capacity is denoted by $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$.

This paper presents the exact conditions on $\overleftarrow{n}_{11}$ (resp. $\overleftarrow{n}_{22}$) for observing an improvement in the capacity region $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$ (resp. $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, \overleftarrow{n}_{22})$) when compared to $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, 0)$, for any 4-tuple $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}) \in \mathbb{N}^4$. More specifically, it is shown that there exists a threshold for the number of bit-pipes in the feedback link of transmitter-receiver pair 1 (resp. 2), beyond which the capacity region of the two-user LD-IC-NOF can be enlarged, i.e., $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, 0) \subset \mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$ (resp. $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, 0) \subset \mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, \overleftarrow{n}_{22})$), with strict inclusion. The exact conditions on $\overleftarrow{n}_{11}$ (resp. $\overleftarrow{n}_{22}$) to observe an improvement on a single rate or the sum-rate capacity, for any 4-tuple $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}) \in \mathbb{N}^4$ are also presented in this paper. Surprisingly, these values can be expressed in closed-form using relatively simple expressions that depend on some of the parameters $\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}$ and n_{21}.

Based on these results, several relevant engineering questions arise in this setting. For instance,

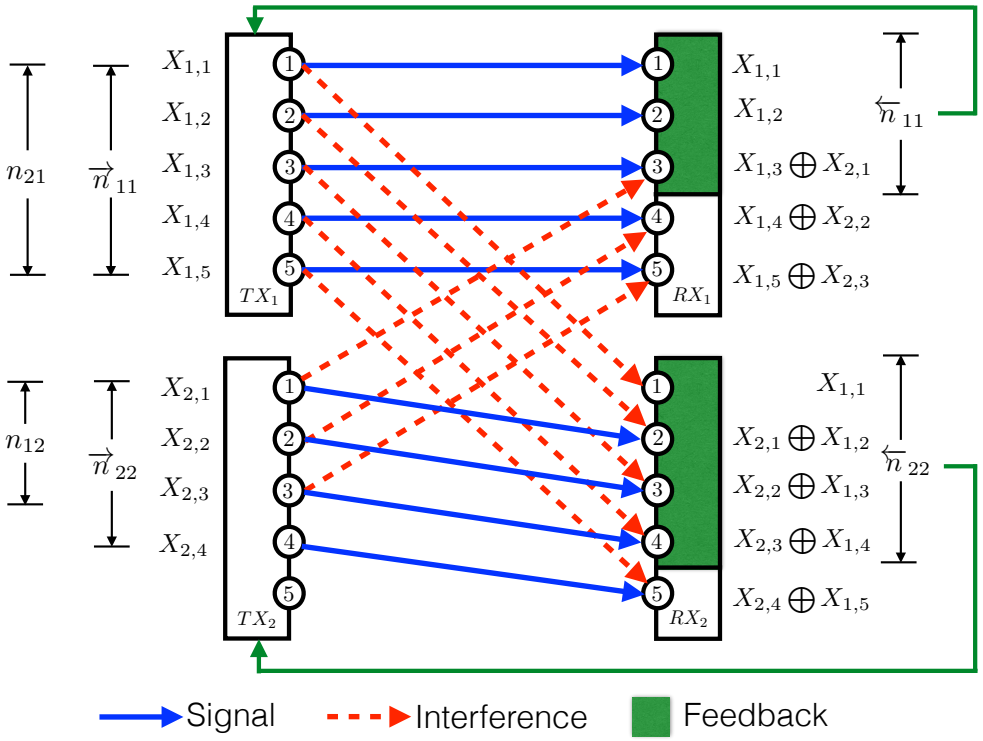

Fig. 1. Two-user linear deterministic interference channel with noisy channel-output feedback. The bit-pipe line number 1 represents the most significant bit.

in which of the two transmitter-receiver pairs must the feedback link be implemented if the objective is to improve: (a) the individual rate of the transmitter-receiver pair in which feedback is implemented; (b) the individual rate of the other transmitter-receiver pair; or (c) the sum-rate of both transmitter-receiver pairs. In each of these scenarios, the feedback SNR, either $\overleftarrow{n}_{11}$ or $\overleftarrow{n}_{22}$, must be bigger than a given threshold for the improvement to be observed. Interestingly, the answer is completely different for each of these scenarios. In addition to the results described above, the exact values of $\overleftarrow{n}_{11}$ or $\overleftarrow{n}_{22}$ for which feedback does not enlarge the capacity region are also identified.

II. Linear Deterministic Interference Channel with Noisy Channel-Output Feedback

Consider the two-user LD-IC-NOF, with parameters $\overrightarrow{n}_{11}$, $\overrightarrow{n}_{22}$, n_{12}, n_{21}, $\overleftarrow{n}_{11}$ and $\overleftarrow{n}_{22}$ described in Fig. 1. The parameters $\overrightarrow{n}_{ii}$, n_{ij} and $\overleftarrow{n}_{ii}$ with $i \in \{1,2\}$ and $j \in \{1,2\}\backslash\{i\}$, are non-

negative integers. Parameter $\overrightarrow{n}_{ii}$ represents the number of bit-pipes between transmitter i and receiver i; parameter n_{ij} represents the number of bit-pipes between transmitter j and receiver i; and parameter $\overleftarrow{n}_{ii}$ represents the number of bit-pipes between receiver i and transmitter i (feedback). At transmitter i, the channel-input $\boldsymbol{X}_{i,n}$ during channel use n, with $n \in \{1, 2, \ldots, N\}$, is a q-dimensional binary vector $\boldsymbol{X}_{i,n} = \left(X_{i,n}^{(1)}, X_{i,n}^{(2)}, \ldots, X_{i,n}^{(q)}\right)^{\mathsf{T}}$, with

$$q = \max\left(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}\right), \tag{1}$$

and N the block-length. At receiver i, the channel-output $\overrightarrow{\boldsymbol{Y}}_{i,n}$ during channel use n is also a q-dimensional binary vector $\overrightarrow{\boldsymbol{Y}}_{i,n} = \left(\overrightarrow{Y}_{i,n}^{(1)}, \overrightarrow{Y}_{i,n}^{(2)}, \ldots, \overrightarrow{Y}_{i,n}^{(q)}\right)^{\mathsf{T}}$. The input-output relation during channel use n is given by

$$\overrightarrow{\boldsymbol{Y}}_{i,n} = \boldsymbol{S}^{q-\overrightarrow{n}_{ii}} \boldsymbol{X}_{i,n} + \boldsymbol{S}^{q-n_{ij}} \boldsymbol{X}_{j,n}, \tag{2}$$

and the feedback signal $\overleftarrow{\boldsymbol{Y}}_{i,n}$ available at transmitter i at the end of channel use n satisfies

$$\left((0, \ldots, 0), \overleftarrow{\boldsymbol{Y}}_{i,n}^{\mathsf{T}}\right)^{\mathsf{T}} = \boldsymbol{S}^{\left(\max(\overrightarrow{n}_{ii}, n_{ij}) - \overleftarrow{n}_{ii}\right)^{+}} \overrightarrow{\boldsymbol{Y}}_{i,n-d}, \tag{3}$$

where d is a finite delay, additions and multiplications are defined over the binary field, and $\boldsymbol{S}$ is a $q \times q$ lower shift matrix of the form:

$$\boldsymbol{S} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}. \tag{4}$$

The dimension of the vector $(0, \ldots, 0)$ in (3) is $q - \min\left(\overleftarrow{n}_{ii}, \max(\overrightarrow{n}_{ii}, n_{ij})\right)$ and the vector $\overleftarrow{\boldsymbol{Y}}_{i,n}$ represents the $\min\left(\overleftarrow{n}_{ii}, \max(\overrightarrow{n}_{ii}, n_{ij})\right)$ least significant bits of $\boldsymbol{S}^{\left(\max(\overrightarrow{n}_{ii}, n_{ij}) - \overleftarrow{n}_{ii}\right)^{+}} \overrightarrow{\boldsymbol{Y}}_{i,n-d}$. Without any loss of generality, the feedback delay is assumed to be equal to 1 channel use. Transmitter i sends the message index W_i by sending the codeword $\boldsymbol{X}_i = (\boldsymbol{X}_{i,1}, \boldsymbol{X}_{i,2}, \ldots, \boldsymbol{X}_{i,N}) \in \mathcal{X}_i^N$. The encoder of transmitter i can be modeled as a set of deterministic mappings $f_i^{(1)}$, $f_i^{(2)}, \ldots, f_i^{(N)}$, with $f_i^{(1)} : \mathcal{W}_i \to \{0,1\}^q$ and for all $n \in \{2, 3, \ldots, N\}$, $f_i^{(n)} : \mathcal{W}_i \times \{0,1\}^{q(n-1)} \to \{0,1\}^q$, such that

$$\boldsymbol{X}_{i,1} = f_i^{(1)}\left(W_i\right) \text{ and} \tag{5}$$

$$\boldsymbol{X}_{i,n} = f_i^{(n)}\left(W_i, \overleftarrow{\boldsymbol{Y}}_{i,1}, \overleftarrow{\boldsymbol{Y}}_{i,2}, \ldots, \overleftarrow{\boldsymbol{Y}}_{i,n-1}\right). \tag{6}$$

Assume that during a given communication, T blocks are transmitted. Hence, the decoder of receiver i is defined by a deterministic function $\psi_i : \{0,1\}^{q\times N\times T} \to \mathcal{W}_i^T$. At the end of the communication, receiver i uses the sequence $\left(\overrightarrow{\boldsymbol{Y}}_{i,1}, \overrightarrow{\boldsymbol{Y}}_{i,2}, \ldots, \overrightarrow{\boldsymbol{Y}}_{i,NT}\right)$ to obtain an estimation of the message indices. The decoding error probability in the two-user LD-IC-NOF during block t, denoted by $P_e^{(t)}(N)$, is given by

$$P_e^{(t)}(N)=\max\left(\Pr\left(\widehat{W_1}^{(t)} \neq W_1^{(t)}\right), \Pr\left(\widehat{W_2}^{(t)} \neq W_2^{(t)}\right)\right). \tag{7}$$

A rate pair $(R_1, R_2) \in \mathbb{R}_+^2$ is said to be achievable if it satisfies the following definition.

Definition 1 (Achievable Rate Pairs): The rate pair $(R_1, R_2) \in \mathbb{R}_+^2$ is achievable if there exists at least one pair of codebooks $\mathcal{X}_1^N$ and $\mathcal{X}_2^N$ with codewords of size $q\times N$, and the corresponding encoding functions $f_1^{(1)}, f_1^{(2)}, \ldots, f_1^{(N)}$ and $f_2^{(1)}, f_2^{(2)}, \ldots, f_2^{(N)}$ such that the average bit error probability can be made arbitrarily small by letting the block length N grow to infinity.

The information capacity region is the closure of all information achievable rate pairs (R_1, R_2). Denote by $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ the capacity region of the two-user LD-IC-NOF with parameters $\overrightarrow{n}_{11}$, $\overrightarrow{n}_{22}$, n_{12}, n_{21}, $\overleftarrow{n}_{11}$, and $\overleftarrow{n}_{22}$. Lemma 1 fully characterizes the set $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$.

Lemma 1 (Theorem 1 in [8]): The capacity region $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ of the two-user LD-IC-NOF is the set of non-negative rate pairs (R_1, R_2) that satisfy $\forall i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$:

$$R_i \leqslant \min\left(\max\left(\overrightarrow{n}_{ii}, n_{ji}\right), \max\left(\overrightarrow{n}_{ii}, n_{ij}\right)\right), \tag{8a}$$

$$R_i \leqslant \min\left(\max\left(\overrightarrow{n}_{ii}, n_{ji}\right), \max\left(\overrightarrow{n}_{ii}, \overleftarrow{n}_{jj} - \left(\overrightarrow{n}_{jj} - n_{ji}\right)^+\right)\right), \tag{8b}$$

$$R_1 + R_2 \leqslant \min\left(\max\left(\overrightarrow{n}_{22}, n_{12}\right) + \left(\overrightarrow{n}_{11} - n_{12}\right)^+, \max\left(\overrightarrow{n}_{11}, n_{21}\right) + \left(\overrightarrow{n}_{22} - n_{21}\right)^+\right), \tag{8c}$$

$$\begin{aligned} R_1 + R_2 \leqslant & \max\left(\left(\overrightarrow{n}_{11} - n_{12}\right)^+, n_{21}, \overrightarrow{n}_{11} - \left(\max\left(\overrightarrow{n}_{11}, n_{12}\right) - \overleftarrow{n}_{11}\right)^+\right) \\ & + \max\left(\left(\overrightarrow{n}_{22} - n_{21}\right)^+, n_{12}, \overrightarrow{n}_{22} - \left(\max\left(\overrightarrow{n}_{22}, n_{21}\right) - \overleftarrow{n}_{22}\right)^+\right), \end{aligned} \tag{8d}$$

$$\begin{aligned} 2R_i + R_j \leqslant & \max\left(\overrightarrow{n}_{ii}, n_{ji}\right) + \left(\overrightarrow{n}_{ii} - n_{ij}\right)^+ \\ & + \max\left(\left(\overrightarrow{n}_{jj} - n_{ji}\right)^+, n_{ij}, \overrightarrow{n}_{jj} - \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^+\right). \end{aligned} \tag{8e}$$

III. PRELIMINARIES

A. Definitions

Let $\alpha_i \in \mathbb{Q}$ be the interference regime of transmitter-receiver pair i, with $i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$,

$$\alpha_i = \frac{n_{ij}}{\overrightarrow{n}_{ii}}. \tag{9}$$

For each transmitter-receiver pair i, there exist five possible interference regimes (IRs): very weak IR (VWIR), i.e., $\alpha_i \leqslant \frac{1}{2}$, weak IR (WIR), i.e., $\frac{1}{2} < \alpha_i \leqslant \frac{2}{3}$, moderate IR (MIR), i.e., $\frac{2}{3} < \alpha_i \leqslant 1$, strong IR (SIR), i.e., $1 < \alpha_i \leqslant 2$ and very strong IR (VSIR), i.e., $\alpha_i > 2$ [11]. The scenarios in which the desired signal is stronger than or equal to the interference ($\alpha_i \leqslant 1$), namely VWIR, WIR and MIR, are referred to as low-interference regimes (LIRs). Conversely, the scenarios in which the desired signal is weaker than the interference ($\alpha_i > 1$), namely SIR and VSIR, are referred to as high-interference regimes (HIRs). In the two-user LD-IC, up to twenty-five possible interference regimes might emerge, given α_1 and α_2. However, only twelve cases are of real interest. This is because the transmitter-receiver pairs can be indifferently labeled and thus, for instance, studying the case in which $\alpha_1 \leqslant \frac{1}{2}$ and $\alpha_2 > 2$ is the same as studying the case in which $\alpha_1 > 2$ and $\alpha_2 \leqslant \frac{1}{2}$.

The main results of this paper are presented using a list of events (Boolean variables) that are fully determined by the parameters $\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}$, and n_{21}. For instance, given the parameters $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21})$, the events (10)-(21) describe some combinations of interference regimes that are particularly interesting. Let $i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$ and define the following events:

$$E_1: \quad \alpha_1 \leqslant 1 \ \wedge \ \alpha_2 \leqslant 1, \ (i \text{ and } j \text{ in LIR}) \tag{10}$$

$$E_{2,i}: \quad \alpha_i \leqslant \frac{1}{2} \ \wedge \ 1 < \alpha_j \leqslant 2, \ (i \text{ in VWIR and } j \text{ in SIR}) \tag{11}$$

$$E_{3,i}: \quad \alpha_i \leqslant \frac{1}{2} \ \wedge \ \alpha_j > 2, \ (i \text{ in VWIR and } j \text{ in VSIR}) \tag{12}$$

$$E_{4,i}: \quad \frac{1}{2} < \alpha_i \leqslant \frac{2}{3} \ \wedge \ \alpha_j > 1, \ (i \text{ in WIR and } j \text{ in HIR}) \tag{13}$$

$$E_{5,i}: \quad \frac{2}{3} < \alpha_i \leqslant 1 \ \wedge \ 1 < \alpha_j \leqslant 2, \ (i \text{ in MIR and } j \text{ in SIR}) \tag{14}$$

$$E_{6,i}: \quad \frac{2}{3} < \alpha_i \leqslant 1 \ \wedge \ \alpha_j > 2, \ (i \text{ in MIR and } j \text{ in VSIR}) \tag{15}$$

$$E_{7,i}: \quad \frac{1}{2} < \alpha_i \leqslant 1 \ \wedge \ \alpha_j > 1, \ (i \text{ in WIR or MIR and } j \text{ in HIR}) \tag{16}$$

$$E_{8,i}: \quad \alpha_i > 1 \ \wedge \ \alpha_j \leqslant 1, \ (i \text{ in HIR and } j \text{ in LIR}) \tag{17}$$

$$E_9: \quad \alpha_1 \leqslant \frac{2}{3} \ \wedge \ \alpha_2 \leqslant \frac{2}{3}, \ (i \text{ and } j \text{ in VWIR or WIR}) \tag{18}$$

$$E_{10,i}: \quad \alpha_i \leqslant \frac{2}{3} \ \wedge \ \frac{2}{3} < \alpha_j \leqslant 1, \ (i \text{ in VWIR or WIR and } j \text{ in MIR}) \tag{19}$$

$$E_{11,i}: \quad \frac{2}{3} < \alpha_i \leqslant 1 \ \wedge \ \alpha_j \leqslant \frac{2}{3}, \ (i \text{ in MIR and } j \text{ in VWIR or WIR}) \tag{20}$$

$$E_{12,i}: \quad \alpha_i > 2 \ \wedge \ \alpha_j > 1, \ (i \text{ in VSIR and } j \text{ in HIR}). \tag{21}$$

Some other auxiliary events are considered. The event in which the signal from transmitter i is stronger in its intended receiver than in its non-intended receiver is denoted by $E_{13,i}$, i.e.,

$$E_{13,i}: \quad \overrightarrow{n}_{ii} > n_{ji}. \tag{22}$$

The event in which the sum of the number of bit-pipes in the direct links is bigger than the sum of the number of bit-pipes in the cross-interference links is denoted by E_{14}, i.e.,

$$E_{14}: \quad \overrightarrow{n}_{11} + \overrightarrow{n}_{22} > n_{12} + n_{21}. \tag{23}$$

The event in which the number of bit-pipes in the direct link j is bigger than the sum of bit-pipes in both cross-interference links is denoted by

$$E_{15,i}: \quad \overrightarrow{n}_{jj} > n_{ij} + n_{ji}. \tag{24}$$

The event in which the sum of the number of bit-pipes in the direct links is bigger than the sum of the number of bit-pipes in one cross-interference link and twice the number of the bit-pipes

in the other cross-interference link is denoted by

$$E_{16,i}\colon \quad \overrightarrow{n}_{ii} + \overrightarrow{n}_{jj} > n_{ij} + 2n_{ji}. \tag{25}$$

Finally, the event in which the sum of the number of bit-pipes in the direct links is bigger than the number of bit-pipes in one cross-interference link is denoted by

$$E_{17,i}\colon \quad \overrightarrow{n}_{ii} + \overrightarrow{n}_{jj} < n_{ij}. \tag{26}$$

In the following, given an event, e.g. $E_{13,i} : \overrightarrow{n}_{ii} > n_{ji}$, the notation $\widetilde{E}_{13,i}$ implies $\overrightarrow{n}_{ii} < n_{ji}$; $\overline{E}_{13,i}$ implies $\overrightarrow{n}_{ii} \leqslant n_{ji}$; and $\check{E}_{13,i}$ implies $\overrightarrow{n}_{ii} \geqslant n_{ji}$.

Combining the events (10)-(26), five main events are identified:

$$\begin{aligned} S_{1,i}\colon \quad & (E_1 \wedge E_{13,i}) \vee (E_{2,i} \wedge E_{13,i}) \vee (E_{3,i} \wedge E_{13,i} \wedge E_{14}) \vee (E_{4,i} \wedge E_{13,i} \wedge E_{14}) \\ & \vee (E_{5,i} \wedge E_{13,i} \wedge E_{14}) \vee \left(E_{6,i} \wedge \widetilde{E}_{13,j} \wedge E_{14}\right), \end{aligned} \tag{27}$$

$$S_{2,i}\colon \quad \left(E_{3,i} \wedge \widetilde{E}_{13,j} \wedge \overline{E}_{14}\right) \vee \left(E_{7,i} \wedge \widetilde{E}_{13,j} \wedge \overline{E}_{14}\right) \vee \left(\overline{E}_1 \wedge \widetilde{E}_{13,j}\right), \tag{28}$$

$$\begin{aligned} S_{3,i}\colon \quad & \left(E_1 \wedge \overline{E}_{13,i}\right) \vee \left(E_{2,i} \wedge \overline{E}_{13,i}\right) \vee \left(E_{3,i} \wedge \check{E}_{13,j} \wedge \overline{E}_{13,i}\right) \vee \left(E_{4,i} \wedge \check{E}_{13,j} \wedge \overline{E}_{13,i}\right) \\ & \vee \left(E_{5,i} \wedge \check{E}_{13,j} \wedge \overline{E}_{13,i}\right) \vee \left(E_{6,i} \wedge \check{E}_{13,j}\right) \vee \left(\widetilde{E}_1 \wedge \check{E}_{13,j}\right) \vee (E_{8,i}), \end{aligned} \tag{29}$$

$$\begin{aligned} S_{4,i}\colon \quad & (E_9 \wedge E_{13,i} \wedge E_{13,j}) \vee (E_{10,i} \wedge E_{13,i} \wedge E_{13,j} \wedge E_{16,i} \wedge E_{16,j}) \\ & \vee (E_{11,i} \wedge E_{13,i} \wedge E_{13,j} \wedge E_{16,i} \wedge E_{16,j}), \end{aligned} \tag{30}$$

$$S_{5,i}\colon \quad (E_{12,i} \wedge E_{17,i} \wedge E_{17,j}) \vee (E_{12,j} \wedge E_{17,i} \wedge E_{17,j}). \tag{31}$$

For all $i \in \{1, 2\}$ the events $S_{1,i}$, $S_{2,i}$ and $S_{3,i}$ exhibit the property stated by the following lemma.

Lemma 2: For all $i \in \{1, 2\}$ and for all $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}) \in \mathbb{N}^4$, only one of the events among $S_{1,i}$, $S_{2,i}$ and $S_{3,i}$ is true.

Proof: The proof follows from verifying that for all $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$, the events (27)-(29) are mutually exclusive. For instance, consider that the event $(E_1 \wedge E_{13,i})$ in (27) is true. Then, $S_{1,i}$ is true and $E_{2,i}$, $E_{3,i}$, $E_{4,i}$, $E_{5,i}$, $E_{6,i}$, $E_{7,i}$ and $E_{8,i}$ hold false, which implies that $S_{2,i}$ and $S_{3,i}$ hold false as well, since all events in (28) and (29) are false. The same verification can be made for all the remaining events in (27). This proves that if $S_{1,i}$ is true then $S_{2,i}$ and $S_{3,i}$ hold simultaneously false. The same verification can be done for showing that when $S_{2,i}$ holds true (resp. $S_{3,i}$), both events $S_{1,i}$ and $S_{3,i}$ (resp. $S_{1,i}$ and $S_{2,i}$) hold simultaneously false. Finally

following the same reasoning it can be verified that if any pair of the events $\{S_{1,i}, S_{2,i}, S_{3,i}\}$ is false, the remaining event is necessarily true. This completes the proof. ■

For all $i \in \{1,2\}$ the events $S_{4,i}$ and $S_{5,i}$ exhibit the property stated by the following lemma.

Lemma 3: For all $i \in \{1,2\}$ if one of the events $S_{4,i}$ or $S_{5,i}$ holds true, then the other necessarily holds false.

Proof: The proof of Lemma 3 follows along the same lines of the proof of Lemma 2. ■

Note that Lemma 3 does not exclude the case in which both $S_{4,i}$ and $S_{5,i}$ are simultaneously false.

B. Rate Improvement Metrics

The rate improvements are given in terms of the following metrics [10]: (a) maximum individual rate improvements Δ_1 and Δ_2; and (b) maximum sum-rate improvement Σ, with $\Delta_i \in \mathbb{R}_+$ and $\Sigma \in \mathbb{R}_+$ for $i \in \{1,2\}$.

Let $\mathcal{C}_1 = \mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ and $\mathcal{C}_2 = \mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, 0)$ be the capacity region with noisy channel-output feedback and without feedback, respectively. In order to formally define Δ_1, Δ_2 and Σ, consider a two-user LD-IC-NOF with parameters $\overrightarrow{n}_{11}$, $\overrightarrow{n}_{22}$, n_{12}, n_{21}, $\overleftarrow{n}_{11}$, and $\overleftarrow{n}_{22}$. The maximum improvement of the individual rate R_i, $\Delta_i(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$, due to the effect of channel-output feedback with respect to the case without feedback is

$$\Delta_i(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22}) = \max_{R_j>0} \left\{ \sup_{(R_i,R_j)\in\mathcal{C}_1} \{R_i\} - \sup_{(R_i^\dagger,R_j)\in\mathcal{C}_2} \{R_i^\dagger\} \right\}, \tag{32}$$

and the maximum improvement of the sum rate $\Sigma(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ with respect to the case without feedback is

$$\Sigma(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22}) = \sup_{(R_1,R_2)\in\mathcal{C}_1} \left\{ R_1 + R_2 \right\} - \sup_{(R_1^\dagger,R_2^\dagger)\in\mathcal{C}_2} \left\{ R_1^\dagger + R_2^\dagger \right\}. \tag{33}$$

In the following, when feedback is exclusively used by transmitter-receiver pair i, i.e., $\overleftarrow{n}_{ii} > 0$ and $\overleftarrow{n}_{jj} = 0$, then the maximum improvement of the individual rate of transmitter-receiver k, with $k \in \{1,2\}$, and the maximum improvement of the sum rate are denoted by $\Delta_k(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii})$ and $\Sigma(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii})$, respectively. Hence, this notation $\Delta_k(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii})$ replaces either $\Delta_k(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$ or $\Delta_k(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, \overleftarrow{n}_{22})$, when $i = 1$ or $i = 2$, respectively. The same holds for the notation $\Sigma(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii})$

that replaces $\Sigma(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$ or $\Sigma(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, \overleftarrow{n}_{22})$, when $i = 1$ or $i = 2$, respectively.

IV. Main Results

A. Enlargement of the Capacity Region

In this subsection, the capacity region of a two-user LD-IC-NOF with parameters $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21})$, when feedback is available only at transmitter-receiver pair i, i.e., $\overleftarrow{n}_{ii} > 0$ and $\overrightarrow{n}_{jj} = 0$, is denoted by $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii})$ instead of $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$ or $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, \overleftarrow{n}_{22})$, when $i = 1$ or $i = 2$, respectively. Following this notation, Theorem 1 identifies the exact values of $\overleftarrow{n}_{ii}$ for which the strict inclusion $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, 0) \subset \mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii})$ holds, with $i \in \{1, 2\}$.

Theorem 1: Let $i \in \{1, 2\}$, $j \in \{1, 2\} \setminus \{i\}$ and $\overleftarrow{n}^{*}_{ii} \in \mathbb{N}$ be

$$\overleftarrow{n}^{*}_{ii} = \begin{cases} \max\left(n_{ji}, (\overrightarrow{n}_{ii} - n_{ij})^{+}\right) & \text{if } S_{1,i} = \text{True} \\ \overrightarrow{n}_{jj} + (\overrightarrow{n}_{ii} - n_{ij})^{+} & \text{if } S_{2,i} = \text{True}. \end{cases} \tag{34}$$

Assume that $S_{3,i} = \text{True}$. Then, for all $\overleftarrow{n}_{ii} \in \mathbb{N}$, $\mathcal{C}\Big(\overrightarrow{n}_{11},\ \overrightarrow{n}_{22},\ n_{12},\ n_{21},\ 0,\ 0\Big) = \mathcal{C}\Big(\overrightarrow{n}_{11},\ \overrightarrow{n}_{22},\ n_{12},\ n_{21},\ \overleftarrow{n}_{ii}\Big)$. Assume that either $S_{1,i} = \text{True}$ or $S_{2,i} = \text{True}$. Then, for all $\overleftarrow{n}_{ii} \leqslant \overleftarrow{n}^{*}_{ii}$, $\mathcal{C}\Big(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, 0\Big) = \mathcal{C}\Big(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}\Big)$ and for all $\overleftarrow{n}_{ii} > \overleftarrow{n}^{*}_{ii}$, $\mathcal{C}\Big(\overrightarrow{n}_{11},\ \overrightarrow{n}_{22},\ n_{12},\ n_{21},\ 0,\ 0\Big) \subset \mathcal{C}\Big(\overrightarrow{n}_{11},\ \overrightarrow{n}_{22},\ n_{12},\ n_{21},\ \overleftarrow{n}_{ii}\Big)$.

Proof: The proof of Theorem 1 is presented in Appendix A. ■

Theorem 1 shows that under event $S_{3,i}$ in (29), implementing feedback in transmitter-receiver pair i does not bring any capacity region enlargement. Alternatively, under events $S_{1,i}$ in (27) and $S_{2,i}$ in (28), the capacity region can be enlarged when $\overleftarrow{n}_{ii} > \overleftarrow{n}^{*}_{ii}$. That is, there exists a threshold on the SNR of the feedback link beyond which it is possible to observe a capacity region enlargement. Consider the following example.

Example 1: Consider a case in which transmitter-receiver pair 1 is in SIR ($1 < \alpha_1 \leqslant 2$), transmitter-receiver pair 2 is in VWIR ($\alpha_2 \leqslant \frac{1}{2}$) and $\overrightarrow{n}_{22} > n_{12}$. In this case, the event $E_{8,1}$ holds true (and so does $S_{3,1}$). Thus, implementing feedback in the transmitter-receiver pair 1 does not enlarge the capacity region. Note also that the events $E_{2,2}$ and $E_{13,2}$ hold true (and so does $S_{1,2}$). In view of this, implementing feedback in transmitter-receiver pair 2 enlarges the capacity region whenever the feedback link 2 satisfies that $\overleftarrow{n}_{22} > \max\left(n_{12}, (\overrightarrow{n}_{22} - n_{21})^{+}\right)$.

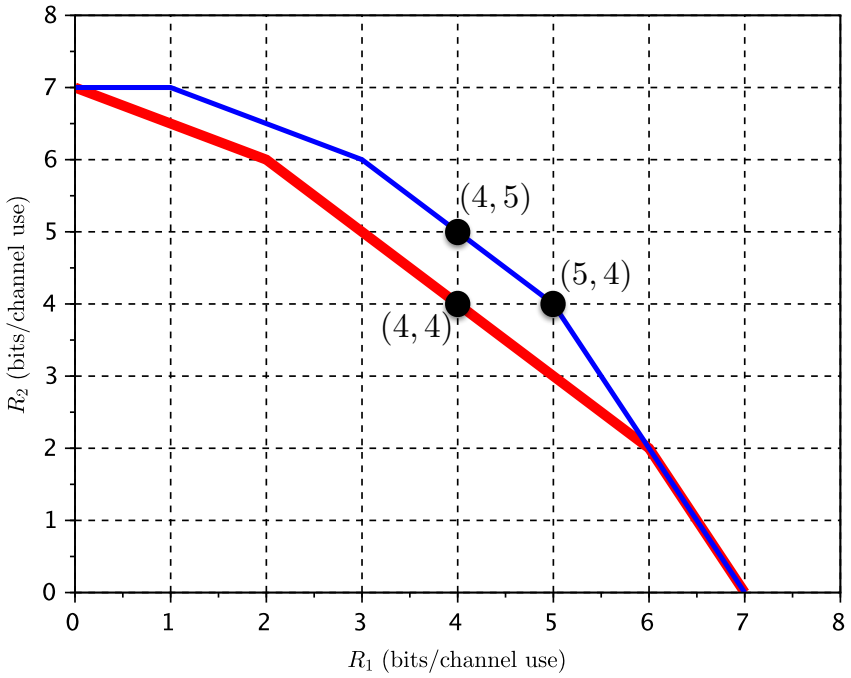

Fig. 2. Capacity regions of $\mathcal{C}(7,7,3,5,0,0)$ (thick red line) and $\mathcal{C}(7,7,3,5,6,0)$ (thin blue line).

It is important to highlight that in the cases in which feedback might potentially enlarge the capacity region of the two-user LD-IC-NOF, that is, in events $S_{1,1}$, $S_{2,1}$, $S_{1,2}$ or $S_{2,2}$, the following is always true for all $i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$:

$$\overleftarrow{n}^{*}_{ii} > (\overrightarrow{n}_{ii} - n_{ij})^{+}. \tag{35}$$

The inequality in (35) unveils the fact that feedback is useful for enlarging the capacity region of the two-user LD-IC-NOF when at least one receiver is capable of feeding back to its intended transmitter at least part of the information transmitted by the other transmitter. Essentially, when $\overleftarrow{n}_{ii} \leqslant (\overrightarrow{n}_{ii} - n_{ij})^{+}$, the noise over the feedback link i impairs all the bit-pipes that contain information sent by transmitter j. This implies that transmitter i cannot obtain any information sent by transmitter j, see for instance Fig. 1. Another interesting observation is that the threshold $\overleftarrow{n}^{*}_{ii}$ beyond which feedback is useful is different under event $S_{1,i}$ in (27) and event $S_{2,i}$ in (28). In general when $S_{1,i}$ holds true, the enlargement of the capacity region is due to the fact that feedback allows using *interference as side information* [12]. Alternatively, when $S_{2,i}$ in (28) holds true, the enlargement of the capacity region occurs thanks to the fact that some of the bits

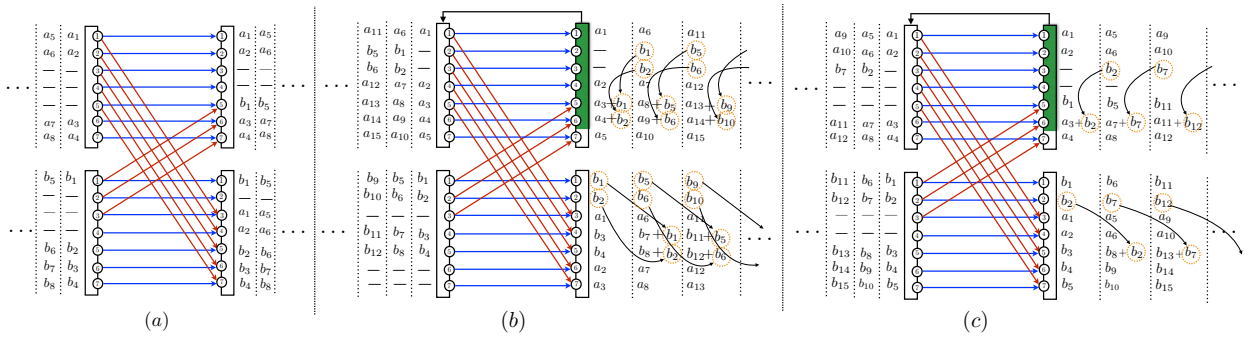

Fig. 3. (a) Achievability of the rate pair $(4,4)$ in an LD-IC with parameters $\overrightarrow{n}_{11}=7$, $\overrightarrow{n}_{22}=7$, $n_{12}=3$, $n_{21}=5$, $\overleftarrow{n}_{11}=0$ and $\overleftarrow{n}_{22}=0$. (b) Achievability of the rate pair $(5,4)$ in an LD-IC with parameters $\overrightarrow{n}_{11}=7$, $\overrightarrow{n}_{22}=7$, $n_{12}=3$, $n_{21}=5$, $\overleftarrow{n}_{11}=6$ and $\overleftarrow{n}_{22}=0$. (c) Achievability of the rate pair $(4,5)$ in an LD-IC with parameters $\overrightarrow{n}_{11}=7$, $\overrightarrow{n}_{22}=7$, $n_{12}=3$, $n_{21}=5$, $\overleftarrow{n}_{11}=6$ and $\overleftarrow{n}_{22}=0$.

that cannot be transmitted directly from transmitter j to receiver j, can arrive to receiver j via an alternative path: transmitter j - receiver i - transmitter i - receiver j. Consider the following examples.

Example 2: Consider the parameters $\overrightarrow{n}_{11}=7$, $\overrightarrow{n}_{22}=7$, $n_{12}=3$, *and* $n_{21}=5$. *This implies that both* $S_{1,1}$ *and* $S_{1,2}$ *hold true. Hence, according to Theorem 1, it follows that the capacity region can be enlarged by either using feedback in transmitter-receiver pair* 1 *or in transmitter-receiver pair* 2, *with* $\overleftarrow{n}_{11}>\max\left(n_{21},(\overrightarrow{n}_{11}-n_{12})^{+}\right)=5$ *or* $\overleftarrow{n}_{22}>\max\left(n_{12},(\overrightarrow{n}_{22}-n_{21})^{+}\right)=3$, *respectively. Figure 2 shows the capacity region of* $\mathcal{C}(7,7,3,5,0,0)$ *(thick red line) and* $\mathcal{C}(7,7,3,5,6,0)$ *(thin blue line).*

Note that in example 2, the capacity region is enlarged when $\overleftarrow{n}_{11}=6$, with respect to the case in which $\overleftarrow{n}_{11}=0$. Figure 3a shows the achievability of $(4,4)$ without feedback, whereas Figures 3b and 3c show the achievability of $(5,4)$ and $(4,5)$ with feedback in transmitter-receiver pair 1, respectively. More specifically, when feedback is used at transmitter-receiver pair 1 and $\overleftarrow{n}_{11}>\max\left(n_{21},(\overrightarrow{n}_{11}-n_{12})^{+}\right)$, transmitter 1 obtains part of the information sent by transmitter 2. This information can be re-transmitted by transmitter 1 to cancel the interference it produced at receiver 1 when it was first transmitted by transmitter 2. Interestingly, the interference perceived at receiver 2 due to this re-transmission can be cancelled given that this information was reliably decoded when it was first sent by transmitter 2. This allows transmitter-receiver pair 1 or 2 to improve its individual rate.

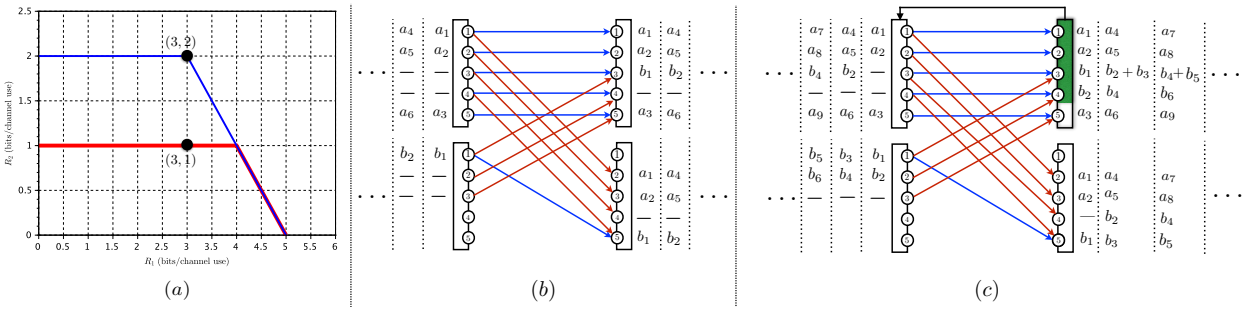

Fig. 4. (a) Capacity regions of $\mathcal{C}(5,1,3,4,0,0)$ (thick red line) and $\mathcal{C}(5,1,3,4,4,0)$ (thin blue line). (b) Achievability of the rate pair $(3,1)$ in an LD-IC with parameters $\overrightarrow{n}_{11}=5$, $\overrightarrow{n}_{22}=1$, $n_{12}=3$, $n_{21}=4$, $\overleftarrow{n}_{11}=0$ and $\overleftarrow{n}_{22}=0$ (no feedback links). (c) Achievability of the rate pair $(3,2)$ in an LD-IC with parameters $\overrightarrow{n}_{11}=5$, $\overrightarrow{n}_{22}=1$, $n_{12}=3$, $n_{21}=4$, $\overleftarrow{n}_{11}=4$ and $\overleftarrow{n}_{22}=0$.

Example 3: Consider the parameters $\overrightarrow{n}_{11}=5$, $\overrightarrow{n}_{22}=1$, $\overrightarrow{n}_{12}=3$, *and* $\overrightarrow{n}_{21}=4$. *In this case,* $S_{2,1}$ *in* (28) *and* $S_{3,2}$ *in* (29) *hold true. Hence from Theorem 1, it follows that the capacity region cannot be enlarged by using feedback in transmitter-receiver pair* 2 *(*$S_{3,2}$ *holds true). At the same time, it follows that the capacity region can be enlarged using feedback in transmitter-receiver pair* 1, *whenever* $\overleftarrow{n}_{11} > \overrightarrow{n}_{22} + (\overrightarrow{n}_{11} - n_{12})^+ = 3$ *(*$S_{2,1}$ *holds true).*

Figure 4a shows the capacity region of $\mathcal{C}(5,1,3,4,0,0)$ (thick red line) and $\mathcal{C}(5,1,3,4,4,0)$ (thin blue line). Note that the capacity region is indeed enlarged when $\overleftarrow{n}_{11}=4$, with respect to $\overleftarrow{n}_{11}=0$. Figure 4b and Figure 4c show the achievability of the rate pairs $(3,1)$ and $(3,2)$ without and with feedback, respectively. Note that the main difference between both schemes is that the existence of a feedback link between transmitter 1 and receiver 1, with $\overleftarrow{n}_{11}=4$, allows transmitter 2 to send an additional bit per channel use, i.e., bits $b_2, b_4, b_6, \ldots$, via the alternative path (transmitter 2 - receiver 1 - transmitter 1 - receiver 2). This accounts for transmitter-receiver pair 2 increasing its rate from one to two bits per channel use.

B. Improvement of the Individual Rate R_i by Using Feedback in Link i

Implementing channel output feedback in transmitter-receiver pair i might allow increasing the individual rate of either transmitter-receiver pair i or j. Theorem 2 identifies the exact values of $\overleftarrow{n}_{ii}$ for which the individual rate R_i can be improved, given the parameters $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21})$ and $\overleftarrow{n}_{jj}=0$ in the two-user LD-IC-NOF.

Theorem 2: Let $i \in \{1,2\}$, $j \in \{1,2\} \setminus \{i\}$ *and* $\overleftarrow{n}_{ii}^{\dagger} \in \mathbb{N}$ *be*

$$\overleftarrow{n}_{ii}^{\dagger} = \max\left(n_{ji}, (\overrightarrow{n}_{ii} - n_{ij})^{+}\right). \tag{36}$$

Assume that either $S_{2,i} = \text{True}$ *or* $S_{3,i} = \text{True}$. *Then, for all* $\overleftarrow{n}_{ii} \in \mathbb{N}$, $\Delta_i(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}) = 0$. *Assume that* $S_{1,i} = \text{True}$. *Then, when* $\overleftarrow{n}_{ii} \leqslant \overleftarrow{n}_{ii}^{\dagger}$, *it holds that* $\Delta_i(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}) = 0$; *and when* $\overleftarrow{n}_{ii} > \overleftarrow{n}_{ii}^{\dagger}$, *it holds that* $\Delta_i(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}) > 0$.

Proof: The proof of Theorem 2 is presented in Appendix B. ■

Theorem 2 highlights that under events $S_{2,i}$ in (28) and $S_{3,i}$ in (29), the individual rate R_i cannot be improved by using feedback in transmitter-receiver pair i, i.e., $\Delta_i(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}) = 0$. Alternatively, under event $S_{1,i}$ in (27), the individual rate R_i can be improved, i.e., $\Delta_i\Big(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}\Big) > 0$, whenever $\overleftarrow{n}_{ii} > \max\left(n_{ji}, (\overrightarrow{n}_{ii} - n_{ij})^{+}\right)$.

An example in which for all $(R_1', R_2') \in \mathcal{C}\left(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, 0\right)$ there always exists a rate pair $(R_1, R_2) \in \mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$ such that $R_1' \leqslant R_1$ and $R_2' = R_2$, is presented in Fig. 2.

It is worth noting that under event $S_{3,i}$ in (29), the capacity region cannot be improved via feedback (Theorem 1) and thus, none of the individual rates can be improved as suggested by Theorem 2. Alternatively, under event $S_{2,i}$ in (28), the capacity region can be enlarged (Theorem 1) but the individual rate R_i cannot be improved (Theorem 2). This implies that the capacity improvement occurs due to the fact that R_j can be improved. More specifically, in this case: $\Delta_i(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}) = 0$ and $\Delta_j(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}) > 0$. This implies that using feedback in transmitter-receiver pair i is exclusively beneficial for transmitter-receiver pair j, as shown in the following section.

C. Improvement of the Individual Rate R_j by Using Feedback in Link i

Implementing channel output feedback in transmitter-receiver pair i might allow increasing the individual rate of transmitter-receiver pair i or j; or both individual rates. This reveals the altruistic nature of implementing feedback as suggested in [13], [14]. Theorem 3 identifies the exact values of $\overleftarrow{n}_{ii}$ for which the individual rate R_j can be improved by using feedback in transmitter-receiver pair i, given the parameters $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21})$ and $\overleftarrow{n}_{jj} = 0$ in the two-user LD-IC-NOF.

Theorem 3: Let $i \in \{1,2\}$, $j \in \{1,2\} \setminus \{i\}$ *and* $\overleftarrow{n}_{ii}^{\ddagger} \in \mathbb{N}$ *be*

$$\overleftarrow{n}_{ii}^{\ddagger} = \begin{cases} \max\left(n_{ji}, (\overrightarrow{n}_{ii} - n_{ij})^{+}\right) & \text{if } S_{1,i} = \text{True} \\ \overrightarrow{n}_{jj} + (\overrightarrow{n}_{ii} - n_{ij})^{+} & \text{if } S_{2,i} = \text{True}. \end{cases} \tag{37}$$

Assume that $S_{3,i} = \text{True}$. *Then, for all* $\overleftarrow{n}_{ii} \in \mathbb{N}$, $\Delta_j(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}) = 0$. *Assume that either* $S_{1,i} = \text{True}$ *or* $S_{2,i} = \text{True}$. *Then, when* $\overleftarrow{n}_{ii} \leqslant \overleftarrow{n}_{ii}^{\ddagger}$, *it holds that* $\Delta_j(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}) = 0$; *and when* $\overleftarrow{n}_{ii} > \overleftarrow{n}_{ii}^{\ddagger}$, *it holds that* $\Delta_j(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}) > 0$.

Proof: The proof of Theorem 3 follows along the same lines of the proof on Theorem 2 in Appendix B. The difference is that the analysis is focused on the individual rate R_2. ■

Theorem 3 shows that under event $S_{3,i}$ in (29), implementing feedback in transmitter-receiver pair i does not bring any improvement on the rate R_j. This is in line with the results of Theorem 1 that states that under event $S_{3,i}$ in (29), implementing feedback in transmitter-receiver pair i does not enlarge the capacity region. In contrast, under events $S_{1,i}$ in (27) and $S_{2,i}$ in (28), the individual rate R_j can be improved ($\Delta_j(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}) > 0$) whenever $\overleftarrow{n}_{ii} > \overleftarrow{n}_{ii}^{\ddagger}$. It is important to highlight that under event $S_{1,i}$, the threshold on $\overleftarrow{n}_{ii}$ for increasing the individual rate R_i i.e., ($\overleftarrow{n}_{ii}^{\dagger}$), and R_j i.e., ($\overleftarrow{n}_{ii}^{\ddagger}$), are identical, see Theorem 2 and Theorem 3. This shows that in this case, the use of feedback in transmitter-receiver pair i, with $\overleftarrow{n}_{ii} > \overleftarrow{n}_{ii}^{\dagger} = \overleftarrow{n}_{ii}^{\ddagger}$, simultaneously improves both individual rates. Under event $S_{2,i}$, using feedback in transmitter-receiver pair i, with $\overleftarrow{n}_{ii} > \overleftarrow{n}_{ii}^{\ddagger}$, exclusively benefits transmitter-receiver pair j, which can improve its own individual rate. An example in which for all $(R_1', R_2') \in \mathcal{C}\,(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, 0)$ there always exists a rate pair $(R_1, R_2) \in \mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$ such that $R_2' < R_2$, is presented in Fig. 4a.

D. Improvement of the Sum-Capacity

Implementing channel output feedback in transmitter-receiver pair i increases the sum-capacity under certain conditions. Theorem 4 identifies the exact values of $\overleftarrow{n}_{ii}$ for which the sum-capacity can be improved, for parameters $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21})$ and $\overleftarrow{n}_{jj} = 0$ in the two-user LD-IC-NOF.

Theorem 4: Let $i \in \{1,2\}$, $j \in \{1,2\} \setminus \{i\}$ *and* $\overleftarrow{n}_{ii}^{+} \in \mathbb{N}$ *be*

$$\overleftarrow{n}_{ii}^{+} = \begin{cases} \max\left(n_{ji}, (\overrightarrow{n}_{ii} - n_{ij})^{+}\right) & \text{if } S_{4,i} = \text{True} \\ \overrightarrow{n}_{jj} + (\overrightarrow{n}_{ii} - n_{ij})^{+} & \text{if } S_{5,i} = \text{True}. \end{cases} \tag{38}$$

Assume that $S_{4,i} = \text{False}$ *and* $S_{5,i} = \text{False}$. *Then, for all* $\overleftarrow{n}_{ii} \in \mathbb{N}$, $\Sigma(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}) = 0$. *Assume that* $S_{4,i} = \text{True}$ *or* $S_{5,i} = \text{True}$. *Then, when* $\overleftarrow{n}_{ii} \leqslant \overleftarrow{n}_{ii}^{+}$, *it holds that*

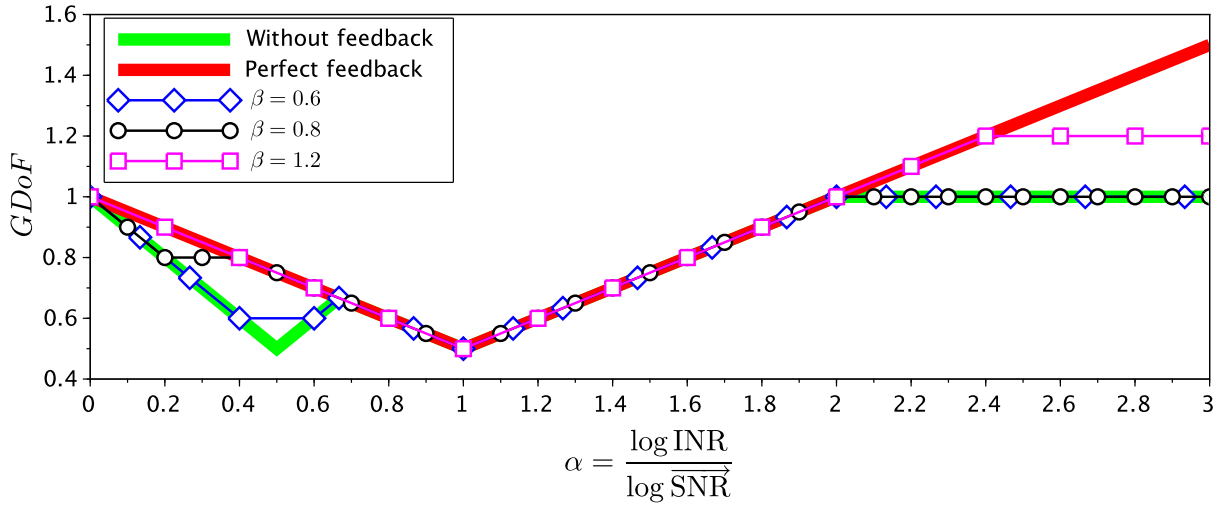

Fig. 5. Generalized Degrees of Freedom (GDoF) as a function of parameters α and β, with $0 \leqslant \alpha \leqslant 3$ and $\beta \in \{\frac{3}{5}, \frac{4}{5}, \frac{6}{5}\}$, of the symmetric LD-IC-NOF. The plot without feedback is obtained from [11] and the plot with perfect-output feedback is obtained from [3].

$\Sigma(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}) = 0$*; and when* $\overleftarrow{n}_{ii} > \overleftarrow{n}^{+}_{ii}$*, it holds that* $\Sigma(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}) > 0$.

Proof: The proof of Theorem 4 is presented in Appendix C. ■

Theorem 4 identifies the conditions under which implementing feedback in transmitter-receiver pair i improves the sum-capacity whenever $\overleftarrow{n}_{ii} > \overleftarrow{n}^{+}_{ii}$, that is, $\Sigma(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}) > 0$. Theorem 4 highlights that one of the necessary but not sufficient conditions for improving the sum-capacity by implementing feedback in transmitter-receiver pair i is that either (a) at least one transmitter-receiver pair must be in VWIR or WIR; or (b) both transmitter-receiver pairs must be in VSIR. This follows immediately from observing that for $S_{4,i}$ or $S_{5,i}$ to hold true, at least one of the events E_9, $E_{10,i}$, $E_{11,i}$ or E_{12} must hold true. An example in which $C_{\mathrm{sum}}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, 0) < C_{\mathrm{sum}}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$, with $\overleftarrow{n}_{11} > \overleftarrow{n}^{+}_{11}$, is presented in Fig. 2.

Interestingly, Theorem 4 shows that if at least one transmitter-receiver pair is in SIR, then the sum-capacity cannot be improved. Finally, note that the thresholds $\overleftarrow{n}^{+}_{ii}$ in the events $S_{4,i}$ and $S_{5,i}$ coincide with those observed in Theorem 1.

V. Generalized Degrees of Freedom

This section focuses on the analysis of the number of GDoF of the LD-IC-NOF for studying the case in which feedback is simultaneously implemented in both transmitter-receiver pairs. Moreover, the analysis is only performed for the symmetric case, i.e., $\overrightarrow{n} = \overrightarrow{n}_{11} = \overrightarrow{n}_{22}$, $m = n_{12} = n_{21}$, and $\overleftarrow{n} = \overleftarrow{n}_{11} = \overleftarrow{n}_{22}$, with $(\overrightarrow{n}, m, \overleftarrow{n}) \in \mathbb{N}^3$. The results in Lemma 1 allow a more general analysis of the GDoF, e.g., non-symmetric case. However, the symmetric case captures some of the most important insights about how the capacity region is enlarged when feedback is used in both transmitter-receiver pairs.

Essentially, given the parameters $\overrightarrow{n}$, m and $\overleftarrow{n}$, with $\alpha = \frac{m}{\overrightarrow{n}}$ and $\beta = \frac{\overleftarrow{n}}{\overrightarrow{n}}$, the number of GDoF, denoted by $D(\alpha, \beta)$, is the ratio between the symmetric capacity, i.e., $C_{\text{sym}}(\overrightarrow{n}, m, \overleftarrow{n}) = \sup\{R : (R, R) \in \mathcal{C}(\overrightarrow{n}, \overrightarrow{n}, m, m, \overleftarrow{n}, \overleftarrow{n})\}$, and the individual interference-free point-to-point capacity, i.e., $\overrightarrow{n}$, when $(\overrightarrow{n}, m, \overleftarrow{n}) \to (\infty, \infty, \infty)$ at constant ratios $\alpha = \frac{m}{\overrightarrow{n}}$ and $\beta = \frac{\overleftarrow{n}}{\overrightarrow{n}}$. More specifically, the number of GDoF is

$$D(\alpha, \beta) = \lim_{\overrightarrow{n}, m, \overleftarrow{n} \to \infty} \frac{C_{\text{sym}}(\overrightarrow{n}, m, \overleftarrow{n})}{\overrightarrow{n}}. \tag{39}$$

Theorem 5 determines the number of GDoF for the two-user LD-IC-NOF.

Theorem 5: The number of GDoF for the two user symmetric LD-IC-NOF with parameters α and β is given by

$$\begin{aligned} D(\alpha,\beta)=\min\Bigg(& \max(1,\alpha), \max\left(1, \beta-(1-\alpha)^+\right), \frac{1}{2}\left(\max(1,\alpha)+(1-\alpha)^+\right), \\ & \max\left((1-\alpha)^+, \alpha, 1-(\max(1,\alpha)-\beta)^+\right), \\ & \frac{1}{3}\Bigg(\max(1,\alpha)+(1-\alpha)^+ + \max\left(\left(1-\alpha\right)^+, \alpha, 1-\left(\max\left(1,\alpha\right)-\beta\right)^+\right)\Bigg). \end{aligned} \tag{40}$$

Proof: The proof of Theorem 5 is presented in Appendix D. ■

The result in Theorem 5 can also be obtained from Theorem 1 in [7]. The following properties are a direct consequence of Theorem 5.

Corollary 1: The number of GDoF for the two user symmetric LD-IC-NOF with parameters

α and β satisfies the following properties:

$$\forall \alpha \in \left[0, \frac{2}{3}\right] \text{ and } \beta \leqslant 1, \quad \max\left(\frac{1}{2}, \beta\right) \leqslant D(\alpha, \beta) \leqslant 1, \tag{41a}$$

$$\forall \alpha \in \left[0, \frac{2}{3}\right] \text{ and } \beta > 1, \quad D(\alpha, \beta) = 1 - \frac{\alpha}{2}, \tag{41b}$$

$$\forall \alpha \in \left(\frac{2}{3}, 2\right] \text{ and } \beta \in [0, \infty), \quad D(\alpha, 0) = D(\alpha, \beta) = D(\alpha, \max(1, \alpha)), \tag{41c}$$

$$\forall \alpha \in (2, \infty) \text{ and } \beta \geqslant 1, \quad 1 \leqslant D(\alpha, \beta) \leqslant \min\left(\frac{\alpha}{2}, \beta\right), \tag{41d}$$

$$\forall \alpha \in (2, \infty) \text{ and } \beta < 1, \quad D(\alpha, \beta) = 1. \tag{41e}$$

Properties (41a) and (41b) highlight the fact that the existence of feedback links in the symmetric LD-IC in the VWIR and WIR does not have any impact in the GDoF when $\beta \leqslant \frac{1}{2}$, and the GDoF is equal to the case with perfect-output feedback when $\beta > 1$. Property (41c) underlines that in the symmetric LD-IC in MIR and SIR, the number of GDoF is identical in both extreme cases: without feedback ($\beta = 0$) and with perfect-output feedback $\big(\beta = \max(1, \alpha)\big)$. Finally, from (41d) and (41e), it follows that for observing an improvement in the GDoF of the LD-IC-NOF in VSIR, the following condition must be met: $\beta > 1$. That is, the number of bit-pipes in the feedback links must be strictly bigger than the number of bit-pipes in the direct links.
Figure 5 shows the number of GDoF for the two user symmetric LD-IC-NOF for the case in which $0 \leqslant \alpha \leqslant 3$ and $\beta \in \{\frac{3}{5}, \frac{4}{5}, \frac{6}{5}\}$.

VI. Conclusions

This paper presented the exact conditions on the feedback parameters $\overleftarrow{n}_{11}$ and $\overleftarrow{n}_{22}$, beyond which the capacity region of the two-user LD-IC-NOF can be enlarged for any 4-tuple $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}) \in \mathbb{N}^4$. More specifically, the exact values of $\overleftarrow{n}_{11}$ (resp. $\overleftarrow{n}_{22}$) for which $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, 0) \subset \mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$ $\big($resp. $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, 0) \subset \mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, \overleftarrow{n}_{22})\big)$, with strict inclusion. The exact conditions on $\overleftarrow{n}_{11}$ (resp. $\overleftarrow{n}_{22}$) to observe an improvement on a single rate or the sum-rate capacity, for any 4-tuple $\Big(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}\Big) \in \mathbb{N}^4$ were also presented. Interestingly, there exist conditions in the two-user LD-IC-NOF in which the use of feedback does not enlarge the capacity region. The gain in the capacity of the two user LD-IC-NOF under symmetric conditions and for the VSIR can be bounded by the noise in the feedback links. Finally, closed-form expressions for the number of GDoF are

presented in the case of symmetric LD-IC-NOF. As observed before in [3] and [11], the number of GDoF is invariant with respect to the use of output feedback links in both moderate and strong interference regimes. In all the other interference regimes, the exact improvement on the GDoF is perfectly identified.

APPENDIX A
PROOF OF THEOREM 1: ENLARGEMENT OF THE CAPACITY REGION BY USING FEEDBACK IN ONE TRANSMITTER-RECEIVER PAIR

The proof of Theorem 1 is obtained by comparing $\mathcal{C}(\overrightarrow{n}_{11},\ \overrightarrow{n}_{22},\ n_{12},\ n_{21},\ \overleftarrow{n}_{11}, 0)$ $\big($resp. $\mathcal{C}(\overrightarrow{n}_{11},\ \overrightarrow{n}_{22},\ n_{12},\ n_{21},\ 0,\ \overleftarrow{n}_{22})\big)$ and $\mathcal{C}(\overrightarrow{n}_{11},\ \overrightarrow{n}_{22},\ n_{12},\ n_{21},\ 0,\ 0)$, for all possible parameters $\overrightarrow{n}_{11},\ \overrightarrow{n}_{22},\ n_{12},\ n_{21}$, and $\overleftarrow{n}_{11}$ (resp. $\overrightarrow{n}_{11},\ \overrightarrow{n}_{22},\ n_{12},\ n_{21}$, and $\overleftarrow{n}_{22}$). More specifically, for each tuple $\big(\overrightarrow{n}_{11},\ \overrightarrow{n}_{22},\ n_{12},\ n_{21}\big)$, the exact value $\overleftarrow{n}^*_{11}$ (resp $\overleftarrow{n}^*_{22}$) for which any $\overleftarrow{n}_{11} > \overleftarrow{n}^*_{11}$ (resp $\overleftarrow{n}_{22} > \overleftarrow{n}^*_{22}$) ensures $\mathcal{C}(\overrightarrow{n}_{11},\ \overrightarrow{n}_{22},\ n_{12},\ n_{21},\ 0,0) \subset \mathcal{C}(\overrightarrow{n}_{11},\ \overrightarrow{n}_{22},\ n_{12},\ n_{21},\ \overleftarrow{n}_{11}, 0)$ (resp. $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, 0) \subset \mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, \overleftarrow{n}_{22})$) is calculated. This procedure is tedious and repetitive, and thus, in this appendix only one combination of interference regimes is fully studied, e.g., VWIR - VWIR.

Proof:

Consider that both transmitter-receiver pairs are in VWIR, that is,

$$\alpha_1 = \frac{n_{12}}{\overrightarrow{n}_{11}} \leqslant \frac{1}{2} \text{ and } \alpha_2 = \frac{n_{21}}{\overrightarrow{n}_{22}} \leqslant \frac{1}{2}. \tag{42}$$

Under conditions (42), $\mathcal{C}(\overrightarrow{n}_{11},\ \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, 0)$ is the set of non-negative rate pairs (R_1, R_2) that satisfy

$$R_1 \leqslant \overrightarrow{n}_{11} \triangleq \theta_1, \tag{43a}$$

$$R_2 \leqslant \overrightarrow{n}_{22} \triangleq \theta_2, \tag{43b}$$

$$R_1 + R_2 \leqslant \min\left(\max\left(\overrightarrow{n}_{22}, n_{12}\right) + \overrightarrow{n}_{11} - n_{12}, \max\left(\overrightarrow{n}_{11}, n_{21}\right) + \overrightarrow{n}_{22} - n_{21}\right) \triangleq \theta_3, \tag{43c}$$

$$R_1 + R_2 \leqslant \max\left(\overrightarrow{n}_{11} - n_{12}, n_{21}\right) + \max\left(\overrightarrow{n}_{22} - n_{21}, n_{12}\right) \triangleq \theta_4, \tag{43d}$$

$$2R_1 + R_2 \leqslant \max\left(\overrightarrow{n}_{11}, n_{21}\right) + \overrightarrow{n}_{11} - n_{12} + \max\left(\overrightarrow{n}_{22} - n_{21}, n_{12}\right) \triangleq \theta_5, \tag{43e}$$

$$R_1 + 2R_2 \leqslant \max\left(\overrightarrow{n}_{22}, n_{12}\right) + \overrightarrow{n}_{22} - n_{21} + \max\left(n_{21}, \overrightarrow{n}_{11} - n_{12}\right) \triangleq \theta_6. \tag{43f}$$

Note that for all $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{22}) \in \mathbb{N}^5$ and $\overleftarrow{n}_{11} > \max\left(\overrightarrow{n}_{11}, n_{12}\right)$, it follows that $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22}) = \mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \max(\overrightarrow{n}_{11}, n_{12}), \overleftarrow{n}_{22})$. Hence, in the

following, the analysis is restricted to the following condition:

$$\overleftarrow{n}_{11} \leqslant \max\left(\overrightarrow{n}_{11}, n_{12}\right). \tag{44}$$

Under conditions (42) and (44), $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$ is the set of non-negative rate pairs (R_1, R_2) that satisfy

$$R_1 \leqslant \overrightarrow{n}_{11}, \tag{45a}$$

$$R_2 \leqslant \overrightarrow{n}_{22}, \tag{45b}$$

$$R_1 + R_2 \leqslant \min\left(\max\left(\overrightarrow{n}_{22}, n_{12}\right) + \overrightarrow{n}_{11} - n_{12}, \max\left(\overrightarrow{n}_{11}, n_{21}\right) + \overrightarrow{n}_{22} - n_{21}\right), \tag{45c}$$

$$R_1 + R_2 \leqslant \max\left(\overrightarrow{n}_{11} - n_{12}, n_{21}, \overleftarrow{n}_{11}\right) + \max\left(\overrightarrow{n}_{22} - n_{21}, n_{12}\right) \triangleq \theta_7, \tag{45d}$$

$$2R_1 + R_2 \leqslant \max\left(\overrightarrow{n}_{11}, n_{21}\right) + \overrightarrow{n}_{11} - n_{12} + \max\left(\overrightarrow{n}_{22} - n_{21}, n_{12}\right), \tag{45e}$$

$$R_1 + 2R_2 \leqslant \max\left(\overrightarrow{n}_{22}, n_{12}\right) + \overrightarrow{n}_{22} - n_{21} + \max\left(\overrightarrow{n}_{11} - n_{12}, n_{21}, \overleftarrow{n}_{11}\right) \triangleq \theta_8. \tag{45f}$$

When comparing $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, 0)$ and $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$, note that (43a), (43b), (43c), and (43e) are equivalent to (45a), (45b), (45c), and (45e), respectively. Under these observations, the region $\mathcal{C}(\overrightarrow{n}_{11},\ \overrightarrow{n}_{22},\ n_{12},\ n_{21},\ \overleftarrow{n}_{11},\ 0)$ is greater than the region $\mathcal{C}(\overrightarrow{n}_{11},\ \overrightarrow{n}_{22},\ n_{12},\ n_{21},\ 0,\ 0)$ if at least one of the following conditions is true:

$$\min(\theta_3, \theta_4) < \min(\theta_3, \theta_7) < \theta_1 + \theta_2, \tag{46a}$$

$$\theta_6 < \min\left(\theta_8, \theta_1 + 2\theta_2, \theta_2 + \theta_3, \theta_2 + \theta_4\right) < \min\left(\theta_1 + 2\theta_2, \theta_2 + \theta_3, \theta_2 + \theta_7\right). \tag{46b}$$

Condition (46a) implies that the active sum-rate bound in $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$ is greater than the active sum-rate bound in $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, 0)$. Condition (46b) implies that the active weighted sum-rate bound on $R_1 + 2R_2$ in $\mathcal{C}(\overrightarrow{n}_{11},\ \overrightarrow{n}_{22},\ n_{12},\ n_{21},\overleftarrow{n}_{11},0)$ is greater than the active weighted sum-rate bound on $R_1 + 2R_2$ in $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, 0)$.

To simplify the inequalities containing the operator $\max(\cdot, \cdot)$ in (45) and (43), the following 4 cases are identified:

$$\text{Case } 1: \overrightarrow{n}_{11} - n_{12} < n_{21} \text{ and } \overrightarrow{n}_{22} - n_{21} < n_{12}; \tag{47}$$

$$\text{Case } 2: \overrightarrow{n}_{11} - n_{12} < n_{21} \text{ and } \overrightarrow{n}_{22} - n_{21} > n_{12}; \tag{48}$$

$$\text{Case } 3\text{: } \overrightarrow{n}_{11} - n_{12} > n_{21} \text{ and } \overrightarrow{n}_{22} - n_{21} < n_{12};\ \text{and} \tag{49}$$

$$\text{Case } 4\text{: } \overrightarrow{n}_{11} - n_{12} > n_{21} \text{ and } \overrightarrow{n}_{22} - n_{21} > n_{12}. \tag{50}$$

Case 1: Under assumptions (42) and (47), this case is not possible by contradiction.

Case 2: Under assumptions (42) and (48), this case is possible.

Plugging (48) into (45) yields:

$$R_1 + R_2 \leqslant \min\left(\overrightarrow{n}_{22} + \overrightarrow{n}_{11} - n_{12}, \max\left(\overrightarrow{n}_{11}, n_{21}\right) + \overrightarrow{n}_{22} - n_{21}\right), \tag{51a}$$

$$R_1 + R_2 \leqslant \max\left(n_{21}, \overleftarrow{n}_{11}\right) + \overrightarrow{n}_{22} - n_{21}, \tag{51b}$$

$$R_1 + 2R_2 \leqslant 2\overrightarrow{n}_{22} - n_{21} + \max\left(n_{21}, \overleftarrow{n}_{11}\right). \tag{51c}$$

Plugging (48) into (43) yields:

$$R_1 + R_2 \leqslant \overrightarrow{n}_{22}, \tag{52a}$$

$$R_1 + 2R_2 \leqslant 2\overrightarrow{n}_{22}. \tag{52b}$$

To simplify the inequalities containing the operator $\max(\cdot,\cdot)$ in (51), the following 2 cases are identified:

$$\text{Case 2a}: \overrightarrow{n}_{11} > n_{21}; \text{ and} \tag{53}$$

$$\text{Case 2b}: \overrightarrow{n}_{11} \leqslant n_{21}. \tag{54}$$

Case 2a: Plugging (53) into (51) yields:

$$R_1 + R_2 \leqslant \overrightarrow{n}_{11} + \overrightarrow{n}_{22} - n_{21}, \tag{55a}$$

$$R_1 + R_2 \leqslant \max\left(n_{21}, \overleftarrow{n}_{11}\right) + \overrightarrow{n}_{22} - n_{21}, \tag{55b}$$

$$R_1 + 2R_2 \leqslant 2\overrightarrow{n}_{22} - n_{21} + \max\left(n_{21}, \overleftarrow{n}_{11}\right). \tag{55c}$$

Comparing inequalities (55a) and (55b) with inequality (52a), it can be verified that $\min\Big(\overrightarrow{n}_{11} + \overrightarrow{n}_{22} - n_{21},\ \max\left(n_{21},\ \overleftarrow{n}_{11}\right) + \overrightarrow{n}_{22} - n_{21}\Big) > \overrightarrow{n}_{22}$, i.e., condition (46a) holds, when $\overleftarrow{n}_{11} > n_{21}$. Comparing inequalities (55c) and (52b), it can be verified that $2\overrightarrow{n}_{22} - n_{21} + \max\left(n_{21}, \overleftarrow{n}_{11}\right) > 2\overrightarrow{n}_{22}$, i.e., condition (46b) holds, when $\overleftarrow{n}_{11} > n_{21}$. Therefore, $\overleftarrow{n}^{*}_{11} = n_{21}$ under assumptions (42), (44), (48), and (53).

Case 2b: Plugging (54) into (51) yields:

$$R_1 + R_2 \leqslant \overrightarrow{n}_{22}, \tag{56a}$$

$$R_1 + R_2 \leqslant \max\left(n_{21}, \overleftarrow{n}_{11}\right) + \overrightarrow{n}_{22} - n_{21}, \tag{56b}$$

$$R_1 + 2R_2 \leqslant 2\overrightarrow{n}_{22} - n_{21} + \max\left(n_{21}, \overleftarrow{n}_{11}\right). \tag{56c}$$

Comparing inequalities (56a) and (56b) with inequality (52a), it can be verified that $\min\Big(\overrightarrow{n}_{22}, \max\left(n_{21}, \overleftarrow{n}_{11}\right)+\overrightarrow{n}_{22}-n_{21}\Big) = \overrightarrow{n}_{22}$, i.e., condition (46a) does not hold, for all $\overleftarrow{n}_{11} \in \mathbb{N}$. Comparing inequalities (56c) and (52b) it can be verified that $2\overrightarrow{n}_{22}-n_{21}+\max\left(n_{21}, \overleftarrow{n}_{11}\right) > 2\overrightarrow{n}_{22}$, when $\overleftarrow{n}_{11} > n_{21}$, which implies that $\overleftarrow{n}_{11} > \max\left(\overrightarrow{n}_{11}, n_{12}\right)$. However, under the assumptions (42), (44), (48), and (54), the bounds (52b) and (56c) are not active. Hence, condition (46b) does not hold. Therefore, for all $\overleftarrow{n}_{11} \in \mathbb{N}$, the capacity region cannot be enlarged under assumptions (42), (44), (48), and (54).

Case 3: Under assumptions (42) and (49), this case is possible.

Plugging (49) into (45) yields:

$$R_1 + R_2 \leqslant \min\left(\max\left(\overrightarrow{n}_{22}, n_{12}\right) + \overrightarrow{n}_{11} - n_{12}, \overrightarrow{n}_{11} + \overrightarrow{n}_{22} - n_{21}\right), \tag{57a}$$

$$R_1 + R_2 \leqslant \max\left(\overrightarrow{n}_{11} - n_{12}, \overleftarrow{n}_{11}\right) + n_{12}, \tag{57b}$$

$$R_1 + 2R_2 \leqslant \max\left(\overrightarrow{n}_{22}, n_{12}\right) + \overrightarrow{n}_{22} - n_{21} + \max\left(\overrightarrow{n}_{11} - n_{12}, \overleftarrow{n}_{11}\right). \tag{57c}$$

Plugging (49) into (43) yields:

$$R_1 + R_2 \leqslant \overrightarrow{n}_{11}, \tag{58a}$$

$$R_1 + 2R_2 \leqslant \max\left(\overrightarrow{n}_{22}, n_{12}\right) + \overrightarrow{n}_{22} - n_{21} + \overrightarrow{n}_{11} - n_{12}. \tag{58b}$$

To simplify the inequalities containing the operator $\max(\cdot,\cdot)$ in (57) and (58), the following 2 cases are identified:

$$\text{Case 3a} : \overrightarrow{n}_{22} > n_{12}; \text{ and} \tag{59}$$

$$\text{Case 3b} : \overrightarrow{n}_{22} \leqslant n_{12}. \tag{60}$$

Case 3a: Plugging (59) into (57) yields:

$$R_1 + R_2 \leqslant \overrightarrow{n}_{22} + \overrightarrow{n}_{11} - n_{12}, \tag{61a}$$

$$R_1 + R_2 \leqslant \max\left(\overrightarrow{n}_{11} - n_{12}, \overleftarrow{n}_{11}\right) + n_{12}, \tag{61b}$$

$$R_1 + 2R_2 \leqslant 2\overrightarrow{n}_{22} - n_{21} + \max\left(\overrightarrow{n}_{11} - n_{12}, \overleftarrow{n}_{11}\right). \tag{61c}$$

Plugging (59) into (58) yields:

$$R_1 + R_2 \leqslant \overrightarrow{n}_{11}, \tag{62a}$$

$$R_1 + 2R_2 \leqslant 2\overrightarrow{n}_{22} - n_{21} + \overrightarrow{n}_{11} - n_{12}. \tag{62b}$$

Comparing inequalities (61a) and (61b) with inequality (62a), it can be verified that $\min\Big(\overrightarrow{n}_{22}+\overrightarrow{n}_{11}-n_{12},\max\left(\overrightarrow{n}_{11}-n_{12},\overleftarrow{n}_{11}\right)+n_{12}\Big)>\overrightarrow{n}_{11}$, i.e., condition (46a) holds, when $\overleftarrow{n}_{11}>\overrightarrow{n}_{11}-n_{12}$. Comparing inequalities (61c) and (62b), it can be verified that $2\overrightarrow{n}_{22}-n_{21}+\max\left(\overrightarrow{n}_{11}-n_{12},\overleftarrow{n}_{11}\right)>2\overrightarrow{n}_{22}-n_{21}+\overrightarrow{n}_{11}-n_{12}$, i.e., condition (46b) holds, when $\overleftarrow{n}_{11}>\overrightarrow{n}_{11}-n_{12}$. Therefore, $\overleftarrow{n}^{*}_{11}=\overrightarrow{n}_{11}-n_{12}$ under assumptions (42), (44), (49), and (59).

Case 3b: Plugging (60) into (57) yields:

$$R_1+R_2\leqslant\overrightarrow{n}_{11}, \tag{63a}$$

$$R_1+R_2\leqslant\max\left(\overrightarrow{n}_{11}-n_{12},\overleftarrow{n}_{11}\right)+n_{12}, \tag{63b}$$

$$R_1+2R_2\leqslant n_{12}+\overrightarrow{n}_{22}-n_{21}+\max\left(\overrightarrow{n}_{11}-n_{12},\overleftarrow{n}_{11}\right). \tag{63c}$$

Plugging (59) into (58) yields:

$$R_1+R_2\leqslant\overrightarrow{n}_{11}, \tag{64a}$$

$$R_1+2R_2\leqslant\overrightarrow{n}_{22}-n_{21}+\overrightarrow{n}_{11}. \tag{64b}$$

Comparing inequalities (63a) and (63b) with inequality (64a), it can be verified that $\min\Big(\overrightarrow{n}_{11},\max\left(\overrightarrow{n}_{11}-n_{12},\overleftarrow{n}_{11}\right)+n_{12}\Big)=\overrightarrow{n}_{11}$, i.e., condition (46a) does not hold, for all $\overleftarrow{n}_{11}\in\mathbb{N}$. Comparing inequalities (63c) and (64b), it can be verified that $n_{12}+\overrightarrow{n}_{22}-n_{21}+\max\left(\overrightarrow{n}_{11}-n_{12},\overleftarrow{n}_{11}\right)>\overrightarrow{n}_{22}-n_{21}+\overrightarrow{n}_{11}$, i.e., condition (46b) holds, when $\overleftarrow{n}_{11}>\overrightarrow{n}_{11}-n_{12}$. Therefore, $\overleftarrow{n}^{*}_{11}=\overrightarrow{n}_{11}-n_{12}$ under conditions (42), (44), (49), and (60).

Case 4: Under conditions (42) and (50), this case is possible.

Plugging (50) into (45) yields:

$$R_1+R_2\leqslant\min\left(\overrightarrow{n}_{22}+\overrightarrow{n}_{11}-n_{12},\overrightarrow{n}_{11}+\overrightarrow{n}_{22}-n_{21}\right), \tag{65a}$$

$$R_1+R_2\leqslant\max\left(\overrightarrow{n}_{11}-n_{12},\overleftarrow{n}_{11}\right)+\overrightarrow{n}_{22}-n_{21}, \tag{65b}$$

$$R_1+2R_2\leqslant2\overrightarrow{n}_{22}-n_{21}+\max\left(\overrightarrow{n}_{11}-n_{12},\overleftarrow{n}_{11}\right). \tag{65c}$$

Plugging (50) into (43) yields:

$$R_1+R_2\leqslant\overrightarrow{n}_{11}-n_{12}+\overrightarrow{n}_{22}-n_{21}, \tag{66a}$$

$$R_1+2R_2\leqslant2\overrightarrow{n}_{22}-n_{21}+\overrightarrow{n}_{11}-n_{12}. \tag{66b}$$

Comparing inequalities (65a) and (65b) with inequality (66a), it can be verified that $\min\Big(\min\left(\overrightarrow{n}_{22}+\overrightarrow{n}_{11}-n_{12},\overrightarrow{n}_{11}+\overrightarrow{n}_{22}-n_{21}\right),\max\left(\overrightarrow{n}_{11}-n_{12},\overleftarrow{n}_{11}\right)+\overrightarrow{n}_{22}-n_{21}\Big)>\overrightarrow{n}_{11}-n_{12}+\overrightarrow{n}_{22}-n_{21}$,

i.e., condition (46a) holds, when $\overleftarrow{n}_{11} > \overrightarrow{n}_{11} - n_{12}$. Comparing inequalities (65c) and (66b), it can be verified that: $2\overrightarrow{n}_{22} - n_{21} + \max\left(\overrightarrow{n}_{11} - n_{12},\, \overleftarrow{n}_{11}\right) > 2\overrightarrow{n}_{22} - n_{21} + \overrightarrow{n}_{11} - n_{12}$, i.e., condition (46b) holds, when $\overleftarrow{n}_{11} > \overrightarrow{n}_{11} - n_{12}$. Therefore, $\overleftarrow{n}^{*}_{11} = \overrightarrow{n}_{11} - n_{12}$ under conditions (42), (44), and (50).

From all the observations above, when both transmitter-receiver pairs are in VWIR (event E_1 is True), it follows that when $\overleftarrow{n}_{11} > \overleftarrow{n}^{*}_{11}$ and $\overrightarrow{n}_{11} > n_{21}$ (event $E_{13,1}$ is True) with with $\overleftarrow{n}^{*}_{11} = \max\left(\overrightarrow{n}_{11} - n_{12}, n_{21}\right)$, then $\mathcal{C}(\overrightarrow{n}_{11},\, \overrightarrow{n}_{22},\, n_{12},\, n_{21},\, 0,\, 0) \subset \mathcal{C}(\overrightarrow{n}_{11},\, \overrightarrow{n}_{22},\, n_{12},\, n_{21},\, \overleftarrow{n}_{11},\, 0)$. Otherwise $\mathcal{C}(\overrightarrow{n}_{11},\, \overrightarrow{n}_{22},\, n_{12},\, n_{21},\, 0,\, 0) = \mathcal{C}(\overrightarrow{n}_{11},\, \overrightarrow{n}_{22},\, n_{12},\, n_{21},\, \overleftarrow{n}_{11},\, 0)$. Note that when events E_1 and $E_{13,1}$ hold simultaneously true, then the event $S_{1,1}$ is true, which verifies the statement of Theorem 1. The same procedure can be applied for all the other combinations of interference regimes. This completes the proof. ■

APPENDIX B
PROOF OF THEOREM 2: IMPROVEMENT OF THE INDIVIDUAL RATE R_i BY USING FEEDBACK IN LINK i

The proof of Theorem 2 is obtained by comparing $\mathcal{C}(\overrightarrow{n}_{11},\, \overrightarrow{n}_{22},\, n_{12},\, n_{21},\, \overleftarrow{n}_{11}, 0)$ $\big($resp. $\mathcal{C}(\overrightarrow{n}_{11},\, \overrightarrow{n}_{22},\, n_{12},\, n_{21},\, 0,\, \overleftarrow{n}_{22})\big)$ and $\mathcal{C}(\overrightarrow{n}_{11},\, \overrightarrow{n}_{22},\, n_{12},\, n_{21},\, 0,\, 0)$, for all possible parameters $\overrightarrow{n}_{11}$, $\overrightarrow{n}_{22}$, n_{12}, n_{21}, and $\overleftarrow{n}_{11}$ (resp. $\overrightarrow{n}_{11}$, $\overrightarrow{n}_{22}$, n_{12}, n_{21}, and $\overleftarrow{n}_{22}$). More specifically, for each tuple $\big(\overrightarrow{n}_{11},\, \overrightarrow{n}_{22},\, n_{12},\, n_{21}\big)$, the exact value $\overleftarrow{n}^{\dagger}_{11}$ (resp $\overleftarrow{n}^{\dagger}_{22}$) for which any $\overleftarrow{n}_{11} > \overleftarrow{n}^{\dagger}_{11}$ (resp $\overleftarrow{n}_{22} > \overleftarrow{n}^{\dagger}_{22}$) ensures an improvement on R_1 (resp. R_2) , i.e., $\Delta_1(\overrightarrow{n}_{11},\, \overrightarrow{n}_{22},\, n_{12},\, n_{21},\, \overleftarrow{n}_{11},\, 0) > 0$ (resp. $\Delta_2(\overrightarrow{n}_{11},\, \overrightarrow{n}_{22},\, n_{12},\, n_{21},\, 0,\, \overleftarrow{n}_{22}) > 0$), is calculated. This procedure is tedious and repetitive, and thus, in this appendix only one combination of interference regimes is fully studied, e.g., VWIR - VWIR.

Proof:

Consider that both transmitter-receiver pairs are in VWIR, i.e., conditions (42) hold. Under these conditions, the capacity regions $\mathcal{C}(\overrightarrow{n}_{11},\, \overrightarrow{n}_{22},\, n_{12},\, n_{21},\, 0,\, 0)$ and $\mathcal{C}(\overrightarrow{n}_{11},\, \overrightarrow{n}_{22},\, n_{12},\, n_{21},\, \overleftarrow{n}_{11},\, 0)$ are given by (43) and (45), respectively. When comparing $\mathcal{C}(\overrightarrow{n}_{11},\, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, 0)$ and $\mathcal{C}(\overrightarrow{n}_{11},\, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$, note that (43a), (43b), (43c), and (43e) are equivalent to (45a), (45b), (45c), and (45e), respectively. In this case any improvement on R_1 is produced by an improvement on $R_1 + R_2$ (condition (46a)) or $2R_1 + R_2$ (condition (46a)), and thus, the proof of Theorem 2 in these particular interference regimes follows exactly the same steps in Theorem

1. This completes the proof. ■

APPENDIX C
PROOF OF THEOREM 4: IMPROVEMENT OF THE SUM-RATE CAPACITY BY USING FEEDBACK IN ONE TRANSMITTER-RECEIVER PAIR

The proof of Theorem 4 is obtained by comparing $\mathcal{C}(\overrightarrow{n}_{11},\ \overrightarrow{n}_{22},\ n_{12},\ n_{21},\ \overleftarrow{n}_{11}, 0)$ $\big($resp. $\mathcal{C}(\overrightarrow{n}_{11},\ \overrightarrow{n}_{22},\ n_{12},\ n_{21},\ 0,\ \overleftarrow{n}_{22})\big)$ and $\mathcal{C}(\overrightarrow{n}_{11},\ \overrightarrow{n}_{22},\ n_{12},\ n_{21},\ 0,\ 0)$, for all possible parameters $\overrightarrow{n}_{11}$, $\overrightarrow{n}_{22}$, n_{12}, n_{21}, and $\overleftarrow{n}_{11}$ (resp. $\overrightarrow{n}_{11}$, $\overrightarrow{n}_{22}$, n_{12}, n_{21}, and $\overleftarrow{n}_{22}$). More specifically, for each tuple $\big(\overrightarrow{n}_{11},\ \overrightarrow{n}_{22},\ n_{12},\ n_{21}\big)$, the exact value $\overleftarrow{n}_{11}^{+}$ (resp $\overleftarrow{n}_{22}^{+}$) for which any $\overleftarrow{n}_{11} > \overleftarrow{n}_{11}^{+}$ (resp $\overleftarrow{n}_{22} > \overleftarrow{n}_{22}^{+}$) ensures an improvement on R_1+R_2, i.e., $\Sigma(\overrightarrow{n}_{11},\ \overrightarrow{n}_{22},\ n_{12},\ n_{21},\ \overleftarrow{n}_{11}, 0) > 0$ (resp. $\Sigma(\overrightarrow{n}_{11},\ \overrightarrow{n}_{22},\ n_{12},\ n_{21},\ 0,\ \overleftarrow{n}_{22}) > 0$), is calculated. This procedure is tedious and repetitive, and thus, in this appendix only one combination of interference regimes is fully studied, e.g., VWIR - VWIR.

Proof:

Consider that both transmitter-receiver pairs are in VWIR, i.e., conditions (42) hold. Under these conditions, the capacity regions $\mathcal{C}(\overrightarrow{n}_{11},\ \overrightarrow{n}_{22},\ n_{12},\ n_{21},\ 0,\ 0)$ and $\mathcal{C}(\overrightarrow{n}_{11},\ \overrightarrow{n}_{22},\ n_{12},\ n_{21},\ \overleftarrow{n}_{11},\ 0)$ are given by (43) and (45), respectively. When comparing $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, 0)$ and $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$, note that (43a), (43b), (43c), and (43e) are equivalent to (45a), (45b), (45c), and (45e), respectively. In this case, the proof is focused on any improvement on $R_1 + R_2$ (condition (46a)), and thus, the proof of Theorem 4 in these particular interference regimes follows exactly the same steps in Theorem 1.

From the analysis presented in Appendix A, it follows that:

Case 2a: condition (46a) holds true, when $\overleftarrow{n}_{11} > n_{21}$ under assumptions (42), (44), (48), and (53).

Case 2b: condition (46a) does not hold true, under assumptions (42), (48), and (54).

Case 3a: condition (46a) holds true, when $\overleftarrow{n}_{11} > \overrightarrow{n}_{11} - n_{12}$ under assumptions (42), (44), (49), and (59).

Case 3b: condition (46a) does not hold true, when $\overleftarrow{n}_{11} > \overrightarrow{n}_{11} - n_{12}$ under assumptions (42), (44), (49), and (60).

Case 4: condition (46a) holds true, when $\overleftarrow{n}_{11} > \overrightarrow{n}_{11} - n_{12}$ under assumptions (42), (44), and (50).

From all the observations above, when both transmitter-receiver pairs are in VWIR (event E_9 is True), it follows that when $\overleftarrow{n}_{11} > \overleftarrow{n}^{+}_{11}$, $\overrightarrow{n}_{11} > n_{21}$ (event $E_{13,1}$ is True) and $\overrightarrow{n}_{22} > n_{12}$ (event $E_{13,2}$ is True) with $\overleftarrow{n}^{+}_{11} = \max(\overrightarrow{n}_{11} - n_{12}, n_{21})$, then $\Sigma(\overrightarrow{n}_{11},\ \overrightarrow{n}_{22},\ n_{12},\ n_{21},\ \overleftarrow{n}_{11},\ 0) > 0$. Otherwise $\Sigma(\overrightarrow{n}_{11},\ \overrightarrow{n}_{22},\ n_{12},\ n_{21},\ \overleftarrow{n}_{11},\ 0) = 0$. Note that when events E_9, $E_{13,1}$ and $E_{13,2}$ hold simultaneously true, then the event $S_{4,1}$ is true, which verifies the statement of Theorem 4. The same procedure can be applied for all the other combinations of interference regimes. This completes the proof. ■

APPENDIX D
PROOF OF THEOREM 5: GENERALIZED DEGREES OF FREEDOM

This appendix provides a proof to Theorem 5 for the two user LD-IC-NOF.

Proof:

Under symmetric conditions i.e., $\overrightarrow{n} = \overrightarrow{n}_{11} = \overrightarrow{n}_{22}$, $m = n_{12} = n_{21}$ and $\overleftarrow{n} = \overleftarrow{n}_{11} = \overleftarrow{n}_{22}$, from (8a) and (8b) with $i = 1$ and $j = 2$, it follows that:

$$R_1 \leqslant \min\left(\max(\overrightarrow{n}, m), \max\left(\overrightarrow{n}, \overleftarrow{n} - (\overrightarrow{n} - m)^+\right)\right) \triangleq a_1; \tag{67}$$

from (8c) and (8d), it follows that:

$$\begin{aligned} R_1 + R_2 \leqslant & \min\left(\max(\overrightarrow{n}, m) + (\overrightarrow{n} - m)^+, 2\max\left((\overrightarrow{n} - m)^+, m, \overrightarrow{n} - (\max(\overrightarrow{n}, m) - \overleftarrow{n})^+\right)\right) \\ \triangleq & a_2; \end{aligned} \tag{68}$$

and from (8e) with $i = 1$ and $j = 2$, it follows that:

$$2R_1 + R_2 \leqslant \max(\overrightarrow{n}, m) + (\overrightarrow{n} - m)^+ + \max\left((\overrightarrow{n} - m)^+, m, \overrightarrow{n} - (\max(\overrightarrow{n}, m) - \overleftarrow{n})^+\right) \triangleq a_3 \tag{69}$$

The symmetric sum-capacity, $C_{\text{sym}}(\overrightarrow{n}, m, \overleftarrow{n}) = \sup\{R : (R, R) \in \mathcal{C}(\overrightarrow{n}, \overrightarrow{n}, m, m, \overleftarrow{n}, \overleftarrow{n})\}$, can be obtained from (67), (68) and (69) as follows

$$\begin{aligned} C_{\text{sym}} = & \min\left(a_1, \frac{a_2}{2}, \frac{a_3}{2}, \frac{a_4}{3}\right) \\ = & \min\Bigg(\max(\overrightarrow{n}, m), \max\left(\overrightarrow{n}, \overleftarrow{n} - (\overrightarrow{n} - m)^+\right), \frac{1}{2}\left(\max(\overrightarrow{n}, m) + (\overrightarrow{n} - m)^+\right), \\ & \max\left((\overrightarrow{n} - m)^+, m, \overrightarrow{n} - (\max(\overrightarrow{n}, m) - \overleftarrow{n})^+\right), \frac{1}{3}\Big(\max(\overrightarrow{n}, m) + (\overrightarrow{n} - m)^+ \\ & \max\left((\overrightarrow{n} - m)^+, m, \overrightarrow{n} - (\max(\overrightarrow{n}, m) - \overleftarrow{n})^+\right)\Big)\Bigg). \end{aligned} \tag{70}$$

Plugging (70) into (39) yields

$$D_{\text{sym}}(\alpha,\beta)=\min\Bigg(\max\left(1,\alpha\right),\max\left(1,\beta-\left(1-\alpha\right)^{+}\right),\frac{1}{2}\left(\max\left(1,\alpha\right)+\left(1-\alpha\right)^{+}\right),$$
$$\max\Big(\left(1-\alpha\right)^{+},\alpha,1-\left(\max\left(1,\alpha\right)-\beta\right)^{+}\Big),\frac{1}{3}\Big(\max\left(1,\alpha\right)+\left(1-\alpha\right)^{+}$$
$$+\max\Big(\left(1-\alpha\right)^{+},\alpha,1-\left(\max\left(1,\alpha\right)-\beta\right)^{+}\Big)\Big)\Bigg). \tag{71}$$

where $\alpha=\frac{n}{\overrightarrow{n}}$ and $\beta=\frac{\overleftarrow{n}}{\overrightarrow{n}}$ and this completes the proof. ■

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