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On the spectrum of non-selfadjoint Schrödinger operators with compact resolvent

Y. Almog and B. Helffer

Abstract

We determine the Schatten class for the compact resolvent of Dirichlet realizations, in unbounded domains, of a class of non-selfadjoint differential operators. This class consists of operators that can be obtained via analytic dilation from a Schrödinger operator with magnetic field and a complex electric potential. As an application, we prove, in a variety of examples motivated by Physics, that the system of generalized eigenfunctions associated with the operator is complete, or at least the existence of an infinite discrete spectrum.

1 Introduction

The theory of non-selfadjoint differential operators is at a much less developed state than that of selfadjoint theory. The lack of variational methods makes it difficult, in many interesting cases, to determine whether a non-selfadjoint operator \mathcal{P} possesses a complete systems of generalized eigenvectors (by which we mean that the vector space they span is dense), or even if the spectrum is non-empty. In addition, the definition of a closed extension of the differential operator, is not always a straightforward matter. (There are, of course, other questions of interest, such as the effectiveness of the Fourier expansion [14, 7] which we do not address here.) There is, however, significant interest in these questions (cf. for instance [8, 15, 19, 1, 22] to name just a few references).

In recent contributions we have considered, together with X.B. Pan, similar questions for a well defined closed extension $\overline{\mathcal{A}}$ (or \mathcal{A}^D) of the differential operator

$$\mathcal{A} = -\left(\nabla - i\frac{x^2}{2}\hat{1}_y\right)^2 + icy, \quad (1.1)$$

where \hat{i}_y is a unit vector in the y direction. We studied the spectrum of this extension both in the entire (x, y) plane [3], where we show that $\sigma(\overline{\mathcal{A}}) = \emptyset$, and for the Dirichlet realization \mathcal{A}^D of \mathcal{A} in the half plane

$$\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}.$$

In the latter case we show that the spectrum is not empty in the limit $c \rightarrow 0$ [5], and in the limit $c \rightarrow \infty$ [4], where our techniques involve analytic dilation. The existence of a non-empty spectrum for general values of c remains an open question.

In another contribution [2] we show that the normal state for a superconductor in the presence of an electric current, and the magnetic field it induces becomes locally unstable, under some additional conditions which are omitted here, whenever $(\mathcal{A} - \lambda)^{-1}$ becomes unbounded, where λ is a function of the electric current. The analysis in [2] rests on some assumptions on the normal electric potential, that are not the most general ones. Thus, for instance, it assumes a non vanishing electric potential gradient. If the boundary conditions are such that this assumption is violated, one has to analyze the spectrum of different linear differential operators instead of (1.1), that are still of some physical interest.

Let then $\Omega \subseteq \mathbb{R}^d$ be (possibly) unbounded, $\partial\Omega \in C^{2,\alpha}$ for some $\alpha > 0$, $A = (A_1, \dots, A_d) \in C^2(\overline{\Omega}, \mathbb{R}^d)$. Let $B(x)$ denote the anti-symmetric matrix associated with curl A :

$$B_{jk} = \partial_k A_j - \partial_j A_k.$$

We attempt to prove existence of a non-empty spectrum, and when possible, to prove completeness of the system of (generalized) eigenvectors of the operator $-\Delta_A + V$ in $L^2(\Omega, \mathbb{C})$, where V may be complex valued. Since in some of the examples below we use analytic dilation, we consider a more general class of operators. (Note that analytical dilation can be applied, in general, only in domains that are invariant under real dilation). In particular, we consider here the operator

$$\mathcal{A}_\alpha = -e^{-i\frac{\pi}{m(k+1)}} \frac{\partial^2}{\partial x^2} - e^{i\frac{\pi}{2(k+1)}} \left(\frac{\partial}{\partial y} - i\frac{x^m}{m} \right)^2 + e^{i\frac{\pi}{2(k+1)}} y^{2k}, \quad (1.2)$$

obtained via analytic dilation of the operator:

$$\mathcal{A}_0 = -\frac{\partial^2}{\partial x^2} - \left(\frac{\partial}{\partial y} - i\frac{x^m}{m} \right)^2 + iy^{2k}. \quad (1.3)$$

We begin by defining the class of operators considered in this work. Let then

$$(\alpha_1, \dots, \alpha_d) \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)^d$$

and

$$K \stackrel{def}{=} \min_{1 \leq k \leq d} \cos(2\alpha_k) > 0. \quad (1.4)$$

Suppose that

$$V = V_1 + V_2$$

where $V_1 \in C^1(\overline{\Omega}, \mathbb{C})$ satisfies

$$\Re V_1 \geq -\lambda^*, \quad (1.5)$$

for some $\lambda^* \in \mathbb{R}$.

Suppose further that there exists a constant C such that

$$|\nabla V_1| + \max_{(k,l) \in \{1, \dots, d\}^2} |\nabla B_{kl}| \leq C m_{B, V_1} \text{ in } \Omega, \quad (1.6)$$

where

$$m_{B, V_1} = \sqrt{|V_1|^2 + |B|^2 + 1}, \quad (1.7)$$

Assume in addition that $V_2 \in L_{loc}^\infty(\overline{\Omega}, \mathbb{C})$ is such that for every $\epsilon > 0$, there exists C_ϵ for which

$$|V_2| \leq C_\epsilon + \epsilon m_{B, V_1} \text{ in } \Omega. \quad (1.8)$$

Finally, to assure compactness of the resolvent, we assume that

$$m_{B, V_1} \xrightarrow{|x| \rightarrow +\infty} +\infty \text{ in } \Omega. \quad (1.9)$$

Consider then the operator initially defined on $C_0^\infty(\Omega)$ by:

$$\mathcal{P} = \mathcal{P}_0 + V, \quad (1.10)$$

where

$$\mathcal{P}_0 = - \sum_{k=1}^d e^{2i\alpha_k} \partial_{A_k}^2 \quad (1.11)$$

and

$$\partial_{A_k} := \frac{\partial}{\partial x_k} - iA_k.$$

We consider here the Dirichlet realization \mathcal{P}^D of \mathcal{P} in Ω , i.e., some closed extension of \mathcal{P} which should be defined properly on a subspace of functions satisfying a Dirichlet condition on $\partial\Omega$, or on the entire space \mathbb{R}^d . Our definition, based on some generalization of the Friedrichs extension in a non necessarily coercive case, will coincide with the standard notion when Ω is bounded, $\Omega = \mathbb{R}^d$ or when the operator is selfadjoint and semi-bounded.

Before stating our main result, we recall that, if \mathcal{H} is a Hilbert space and $p > 0$ the Schatten class $C^p(H)$ denotes the set of compact operators \mathcal{T} such that

$$\|\mathcal{T}\|_p = \left(\sum_{n=1}^{\infty} \mu_n(\mathcal{T})^p \right)^{1/p} < \infty,$$

where $\mu_n(\mathcal{T})$ are the eigenvalues of $(\mathcal{T}^*\mathcal{T})^{1/2}$ repeated according to their multiplicity [10, 11]. For $1 \leq p$, C^p is a Banach space with $\|\cdot\|_p$ as its norm. For $0 < p < 1$ C^p is still well defined, but is not a Banach space and $\|\Delta\|_p$ is not a norm.

Our main result follows

Theorem 1.1. *Under the above assumptions we have, for the Dirichlet realizations of \mathcal{P} and $-\Delta_A + |V|$, that for every $\lambda \in \rho(-\Delta_A + |V|) \cap \rho(\mathcal{P})$*

$$(-\Delta_A + |V| - \lambda)^{-1} \in C^p(L^2(\Omega, \mathbb{C})) \Rightarrow (\mathcal{P} - \lambda)^{-1} \in C^p(L^2(\Omega, \mathbb{C})). \quad (1.12)$$

The optimality of this result is also of interest. In this direction we prove the following

Theorem 1.2. *With the notation and assumptions of the previous theorem,*

- *When Ω is either the entire space or the half-space in \mathbb{R}^d we have*

$$(\mathcal{P} - \lambda)^{-1} \in C^p(L^2(\Omega, \mathbb{C})) \Rightarrow (-\Delta_A + |V| - \lambda)^{-1} \in C^p(L^2(\Omega, \mathbb{C})) \quad (1.13)$$

for every $\lambda \in \rho(-\Delta_A + |V|) \cap \rho(\mathcal{P})$.

- *For a general domain Ω and for any $\alpha \in [-\pi, \pi)$, the Dirichlet realizations of $-e^{i\alpha}\Delta_A + V$ and $-\Delta_A + |V|$ satisfy*

$$(-e^{i\alpha}\Delta_A + V - \lambda)^{-1} \in C^p(L^2(\Omega, \mathbb{C})) \Rightarrow (-\Delta_A + |V| - \lambda)^{-1} \in C^p(L^2(\Omega, \mathbb{C})), \quad (1.14)$$

for every $\lambda \in \rho(-\Delta_A + |V|) \cap \rho(-e^{i\alpha}\Delta_A + V)$.

In this last case, we consequently have an equivalence, in the sense of Schatten classes of the resolvents, between the Dirichlet realizations of \mathcal{P} and $-\Delta_A + |V|$.

The complementary question is, naturally, to which Schatten class does the resolvent of $-\Delta_A + |V|$ belong? The following theorem provides a satisfactory answer.

Theorem 1.3. *Suppose that for some $p > 0$*

$$\int_{\Omega \times \mathbb{R}^d} (|\xi|^2 + m_{B,V_1})^{-p} dx d\xi < \infty. \quad (1.15)$$

Then, $(-\Delta_A + |V| + 1)^{-1} \in C^p(L^2(\Omega, \mathbb{C}))$.

Once the Schatten class for a compact operator has been obtained, one can use the following fundamental result in operator theory to prove completeness of its system of eigenvectors (cf. for instance Theorem X.3.1 in [11], or Corollary XI.9.31 in [10]).

Theorem 1.4. *Let \mathcal{H} denote a Hilbert space, and $\mathcal{A} \in C^p(\mathcal{H})$ be a compact operator for some $p > 0$. Assume that its numerical range*

$$W_{\mathcal{A}} = \{ \langle \mathcal{A}\varphi, \varphi \rangle \mid \varphi \in \mathcal{H}, \|\varphi\| = 1 \}$$

lies inside a closed angle with vertex at zero and opening π/p .

Let $\text{Span}(\mathcal{A})$ denote the closure of the vector space generated by the generalized eigenfunctions. Then, $\text{Span}(\mathcal{A}) = \mathcal{H}$ is complete.

We apply this statement to the resolvent of \mathcal{P} . Note that

$$\begin{aligned} W_{(\mathcal{P}-\lambda)^{-1}} &= \{ \langle (\mathcal{P} - \lambda)^{-1}\varphi, \varphi \rangle \mid \varphi \in \mathcal{H}, \|\varphi\| = 1 \} \\ &= \{ \langle \psi, (\mathcal{P} - \lambda)\psi \rangle \mid \psi \in D(\mathcal{P}), \|(\mathcal{P} - \lambda)\psi\| = 1 \}. \end{aligned}$$

Hence, if for some $\lambda \in \rho(\mathcal{P})$, $W_{\mathcal{P}-\lambda}$ lies in a closed angle with vertex at zero and opening π/p , then so does $W_{(\mathcal{P}-\lambda)^{-1}}$ and it would follow immediately that \mathcal{P} has a complete system of eigenfunctions.

The rest of the contribution is arranged as follows.

In the next section, we define the Dirichlet realization of \mathcal{P} and prove that its resolvent is compact.

In Section 3 we prove Theorems 1.1-1.3.

Finally, in the last section, we use these results, together with Theorem 1.4 to prove completeness of the system of (generalized)-eigenvectors, or at least existence of a non-empty spectrum, for a few particular cases of (1.10) motivated by superconductivity problems.

2 Definition of the Dirichlet realization

2.1 Preliminaries

As \mathcal{P} is defined by (1.10) for smooth functions only, we seek a closed extension \mathcal{P}^D corresponding to its Dirichlet realization. For unbounded domains the definition of this extension deserves special attention. We thus consider the sesquilinear form

$$(u, v) \mapsto a(u, v) := \sum_{k=1}^d \exp\{-2i\alpha_k\} \langle \partial_{A_k} u, \partial_{A_k} v \rangle_{L^2(\Omega)} + \int_{\Omega} V(x) u(x) \bar{v}(x) dx \quad (2.1)$$

initially defined on $C_0^\infty(\Omega) \times C_0^\infty(\Omega)$.

As is the common practice in such cases, it is useful to consider instead, for some, sufficiently large, $\gamma > 0$

$$(u, v) \mapsto a_\gamma(u, v) := \sum_k \exp\{-2i\alpha_k\} \langle \partial_{A_k} u, \partial_{A_k} v \rangle_{L^2(\Omega)} + \int_{\Omega} V(x) u(x) \bar{v}(x) dx + \gamma \int u \bar{v} dx, \quad (2.2)$$

to assure some coercivity.

The Friedrichs extension of (2.2) is a continuous sesquilinear form on $\mathcal{V} \times \mathcal{V}$, where

$$\mathcal{V} = \{u \in H_{0,A}^1(\Omega) \mid |V_1|^{1/2} u \in L^2(\Omega, \mathbb{C})\}, \quad (2.3)$$

and $H_{0,A}^1(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in the magnetic Sobolev space

$$H_A^1(\Omega) = \{u \in L^2(\Omega), \nabla_A u \in L^2(\Omega)\}. \quad (2.4)$$

When Ω is bounded, $\alpha_k = 0$ for all $1 \leq k \leq d$, and V is real, the Dirichlet realization of \mathcal{P} can be easily obtained by applying the Lax-Milgram Theorem (and the Friedrichs extension construction). When V is complex valued and $\Im V \geq 0$, the same method prevails possible if we employ a minor generalization of the Lax Milgram Theorem, where hermitianity for the sesquilinear form is no longer assumed [4]. When $\Im V$ has no definite sign and is not bounded by $\Re V$ or the magnetic field, a more elaborate generalization of the Lax-Milgram Theorem is needed. In particular, it is a necessary to replace the standard requirement for \mathcal{V} -ellipticity (or coercivity) of (2.2) by a weaker one. This is the object of the next subsection.

2.2 A generalized Lax-Milgram Theorem

Let \mathcal{V} denote a Hilbert space. Consider a continuous sesquilinear form a defined on $\mathcal{V} \times \mathcal{V}$:

$$(u, v) \mapsto a(u, v).$$

Recall that for a sesquilinear form continuity means that for some $C > 0$

$$|a(u, v)| \leq C \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}}, \forall u, v \in \mathcal{V}. \quad (2.5)$$

We denote the associated linear map by $\mathcal{A} \in \mathfrak{L}(\mathcal{V})$, i.e.,

$$a(u, v) = \langle \mathcal{A}u, v \rangle_{\mathcal{V}}. \quad (2.6)$$

Theorem 2.1.

Let a be a continuous sesquilinear form on $\mathcal{V} \times \mathcal{V}$. If a satisfies, for some $\Phi_1, \Phi_2 \in \mathcal{L}(\mathcal{V})$

$$|a(u, u)| + |a(u, \Phi_1(u))| \geq \alpha \|u\|_{\mathcal{V}}^2, \forall u \in \mathcal{V}. \quad (2.7)$$

$$|a(u, u)| + |a(\Phi_2(u), u)| \geq \alpha \|u\|_{\mathcal{V}}^2, \forall u \in \mathcal{V}. \quad (2.8)$$

then \mathcal{A} , as defined in (2.6), is a continuous isomorphism from \mathcal{V} onto \mathcal{V} . Moreover \mathcal{A}^{-1} is continuous.

Proof. The proof is standard but we detail it in order to show how the assumptions appear. We split the proof into two different steps.

Step 1: \mathcal{A} is injective, and has a closed range.

Choose $u \in \mathcal{V}$, such that $\mathcal{A}u = 0$. This implies

$$\langle \mathcal{A}u, u \rangle = 0 \text{ and } \langle \mathcal{A}u, \Phi_1(u) \rangle = 0. \quad (2.9)$$

It, however, follows from (2.7) that

$$|\langle \mathcal{A}u, u \rangle_{\mathcal{V}}| + |\langle \mathcal{A}u, \Phi_1(u) \rangle_{\mathcal{V}}| \geq \alpha \|u\|_{\mathcal{V}}^2, \forall u \in \mathcal{V}.$$

Hence,

$$(1 + \|\Phi_1\|) \|\mathcal{A}u\|_{\mathcal{V}} \cdot \|u\|_{\mathcal{V}} \geq \alpha \|u\|_{\mathcal{V}}^2, \forall u \in \mathcal{V},$$

and consequently, for some $\tilde{\alpha} > 0$

$$\|\mathcal{A}u\|_{\mathcal{V}} \geq \tilde{\alpha} \|u\|_{\mathcal{V}}, \forall u \in \mathcal{V}, \quad (2.10)$$

from which injectivity readily follows. Closedness of the range easily follows from (2.10) and the continuity of \mathcal{A} .

Step 2: $\mathcal{A}(\mathcal{V})$ is dense in \mathcal{V} , and \mathcal{A}^{-1} is continuous.

Consider $u \in \mathcal{V}$ such that $\langle \mathcal{A}v, u \rangle_{\mathcal{V}} = 0, \forall v \in \mathcal{V}$. In particular, we can choose $v_1 = u$ and $v_2 = \Phi_2(u)$ to obtain $a(u, u) = 0$ and $a(\Phi_2(u), u) = 0$. Hence, by (2.8) we must have $u = 0$. Thus, \mathcal{A} is a bijection, $\mathcal{A}^{-1} : \mathcal{V} \rightarrow \mathcal{V}$ exists and is continuous by (2.10). ■

We now consider two Hilbert spaces \mathcal{V} and \mathcal{H} such that $\mathcal{V} \subset \mathcal{H}$, and that for some $C > 0$ and any $u \in \mathcal{V}$, we have

$$\|u\|_{\mathcal{H}} \leq C\|u\|_{\mathcal{V}}. \quad (2.11)$$

Suppose further that

$$\mathcal{V} \text{ is dense in } \mathcal{H}. \quad (2.12)$$

Let

$$D(S) = \{u \in \mathcal{V} \mid v \mapsto a(u, v) \text{ is continuous on } \mathcal{V} \text{ in the norm of } \mathcal{H}\}. \quad (2.13)$$

We can now define the operator $S : D(S) \rightarrow \mathcal{H}$ by

$$a(u, v) = \langle Su, v \rangle_{\mathcal{H}}, \forall u \in D(S) \text{ and } \forall v \in \mathcal{V}. \quad (2.14)$$

We can now prove

Theorem 2.2. *Let a be a continuous sesquilinear form satisfying (2.7) and (2.8). Suppose, in addition, that $\mathcal{V} \subset \mathcal{H}$ and that (2.11) and (2.12) hold. Assume further that Φ_1 and Φ_2 extend into a continuous linear maps in $\mathcal{L}(\mathcal{H})$. Let S be defined by (2.13)-(2.14). Then*

1. S is bijective from $D(S)$ onto \mathcal{H} and $S^{-1} \in \mathfrak{L}(\mathcal{H})$.
2. $D(S)$ is dense in both \mathcal{V} and \mathcal{H}
3. S is closed.
4. Let b denote the conjugate sesquilinear form of a , i.e.

$$(u, v) \mapsto b(u, v) := \overline{a(v, u)}. \quad (2.15)$$

Let S_1 denote the closed linear operator associated with b by the same construction . Then

$$S^* = S_1 \text{ and } S_1^* = S . \quad (2.16)$$

Proof. We show first that S is injective. A direct proof is immediate but this is also a consequence of the observation that, for all $u \in D(S)$, of

$$\begin{aligned} \alpha \|u\|_{\mathcal{H}}^2 &\leq C \alpha \|u\|_{\mathcal{V}}^2 \\ &\leq C (|a(u, u)| + |a(u, \Phi_1(u))|) \\ &= C |\langle Su, u \rangle_{\mathcal{H}}| + |\langle Su, \Phi_1(u) \rangle_{\mathcal{H}}| \\ &\leq \hat{C} \|Su\|_{\mathcal{H}} \cdot \|u\|_{\mathcal{H}} , \end{aligned}$$

which leads to the two estimates (to be used later)

$$\alpha \|u\|_{\mathcal{H}} \leq C \|Su\|_{\mathcal{H}} , \quad \forall u \in D(S) , \quad (2.17)$$

and

$$\alpha \|u\|_{\mathcal{V}} \leq C \|Su\|_{\mathcal{H}} \|u\|_{\mathcal{H}} , \quad \forall u \in D(S) . \quad (2.18)$$

Injectivity easily follows.

We next prove the surjectivity of S in the following direct manner. To this end let $h \in \mathcal{H}$. Then, choose $w \in \mathcal{V}$ is such that

$$\langle h, v \rangle_{\mathcal{H}} = \langle w, v \rangle_{\mathcal{V}} , \quad \forall v \in \mathcal{V} , \quad (2.19)$$

(that such a choice is possible follows from Riesz's theorem). We can then set $u = \mathcal{A}^{-1}w \in \mathcal{V}$, which must clearly satisfy

$$a(u, v) = \langle w, v \rangle_{\mathcal{V}} . \quad (2.20)$$

Since by (2.19) and (2.20) we have that

$$|a(u, v)| = |\langle h, v \rangle_{\mathcal{H}}| \leq \|h\|_{\mathcal{H}} \|v\|_{\mathcal{H}} , \quad \forall v \in \mathcal{V} ,$$

we obtain simultaneously that $u \in D(S)$ and that

$$Su = h .$$

The continuity of S^{-1} is a consequence of (2.17).

We next prove the density of $D(S_1)$ in \mathcal{V} . We first note that the sesquilinear form b , defined in (2.15), is continuous and satisfies (2.7) and (2.8) (with switched roles for Φ_1 and Φ_2). Hence, we may conclude that S_1 possesses, just like S , a bounded inverse. Let $h \in \mathcal{V}$ be such that

$$\langle u, h \rangle_{\mathcal{V}} = 0, \forall u \in D(S_1).$$

Let further $f = A^{-1}h \in \mathcal{V}$. We then get

$$0 = \langle u, h \rangle_{\mathcal{V}} = \langle u, Af \rangle_{\mathcal{V}} = \overline{\langle Af, u \rangle_{\mathcal{V}}} = \overline{a(f, u)} = b(u, f) = \langle S_1 u, f \rangle_{\mathcal{H}}.$$

Using the surjectivity of S_1 on \mathcal{H} , we obtain $f = 0$ and consequently $h = 0$. The density of $D(S_1)$ in \mathcal{H} is then deduced from the assumption of density of \mathcal{V} in \mathcal{H} .

As above, but this time in an opposite direction, we can now claim that since a , just like b , is continuous and satisfies (2.7) and (2.8), $D(s)$ must be dense in both \mathcal{H} and \mathcal{V} . ■

2.3 Definition of \mathcal{P}^D

We return to the operator \mathcal{P} introduced in (1.10) on $C_0^\infty(\Omega)$ and describe how the previous abstract theory applies to the construction \mathcal{P}^D .

2.3.1 The case $V_2 = 0$

Let $\mathcal{H} = L^2(\Omega, \mathbb{C})$ and \mathcal{V} as introduced in (2.3). Initially, we equip \mathcal{V} with the norm:

$$u \mapsto \|u\|_{\mathcal{V}} := \sqrt{\|u\|_{H_A^1}^2 + \int \sqrt{|V_1|^2 + 1} |u(x)|^2 dx}.$$

We later prove (see (2.26)) that

$$u \mapsto \|u\|_{\mathcal{V}, B} := \sqrt{\|u\|_{H_A^1}^2 + \int m_{B, V_1} |u(x)|^2 dx},$$

is an equivalent norm on \mathcal{V} .

To apply the previous results to the sesquilinear form introduced in (2.2) we need to establish first that a satisfies (2.7) and (2.8). To this end we set $\Phi_1(u) = \Phi_2(u) = \phi_1 u$, where

$$\phi_1 = \frac{\Im V_1}{m_{B,V_1}}.$$

Clearly Φ_1 belongs to $\mathfrak{L}(\mathcal{V})$ and $\mathfrak{L}(\mathcal{H})$, since it is a multiplication operator by a function in $W^{1,\infty}(\Omega)$. Note that by (1.8) $\nabla \phi_1$ belongs to $L^\infty(\Omega)$.

It can be easily verified that a is continuous on $\mathcal{V} \times \mathcal{V}$. To use Theorem 2.2, we thus need to establish (2.7) and (2.8). We first observe that for any $u \in \mathcal{V}$, we have:

$$\Re a_\gamma(u, u) \geq K \int_{\Omega} |\nabla_A u(x)|^2 dx + \int_{\Omega} (\Re V_1(x) + \gamma) |u(x)|^2 dx, \quad (2.21)$$

where K is defined in (1.4). Furthermore,

$$\Im a(u, \phi_1 u) = \Im \sum_{k=1}^d \int_{\Omega} e^{2i\alpha_k} \partial_{A_k} u(x) \overline{\partial_{A_k}(\phi_1 u)} dx + \int_{\Omega} |\Im V_1|^2 (m_{B,V_1})^{-1} |u|^2 dx.$$

After some simple manipulation we arrive at

$$\begin{aligned} \Im a(u, \phi_1 u) &= \Im \sum_{k=1}^d \int_{\Omega} e^{2i\alpha_k} \partial_{A_k} u(x) \partial_k \phi_1 \bar{u} dx \\ &\quad + \int_{\Omega} |\Im V_1|^2 (m_{B,V_1})^{-1} |u|^2 dx + \sum_{k=1}^d \int_{\Omega} \phi_1 \sin(2\alpha_k) |\partial_{A_k} u|^2 dx. \end{aligned}$$

Clearly, as $\|\phi_1\|_{\infty} \leq 1$, we have

$$\left| \Im \sum_{k=1}^d \int_{\Omega} e^{2i\alpha_k} \partial_{A_k} u(x) \partial_k \phi_1 \bar{u} dx \right| \leq \epsilon \|\nabla_A u\|_2^2 + C_\epsilon \|u\|_2^2.$$

Consequently, there exists γ_0 and C such that for $\gamma > \gamma_0$

$$\begin{aligned} C (\Im a_\gamma(u, \phi_1 u) + \Re a_\gamma(u, u)) &\geq \|\nabla_A u(x)\|_2^2 dx \\ &\quad + \int_{\Omega} \Re V_1(x) |u(x)|^2 dx + \int_{\Omega} |\Im V_1|^2 (m_{B,V_1})^{-1} |u|^2 dx + \int_{\Omega} |u(x)|^2 dx. \end{aligned} \quad (2.22)$$

To complete the proof of (2.7) we need an estimate for $\|B m_{B,V}^{-1/2} u\|_2$. To this end we introduce the operator identity ($[\cdot, \cdot]$ being the Poisson bracket)

$$B_{k\ell} = i[\partial_{A_k}, \partial_{A_\ell}]. \quad (2.23)$$

We then use (2.23) to obtain

$$\begin{aligned} & \int_{\Omega} B_{k\ell}(x)^2 (m_{\mathbf{B},v})^{-1} |u|^2 dx \\ &= i \int_{\Omega} \{[\partial_{A_k}, \partial_{A_\ell}] u\} \cdot B_{k\ell}(x) (m_{B,V_1})^{-1} \bar{u} dx \\ &\leq C \left\{ \|\partial_{A_\ell} u\|_2 \|\partial_{A_k} u\|_2 + (\|\partial_{A_k} u\|_2 \right. \\ &\quad \left. + \|\partial_{A_\ell} u\|_2) \|u\|_2 \sup_{x \in \Omega} |\nabla(B_{k\ell}(m_{B,V_1})^{-1})| \right\}. \end{aligned}$$

As before this leads to

$$\int_{\Omega} B_{k\ell}(x)^2 (m_{B,V})^{-1} |u|^2 dx \leq C(\|\nabla_A u\|_2^2 + \|u\|_2^2) \leq \tilde{C} \Re a(u, u). \quad (2.24)$$

We can now deduce from (2.24) and (2.22) that for some $C > 0$ and $\gamma_0 > 0$, we have for all $\gamma > \gamma_0$,

$$\|u\|_{\mathcal{V},B}^2 \leq C(|\Im a_\gamma(u, \phi_1 u)| + |\Re a_\gamma(u, u)|), \quad (2.25)$$

establishing both (2.7) and (2.8) and also

$$\|u\|_{\mathcal{V},B} \leq \tilde{C} \|u\|_{\mathcal{V}}. \quad (2.26)$$

Hence, the linear operator S_γ associated with a_γ can be defined over the set (2.13), and is an isomorphism from $D(S_\gamma)$ into $L^2(\Omega, \mathbb{C})$. It can be easily verified that $a_\gamma(u, v) = \langle \mathcal{P}_\gamma u, v \rangle$ for all $u \in C_0^\infty(\Omega)$ (with $\mathcal{P}_\gamma = \mathcal{P} + \gamma$) and hence $S_\gamma|_{C_0^\infty(\Omega)} = \mathcal{P}_\gamma$. We can then define the extension of \mathcal{P} on $D(S_\gamma)$ by

$$\mathcal{P}^D = S_\gamma - \gamma I.$$

We have, hence, defined a closed extension of \mathcal{P} , on a set of functions satisfying, as $D(S_\gamma) \subset \mathcal{V}$, a Dirichlet boundary condition.

As a matter of fact, it can be easily verified that

$$D(\mathcal{P}^D) = D(S_\gamma) = \{u \in \mathcal{V} \mid \mathcal{P}u \in L^2(\Omega)\} \quad (2.27)$$

where $\mathcal{P}u$ is defined as a distribution on Ω .

Hence, \mathcal{P}^D defines the Dirichlet realization of \mathcal{P} in Ω .

Additionally, as $D(S_\gamma) \subset \mathcal{V}$ which is compactly embedded in $L^2(\Omega, \mathbb{C})$ in view of (1.9), it follows that \mathcal{P}^D has a compact resolvent.

2.3.2 The general case

We conclude this section by establishing the same results for \mathcal{P} when V_2 is not necessarily 0 but satisfies (1.8). We define to this end the sesquilinear form $b_\gamma : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{C}$

$$\begin{aligned} b_\gamma(u, v) &:= \sum_k e^{2i\alpha_k} \int_\Omega \partial_{A_k} u(x) \overline{\partial_{A_k} v(x)} dx + \int_\Omega (V(x) + \gamma) u(x) \bar{v}(x) dx \\ &= a_\gamma(u, v) + \int_\Omega V_2(x) u(x) \bar{v}(x) dx. \end{aligned}$$

In view of (1.8), b_γ is continuous. Furthermore, for any $u \in \mathcal{V}$ we have by (2.25) that

$$\begin{aligned} \|u\|_{H_A^1}^2 + \int m_{B, V_1} |u(x)|^2 dx \\ \leq C(\Im a_\gamma(u, \phi_1 u) + \Re a_\gamma(u, u)) \leq C(\Im b_\gamma(u, \phi_1 u) + \Re b_\gamma(u, u)) + C\langle |V_2| u, u \rangle. \end{aligned}$$

We thus obtain, by (1.8), the existence of γ_1 and C such that:

$$\|u\|_{\mathcal{V}, B} \leq C(|\Im b_\gamma(u, \phi_1 u)| + \Re b_{\gamma+\gamma_1}(u, u)).$$

It therefore follows, that we can apply to $b_{\gamma+\gamma_1}$ the same construction which was applied to a_γ , to obtain the same domain in the general case (and the same form domain) and that its resolvent is compact.

3 The Schatten class

In this section we attempt to obtain the optimal value of p for the Schatten class of the resolvent of the Dirichlet realization \mathcal{P}^D . We begin by showing that if $(-\Delta_A + |V| + 1)^{-1} \in C^p$ then $(\mathcal{P}^D - \lambda)^{-1} \in C^p$, thereby allowing us to use techniques from selfadjoint theory. Then, we provide a criterion on V and B which can be used to determine whether the resolvent of \mathcal{P}^D is in a given Schatten class. For convenience of notation we omit from now on the superscript D and write \mathcal{P} instead of \mathcal{P}^D .

3.1 Comparison with a selfadjoint problem

We begin with the following comparison result

Proposition 3.1. *Let $\{\mu_n\}_{n=1}^\infty$ denote the n 'th eigenvalue of $(\mathcal{P}^* \mathcal{P})^{1/2}$, where $\mathcal{P}^* \mathcal{P}$ is the linear operator associated with the sesquilinear form $q : D(\mathcal{P}) \times D(\mathcal{P}) \rightarrow \mathbb{C}$ given by*

$$q(u, v) = \langle \mathcal{P}u, \mathcal{P}v \rangle.$$

The domain of $\mathcal{P}^*\mathcal{P}$ is given by

$$D(\mathcal{P}^*\mathcal{P}) = \{u \in D(\mathcal{P}) \mid \mathcal{P}u \in D(\mathcal{P}^*)\}.$$

Consider then the Dirichlet realization in Ω of $-\Delta_A + m_{B,V_1}$, and let

$$\sigma(-\Delta_A + m_{B,V_1}) = \{\nu_j\}_{j=1}^\infty.$$

Then, there exists $C > 0$ such that

$$\nu_n \leq C(1 + \mu_n). \quad (3.1)$$

Proof. For any $u \in \mathcal{V}$ we have

$$\langle (-\Delta_A + m_{B,V_1})u, u \rangle \leq C (|b(u, \phi_1 u)| + |b(u, u)| + \|u\|_2^2). \quad (3.2)$$

Hence, for any $u \in D(\mathcal{P})$, we can write

$$\langle (-\Delta_A + m_{B,V_1})u, u \rangle \leq C (|\langle \mathcal{P}u, \phi_1 u \rangle| + |\langle \mathcal{P}u, u \rangle| + \|u\|_2^2). \quad (3.3)$$

Consequently, for all $u \in D(\mathcal{P})$,

$$\langle (-\Delta_A + m_{B,V_1})u, u \rangle \leq C (\|\mathcal{P}u\|_2 \|u\|_2 + \|u\|_2^2). \quad (3.4)$$

It thus follows that for each $j \geq 1$ there exists a vector space E_j of dimension j in $D(\mathcal{P}^*\mathcal{P})$ (hence also in $D(\mathcal{P})$) such that, for all $u \in E_j$, we have

$$\langle (-\Delta_A + m_{B,V_1})u, u \rangle \leq C(\mu_j + 1) \|u\|_2^2. \quad (3.5)$$

By Proposition 11.9 in [13] applied to the Dirichlet realization of $(-\Delta_A + m_{B,V_1})$ (observing that the domain of the operator can be replaced by the form domain \mathcal{V} of this Dirichlet realization and that $D(\mathcal{P}) \subset \mathcal{V}$), we then obtain (3.1). ■

3.2 Proof of Theorem 1.3

By (2.26) we have that

$$\langle u, (-\Delta_A + m_{B,V_1})u \rangle \leq \tilde{C} \langle u, (-\Delta_A + |V| + 1)u \rangle.$$

Thus, by the min-max principle, the resolvents of the operators $-\Delta_A + m_{B,V_1}$ and $-\Delta_A + |V| + 1$ always belong to the same Schatten class. We therefore begin by restating the theorem in the following equivalent form

Theorem 3.2. *Suppose that*

$$\int_{\Omega \times \mathbb{R}^d} (|\xi|^2 + m_{B, V_1})^{-p} dx d\xi < \infty. \quad (3.6)$$

Then, $(-\Delta_A + m_{B, V_1})^{-1} \in C^p(L^2(\Omega, \mathbb{C}))$.

We note that by (3.1) it follows that whenever (3.6) is satisfied, then $(\mathcal{P} - \lambda)^{-1} \in C^p$. Note further that (3.6) can hold only if $p > d/2$.

Proof. For $\Omega = \mathbb{R}^d$, (3.6) has been established in [6]. To extend it to the Dirichlet realization of $-\Delta_A + m_{B, V_1}$ for general domains we extend

$$U = m_{B, V_1}$$

to \mathbb{R}^d in the following manner:

$$U_{\rho, M} = \begin{cases} U & \text{in } \Omega \\ \rho(1 + x^2)^M & \text{in } \Omega^c, \end{cases}$$

where $\rho \geq 1$ and M is chosen so that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} (|\xi|^2 + (1 + x^2)^M)^{-p} dx d\xi < \infty.$$

or equivalently (recall that $p > d/2$)

$$M > \frac{d}{2p - d}.$$

Let

$$\sigma(-\Delta_A + U_{\rho, M}) = \{\mu_j\}_{j=1}^{\infty}, \quad \sigma(-\Delta_A + U) = \{\lambda_j\}_{j=1}^{\infty},$$

where $-\Delta_A + U_{\rho, M}$ is the unique self-adjoint extension on \mathbb{R}^d (by Kato's theorem) and $-\Delta_A + U$ is the Dirichlet realization in Ω .

It can be easily verified by comparison of the form domains that

$$\mu_j \leq \lambda_j, \quad \forall j \geq 1.$$

From Theorem 2.1 in [6] we then get, for any $\rho \geq 1$,

$$\begin{aligned} & \text{Trace}(-\Delta_A + U + 1)^{-p} \\ & \leq \text{Trace}(-\Delta_A + U_{\rho, M} + 1)^{-p} \\ & \leq (2\pi)^{-n} \int_{\mathbb{R}^d \times \mathbb{R}^d} (\xi^2 + U_{\rho, M} + 1)^{-p} dx d\xi. \end{aligned} \quad (3.7)$$

Taking the limit $\rho \rightarrow \infty$ yields

$$\text{Trace}(-\Delta_A + U + 1)^{-p} \leq (2\pi)^{-n} \int_{\Omega \times \mathbb{R}^d} (\xi^2 + U + 1)^{-p} dx d\xi.$$

It follows that if (3.6) holds true, then $(-\Delta_A + U)^{-1} \in C^p(L^2(\Omega, \mathbb{C}))$. ■

3.3 On the optimality of the criterion (3.6)

Let

$$\mathcal{D} = \{u \in C^\infty(\bar{\Omega}, \mathbb{C}) \cap H_0^1(\Omega, \mathbb{C}) \mid \text{Supp } u \text{ is a compact subset of } \bar{\Omega}\}.$$

It is clear that \mathcal{D} is in $D(\mathcal{P}^D)$. We first show

Lemma 3.3. *Let $D(\mathcal{P}^D)$ be given by (2.27). Then, \mathcal{D} is dense in $D(\mathcal{P}^D)$ under the norm $\|\mathcal{P} \cdot\|_2 + \|\cdot\|_{\mathcal{V}}$.*

Proof. Let $u \in D(\mathcal{P}^D)$. By (2.18), (2.8) and (2.26) we have

$$\|u\|_{\mathcal{V}, B} \leq C (\|u\|_2 + \|\mathcal{P}u\|_2). \quad (3.8)$$

Let $\eta \in C^\infty(\mathbb{R}_+, [0, 1])$ denote a cutoff function satisfying

$$\eta(t) = \begin{cases} 1 & t < 1 \\ 0 & t > 2. \end{cases}$$

Denote then by $\eta_k : \Omega \rightarrow \mathbb{R}_+$ the restriction to Ω of the cutoff function defined by $\eta_k(x) = \eta(|x|/k)$ for all $x \in \Omega$. By our assumption on $\partial\Omega$ we have that $\eta_k \in C^2(\bar{\Omega})$ for all $k \geq 1$.

Next, define the sequence $\{v_k\}_{k=1}^\infty$ through $v_k = \eta_k u$. Using the local regularity of the Dirichlet problem we easily conclude that $v_k \in H^2(\Omega, \mathbb{C})$ has compact support, and it can be readily verified that $\|v_k - u\|_{\mathcal{V}} \rightarrow 0$ as $k \rightarrow +\infty$.

We now prove that $v_k \rightarrow u$ in the graph norm. To prove that

$$\|\mathcal{P}(v_k - u)\|_2 \rightarrow 0 \quad (3.9)$$

we first compute

$$\mathcal{P}(v_k - u) = (\eta_k - 1)\mathcal{P}u + u \sum_{m=1}^d e^{2i\alpha_m} \frac{\partial^2 \eta_k}{\partial x_m^2} + 2 \sum_{m=1}^d e^{2i\alpha_m} \frac{\partial \eta_k}{\partial x_m} \partial_{A_m} u.$$

The first two terms on the right-hand-side tend to 0 (in L^2 sense) since both u and $\mathcal{P}u$ are in $L^2(\Omega, \mathbb{C})$. For the last term we have

$$\left\| \sum_{m=1}^d e^{2i\alpha_m} \frac{\partial \eta_k}{\partial x_m} \partial_{A_m} u \right\|_2 \leq \frac{C}{k} \|\nabla_{A_m} u\|_2.$$

Hence, by (3.8), we obtain (3.9).

The result of the foregoing discussion is that

$$\tilde{\mathcal{D}} = \{u \in H^2(\bar{\Omega}, \mathbb{C}) \cap H_0^1(\Omega, \mathbb{C}) \mid \text{Supp } u \text{ is a compact subset of } \bar{\Omega}\},$$

is dense in $D(\mathcal{P}^D)$ with respect to the norm introduced in the lemma. One can now complete the proof invoking standard arguments that show that \mathcal{D} is dense in $\tilde{\mathcal{D}}$ under the same norm. ■

We continue this subsection by the following lemma.

Lemma 3.4. *Under Assumptions (1.5)-(1.11), there exists $C(\Omega, \mathcal{P})$ such that for all $u \in \mathcal{D}$ we have*

$$\|Bu\|_2^2 \leq C(\|\Delta_A u\|_2^2 + \|V_1 u\|_2^2 + \|u\|_2^2). \quad (3.10)$$

Proof. Let $u \in \mathcal{D}$. We use the identity (2.23) to obtain, after integration by parts

$$\|B_{km} u\|_2^2 = \Im \langle [\partial_{A_k}, \partial_{A_m}] u, B_{km} u \rangle \leq |\langle \partial_{A_m} u, \partial_{A_k} (B_{km} u) \rangle| + |\langle \partial_{A_k} u, \partial_{A_m} (B_{km} u) \rangle|. \quad (3.11)$$

As

$$\langle \partial_{A_m} u, \partial_{A_k} (B_{km} u) \rangle = \langle B_{km} \partial_{A_m} u, \partial_{A_k} u \rangle + \langle \partial_{A_m} u, u \partial_k B_{km} \rangle,$$

it follows from (1.6) and (3.11) that

$$\|B_{km} u\|_2^2 \leq 2 \| |B_{km}|^{1/2} \partial_{A_m} u \|_2 \| |B_{km}|^{1/2} \partial_{A_k} u \|_2 + C \|m_{B, V_1} u\|_2 (\|\partial_{A_m} u\|_2 + \|\partial_{A_k} u\|_2).$$

Consequently, for any $\epsilon > 0$,

$$\begin{aligned} \|B_{km} u\|_2^2 &\leq 2 \| |B_{km}|^{1/2} \partial_{A_m} u \|_2 \| |B_{km}|^{1/2} \partial_{A_k} u \|_2 \\ &\quad + C \left[\frac{\epsilon}{2} \|m_{B, V_1} u\|_2^2 + \frac{1}{4\epsilon} (\|\partial_{A_m} u\|_2^2 + \|\partial_{A_k} u\|_2^2 + \|u\|_2^2) \right]. \end{aligned}$$

Hence, for all $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$\begin{aligned} & \|B_{km}u\|_2^2 - \epsilon \|Bu\|_2^2 \\ & \leq C_\epsilon (\| |B_{km}|^{1/2} \partial_{A_m} u \|_2^2 + \| |B_{km}|^{1/2} \partial_{A_k} u \|_2^2 + \|V_1 u\|_2^2 \\ & \quad + \|\partial_{A_m} u\|_2^2 + \|\partial_{A_k} u\|_2^2 + \|u\|_2^2) . \end{aligned}$$

Summing over k and m we then obtain, using the standard inequality

$$\|\nabla_A u\|_2^2 \leq \frac{1}{2} (\|\Delta_A u\|_2^2 + \|u\|_2^2), \quad (3.12)$$

and taking above a sufficiently small ϵ ,

$$\|Bu\|_2^2 \leq C (\| |B|^{1/2} \nabla_A u \|_2^2 + \|\Delta_A u\|_2^2 + \|V_1 u\|_2^2 + \|u\|_2^2),$$

and hence,

$$\|m_{B,V_1} u\|_2^2 \leq C (\| |B|^{1/2} \nabla_A u \|_2^2 + \|\Delta_A u\|_2^2 + \|V_1 u\|_2^2 + \|u\|_2^2). \quad (3.13)$$

Next, we use integration by parts to show that

$$-\langle m_{B,V_1} u, \Delta_A u \rangle = \langle m_{B,V_1} \nabla_A u, \nabla_A u \rangle + \langle u \nabla m_{B,V_1}, \nabla_A u \rangle.$$

By (1.6) we have that

$$|\nabla m_{B,V_1}| \leq C m_{B,V_1}.$$

Consequently,

$$-\langle m_{B,V_1} u, \Delta_A u \rangle \geq \|m_{B,V_1}^{1/2} \nabla_A u\|_2^2 - C \|m_{B,V_1} u\|_2 \|\nabla_A u\|_2.$$

With the aid of (3.12), we then obtain that

$$\|m_{B,V_1} u\|_2 \|\Delta_A u\|_2 \geq \|m_{B,V_1}^{1/2} \nabla_A u\|_2^2 - C \|m_{B,V_1} u\|_2 (\|\Delta_A u\|_2 + \|u\|_2).$$

Hence,

$$\|m_{B,V_1}^{1/2} \nabla_A u\|_2^2 \leq C \|m_{B,V_1} u\|_2 (\|\Delta_A u\|_2 + \|u\|_2),$$

from which we conclude that

$$\|m_{B,V_1}^{1/2} \nabla_A u\|_2^2 \leq \epsilon \|m_{B,V_1} u\|_2^2 + C_\epsilon (\|\Delta_A u\|_2^2 + \|u\|_2^2), \quad (3.14)$$

which, combined with (3.13), easily yields (3.10). ■

We continue with the following comparison result

Lemma 3.5. *Let $\mathcal{A} = -e^{i\alpha}\Delta_A + V$, and let $\{\kappa_n\}_{n=1}^\infty$ denote the n 'th eigenvalue of $(\mathcal{A}^*\mathcal{A})^{1/2}$. Let further*

$$\sigma((-\Delta_A + |V|)) = \{\nu_j\}_{j=1}^\infty.$$

Then, there exists $C > 0$ and $n_0 > 0$, such that for all $n > n_0$ we have

$$\kappa_n \leq C(1 + \nu_n). \quad (3.15)$$

Proof. Clearly, for every $u \in \mathcal{D}$,

$$\|(-\Delta_A + |V|)u\|_2^2 = \|\Delta_A u\|_2^2 + \|Vu\|_2^2 - 2\Re\langle |V|u, \Delta_A u \rangle. \quad (3.16)$$

As

$$2\Re\langle (|V_1 + V_2| - |V_1|)u, \Delta_A u \rangle \leq 2\| |V_2|u \|_2 \|\Delta_A u\|_2,$$

it follows, after integration by parts, that

$$2\Re\langle |V|u, \Delta_A u \rangle \leq -2\langle |V_1|\nabla_A u, \nabla_A u \rangle + 2|\Re\langle u\nabla|V_1|, \nabla_A u \rangle| + 2\| |V_2|u \|_2 \|\Delta_A u\|_2. \quad (3.17)$$

For every $\epsilon > 0$ we have, by (1.8), that

$$\begin{aligned} 2|\Re\langle u\nabla|V_1|, \nabla_A u \rangle| &\leq \epsilon\|u\nabla V_1\|_2^2 + \frac{1}{\epsilon}\|\nabla_A u\|_2^2 \\ &\leq C\epsilon(\|V_1 u\|_2^2 + \|B|u\|_2^2) - \frac{1}{\epsilon}\Re\langle u, \Delta_A u \rangle. \end{aligned} \quad (3.18)$$

By (3.10), for each positive ϵ , there exists C_ϵ such that

$$2|\Re\langle u\nabla|V_1|, \nabla_A u \rangle| \leq \epsilon(\|V_1 u\|_2^2 + \|\Delta_A u\|_2^2) + C_\epsilon\|u\|_2^2. \quad (3.19)$$

For the last term on the right-hand-side we have, by (1.8), that for any $\epsilon > 0$ there exists $C_\epsilon > 0$ for which

$$2\| |V_2|u \|_2 \|\Delta_A u\|_2 \leq \epsilon(\|V_1 u\|_2^2 + \|B|u\|_2^2 + \|\Delta_A u\|_2^2) + C_\epsilon\|u\|_2^2.$$

With the aid of (3.10), we then conclude that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$2\| |V_2|u \|_2 \|\Delta_A u\|_2 \leq \epsilon(\|\Delta_A u\|_2^2 + \|Vu\|_2^2) + C_\epsilon\|u\|_2^2.$$

Substituting the above together with (3.19) into (3.17) yields that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$2\Re\langle |V|u, \Delta_A u \rangle \leq \epsilon (\| |V|u \|_2^2 + \| \Delta_A u \|_2^2) + C_\epsilon \|u\|_2^2.$$

Substituting the above into (3.16) we obtain, if we choose ϵ sufficiently small,

$$\| \Delta_A u \|_2^2 + \|Vu\|_2^2 \leq C (\|(-\Delta_A + |V|)u\|_2^2 + \|u\|_2^2). \quad (3.20)$$

From here we easily obtain that there exists $C > 0$ such that for all $u \in \mathcal{D}$

$$\|(-e^{i\alpha} \Delta_A + V)u\|_2^2 \leq C (\|(-\Delta_A + |V|)u\|_2^2 + \|u\|_2^2). \quad (3.21)$$

We now use Lemma 3.3 to conclude that \mathcal{D} is dense in $D(-\Delta_A + |V|)$. Hence for all $n \geq 1$ there exist $\{u_k\}_{k=1}^n \subset \mathcal{D}$ such that

$$\langle (-\Delta_A + |V|)u, u \rangle \leq 2\nu_n \|u\|_2^2, \quad \forall u \in \text{span}\{u_k\}_{k=1}^n.$$

We can now obtain (3.15) by using once again Proposition 11.9 in [13] (as in the proof of (3.1)) together with (3.21) and the fact that $\mathcal{D} \subset D(-e^{i\alpha} \Delta_A + V)$.

■

Remark 3.6. *The above lemma, together with (3.1) proves that for any $p > 0$ and $\lambda \in \rho(-e^{i\alpha} \Delta_A + V) \cap \rho(-\Delta_A + |V|)$,*

$$(-\Delta_A + |V| - \lambda)^{-1} \in C^p \Leftrightarrow (-e^{i\alpha} \Delta_A + V - \lambda)^{-1} \in C^p, \quad (3.22)$$

whereas Theorem 3.2 provides us only with an upper bound for the optimal value of p . While the Dirichlet realization of $-\Delta_A + |V|$ in Ω is self-adjoint, the authors are unaware of an asymptotic expansion for the counting function associated with it in the necessary generality. There is, however, reason to believe that (3.6) is optimal. An example where such an asymptotic expansion has been derived is given in [16, 17] for the case $\Omega = \mathbb{R}^d$, and where A and V are polynomials.

We now prove optimality in the general case for the particular case when Ω is the half-space in \mathbb{R}^d .

Lemma 3.7. *Let $\Omega = \mathbb{R}_+^d$ where*

$$\mathbb{R}_+^d = \{x \in \mathbb{R}^d \mid x_d > 0\}.$$

Let further

$$\mathcal{D}_1 = \{u \in \mathcal{D}, |\mathcal{P}u \in \mathcal{D}\}.$$

Then, there exists $C(\Omega, \mathcal{P}) > 0$, such that for all $u \in \mathcal{D}_1$ we have

$$\|\mathcal{P}u\|_2^2 \leq C[\|(\Delta_A + |V|)u\|_2^2 + \|u\|_2^2]. \quad (3.23)$$

Proof. Clearly,

$$\|\mathcal{P}_0 u\|_2^2 = \sum_{k,m=1}^d e^{2i(\alpha_k - \alpha_m)} \langle \partial_{A_m}^2 u, \partial_{A_k}^2 u \rangle.$$

We now write

$$\langle \partial_{A_m}^2 u, \partial_{A_k}^2 u \rangle = -\langle \partial_{A_k} \partial_{A_m}^2 u, \partial_{A_k} u \rangle + \int_{\partial\Omega} \partial_{A_m}^2 u \cdot \overline{\partial_{A_k} u} \nu_k ds.$$

Since $u \in \mathcal{D}_1$, we have that $\mathcal{P}_0 u = 0$ on $\partial\Omega$, and hence

$$\sum_{m=1}^d e^{2i(\alpha_k - \alpha_m)} \int_{\partial\Omega} \partial_{A_m}^2 u \cdot \overline{\partial_{A_k} u} \nu_k ds = 0, \forall 1 \leq k \leq d.$$

Consequently,

$$\|\mathcal{P}_0^2 u\|_2^2 = - \sum_{k,m=1}^d e^{2i(\alpha_k - \alpha_m)} \langle \partial_{A_k} \partial_{A_m}^2 u, \partial_{A_k} u \rangle. \quad (3.24)$$

Next we write

$$\begin{aligned} \langle \partial_{A_k} \partial_{A_m}^2 u, \partial_{A_k} u \rangle &= \langle \partial_{A_m} \partial_{A_k} \partial_{A_m} u, \partial_{A_k} u \rangle + i \langle B_{km} \partial_{A_m} u, \partial_{A_k} u \rangle = \\ &= \langle \partial_{A_m}^2 \partial_{A_k} u, \partial_{A_k} u \rangle + i (\langle B_{km} \partial_{A_m} u, \partial_{A_k} u \rangle + \langle \partial_{A_m} B_{km} u, \partial_{A_k} u \rangle). \end{aligned} \quad (3.25)$$

Integration by parts yields

$$-\langle \partial_{A_m}^2 \partial_{A_k} u, \partial_{A_k} u \rangle = \|\partial_{A_m} \partial_{A_k} u\|_2^2 + \int_{\partial\Omega} \nu_m \partial_{A_m} \partial_{A_k} u \overline{\partial_{A_k} u} ds. \quad (3.26)$$

As $u = 0$ on the boundary, the surface integral on the right-hand-side vanishes whenever $(m, k) \neq (d, d)$. If $m = k = d$, we use the fact that $\mathcal{P}_0^2 u = 0$ on the boundary to obtain

$$\int_{\partial\Omega} \partial_{A_d} \partial_{A_d} u \overline{\partial_{A_d} u} ds = \sum_{n=1}^{d-1} e^{2i(\alpha_d - \alpha_n)} \int_{\partial\Omega} \partial_{A_n} \partial_{A_n} u \overline{\partial_{A_d} u} ds = 0. \quad (3.27)$$

Combining the above with (3.25), (3.26), and (3.24), yields

$$\begin{aligned} \|\mathcal{P}_0 u\|_2^2 = \Re \sum_{k,m=1}^d e^{2i(\alpha_k - \alpha_m)} [& \|\partial_{A_m} \partial_{A_k} u\|_2^2 \\ & - i(\langle B_{km} \partial_{A_m} u, \partial_{A_k} u \rangle + \langle \partial_{A_m} B_{km} u, \partial_{A_k} u \rangle)]. \end{aligned} \quad (3.28)$$

We proceed by estimating the two rightmost terms in (3.28). To this end we use (3.14) to obtain that

$$|\langle B_{km} \partial_{A_m} u, \partial_{A_k} u \rangle| \leq \| |B|^{1/2} \nabla_{A_k} u \|_2^2 \leq C(\|\Delta_{A_k} u\|_2^2 + \|Vu\|_2^2 + \|u\|_2^2). \quad (3.29)$$

For the last term on the right-hand-side we have

$$\langle \partial_{A_m} B_{km} u, \partial_{A_k} u \rangle = \langle B_{km} \partial_{A_m} u, \partial_{A_k} u \rangle + \langle u \partial_m B_{km}, \partial_{A_k} u \rangle.$$

The first term on the right-hand-side of the above equation has already been estimated by (3.29). For the second term we use (1.6) and (3.10) to obtain

$$|\langle u \partial_m B_{km}, \partial_{A_k} u \rangle| \leq C \|m_{B,V} u\|_2 \|\nabla_{A_k} u\|_2 \leq C(\|\Delta_{A_k} u\|_2^2 + \|Vu\|_2^2 + \|u\|_2^2). \quad (3.30)$$

Substituting the above together with (3.29) into (3.28) yields

$$\|\mathcal{P}_0 u\|_2^2 \leq \Re \sum_{k,m=1}^d e^{2i(\alpha_k - \alpha_m)} \|\partial_{A_m} \partial_{A_k} u\|_2^2 + C(\|\Delta_{A_k} u\|_2^2 + \|Vu\|_2^2 + \|u\|_2^2).$$

Hence,

$$\|\mathcal{P}_0 u\|_2^2 \leq \Re \sum_{k,m=1}^d \|\partial_{A_m} \partial_{A_k} u\|_2^2 + C(\|\Delta_{A_k} u\|_2^2 + \|Vu\|_2^2 + \|u\|_2^2). \quad (3.31)$$

It is easy to show that if $u \in \mathcal{D}_1$ then $\Delta_{A_k} u = 0$ on $\partial\Omega$: As $\partial_{A_n}^2 u = 0$ on $\partial\Omega$ for all $1 \leq n \leq d$, both $\Delta_{A_k} u|_{\partial\Omega} = 0$ and $\mathcal{P}_0 u|_{\partial\Omega} = 0$ are equivalent to $\partial_d^2 u|_{\partial\Omega} = 0$. Hence, we may conclude from (3.28) that

$$\|\Delta_{A_k} u\|_2^2 = \Re \sum_{k,m=1}^d [\|\partial_{A_m} \partial_{A_k} u\|_2^2 - i(\langle B_{km} \partial_{A_m} u, \partial_{A_k} u \rangle + \langle \partial_{A_m} B_{km} u, \partial_{A_k} u \rangle)].$$

Hence, by (3.31)

$$\begin{aligned} \|\mathcal{P}_0 u\|_2^2 &\leq \|\Delta_A u\|_2^2 - \Im \sum_{k,m=1}^d [(\langle B_{km} \partial_{A_m} u, \partial_{A_k} u \rangle + \langle \partial_{A_m} B_{km} u, \partial_{A_k} u \rangle)] \\ &\quad + C(\|\Delta_A u\|_2^2 + \|Vu\|_2^2 + \|u\|_2^2). \end{aligned}$$

Using (3.29) and (3.30) once again yields

$$\|\mathcal{P}_0 u\|_2^2 \leq C(\|\Delta_A u\|_2^2 + \|Vu\|_2^2 + \|u\|_2^2),$$

and hence

$$\|\mathcal{P}u\|_2^2 \leq C(\|\Delta_A u\|_2^2 + \|Vu\|_2^2 + \|u\|_2^2).$$

The proof of (3.30) can now be easily completed with the aid of (3.20)(which is valid for all $u \in \mathcal{D}$). ■

An immediate conclusion is

Corollary 3.8. *Let $\{\mu_n\}_{n=1}^\infty$ and $\{\nu_n\}_{n=1}^\infty$ be defined as in Proposition 3.1, and let $\Omega = \mathbb{R}_+^d$ or $\Omega = \mathbb{R}^d$. Then, there exists $C > 0$ and $n_0 \in \mathbb{N}$ such that*

$$\mu_n \leq C(1 + \nu_n). \quad (3.32)$$

Thus, if the resolvent of \mathcal{P} is in $C^p(L^2(\Omega, \mathbb{C}))$ for some $p > 0$, the the same conclusion follows for the resolvent of $-\Delta_A + |V|$.

Proof. Let $u \in \mathcal{D}_1$. We need to show first that $(-\Delta_A + |V|)u \in \mathcal{D}$, or equivalently that $\Delta_A u|_{\partial\Omega} = 0$ in the case $\Omega = \mathbb{R}_+^d$ (where $\partial\Omega$ is the hyperplane $x_d = 0$). Since $u \in \mathcal{D}$, it follows that $\partial_{A_k}^2 u|_{\partial\Omega} = 0$ for all $1 \leq k \leq d-1$. It is easy to show (see (3.27)) that $\partial_{A_d}^2 u|_{\partial\Omega} = 0$ using the fact that $\mathcal{P}_0 u|_{\partial\Omega} = 0$ for all $u \in \mathcal{D}_1$.

Since by Lemma 3.3 \mathcal{D} is dense in $D(-\Delta_A + |V|)$, it follows that \mathcal{D}_1 is dense in $D((-\Delta_A + |V|)^2)$ under the same graph norm. We can now use statement 2 of Theorem 2.2 with $S = (-\Delta_A + |V|)^2$ and $\mathcal{V} = D(-\Delta_A + |V|)$ to prove that \mathcal{D}_1 is dense in $D(-\Delta_A + |V|)$. The proof of (3.32) can now be completed by using Proposition 11.9 in [13] as in the proof of Lemma 3.5. ■

4 Examples

4.1 One-dimensional examples

4.1.1 The entire real line case.

Consider the case $\Omega = \mathbb{R}$ and

$$\mathcal{L} = -\frac{d^2}{dx^2} + e^{i\theta}v_1(x) + v_2(x), \quad (4.1)$$

where $\theta \in (-\pi, \pi)$, defined initially on $C_0^\infty(\mathbb{R})$.

In the above, $v_1(x) \in C^1(\mathbb{R}, \mathbb{R})$ has the asymptotic behaviour

$$|v_1| \sim |x|^\alpha \quad \text{as } |x| \rightarrow \infty, \quad (4.2)$$

for some $\alpha > 0$,

$$|v_1'| \leq C(1 + |x|^{\alpha-1}), \quad (4.3)$$

and $v_2 \in L_{loc}^\infty(\mathbb{R}, \mathbb{C})$ satisfies

$$v_2 = o(|x|^\alpha) \quad \text{as } |x| \rightarrow \infty. \quad (4.4)$$

Set $\mathcal{P} = i \operatorname{sign} \theta e^{-i\theta} \mathcal{L}$. Then \mathcal{P} meets (1.5)-(1.10), and we may apply Theorems 1.1 and 1.3. Henry [14] considers this example for the case $v(x) = x^{2k}$, $v_2 \equiv 0$.

By Theorem 3.2 it follows that \mathcal{L}^{-1} is in C^p for all $p > 1/2 + 1/\alpha$. As the numerical range of $\mathcal{L} + \mu$, for sufficiently large $\mu \in \mathbb{R}_+$, lies in the sector $\arg \lambda \in [\theta - \pi, \theta]$, Theorem 1.4 can be applied to obtain, for any $\theta \in (-\pi, \pi)$, that $\operatorname{Span}(\mathcal{L}) = L^2(\mathbb{R}, \mathbb{C})$ whenever $\alpha > 2$.

In the particular case where $v_1 \rightarrow +\infty$, as $|x| \rightarrow \infty$, the numerical range lies inside the sector $\arg \lambda \in [0, \theta]$ and hence for any $\theta \in (-\pi, \pi)$ and $\alpha > \frac{2|\theta|}{2\pi - |\theta|}$, $\operatorname{Span}(\mathcal{L}) = L^2(\mathbb{R}, \mathbb{C})$.

The complex cubic oscillator is an example included in the class we introduce in (4.1) (for the general case where v_1 can change its sign between $-\infty$ and ∞) which has been frequently addressed in the literature (cf. [20], [12], [9], [15] to name just a few references). In this case, $\theta = \pi/2$ and

$$v_1(x) = x^3, \quad v_2(x) = \beta_2 x^2 + \beta_1 x,$$

where $\beta_1, \beta_2 \in \mathbb{C}$. In [21], the existence of an infinite sequence of eigenvalues has been established (cf. [20] for more details). Completeness of the system

of eigenfunctions in $L^2(\mathbb{R}, \mathbb{C})$ has been established in [19] for the case $\beta_1 = \beta_2 = 0$. Here we show it in greater generality, without the need to rely on the symmetries of the particular cases addressed in [15, 19].

Another case that has been addressed in [8, 7] is

$$\mathcal{L} = -\frac{d^2}{dx^2} + ce^{i\theta}|x|^\alpha, \quad (4.5)$$

where $c > 0$ and $\theta \in (-\pi/2, \pi/2)$. It is stated in [7], based on [18], that the eigenspace of \mathcal{L} is dense in $L^2(\mathbb{R}, \mathbb{C})$ when either $\alpha \geq 1$, or $\alpha \in (0, 1)$ and

$$\theta \leq \frac{\pi\alpha}{2}.$$

Since (4.5) is a particular case of (4.1), it follows that whenever $\theta \in (-\pi, \pi)$, \mathcal{L}^{-1} must be in C^p for all $p > 1/2 + 1/\alpha$. The numerical range of \mathcal{L} however, is confined in $\arg z \in (0, \theta)$. Hence, the eigenfunctions of \mathcal{L} form a complete system whenever

$$|\theta| < \frac{2\pi\alpha}{\alpha + 2}.$$

In particular, if $\theta \in (-\pi/2, \pi/2)$ completeness of the eigensystem of \mathcal{L} is guaranteed for all $\alpha > 2/3$. Note that for $0 < \alpha < 1$

$$\frac{\pi\alpha}{2} < \frac{2\pi\alpha}{\alpha + 2}.$$

Hence, our method provides greater domains for θ and α where the eigenspace of \mathcal{L} is dense in $L^2(\mathbb{R}, \mathbb{C})$.

4.1.2 The positive real line case:

Here we consider the same differential operator as in (4.1), but this time defined on $C_0^\infty(\mathbb{R}^+)$ and consider the Dirichlet realization. Assuming that v_1 and v_2 satisfy (4.2), (4.3), and (4.4), we have by Theorem 3.2 that $\mathcal{L}^{-1} \in C^p$ whenever $p > 1/2 + 1/\alpha$ as before. However, as in the case where \mathcal{L} is given by (4.5), since every direction outside $[0, \theta]$ is a direction of minimal growth for $(\mathcal{L}_+ - \lambda)^{-1}$. Thus, we have that $\text{Span } \mathcal{L}_+ = L^2(\mathbb{R}, \mathbb{C})$ whenever

$$\theta < \frac{2\alpha\pi}{\alpha + 2}.$$

In particular, for the case $v_1 = x$ and $v_2 = 0$, which is known as the complex Airy's equation, we obtain that $\text{Span } \mathcal{L}_+ = L^2(\mathbb{R}, \mathbb{C})$ whenever $\theta < 2\pi/3$.

4.2 Two-dimensional examples

4.2.1 Electric and magnetic potentials.

Let A and V be such that $V = i\phi$ and $\text{curl } A + i\phi$ is a holomorphic function of $z = x + iy$, x and y being the planar coordinates. Such a choice is in line with the steady state Faraday's law, which in two dimensions read (assuming all constants are equal to 1)

$$\nabla_{\perp} \text{curl } A + \nabla \phi = 0.$$

We further narrow our choice by setting

$$\text{curl } A + i\phi = z^n. \quad (4.6)$$

Consider the differential operator $-\Delta_A + i\phi$ with $\Omega = \mathbb{R}^2$ (which is clearly a particular case of (1.10)). By Theorem 3.2, we have that $\mathcal{P}^{-1} \in C^p$ for all $p > 1 + 2/n$. It can be readily verified that, independently of β , every direction outside $[-\pi/2, \pi/2]$ is a direction of minimal growth for $(\mathcal{P} - \lambda)^{-1}$. Unfortunately, the condition of validity of our theorems will lead to the condition $1 < \frac{1}{1+\frac{2}{n}}$, which is never satisfied. However, if we consider instead of \mathbb{R}^2 , a smooth domain Ω which is contained in a sector whose opening is smaller than $\pi/(n+2)$, and such that the positive real axis is contained in this sector, then the numerical range of \mathcal{P} is contained in a sector whose opening is less than $\pi/(1 + 2/n)$ and we can use Theorem 1.4 to prove completeness of the eigenspace.

4.2.2 Analytically dilated operators

Next, for $m \geq 2$ and $k \geq 1$, let

$$\mathcal{A} = -\frac{\partial^2}{\partial x^2} - \left(\frac{\partial}{\partial y} - i\frac{x^m}{m} \right)^2 + iy^{2k}, \quad (4.7)$$

Note that the numerical range lies in the sector $\arg \lambda \in [0, \pi/2]$. In this case we use analytic dilation, as in [4], to reduce the angle of the sector where the numerical range lies. This would allow us to establish the existence of countable set of eigenvalues,. Thus, we define the dilation operator

$$u \rightarrow (\mathcal{U}u)(x, y) = e^{-i\frac{m-1}{2}\alpha} u(e^{i\alpha}x, e^{-im\alpha}y)$$

where

$$\alpha = -\frac{\pi}{4m(k+1)}, \quad (4.8)$$

to obtain that

$$\mathcal{A}_\alpha = \mathcal{U}^{-1} \mathcal{A} \mathcal{U} = -e^{2i\alpha} \frac{\partial^2}{\partial x^2} - e^{-2im\alpha} \left(\frac{\partial}{\partial y} - i \frac{x^m}{m} \right)^2 + e^{2ikm\alpha + i\pi/2} y^{2k}. \quad (4.9)$$

We next verify \mathcal{A}_α meets the assumptions of Theorem 1.1. Since $V_1 = e^{2ikm\alpha + i\pi/2} y^{2k}$ and

$$2km\alpha + \frac{\pi}{2} = -\frac{k\pi}{2(k+1)} + \frac{\pi}{2} = \frac{\pi}{2(k+1)},$$

we obtain that

$$\Re V_1 = \cos \frac{\pi}{2(k+1)} y^{2k}.$$

Hence the conditions of Theorem 1.1 can be easily verified. We thus obtain that $(\mathcal{A}_\alpha)^{-1}$ is in C^p for all

$$p > 1 + \frac{1}{2k} + \frac{1}{m-1} = \frac{(2k+1)m-1}{2k(m-1)}. \quad (4.10)$$

Next we observe that

$$\langle u, (e^{-2i\alpha} \mathcal{A}_\alpha - \lambda)u \rangle = \|u_x\|_2^2 + e^{-2i(m+1)\alpha} \left(\left\| \left(\partial_y - i \frac{x^m}{m} \right) u \right\|_2^2 + \|y^k u\|_2^2 \right) - \lambda \|u\|_2^2.$$

It can be easily verified from the above and (4.8) that the numerical range of \mathcal{A}_α lies in the sector

$$\arg \lambda \in \left[0, \frac{(m+1)\pi}{2m(k+1)} \right].$$

which is always contained in $[0, 3\pi/8]$ for the assumed range of m and k values. Consequently, the effect of this analytic dilation has permitted us to reduce the angle of the sector in which the numerical range lies. Since

$$\frac{(m+1)}{2m(k+1)} < \frac{2k(m-1)}{(2k+1)m-1}, \quad (4.11)$$

we obtain that $\text{Span } \mathcal{A}_\alpha = L^2(\mathbb{R}^2, \mathbb{C})$. By the same arguments of [4] we then obtain that \mathcal{A} has an infinite sequence of eigenvalues for all $k \geq 1$ and $m \geq 2$.

Remark 4.1.

- *It is not clear whether analytic dilation preserves the completeness of the system of generalized eigenfunctions. Hence, the best we can obtain is the existence of an infinite discrete spectrum for \mathcal{A} .*
- *Had we abandoned analytic dilation, we would have obtained, instead of (4.11), the condition*

$$\frac{1}{2} < \frac{2k(m-1)}{(2k+1)m-1}$$

which implies $m > 2$ and

$$k > \frac{m-1}{2(m-2)} \tag{4.12}$$

to achieve completeness.

- *Note that the results presented in [16] can be applied to (4.9) to conclude the optimality of (4.10), as they can be used to obtain the precise Schatten class of the resolvent of $-\Delta_A + y^{2k}$ in \mathbb{R}^2 .*

4.2.3 A half-plane problem.

Let

$$\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}.$$

Consider the case where $V = e^{i\theta}y$ and $A = x^2\hat{i}_y/2$. We define $\mathcal{P}_+ = -\Delta_A + V$ with $\Omega = \mathbb{R}_+^2$ (which is once again a particular case of (1.10)). Once again we have that $\mathcal{P}_+^{-1} \in C^p$ for all $p > 1 + 2/n$. For $n = 1$ and $\beta = 0$ it can be easily shown that the numerical range of \mathcal{P}_+ is confined within the sector $[0, \theta]$ in \mathbb{C} . Hence, every direction outside $[0, \theta]$ is a direction of minimal growth for $(\mathcal{P}_+ - \lambda)^{-1}$, and consequently, $\text{Span } \mathcal{P}_+ = L^2(\mathbb{R}_+^2, \mathbb{C})$ for all $\theta < \pi/3$.

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