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Small energy traveling waves for the Euler-Korteweg system

Corentin Audiard ^{*†}

January 11, 2017

Abstract

We investigate the existence and properties of traveling waves for the Euler-Korteweg system with general capillarity and pressure. Our main result is the existence in dimension two of waves with arbitrarily small energy. They are obtained as minimizers of a modified energy with fixed momentum. The proof builds upon various ideas developed for the Gross-Pitaevskii equation (and more generally nonlinear Schrödinger equations with non zero limit at infinity). Even in the Schrödinger case, the fact that we work with the hydrodynamical variables and a general pressure law both brings new difficulties and some simplifications. Independently, in dimension one we prove that the criterion for the linear instability of traveling waves from [6] actually implies nonlinear instability.

Résumé

On étudie les ondes progressives des équations d'Euler-Korteweg pour des lois de capillarité et pression générales. Le principal résultat est l'existence en dimension 2 d'ondes d'énergie arbitrairement petite. Elles sont obtenues comme minimiseurs d'une énergie modifiée à moment fixé. La preuve suit plusieurs idées développées pour les équations de Schrödinger non linéaires avec limite non nulle à l'infini. Même dans ces cas, le fait de travailler en variables hydrodynamiques apporte de nouvelles difficultés, mais aussi quelques simplifications. Indépendamment, on montre en dimension un que le critère d'instabilité linéaire des ondes progressives de [6] implique en fait l'instabilité non linéaire.

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1 Introduction

The Euler-Korteweg system is a modification of the usual Euler equations for compressible fluids that includes capillary effects. Mathematically it reads in dimension d as the following system of $d + 1$ equations combining the conservation of mass and momentum

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t u + u \cdot \nabla u + \nabla g(\rho) = \nabla \left(K(\rho) \Delta \rho + \frac{1}{2} K'(\rho) |\nabla \rho|^2 \right), \end{cases} \quad x \in \mathbb{R}^d. \quad (1.1)$$

The variables ρ and u are the density and speed of the fluid, the right hand side of the second line is the so called capillary tensor. The functions K, g are defined on \mathbb{R}^{+*} and are supposed to be smooth and positive. For the equations to make sense, it is necessary that $\rho > 0$ a.e .

The Korteweg tensor was first derived in the work of Dunn and Serrin [20] for models of phase transition, however the equations can appear in very various settings, from water waves (see [13]) to quantum hydrodynamics.

When u is potential (the irrotational case) (1.1) has a hamiltonian structure : indeed if we write $u = \nabla \phi$, the second line of (1.1) rewrites

$$\partial_t \phi + \frac{|\nabla \phi|^2}{2} + g(\rho) = K \Delta \rho + \frac{1}{2} K'(\rho) |\nabla \rho|^2,$$

For G a primitive of g , we define the energy

$$E(\rho, \phi) = \int \frac{K(\rho) |\nabla \rho|^2 + \rho |\nabla \phi|^2}{2} + G(\rho) dx, \quad (1.2)$$

then (1.1) reads

$$\begin{cases} \partial_t \rho - \frac{\delta E}{\delta \phi} = 0, \\ \partial_t \phi + \frac{\delta E}{\delta \rho} = 0. \end{cases} \quad (1.3)$$

In particular, for (ρ, ϕ) a solution with enough integrability and smoothness, $E(\rho, \phi)(t)$ is conserved. One can also check formally the conservation of momentum: if $\lim_{|x| \rightarrow \infty} \rho = \rho_0 \in \mathbb{R}^{+*}$,

$$\frac{d\bar{P}}{dt} := \frac{d}{dt} \int_{\mathbb{R}^d} (\rho - \rho_0) \nabla \phi = 0. \quad (1.4)$$

Concerning the analysis of well-posedness, it was observed in [7] that for smooth solutions without vacuum (1.1) is equivalent to a quasi-linear degenerate Schrödinger equation. Due to this very nonlinear structure, the analysis of the Cauchy problem is quite involved. If $\bar{\rho}, \bar{u}$ is a reference smooth solution, local well-posedness for $(\rho_0, u_0) \in (\bar{\rho}(0) + H^{s+1}) \times (\bar{u} + H^s)$, $s > d/2 + 1$ was obtained in [4]. The energy of the system allows to control at best $(\rho - \bar{\rho}, u - \bar{u})$ in $H^1 \times L^2$, so in any dimension global well-posedness has remained mostly an open problem.

In the special case $K = \kappa/\rho$, with κ a positive constant and $u = \nabla \phi$ is irrotational, up to some rescaling there exists a formal correspondance with the nonlinear Schrödinger equation

$$i\partial_t \psi + \Delta \psi = g(|\psi|^2)\psi, \quad (1.5)$$

through the *Madelung transform* $(\rho, \nabla \phi) \mapsto \psi := \sqrt{\rho} e^{i\phi}$, introduced in [28] (for more details see the review article [14]). We will not dwell upon it but only mention that the nonlinearity g in (1.5) becomes the pressure term, and if ρ vanishes the transform becomes singular. Antonelli and Marcati [1] managed to exploit this correspondance in order to pass from global solutions of NLS (whose existence is standard, see the reference book [16]) to global weak solutions of (1.1). In general such solutions admit vacuum and one can not hope to deduce uniqueness from such arguments. In the special case $g(\rho) = \rho - 1$, (1.1) corresponds to the Gross-Pitaevskii equation which has received a lot of attention over the last fifteen years. In particular, global dispersive solutions of (1.5) were constructed in [23]. Such results were used to construct global *unique* solutions of (1.1) for small irrotational data by the author and B.Haspot in [2].

The result was later extended by the same authors in [3] for general K, g and $d \geq 3$: for initial data near the constant state $(\rho_0, 0)$ with the stability condition $g'(\rho_0) > 0$, the solution is global and converges to a solution of the linearized equation near $(\rho_0, 0)$ (in other words it scatters). The price to pay for this generalization is the necessity to work with much smoother functions, basically: $\rho - \rho_0 \in H^{50}$. The idea behind this result is that the Euler-Korteweg equations (1.1) and the Gross-Pitaevskii equation share the same linearized system (near $(\rho, u) = (\rho_0, 0)$, resp. $\psi = 1$) so that the same small data technics from the field of dispersive equations can be used.

A natural question is then whether such an analogy is still true for *nonlinear* phenomena and in particular the existence for traveling waves which is known for a large class of nonlinear Schrödinger equations. This article gives a partial positive answer: our main result (theorem 1.1) is the existence of small traveling waves in dimension 2. Before turning to a precise statement, let us give some background about this issue.

The existence of *planar* traveling waves, that is solutions of the form $(\rho(x_1 - ct), u(x_1 - ct))$ is a simpler problem as in this case (1.1) can be reduced to a system of two ODEs. Due to

the hamiltonian nature of the equations these ODEs are integrable by quadrature. If g is not monotone (for example with a Van der Waals pressure law) all three types of interesting solutions exist (homoclinic, heteroclinic and periodic). In dimension one, the stability/unstability of such solutions is related to the notion of moment of instability from the seminal paper [22] of Grillakis-Shatah-Strauss. In the absence of global well-posedness result, only conditional stability was derived for the corresponding traveling waves. On this topic, we offer a small contribution with theorem B.4 which states that failure of the stability criterion from [5] implies nonlinear instability. For more details on planar traveling waves we refer to the rich review article [6].

The existence of localized solitary waves In dimension larger than 1 the existence of *localized* traveling waves that depend on $x - \vec{c}t$, with \vec{c} the direction and speed of propagation, has been so far an open problem. Our main result is the existence of small energy traveling waves (see theorem 1.1). The interest is twofold: besides giving global solutions to (1.1), the existence of arbitrarily small solitary waves in dimension 2 is in strong contrast with the scattering of small solutions in dimension ≥ 3 . Note that while our results apply to general K, g , in the special case $K(\rho) = \kappa/\rho, g(\rho) = \rho - 1$, the existence of solitary waves to (1.1) might be deduced thanks to the Madelung transform from the existence of non-vanishing solitary waves to (1.5). However if K is not proportional to $1/\rho$ (even $K \equiv \text{constant}$), new difficulties appear due to the quasilinear nature of (1.1).

Concerning the expected range of speeds, the linearization of the Euler equations (without capillary terms) near $\rho = \rho_0, u = 0$ is

$$\begin{cases} \partial_t \rho + \rho_0 \operatorname{div} u = 0, \\ \partial_t u + g'(\rho_0) \nabla \rho = 0. \end{cases}$$

If $g'(\rho_0) > 0$, ρ satisfies the wave equation $\partial_t^2 \rho - \rho_0 g'(\rho_0) \Delta \rho = 0$, with the so-called *sound speed* $c_s(\rho_0) = \sqrt{\rho_0 g'(\rho_0)}$. By analogy with the Gross-Pitaevskii case, we expect that traveling waves with limit at infinity $(\rho_0, 0)$ can only exist for subsonic speeds $|\vec{c}| \leq c_s$. Obviously, the direction of the speed does not matter, thus from now on we restrict ourselves to $\vec{c} = c \vec{e}_1$.

Some results on (1.5) with nonzero condition at infinity If $g(1) = 0$, a natural problem is the construction of solitary waves such that $\lim_{|x| \rightarrow \infty} |\psi| = 1$. The case of the Gross-Pitaevskii equation $g(\rho) = \rho - 1$ has attracted a lot of attention since the series of papers of Roberts and al [24][25]. Their formal and numerical computations brought a number of conjectures on the existence of branches of solitary waves with speeds c covering the subsonic range $(0, \sqrt{2})$ (the number $\sqrt{2}$ is related to $\sqrt{1 \cdot g'(1)} = 1$ after some rescaling), their stability and limit in the transonic regime. In dimension 2 traveling waves were constructed for any $|c|$ small enough by Béthuel and Saut ([11], 98) with a mountain pass argument. More recently they used with P. Gravejat in [10] a constrained minimization method, that we will also follow.

To shortly describe it, let us introduce the momentum and energy

$$\text{(momentum)} \quad (\vec{P}_{NLS}(\psi) = \frac{1}{2} \operatorname{Re} \int_{\mathbb{R}^d} i \nabla \psi (\bar{\psi} - 1) dx.$$

$$\text{(energy)} \quad E_{NLS}(\psi) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla \psi|^2 + G(|\psi|^2) dx$$

where G is a primitive of g . In the Gross-Pitaevskii case, we simply have $G = (|\psi|^2 - 1)^2/2$. These two quantities are formally conserved by the flow, also it is not hard to check formally

$$c \delta P_{NLS,1}(\psi) = \delta E_{NLS}(\psi) \Leftrightarrow -ic \partial_1 \psi + \Delta \psi = (|\psi|^2 - 1) \psi.$$

It is thus tempting to construct solitary waves as minimizers of the energy with $P_{NLS,1}(\psi) = p$ fixed. However, it is a bit tedious to give a functional framework where both E_{NLS} and \vec{P}_{NLS} make sense, and the existence of a lifting of ψ on subsets of \mathbb{R}^d , while extremely useful, raises significant topological difficulties. Finally, in this approach the speed c is only obtained as a Lagrangian multiplier, which precludes to reach the whole range $c \in (0, \sqrt{2})$. Nevertheless in [10] the authors proved the existence of a branch of solutions parametrized by the momentum $p \in (p_0, \infty)$ in dimension 2 and 3 ($p_0 = 0$ in dimension 2, > 0 in dimension 3). With an alternative approach, Maris [29] obtained the existence of traveling waves for the full range $c \in (0, \sqrt{2})$ in dimension ≥ 3 for a class of equations more general than Gross-Pitaevskii. The proof relied on the minimization of energy with a more subtle constraint based on a Pohozaev type identity. Finally, the construction of solitary waves by minimization with fixed momentum was recently improved by Maris and Chiron [17], giving the precompactness of minimising sequences (which is a classical ingredient for orbital stability).

Remark 1. Note that in the case $K = 1/\rho$, energy and momentum conservation for ψ exactly correspond to energy and momentum conservation in (1.2),(1.4) with $\psi = \sqrt{\rho} e^{i\phi/2}$. Indeed

$$E_{NLS}(\psi) = \frac{1}{2} \int_{\mathbb{R}^d} \frac{1}{4\rho} |\nabla \rho|^2 + \frac{\rho}{4} |\nabla \phi|^2 + G(\rho) dx, \quad \vec{P}_{NLS} = \frac{1}{4} \int \rho \nabla \phi = \frac{1}{4} \int (\rho - 1) \nabla \phi dx.$$

Rescaling, modified energy and main result We construct solutions of (1.1) of the form $(\rho(x_1 - ct, x_2), u(x_1 - ct, x_2))$, the system of partial differential equations to solve is

$$\begin{cases} -c \partial_1 \rho + \operatorname{div}(\rho \nabla \phi) = 0, \\ -c \partial_1 \phi + \frac{|\nabla \phi|^2}{2} - K \Delta \rho - \frac{K' |\nabla \rho|^2}{2} + g(\rho) = 0. \end{cases} \quad (1.6)$$

We will focus on the existence of localized traveling waves near the constant state $(\rho_0, 0)$ with $g(\rho_0) = 0$, $g'(\rho_0) > 0$. We use the following rescaling :

$$\begin{aligned} (\rho, \phi) &= \left(\rho_0 \rho_r \left(\sqrt{\frac{g'(\rho_0)}{\rho_0}} x \right), \phi_r \left(\sqrt{\frac{g'(\rho_0)}{\rho_0}} x \right) \right), \quad K_r(\rho_r) = \frac{K(\rho_0 \rho_r)}{\rho_0}, \\ g_r(\rho_r) &= \frac{g(\rho_0 \rho_r)}{g'(\rho_0) \rho_0}, \quad c_r = \frac{c}{\sqrt{\rho_0 g'(\rho_0)}}. \end{aligned}$$

Then (1.6) is equivalent to

$$\begin{cases} -c_r \partial_1 \rho_r + \operatorname{div}(\rho_r \nabla \phi_r) = 0, \\ -c_r \partial_1 \phi_r + \frac{|\nabla \phi_r|^2}{2} - K_r \Delta \rho_r - \frac{K_r' |\nabla \rho_r|^2}{2} + g_r(\rho) = 0. \end{cases} \quad (1.7)$$

Of course, the point of this rescaling is that the constant state is now 1, $g_r'(1) = 1$ and the sound speed is $\sqrt{1g_r'(1)} = 1$. From now on we drop the r index and work on the rescaled system.

If we define the scalar momentum P as

$$P(\rho, \phi) = \int_{\mathbb{R}^d} (\rho - 1) \partial_1 \phi dx,$$

our starting point is that similarly to NLS, (1.7) can be recast as *

$$c\delta P(\rho, \phi) = \delta E(\rho, \phi).$$

The scalar momentum P is well-defined on¹ $\mathcal{H} := \{(\rho, \phi) \in (1 + H^1) \times \dot{H}^1\}$. On the other hand, the energy has two flaws: depending on G it may not make sense for general $(\rho, u) \in \mathcal{H}$, and even in the simple case $G = (\rho - 1)^2/2$ it satisfies no coercive inequality $E(\rho, \phi) \gtrsim \|(\rho, \phi)\|_{\mathcal{H}}^2$. Since we are interested in the regime $|\rho - 1| \ll 1$, the remedy is to work with a modified energy that we define now. We fix $\chi \in C^\infty(\mathbb{R}^+)$ nondecreasing such that

$$\chi(\rho) = \rho \text{ if } |\rho - 1| < 1/3, \quad \chi|_{]-\infty, 1/2]} = 1/2, \quad \chi|_{[2, \infty[} = 2, \quad (1.8)$$

and define \tilde{G} as follows: since $G''(1) = g'(1) = 1$, $G'(1) = g(1) = 0$, we have $G(\rho) \geq (\rho - 1)^2/3$ on some interval $(1 - \delta, 1 + \delta)$, according to Borel's lemma there exists a smooth extension \tilde{g} of g on $[1 + \delta, 1 + 2\delta]$, $[1 - 2\delta, 1 - \delta]$ such that for any $k \in \mathbb{N}$, $\tilde{g}^{(k)}(1 \pm 2\delta) = \frac{d^k(\rho-1)}{d\rho^k}(1 \pm 2\delta)$, and $\tilde{G} > 0$, then we set $\tilde{g} = \rho - 1$ on $(1 - 2\delta, 1 + 2\delta)^c$, $\tilde{G} = \int_1^\rho \tilde{g}(r) dr$. The function \tilde{G} satisfies

$$\tilde{G} \in C^\infty(\mathbb{R}), \quad \tilde{G}|_{(1-\delta, 1+\delta)} = G, \quad \tilde{G} \gtrsim (\rho - 1)^2, \quad |\tilde{G}'| \lesssim |\rho - 1|. \quad (1.9)$$

Now let us set

$$\tilde{E}(\rho, \phi) = \int_{\mathbb{R}^d} \frac{1}{2} (\chi(\rho) |\nabla \phi|^2 + K(\chi(\rho)) |\nabla \rho|^2) + \tilde{G}(\rho) dx. \quad (1.10)$$

Obviously, if $\|\rho - 1\|_\infty$ is small enough then $E(\rho, \phi) = \tilde{E}(\rho, \phi)$, and from (1.8), (1.9)

$$\forall (\rho, \phi) \in \mathcal{H}, \quad \tilde{E}(\rho, \phi) \gtrsim \|\rho - 1\|_{H^1}^2 + \|\nabla \phi\|_2^2.$$

If (ρ, ϕ) is a solution of the minimization problem

$$\inf\{\tilde{E}(\rho, \phi), (\rho, \phi) \in \mathcal{H} : P(\rho, \phi) = p\}, \quad (1.11)$$

¹In dimension 2 the space \dot{H}^1 requires a bit of cautiousness, see definition 2.1

it should satisfy the following Euler-Lagrange equations where $\tilde{g} := \tilde{G}'$

$$\exists c : \begin{cases} -c\partial_1\rho + \operatorname{div}(\chi(\rho)\nabla\phi) = 0, \\ -c\partial_1\phi + \chi'(\rho)\frac{|\nabla\phi|^2}{2} - K(\chi(\rho))\Delta\rho - \frac{(K \circ \chi)'|\nabla\rho|^2}{2} + \tilde{g}(\rho) = 0. \end{cases} \quad (1.12)$$

Since a solution of (1.12) such that $\|\rho - 1\|_\infty \ll 1$ is a solution of (1.7), our approach will be to prove that for p small enough, there exists existence of a solution to (1.11), the minimizer is smooth and satisfies $\|\rho - 1\|_\infty \ll 1$. We can now give a precise statement of our result, that we chose to state for the non-rescaled variables in order to underline the role of the physical variables.

Theorem 1.1. *Let $\rho_0 \in \mathbb{R}^{+*}$ such that $g'(\rho_0) > 0$, for $p > 0$ we set*

$$\tilde{E}_{\min}(p) := \inf_{(\rho, \phi) \in (\rho_0 + H^1) \times \dot{H}^1, P(\rho, \phi) = p} \tilde{E}(\rho, \phi). \quad (1.13)$$

Under the assumption $\Gamma := 3 + \frac{\rho_0 g''(\rho_0)}{g'(\rho_0)} \neq 0$, there exists $p_0 > 0$ such that for any $0 \leq p \leq p_0$, the infimum is attained at a minimizer $(\rho_p, \phi_p) \in \cap_{j \geq 0} (\rho_0 + H^j) \times \dot{H}^j$, such that (ρ_p, ϕ_p) is a solution of (1.6) for some $c_p > 0$. Moreover let $c_s = \sqrt{\rho_0 g'(\rho_0)}$, then

$$\exists \alpha, \beta > 0 : \forall 0 \leq p \leq p_0, \quad c_s p - \beta p^3 \leq \tilde{E}_{\min}(p) = E(\rho_p, \phi_p) \leq c_s p - \alpha p^3, \quad (1.14)$$

$$c_s - \beta p^2 \leq c_p \leq c_s - \alpha p^2. \quad (1.15)$$

Remark 2. It is not clear if (ρ_p, ϕ_p) is a constrained minimizer of E .

Remark 3. The assumption $\Gamma \neq 0$ is not technical. In the case of NLS it appears in the recent paper [17] as necessary and sufficient for the strict concavity of $E_{NLS, \min}$ near 0, a condition which is important, if not unavoidable, for the construction of minimizers. If $\Gamma = 0$ scattering of solutions of (1.1) for small data seems expectable but remains so far open, even for NLS.

Idea of proof We first point out that (contrary to the NLS case), it is not easy to get elliptic regularity from equations (1.7), indeed they basically look like $\Delta f = |\nabla f|^2$, and the argument $f \in H^1 \Rightarrow |\nabla f|^2 \in L^1 \Rightarrow f \in W^{2,1} \hookrightarrow H^1$ does not allow to bootstrap trivially regularity. But since the failure is somewhat “critical”, working with $\|(\rho, \phi)\|_{\mathcal{H}} \ll 1$ allows to overcome this issue (this is done in proposition 2.3).

The major issue is of course the defect of compactness in \mathbb{R}^2 . In the spirit of [10], this is overcome by first solving the minimization problem on large tori $\mathbb{T}_n^2 = (\mathbb{R}/2\pi n\mathbb{Z})^2$, on which thanks to the compact embedding $H^1 \hookrightarrow L^2$ the existence of a minimizer is easy. In order to handle smoothness issues, we follow a regularization procedure: we use a “mollified energy”

$$\tilde{E}_\varepsilon^n(\rho, \phi) = \tilde{E} + \frac{\varepsilon}{2} \int_{\mathbb{T}_n^2} |\Delta\phi|^2 + |\Delta\rho|^2 dx$$

and prove that minimizers of small energy satisfy elliptic estimates independent of ε (smallness is essential for this step). Letting $\varepsilon \rightarrow 0$, this provides a solution (ρ_n, ϕ_n) of the minimization problem on \mathbb{T}_n^2 . Next we let $n \rightarrow \infty$, and prove the convergence of (ρ_n, ϕ_n) to (ρ, ϕ) , solution of the constrained minimization problem (1.11).

The main tool to get some compactness of the sequence (ρ_n, ϕ_n) is the strict concavity² of the minimal energy $\tilde{E}_{\min}(p)$ which is obtained by mixing general abstract arguments and *ad hoc* computations. To get a feeling of how concavity is used, consider the following simplified version of dichotomy in Lions's concentration compactness principle: assume that instead of converging to a minimizer, (ρ_n, ϕ_n) splits in two parts, namely there exists functions (ρ^1, ϕ^1) , (ρ^2, ϕ^2) such that $\tilde{E}_{\min}^n(\rho_n, \phi_n) \rightarrow \tilde{E}(\rho^1, \phi^1) + \tilde{E}(\rho^2, \phi^2)$, $P(\rho_n, \phi_n) \rightarrow P(\rho^1, \phi^1) + P(\rho^2, \phi^2)$. Then passing to the limit in n we have

$$\begin{aligned} P(\rho^1, \phi^1) + P(\rho^2, \phi^2) &= p_1 + p_2 = p, \\ \tilde{E}_{\min}(p_1 + p_2) &= \tilde{E}(\rho^1, \phi^1) + \tilde{E}(\rho^2, \phi^2) \geq \tilde{E}_{\min}(p_1) + \tilde{E}_{\min}(p_2). \end{aligned}$$

On the other hand, by strict subadditivity $\tilde{E}_{\min}(p_1 + p_2) < \tilde{E}_{\min}(p_1) + \tilde{E}_{\min}(p_2)$, which is a contradiction. For a remarkably clear and general discussion on this strategy, we refer to the seminal paper of P.L. Lions [27].

Plan of the paper The rest of the article is organized as follows: in section 2, we prove a key elliptic estimate for solutions of (1.12) in a simple case, in section 3 we establish the concavity of \tilde{E}_{\min} and the upper bound $\tilde{E}_{\min} \leq p - \alpha p^3$ from which we deduce its strict subadditivity. These sections are preliminaries to section 4 where we prove the existence of solutions to the minimization problem (1.11): we first study the minimization problem on \mathbb{T}_n^2 for fixed n . We obtain the existence of constrained minimizers for the mollified energy \tilde{E}_ε^n , from which we deduce the existence of smooth minimizers for the nonregularized problem. Letting $n \rightarrow \infty$, we obtain the convergence of minimizers on \mathbb{T}_n^2 to a minimizer on \mathbb{R}^2 with a concentration compactness argument. Finally we complete the a priori estimates of \tilde{E}_{\min} and c thanks to Pohozaev type identities in section 5. The concentration compactness argument relies on a kind of profile decomposition essentially similar to the one in [10], for completeness we prove its existence in the appendix A. In appendix B we discuss the one-dimensional case, where explicit computations allow to observe very strong similarities with one-dimensional NLS, and prove a new nonlinear instability property of some solitary waves.

Notations If $a \leq Cb$, $a, b, C > 0$, C a constant independent of the parameters, we write $a \lesssim b$. If $C_1 a \leq b \leq C_2 a$ with C_1, C_2 positive constants, we write $a \sim b$ if there is no ambiguity with the usual meaning of \sim . We denote the Fourier transform of an application ϕ as $\hat{\phi}$. As mentioned in the introduction, $\mathcal{H} = \{(\rho, \phi) \in (1 + H^1) \times \dot{H}^1\}$.

²Actually the key is not concavity, but a consequence: strict subadditivity.

2 An elliptic estimate

We first clarify our functional framework.

Definition 2.1. *The space \dot{H}^1 is the set of $\phi \in L^2_{loc}$ such that $\nabla\phi \in L^2$ in the distributional sense. We define $L^2_{curl} := \nabla\dot{H}^1$ with norm $\|\nabla\phi\|_{L^2}$.*

We shall need the following standard density result.

Proposition 2.2. *L^2_{curl} coincides with $\{u \in L^2 : \text{curl}(u) = 0\}$, and thus is a Hilbert space, in which $\nabla C_c^\infty(\mathbb{R}^2)$ is dense.*

Proof. For the first part see e.g. [30] or [31]. For the second part it suffices to check $(\nabla C_c^\infty)^\perp = \{0\}$. If $\nabla\phi \in (\nabla C_c^\infty)^\perp$ then $\Delta\phi = 0$ (\mathcal{D}'), and $\nabla\phi \in L^2$ thus $\Delta\phi = 0$ (\mathcal{S}'). Thus $|\xi|^2\hat{\phi} = 0$. This implies that $\hat{\phi}$ is a linear combination the Dirac distribution at 0 and its first order derivatives, equivalently ϕ is a first order polynomial. The condition $\nabla\phi \in L^2$ then implies that ϕ is a constant, so that $u = 0$. \square

Proposition 2.3. *Let $M > 0$, (ρ, ϕ) a solution of (1.12) with $|c| \leq M$. There exists $\varepsilon, C > 0$ depending only on M^3 such that*

$$(\rho, \nabla\phi) \in 1 + H^2 \times H^1 \text{ and } \tilde{E}(\rho, \phi) < \varepsilon \Rightarrow \|\rho - 1\|_\infty < C\sqrt{\tilde{E}(\rho, \phi)}.$$

In particular for $\tilde{E}(\rho, \phi)$ small enough, a (smooth) solution of (1.12) is a traveling wave of the Euler-Korteweg system.

Proof. Setting $u = \nabla\phi$ and denoting χ for $\chi(\rho)$ we have from the first equation

$$\Delta\phi + \frac{\nabla\chi \cdot u}{\chi} - c \frac{\partial_1\rho}{\chi} = 0 \Rightarrow \Delta u + \nabla(\nabla \ln(\chi) \cdot u) - c\nabla\left(\frac{\partial_1\rho}{\chi}\right) = 0$$

Taking the gradient of the equation, the scalar product with u and integrating, we get

$$\int |\nabla u|^2 + \left(\nabla \ln \chi \cdot u - c \frac{\partial_1\rho}{\chi} \right) \text{div} u \, dx = 0$$

so that from Cauchy-Schwarz's inequality

$$\frac{1}{2} \|\nabla u\|_2^2 \leq 2 \|\nabla \ln \chi \cdot u\|_2^2 + 2c \|\partial_1\rho/\chi\|_2^2 \leq 2 \|\nabla \ln \chi\|_4^2 \|u\|_4^2 + 2c \|\partial_1\rho/\chi\|_2^2.$$

If $d = 2$, we use $\|u\|_4 \lesssim \|u\|_{\dot{H}^{1/2}} \lesssim \|u\|_2^{1/2} \|\nabla u\|_2^{1/2}$ so that

$$\begin{aligned} \|u\|_4^2 &\leq C \|u\|_2 \|\nabla u\|_2 \leq \sqrt{2} C \|u\|_2 (\|\nabla \ln \chi\|_4 \|u\|_4 + c \|\partial_1\rho/\chi\|_2), \\ \Rightarrow \|u\|_4^2 &\leq \frac{C}{\sqrt{2}} \|u\|_2^2 \|\nabla \ln \chi\|_4^2 + MC\sqrt{2} \|u\|_2 \|\partial_1\rho/\chi\|_2 \\ &\lesssim \|u\|_2^2 \|\nabla \rho\|_4^2 + \|u\|_2 \|\partial_1\rho\|_2, \end{aligned}$$

³of course it depends also on $K \circ \chi$ and \tilde{G} but it does not matter for the analysis.

where we used $\|\chi'\nabla\rho\|_2 \lesssim \|\nabla\rho\|_2$. Next we rewrite the momentum equation as

$$\Delta\rho = \frac{-c\partial_1\rho}{K \circ \chi} + \frac{\chi'}{2K \circ \chi}|\nabla\phi|^2 - \frac{(K \circ \chi)'}{2K \circ \chi}|\nabla\rho|^2 + \frac{\tilde{g}}{K \circ \chi}.$$

Since K is smooth, positive on $]0, \infty[$, $(K \circ \chi)'$ and $1/K \circ \chi$ are uniformly bounded, and from (1.9) taking the L^2 norm gives

$$\begin{aligned} \|\Delta\rho\|_2 &\lesssim \|\partial_1\rho\|_2 + \|\nabla\phi\|_4^2 + \|\nabla\rho\|_4^2 + \|\rho - 1\|_2 \\ &\lesssim \|\partial_1\rho\|_2 + \|\partial_1\rho\|_2\|u\|_2 + \|\nabla\rho\|_4^2 + \|u\|_2^2\|\nabla\rho\|_4^2 + \|\rho - 1\|_2. \end{aligned}$$

Next we use again Sobolev's embedding $\|\nabla\rho\|_4^2 \lesssim \|\nabla\rho\|_2\|\Delta\rho\|_2$ which gives

$$\|\Delta\rho\|_2 \leq C(\|\partial_1\rho\|_2 + \|\partial_1\rho\|_2\|u\|_2 + \|\rho - 1\|_2 + (1 + \|u\|_2^2)\|\nabla\rho\|_2\|\Delta\rho\|_2).$$

We recall that $\tilde{E}(\rho, \phi) \gtrsim \|\rho - 1\|_{H^1}^2 + \|\nabla\phi\|_2^2$, so that if $\tilde{E}(\rho, \phi)$ is small enough, $C(1 + \|u\|_2^2)\|\nabla\rho\|_2 < 1/2$ and we deduce

$$\|\Delta\rho\|_2 \leq C(\|\partial_1\rho\|_2 + \|\partial_1\rho\|_2\|u\|_2 + \|\rho - 1\|_2) \lesssim \sqrt{\tilde{E}(\rho, \phi)}.$$

From Sobolev's embedding we conclude $\|\rho - 1\|_\infty \lesssim \|\rho - 1\|_{H^2} \lesssim \sqrt{\tilde{E}(\rho, \phi)}$. In particular if the energy is small enough $\chi(\rho) = \rho$, $\tilde{G}(\rho) = G(\rho)$ and ρ is a solution of (1.7). \square

3 Properties of the energy

We recall $\tilde{E}_{\min}(p) = \inf_{P(\rho, \phi)=p} \tilde{E}(\rho, \phi)$. We start with some properties that are true in generic minimization settings (continuity, concavity of \tilde{E}_{\min}) before tackling the strict subadditivity of \tilde{E}_{\min} , where we use the structure of \tilde{E} and P .

Lemma 3.1. *For any $p \geq 0$, there exists a minimising sequence $(\rho_n, \nabla\phi_n) \in (1 + C_c^\infty(\mathbb{R}^2)) \times C_c^\infty(\mathbb{R}^2)$.*

Proof. The case $p = 0$ is obvious. For $p > 0$ it suffices to prove that for any $(\rho - 1, \nabla\phi) \in H^1 \times L^2$, there exists $(\rho_n - 1, \phi_n) \in (C_c^\infty(\mathbb{R}^2))^2$ such that $P(\rho_n, \phi_n) = p$, $\tilde{E}(\rho_n, \phi_n) \rightarrow \tilde{E}(\rho, \phi)$. By density (prop. 2.2), there exists $(r_n - 1, \nabla\psi_n) \in C_c^\infty(\mathbb{R}^2)$ such that

$$\|r_n - \rho\|_{H^1} + \|\nabla\psi_n - \nabla\phi\|_{L^2} \rightarrow_n 0, \quad r_n \rightarrow \rho \text{ a.e. .}$$

Clearly $P(r_n, \psi_n) \rightarrow p$, $\int_{\mathbb{R}^2} \tilde{G}(r_n) \rightarrow \int_{\mathbb{R}^2} \tilde{G}(\rho)dx$, and up to an extraction such that $r_n \rightarrow \rho$ a.e. we have by dominated convergence

$$\begin{aligned} \int_{\mathbb{R}^2} \chi(r_n)|\nabla\psi_n|^2 - \chi(\rho)|\nabla\phi|^2 dx &= \int_{\mathbb{R}^2} (\chi(r_n) - \chi(\rho))|\nabla\phi|^2 + \chi(r_n)(|\nabla\phi|^2 - |\nabla\psi_n|^2)dx, \\ &\rightarrow_n 0, \end{aligned}$$

$$\int_{\mathbb{R}^2} K(\chi(r_n))|\nabla r_n|^2 - K(\chi(\rho))|\nabla \rho|^2 dx \rightarrow_n 0,$$

from which we deduce $E(r_n, \psi_n) - E(\rho, \phi) \rightarrow 0$. Let $\varepsilon_n = p - P(r_n, \psi_n)$, we construct a slight modification (ρ_n, ϕ_n) of (r_n, ψ_n) such that $P(\rho_n, \phi_n) = p$: let $\varphi \in C_c^\infty(\mathbb{R}^2)$, $A := \partial_1 \varphi$ with $\|\partial_1 \varphi\|_2 = 1$. Up to a translation (that depends on n), we can assume $\text{supp}(\varphi) \cap (\text{supp}(1 - r_n) \cup \text{supp}(\psi_n)) = \emptyset$, and define

$$\rho_n = r_n + \text{sign}(\varepsilon_n) \sqrt{|\varepsilon_n|} A, \quad \phi_n = \psi_n + \sqrt{|\varepsilon_n|} \varphi.$$

We conclude

$$\begin{aligned} P(\rho_n, \phi_n) &= P(r_n, \psi_n) + \varepsilon_n = p, \\ \tilde{E}(\rho_n, \phi_n) &= \tilde{E}(r_n, \psi_n) + \frac{1}{2} \varepsilon_n \int_{\mathbb{R}^2} K(\chi(\rho_n)) |\nabla A|^2 \\ &\quad + \chi(\rho_n) |\nabla \varphi|^2 + O(A^2) dx \\ &= \tilde{E}(r_n, \psi_n) + O(\varepsilon_n) \rightarrow_n \tilde{E}(\rho, \phi). \end{aligned}$$

□

Proposition 3.2. *The application $p \in \mathbb{R}^+ \mapsto \tilde{E}_{\min}(p)$ is 1-Lipschitz, concave, non decreasing.*

Proof. We split the proof in three steps:

\tilde{E}_{\min} is Lipschitz Let $p < q$, $\delta > 0$ to be fixed, according to lemma 3.1, there exists $(\rho - 1, \phi) \in (C_c^\infty)^2$ such that $P(\rho, \phi) = p$, $\tilde{E}(\rho, \phi) \leq \tilde{E}_{\min}(p) + \delta$. Combining proposition 3.3 and lemma 3.1 there exists $(\rho_0 - 1, \phi_0) \in (C_c^\infty(\mathbb{R}^2))^2$ such that $P(\rho_0, \phi_0) = q - p$, $\tilde{E}(\rho_0, \phi_0) \leq q - p$. Up to a translation, we can assume $(\rho_0 - 1, \phi_0)$ have disjoint support with $(\rho - 1, \phi)$, so that

$$P(\rho + \rho_0 - 1, \phi + \phi_0) = p + q - p = q, \quad \tilde{E}(\rho + \rho_0 - 1, \phi + \phi_0) \leq \tilde{E}_{\min}(p) + \delta + q - p.$$

Since δ is arbitrary, $\tilde{E}_{\min}(q) - \tilde{E}_{\min}(p) \leq q - p$. The reverse inequality can be obtained with a similar argument (using $-\phi_0$ instead of ϕ_0).

\tilde{E}_{\min} is concave Since \tilde{E}_{\min} is continuous, it suffices to prove that for any $p_1 < p_2 \in [0, p_0]$, $\tilde{E}_{\min}((p_1 + p_2)/2) \geq \frac{\tilde{E}_{\min}(p_1) + \tilde{E}_{\min}(p_2)}{2}$. This relies on a classical reflection argument. For f defined on \mathbb{R}^2 , we define $T_a^+(f)$ (resp $T_a^-(f)$) as the function symmetric with respect to the line $x_2 = a$ and that coincides with f on $x_2 > a$ (resp. $x_2 < a$). The maps T_a^+, T_a^- are linear continuous $H^1 \rightarrow H^1$, and from Lebesgue's dominated convergence theorem $T_a^+ \rightarrow_{+\infty} 0$, $T_a^- \rightarrow_{-\infty} 0$, $a \mapsto T_a^\pm$ is continuous. This also implies

$$\|T_a^+ f\|_{L^2} \rightarrow_{-\infty} 2\|f\|_{L^2}, \quad \|\nabla T_a^+ f\|_{L^2} \rightarrow_{-\infty} 2\|\nabla f\|_{L^2},$$

and the symmetric property for T_a^- . We also note that for any function F , as soon as the integrals make sense

$$\int_{\mathbb{R}^2} F(T_a^+ f) + F(T_a^- f) dx = 2 \int_{\mathbb{R}^2} F(f) dx. \quad (3.1)$$

Now let $\delta > 0$, (ρ, ϕ) be such that $P(\rho, \phi) = \frac{p_1 + p_2}{2}$, $\tilde{E}(\rho, \phi) \leq \tilde{E}_{\min}\left(\frac{p_1 + p_2}{2}\right) + \delta$. Since $\lim_{+\infty} \|P(T_a^+(\rho, \phi))\|_2 = 0 < p_1$, there exists a_1 such that $P(T_{a_1}^+(\rho, \phi)) = p_1$, and from (3.1), $P(T_{a_1}^-(\rho, \phi)) = p_2$. Then using again (3.1)

$$\tilde{E}_{\min}(p_1) + \tilde{E}_{\min}(p_2) \leq \tilde{E}(T_{a_1}^+(\rho, \phi)) + \tilde{E}(T_{a_1}^-(\rho, \phi)) \leq 2\tilde{E}_{\min}\left(\frac{p_1 + p_2}{2}\right) + 2\delta.$$

Since δ is arbitrary, we get $\tilde{E}_{\min}((p_1 + p_2)/2) \geq \frac{\tilde{E}_{\min}(p_1) + \tilde{E}_{\min}(p_2)}{2}$.

$\tilde{E}_{\min}(p)$ is non decreasing Obvious since it is concave and nonnegative. \square

The next proposition gives a sharp upper bound for \tilde{E}_{\min} .

Proposition 3.3. *There exists $p_0 > 0$, $\alpha > 0$ such that*

$$\forall 0 < p < p_0, \exists (\rho_p, \phi_p) \in \mathcal{H} : P(\rho_p, \phi_p) = p, \tilde{E}(\rho_p, \phi_p) \leq p - \alpha p^3.$$

In particular $\tilde{E}_{\min}(p) \leq p - \alpha p^3$. Moreover, up to taking a smaller p_0 if (ρ, ϕ) is a minimiser, then $\|\rho - 1\|_{\infty} \gtrsim p^2$.

Proof. The idea is to construct an approximate minimizer by using the following formal asymptotic (rigorously justified for the Gross-Pitaevskii equation [9]): set $\rho = 1 + \varepsilon^2 A_{\varepsilon}(z_1, z_2)$, $\phi = \varepsilon \varphi_{\varepsilon}(z_1, z_2)$, $z_1 = \varepsilon x_1$, $z_2 = \varepsilon^2 x_2$. If (ρ, ϕ) is a solution of (1.7) with speed $c = \sqrt{1 - \varepsilon^2}$, the mass conservation reads

$$-c \partial_1 A_{\varepsilon} + \partial_1^2 \varphi_{\varepsilon} + \varepsilon^2 (\partial_2^2 \varphi_{\varepsilon} + A_{\varepsilon} \partial_1^2 \varphi_{\varepsilon} + \partial_1 A_{\varepsilon} \partial_1 \varphi_{\varepsilon}) = O(\varepsilon^4).$$

Next using Taylor's expansion $\tilde{g} = \varepsilon^2 A_{\varepsilon} + g''(1) \varepsilon^4 \frac{A_{\varepsilon}^2}{2} + O(\varepsilon^4)$, the momentum equation gives

$$-c \partial_1 \varphi_{\varepsilon} + A_{\varepsilon} + \varepsilon^2 \left(\frac{g''(1) A_{\varepsilon}^2 + (\partial_1 \varphi_{\varepsilon})^2}{2} - K(1) \partial_1^2 A_{\varepsilon} \right) = O(\varepsilon^4).$$

At first order, we have $\partial_1 \phi_{\varepsilon} = A_{\varepsilon} + O(\varepsilon^2)$, next if we multiply the mass equation by c , apply ∂_1 to the momentum equation and add them, we get

$$\partial_1 A_{\varepsilon} + \partial_2^2 \partial_1^{-1} A_{\varepsilon} + (3 + g''(1)) A_{\varepsilon} \partial_1 A_{\varepsilon} - K(1) \partial_1^3 A_{\varepsilon} = O(\varepsilon^2), \quad (3.2)$$

Note that $\gamma := 3 + g''(1)$ is the rescaled version of $\Gamma = 3 + \rho_0 g''(\rho_0)/g'(\rho_0)$ thus by assumption $\gamma \neq 0$. (3.2) is a KP1 type equation, the normalized KP1 equation is

$$\partial_1 w + w \partial_1 w + \partial_2^2 \partial_1^{-1} w - \partial_1^3 w = 0 \quad (3.3)$$

One can pass from a solution of (3.3) to a solution of (3.2) by setting

$$A = \frac{1}{\gamma} w(x_1/\sqrt{K(1)}, x_2/\sqrt{K(1)}). \quad (3.4)$$

In [19], solutions of the KP equation are constructed, any such solution satisfy

$$E_{KP}(w) := \frac{1}{2} \int_{\mathbb{R}^2} |\partial_2 \partial_1^{-1} w|^2 + w^3/3 + |\partial_1 w|^2 dx < \infty, \quad (3.5)$$

Moreover, such solutions are smooth, belong to L^q for any $1 < q \leq \infty$ as well as their gradients, there exists a smooth $v \in L^p$ for any $2 < p \leq \infty$ such that $\partial_1 v = w$, $\nabla v \in L^q$, and $E_{KP}(w) = -\int_{\mathbb{R}^2} w^2/6 < 0$ (see [19] or [10] p.41). Let w be such a solution ⁴, we define A by (3.4), and set $\rho = 1 + \varepsilon^2 A(\varepsilon x_1, \varepsilon^2 x_2)$, $\phi = \varepsilon \partial_1^{-1} A(\varepsilon x_1, \varepsilon^2 x_2)$. Since A is bounded, $|\rho - 1| = O(\varepsilon^2)$ so that E and \tilde{E} coincide for ε small enough. We have $E_{\widetilde{KP}}(A) = \frac{K(1)}{\gamma^2} E_{KP}(w) < 0$, and basic computations give

$$P(\rho, \phi) = \varepsilon^4 \int A^2(\varepsilon x_1, \varepsilon^2 x_2) dx = \varepsilon \|A\|_2^2,$$

$$\begin{aligned} \tilde{E}(\rho, \phi) &= \frac{1}{2} \int_{\mathbb{R}^2} (1 + \varepsilon^2 A)(\varepsilon^4 A^2 + \varepsilon^6 (\partial_2 \partial_1^{-1} A)^2) + K(1 + \varepsilon^2 A)(\varepsilon^6 (\partial_1 A)^2 + \varepsilon^8 (\partial_2 A)^2) \\ &\quad + \varepsilon^4 A^2(\varepsilon x_1, \varepsilon^2 x_2) + (2G(1 + \varepsilon^2 A) - \varepsilon^4 A^2) dx \\ &\leq \varepsilon \|A\|_2^2 + \frac{\varepsilon^3}{2} \int_{\mathbb{R}^2} (\partial_2 \partial_1^{-1} A)^2 + A^3 + K(1)(\partial_1 A)^2(z_1, z_2) + \frac{(\max \tilde{G}''') A^3}{3} dz + R \end{aligned}$$

where $R = \frac{\varepsilon^5}{2} \int A(\partial_2 \partial_1^{-1} A)^2 + \frac{K(1 + \varepsilon^2 A) - 1}{\varepsilon^2} (\partial_1 A)^2 + K(1 + \varepsilon^2 A)(\partial_2 A)^2 dx = O(\varepsilon^5)$. As a consequence by definition of $\tilde{E}_{\min}(p)$

$$\tilde{E}_{\min}(\varepsilon \|A\|_2^2) \leq \tilde{E}(\rho_\varepsilon, \phi_\varepsilon) \leq \varepsilon \|A\|_2^2 + \varepsilon^3 E_{\widetilde{KP}}(A) + C\varepsilon^5,$$

taking $\varepsilon \leq \sqrt{-E_{\widetilde{KP}}/(2C)}$ completes the first part of the proof.

Now if (ρ, ϕ) is a minimiser, from $\int_{\mathbb{R}^2} \tilde{G}(\rho) = (1 + O(\|\rho - 1\|_\infty)) \int_{\mathbb{R}^2} \frac{(\rho-1)^2}{2} dx$ we have

$$\begin{aligned} p = \int_{\mathbb{R}^2} (\rho - 1) \partial_1 \phi &\leq \frac{1}{\inf_{\mathbb{R}^2} \sqrt{\chi(\rho)}} \int_{\mathbb{R}^2} \frac{(\rho - 1)^2 + \chi |\nabla \phi|^2}{2} dx \leq \frac{(1 + O(\|\rho - 1\|_\infty)) \tilde{E}_{\min}(p)}{\inf \sqrt{\chi}} \\ &\leq \frac{(1 + O(\|\rho - 1\|_\infty))(p - \alpha p^3)}{\inf \sqrt{\chi}}. \end{aligned}$$

⁴some optimization can be done by choosing a so-called ground state, but it is not really useful here.

There are two possibilities:

- if $\inf \sqrt{\chi} \leq 1 - \alpha p^2/2$, then $\inf \sqrt{\rho} \leq 1 - \alpha p^2$
- else $1 + O(\|\rho - 1\|_\infty) \geq \inf \sqrt{\chi}/(1 - \alpha p^2) \geq 1 + \alpha p^2/2 + O(p^4)$, then $\|\rho - 1\|_\infty \gtrsim p^2$.

□

As pointed out in the introduction, rather than concavity we will use subadditivity:

Proposition 3.4. *The application $\tilde{E}_{\min} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfies the following properties:*

1. *it is differentiable at $p = 0$ and $\tilde{E}'_{\min}(0) = 1$.*
2. *it is strictly subadditive : $\forall 0 < p_1, p_2$, $\tilde{E}_{\min}(p_1 + p_2) < \tilde{E}_{\min}(p_1) + \tilde{E}_{\min}(p_2)$.*
3. *the application*

$$(p_1, p_2) \in (\mathbb{R}^+)^2 \rightarrow D(p_1, p_2) := \tilde{E}_{\min}(p_1) + \tilde{E}_{\min}(p_2) - \tilde{E}_{\min}(p_1 + p_2)$$

is nonnegative and nondecreasing in both p_1 and p_2 . Moreover

$$p_1, p_2 > 0 \Rightarrow D(p_1, p_2) > 0. \quad (3.6)$$

Proof. 1. From proposition 3.3 we have $\overline{\lim}_0 \frac{\tilde{E}_{\min}(p)}{p} \leq 1$. Conversely, consider a sequence $p_n \rightarrow 0$, and pick approximate minimizers (ρ_n, ϕ_n) such that

$$\forall n \geq 1, P(\rho_n, \phi_n) = p_n, \tilde{E}(\rho_n, \phi_n) \leq \tilde{E}_{\min}(p_n)(1 + 1/n).$$

Then $\|\rho_n - 1\|_{H^1}^2 + \|\nabla \phi_n\|_{L^2}^2 \sim \tilde{E}(\rho_n, \phi_n) \rightarrow_n 0$ and from Young's inequality

$$p_n = P(\rho_n, \phi_n) \leq \int_{\mathbb{R}^2} \frac{(\rho_n - 1)^2}{2\chi(\rho_n)} + \frac{\chi|\nabla \phi_n|^2}{2} dx$$

Since $\tilde{G}''(1) = 1$, $\tilde{G}'(1) = \tilde{G}(1) = 0$ and $\tilde{G} = O(\rho - 1)^2$, from Taylor's expansion we have $\tilde{G}(\rho) - \frac{(\rho-1)^2}{2\chi} = O(\rho - 1)^3$. Combining it with Sobolev's embedding $H^1 \hookrightarrow L^3$

$$\begin{aligned} p_n &\leq \int_{\mathbb{R}^2} \tilde{G}(\rho_n) + \frac{\chi|\nabla \phi_n|^2}{2} + O(\rho_n - 1)^3 dx &\leq \tilde{E}(\rho_n, \phi_n) + O(\tilde{E}(\rho_n, \phi_n))^{3/2} \\ & &\leq \tilde{E}_{\min}(p_n)(1 + O(\tilde{E}_{\min}(p_n))^{1/2})(1 + 1/n) \end{aligned}$$

This readily implies $\underline{\lim}_n \frac{\tilde{E}_{\min}(p_n)}{p_n} \geq 1$, and 1) is thus true.

2. This is a basic concavity argument. First we remark that \tilde{E}_{\min} is not linear on any interval $[0, p]$, since $\tilde{E}'_{\min}(0) = 1$ and $\tilde{E}_{\min}(p) < p$. Assume there exists $p_1 \leq p_2$ such that $\tilde{E}_{\min}(p_1 + p_2) =$

$\tilde{E}_{\min}(p_1) + \tilde{E}_{\min}(p_2)$. For a unified treatment, if $p_1 = p_2$ we write $\frac{\tilde{E}_{\min}(p_2) - \tilde{E}_{\min}(p_1)}{p_2 - p_1}$ for the right derivative of \tilde{E}_{\min} . By concavity and using $\tilde{E}_{\min}(0) = 0$

$$\frac{\tilde{E}_{\min}(p_1)}{p_1} \geq \frac{\tilde{E}_{\min}(p_2) - \tilde{E}_{\min}(p_1)}{p_2 - p_1} \geq \frac{\tilde{E}_{\min}(p_1 + p_2) - \tilde{E}_{\min}(p_2)}{p_1 + p_2 - p_2} = \frac{\tilde{E}_{\min}(p_1)}{p_1}.$$

Therefore for any $p \in [p_1, p_1 + p_2]$, $\frac{\tilde{E}_{\min}(p) - \tilde{E}_{\min}(p_1)}{p - p_1} = \frac{\tilde{E}_{\min}(p_1)}{p_1} \Leftrightarrow \tilde{E}_{\min}(p) = \frac{\tilde{E}_{\min}(p_1)}{p_1}p$. Also

$$\begin{aligned} \forall p \in [0, p_1], \quad \frac{\tilde{E}_{\min}(p_1)}{p_1} &= \frac{\tilde{E}_{\min}(p_1 + p_2) - \tilde{E}_{\min}(p_1)}{p_1 + p_2 - p_1} \leq \frac{\tilde{E}_{\min}(p_1) - \tilde{E}_{\min}(p)}{p_1 - p} \leq \frac{\tilde{E}_{\min}(p_1)}{p_1} \\ &\Rightarrow \frac{\tilde{E}_{\min}(p_1)}{p_1}(p) = \frac{\tilde{E}_{\min}(p_1)}{p_1}p, \end{aligned}$$

Hence, \tilde{E}_{\min} is linear on $[0, p_1 + p_2]$, this is a contradiction.

3. Direct consequence of the subadditivity, and the fact that for any concave function f , $x \rightarrow \frac{f(x+p_2) - f(x)}{p_2}$ is decreasing. \square

Remark 4. The better lower bound $\tilde{E}_{\min}(p) \geq p - \beta p^3$ is based on some Pohozaev's identities, that in turn require the existence of minimizers, therefore their proof is postponed to section 5.

4 Existence of minimizers

The existence is obtained by following the procedure in [10], which is the following

- If one replaces \mathbb{R}^2 by the torus $\mathbb{T}_n^2 = \mathbb{R}^2 / (2n\pi\mathbb{Z})^2$, the existence of a minimiser to (1.11) for any p is easy thanks to elliptic estimates and compact embeddings.
- Any minimiser (ρ_n^p, ϕ_n^p) satisfies $\|\rho_n^p - 1\| \geq Cp^2$, with C independent of n (torus version of proposition 3.3).
- Letting $n \rightarrow \infty$, up to translation and extraction (ρ_n^p, ϕ_n^p) converges to $(\rho^{\tilde{p}}, \phi^{\tilde{p}})$, which is a *non trivial* solution of equation (1.12) with $\tilde{E}(\rho^{\tilde{p}}, \phi^{\tilde{p}}) \leq \tilde{E}_{\min}(p)$.
- The sequence (ρ_n^p, ϕ_n^p) actually converges globally so that $P(\rho^p, \phi^p) = \lim_n P_n(\rho_n^p, \phi_n^p) = p$. This is the most difficult point, which requires a careful analysis of the difference between the energy density $K|\nabla\rho_n^p|^2 + \chi(\rho_n^p)|\nabla\phi_n^p|^2 + \tilde{G}(\rho_n)$ and the momentum density $(\rho_n - 1)\partial_1\phi_n$ on the “vanishing set” $|\rho_n - 1| \ll 1$.

We point out that one of the reasons why F. Bethuel, P. Gravejat and J.C. Saut used the preliminary minimization on the torus was the difficulty to define $P(\rho, \phi)$. This is not an issue

here, however working on the torus is essential to get strong a priori estimates and start a compactness procedure.

In order to use the (torus version of) elliptic estimate in proposition 2.3, we first use the smoothed energy

$$\tilde{E}_n^\varepsilon(\rho, \phi) := \frac{1}{2} \int_{\mathbb{T}_n^2} \rho |\nabla \phi|^2 + K(\chi(\rho)) |\nabla \rho|^2 + 2G(\rho) + \varepsilon((\Delta \phi)^2 + (\Delta \rho)^2) dx,$$

and the notation $(\tilde{E}_n^\varepsilon)_{\min}(p) := \inf_{P_n(\rho, \phi)=p} \tilde{E}_n^\varepsilon(\rho, \phi)$. We provide a collection of lemmas that mimick the situation on \mathbb{R}^2 , without the regularizing terms.

4.1 Minimizers on large tori

The first step is a very rough version of proposition 3.3.

Lemma 4.1. *There exists $M > 0$ such that for any $0 \leq p \leq 1$, $n \geq 5/p^2$, $0 \leq \varepsilon \leq 1$,*

$$(\tilde{E}_n^\varepsilon)_{\min}(p) \leq Mp.$$

Proof. We start with an ansatz similar to proposition 3.3: let $\theta \in C_c^\infty(\mathbb{R}^2)$ such that $\text{supp}(\theta) \subset B(0, 5)$, $\|\partial_1 \theta\|_2 = 1$, $\|\partial_1 \theta\|_\infty \leq 1/2$, set $A = \partial_1 \theta$ and $\rho = 1 + p^2 A(px_1, p^2 x_2)$, $\phi = p\theta(px_1, p^2 x_2)$. Since $\text{supp}(\rho) \cup \text{supp}(\phi) \subset B(0, 5/p^2)$, there is an obvious way to define them as functions $(\rho_n, \phi_n) \in C^\infty(\mathbb{T}_n^2)$ for any $n \geq 5/p^2$. Next using $p \leq 1$, $\chi(\rho) = \rho$, basic computations give

$$\begin{aligned} P_n(\theta, \varphi) &= p, \\ \tilde{E}_n^\varepsilon(\rho_n, \phi_n) &= \int_{\mathbb{R}^2} \frac{1 + p^2 A(z_1, z_2)}{2} (p(\partial_1 \theta)^2 + p^3(\partial_2 \theta)^2) + \frac{K(\rho)}{2} (p^3(\partial_1 A)^2 + p^5(\partial_2 A)^2 + \tilde{G} \\ &\quad + \frac{\varepsilon}{2} (p^3(\partial_1^2 \theta + p^2 \partial_2^2 \theta)^2 + p^5(\partial_1^2 A + \partial_2^2 A)^2) dz \\ &\leq p \left(\frac{3}{4} (1 + \|\partial_2 \theta\|_2^2) + \max_{[1/2, 3/2]} K \frac{\|\partial_1 A\|_2^2 + \|\partial_2 A\|_2^2}{2} + \frac{\max \tilde{G}''}{2} \right. \\ &\quad \left. + \|\partial_1^2 \theta\|_2^2 + \|\partial_2^2 \theta\|_2^2 + \|\partial_1^2 A\|_2^2 + \|\partial_2^2 A\|_2^2 \right) \end{aligned}$$

The constant in factor of p is clearly independent of p, n, ε . □

Lemma 4.2. *For any $p > 0$, $n \geq 1$, $\varepsilon > 0$ the minimization problem*

$$\inf \left\{ \tilde{E}_n^\varepsilon(\rho, \phi) \text{ with } (\rho - 1, \nabla \phi) \in H^2 \times H^1, P_n(\rho, \phi) = \int_{\mathbb{T}_n^2} (\rho - 1) \partial_1 \phi dx = p \right\}.$$

admits a minimizer $(\rho_n^\varepsilon(p) - 1, \phi_n^\varepsilon(p)) \in H^2 \times H^2$, solution of

$$\begin{cases} -c_{n,\varepsilon} \partial_1 \rho_n^\varepsilon + \operatorname{div}(\chi(\rho_n^\varepsilon) \nabla \phi_n^\varepsilon) - \varepsilon \Delta^2 \phi_n^\varepsilon = 0, \\ -c_{n,\varepsilon} \partial_1 \phi_n^\varepsilon + \chi'(\rho_n^\varepsilon) \frac{|\nabla \phi_n^\varepsilon|^2}{2} - K(\chi(\rho_n^\varepsilon)) \Delta \rho_n^\varepsilon - \frac{(K \circ \chi)' |\nabla \rho_n^\varepsilon|^2}{2} + \tilde{g}(\rho_n^\varepsilon) + \varepsilon \Delta^2 \rho_n^\varepsilon = 0, \end{cases} \quad (4.1)$$

Moreover, there exists p_1, M such that for $p \leq p_1$, $n \geq 5/p^2$, $\varepsilon \leq 1$,

$$c_{n,\varepsilon} \leq M, \quad \|\rho_n^\varepsilon(p) - 1\|_{H^2} + \|\nabla \phi_n^\varepsilon(p)\|_{H^1} \leq Mp.$$

Furthermore for any $j \geq 2$, there exists $F_j(p) \rightarrow_0 0$ such that

$$\|(\rho_n^\varepsilon - 1, \nabla \phi_n^\varepsilon)\|_{H^j \times H^{j-1}} \leq F_j(p).$$

Proof. We follow the scheme of proof of proposition 2.3 with a few technical additions. For simplicity of notations, we drop the ε, n indices. If ρ_k, φ_k is a minimizing sequence, by weak compactness and proposition 2.2 we can assume $\rho_k - 1 \rightharpoonup \rho - 1$ (H^2), $\nabla \varphi_k \rightharpoonup \nabla \phi$ (H^1), and from Rellich's compact embedding we have

$$\begin{aligned} \lim_k \int_{\mathbb{T}_n^2} \rho_k |\nabla \phi_k|^2 + K(\chi(\rho_k)) |\nabla \rho_k|^2 + \tilde{G}(\rho_k) dx &= \int_{\mathbb{T}_n^2} \rho |\nabla \phi|^2 + K(\chi(\rho)) |\nabla \rho|^2 + \tilde{G}(\rho) dx, \\ p = \lim \int_{\mathbb{T}_n^2} (\rho_k - 1) \partial_1 \phi_k dx &= \int_{\mathbb{T}_n^2} (\rho - 1) \partial_1 \phi dx. \end{aligned}$$

We combine it with lower semi-continuity to obtain $\lim_k \tilde{E}_n^\varepsilon(\rho_k, \varphi_k) \geq \tilde{E}_n^\varepsilon(\rho, \phi)$, so that (ρ, ϕ) is a minimiser and solves (4.1) for some $c(n, p, \varepsilon)$. By standard elliptic regularity, (ρ, u) is smooth (with norms a priori depending on ε). Multiplying the first equation by ϕ and integrating by parts, we find

$$c \int_{\mathbb{T}_n^2} (\rho - 1) \partial_1 \phi dx = \int \chi(\rho) |\nabla \phi|^2 + \varepsilon |\Delta \phi|^2 dx \leq 2(\tilde{E}_n^\varepsilon)_{\min}(p) \Leftrightarrow 0 < c \leq \frac{2(\tilde{E}_n^\varepsilon)_{\min}(p)}{p}$$

We deduce $n \geq 5/p^2 \Rightarrow c < 2M$ with M the constant of lemma 4.1. With this bound on $c(n, p, \varepsilon)$ we can now obtain uniform elliptic estimates. The same computations as for proposition 2.3 lead to

$$\begin{aligned} \int_{\mathbb{T}_n^2} \chi |\Delta \phi|^2 + \varepsilon |\nabla \Delta \phi|^2 dx &\leq 2c \|\partial_1 \rho\|_2^2 + 2 \|\nabla \chi(\rho)\|_4^2 \|\nabla \phi\|_4^2, \\ \int_{\mathbb{T}_n^2} K(\chi(\rho)) (\Delta \rho)^2 + \varepsilon |\nabla \Delta \rho|^2 dx &\leq (c \|\partial_1 \phi\|_2 + \|(K \circ \chi)'\|_\infty \|\nabla \rho\|_4^2 + \|\rho - 1\|_2) \|\Delta \rho\|_2 \end{aligned}$$

As in proposition 2.3 we use $\|\nabla \psi\|_4^2 \lesssim \|\nabla \psi\|_2 \|\Delta \psi\|_2$ to get for some $C > 0$ independent of $M, \varepsilon, n \geq 5/p^2$

$$\|\Delta \rho\|_2^2 + \|\Delta \phi\|_2^2 \lesssim \|\partial_1 \phi\|_2^2 + \|\partial_1 \rho\|_2^2 + \|\Delta \rho\|_2 \|\rho - 1\|_2 + \|\nabla \phi\|_4^4 + \|\nabla \rho\|_4^4$$

$$\begin{aligned} \Rightarrow (\|\Delta\rho\|_2^2 + \|\Delta\phi\|_2^2)(1 - C(\|\nabla\rho\|_2^2 + \|\nabla\phi\|_2^2)) &\lesssim \|\partial_1\phi\|_2^2 + \|\partial_1\rho\|_2^2 + \|\rho - 1\|_2^2 \\ &\lesssim \tilde{E}_n^\varepsilon(\rho, \phi). \end{aligned}$$

Using $\tilde{E}_n^\varepsilon \leq Mp$ from lemma 4.1, we obtain for $p \ll 1/C$

$$\|\Delta\rho\|_2^2 + \|\Delta\phi\|_2^2 \leq M'p, \quad M' \text{ independent of } p, \varepsilon, n \geq 5/p^2.$$

The estimate for $j = 2$ follows since the energy controls $\|\rho - 1\|_{H^1} + \|\nabla\phi\|_{L^2}$, the case $j > 2$ is a standard bootstrap argument. \square

Proposition 4.3. *Let p_1 as in lemma 4.2. For any $p \leq p_1$, $n \geq 5/p^2$, there exists $(\rho_n, \phi_n) \in C^\infty(\mathbb{T}_n^2)$ such that up to an extraction, for any $j \geq 1$, $\|\rho_n^\varepsilon - \rho_n\|_{H^j} + \|\nabla\phi_n^\varepsilon - \nabla\phi_n\|_{H^{j-1}} \rightarrow_{\varepsilon \rightarrow 0} 0$, (ρ_n, ϕ_n) is a solution of the minimization problem*

$$\inf \left\{ \tilde{E}_n(\rho, \phi) = \int_{\mathbb{T}_n^2} \frac{1}{2} (\chi(\rho)|\nabla\phi|^2 + K(\chi(\rho))|\nabla\rho|^2) + \tilde{G}(\rho) dx, \right. \\ \left. P_n(\rho, \phi) = \int_{\mathbb{T}_n^2} (\rho - 1)\partial_1\phi dx = p_0 \right\}.$$

Moreover, (ρ_n, ϕ_n) is a solution of

$$\forall x \in \mathbb{T}_n^2 \begin{cases} -c_n\partial_1\rho_n + \operatorname{div}(\chi(\rho_n)\nabla\phi_n) = 0, \\ -c_n\partial_1\phi_n + \frac{\chi'|\nabla\phi_n|^2}{2} - K\Delta\rho_n - \frac{K'|\nabla\rho_n|^2}{2} + \tilde{g}(\rho_n) = 0. \end{cases} \quad (4.2)$$

for some $0 \leq c_n \leq M$, M the constant from lemma 4.2 independent of p .

Proof. We fix $p \leq p_1$, $(\rho_n^\varepsilon(p), \phi_n^\varepsilon(p))$ a minimizer. Using the a priori bounds, Rellich's compactness theorem and diagonal extraction we can extract a sequence $\varepsilon_k \rightarrow 0$ with

$$c_n^\varepsilon \rightarrow_\varepsilon c_n \in [0, M], \quad \forall j \geq 1, \quad (\rho_n^{\varepsilon_k}(p) - 1, \nabla\phi_n^{\varepsilon_k}(p)) \rightarrow (\rho_n - 1, \nabla\phi_n) \quad (H^j \times H^{j-1}).$$

Therefore we can pass to the limit in (4.1): since $(\rho_n^\varepsilon, \nabla\phi_n^\varepsilon)$ remains uniformly bounded in $H^4 \times H^3$, the terms $\varepsilon\Delta^2\phi_n^\varepsilon, \varepsilon\Delta^2\rho_n^\varepsilon$ vanish, and $(\rho_n, \nabla\phi_n)$ is a solution of (1.12). Similarly, $\tilde{E}(\rho_n, \phi_n) = \lim_\varepsilon \tilde{E}_{\min}^\varepsilon(p)$, $P(\rho_n, \phi_n) = p$.

To check the minimization property, we prove now $\lim_{\varepsilon \rightarrow 0} (\tilde{E}_n^\varepsilon)_{\min}(p) = (\tilde{E}_n)_{\min}(p)$. Clearly, it suffices to prove \leq . Let $\delta > 0$, $(\rho, \phi) \in \mathcal{H}$ such that $P_n(\rho, \phi) = p$, $\tilde{E}_n(\rho, \phi) \leq (\tilde{E}_n)_{\min}(p) + \delta$. By density of smooth functions, there exists $\rho_k, \phi_k \in C^\infty(\mathbb{T}_n^2)$ such that $\|\rho - \rho_k\|_{H^1} + \|\nabla\phi - \nabla\phi_k\|_2 \rightarrow_k 0$. In particular $P_n(\rho_k, \phi_k) = p_k \rightarrow p$ and (for k large enough so that $p_k \neq 0$) $\|\nabla\phi - \frac{p}{p_k}\nabla\phi_k\|_2 \rightarrow_k 0$. Using Lebesgue's dominated convergence theorem and up to an extraction

$$\tilde{E}_n(\rho_k, \frac{p}{p_k}\phi_k) \rightarrow \tilde{E}_n(\rho, \phi), \quad P_n(\rho_k, \frac{p}{p_k}\phi_k) = p.$$

Let us fix k large enough for which $\tilde{E}_n(\rho_k, \phi_k) \leq \tilde{E}_n(\rho, \phi) + \delta$. Then for $\varepsilon(k, \delta)$ small enough $\tilde{E}_n^\varepsilon(\rho_k, \frac{p}{p_k}\phi_k) \leq \tilde{E}_n(\rho_k, \frac{p}{p_k}\phi_k) + \delta \leq \tilde{E}_n(\rho, \phi) + 2\delta \leq (E_n)_{\min}(p) + 3\delta$. Since δ is arbitrary it ends the argument. \square

Remark 5. Using the identity $c_n p = \int \chi(\rho_n) |\nabla \phi_n|^2$, c_n is actually positive rather than non-negative, but this is not useful here.

4.2 Convergence of minimizers as $n \rightarrow \infty$

We start with the following immediate consequence of lemma 3.1.

Proposition 4.4. *For any $p \geq 0$, $\overline{\lim}_{n \rightarrow \infty} (\tilde{E}_n)_{\min}(p) \leq \tilde{E}_{\min}(p)$.*

This opens the path to the existence of minimizers on \mathbb{R}^2 . In this section, we consider a sequence of minimizers (ρ_n, ϕ_n) of momentum p on \mathbb{T}_n^2 . We identify \mathbb{T}_n^2 as $\Omega_n = [-n\pi, n\pi]^2 \subset \mathbb{R}^2$. For any function ψ_n defined on \mathbb{T}_n^2 , K compact, by “ $\psi_n \rightarrow \psi$ on K ” we implicitly identify ψ_n with the function defined on Ω_n , n large enough so that $K \subset \Omega_n$.

Proposition 4.5. *Let $p \leq p_2 = \min(p_0, p_1)$, with (p_0, p_1) from proposition 3.3 and lemma 4.2, let $(\rho_n(p), \phi_n(p))$ be a minimizer of \tilde{E}_n of momentum p . Assume*

$$\exists \delta > 0 : \forall n \geq 0, |\rho_n(0) - 1| \geq \delta, \quad (4.3)$$

then up to an extraction there exists $(\rho, \nabla \phi) \in (\cap_j H^j)^2$ such that

1. for any $j \geq 1$, any compact $K \subset \mathbb{R}^2$, $\|\rho_n - \rho\|_{H^j(K)} + \|\nabla \phi_n - \nabla \phi\|_{H^{j-1}} \rightarrow 0$, in particular $|\rho(0) - 1| \geq \delta$.
2. (ρ, ϕ) is a solution of (1.12) for some $c = \lim c_n \in]0, M]$, M independent of p .
3. $P(\rho, \phi) > 0$.

Proof. Items 1. and 2. follow from the same argument as for proposition 4.3. As for n large enough $0 \leq c_n \leq M$, we have $0 \leq c \leq M$. However, because the convergence is only *local* we can not pass to the limit in $\tilde{E}(\rho_n, \phi_n)$ and $P(\rho_n, \phi_n)$.

For item 3. we note that the assumption $|\rho_n(0) - 1| \geq \delta$ implies $\rho(0) \neq 1$, and since ρ is smooth $\int_{\mathbb{R}^2} \tilde{G}(\rho) dx > 0$. Since $(\rho, \nabla \phi)$ is a solution of (1.12), it satisfies the identity (5.3) which reads

$$cP(\rho, \phi) = 2 \int \tilde{G}(\rho) dx.$$

The right hand side is positive, and $c \geq 0$, therefore $c > 0$ and $P(\rho, \phi) > 0$. \square

Proposition 4.6. *In proposition 4.5, up to a translation assumption (4.3) is true and*

$$P(\rho, \phi) = \lim_n P_n(\rho_n, \phi_n) = p, \quad (4.4)$$

$$\lim_n (\widetilde{E}_n)_{\min}(p) = \lim_n \widetilde{E}_n(\rho_n, \nabla \phi_n) = \widetilde{E}(\rho, \nabla \phi). \quad (4.5)$$

In view of proposition 4.4, this proposition implies the existence of a solution to (1.12) which is a constrained minimizer to \widetilde{E} . The key is to forbid the following behaviours of the sequence (ρ_n, ϕ_n) :

- dichotomy: the minimizing sequence (ρ_n, ϕ_n) splits in (at least) two profiles whose supports are more and more distant.
- spreading: the total energy “far from the profiles” does not converge to 0, although $\rho_n, \phi_n \rightarrow (1, 0)$ uniformly.

Profile decomposition and proof of proposition 4.6 We note $d(\cdot, \cdot)$ the distance on the torus \mathbb{T}_n^2 , the energy density $\widetilde{e}(\rho, \phi) = \frac{1}{2}(\chi(\rho)|\nabla \phi|^2 + K \circ \chi(\rho)|\nabla \rho|^2 + (\rho - 1)^2)$ and the momentum density $p(\rho_n, \phi_n) = (\rho_n - 1)\partial_1 \phi_n$. For $x \in \mathbb{T}_n^2$, the set $B(x, r)$ is the ball in \mathbb{T}_n^2 . The key lemma is a modification of proposition 4.2 and lemma 5.2 in [10]. For the convenience of the reader we include a proof in the appendix.

Lemma 4.7. *Let (ρ_n, ϕ_n) be a sequence of minimizers of \widetilde{E}_n of momentum p . Up to an extraction, $c_n \rightarrow c \in]0, M]$, M independent of p .*

For $\delta \lesssim p^2$, we set $A_n^\delta = \{x : |\rho_n(x) - 1| \geq \delta\}$. Up to an other extraction there exists a sequence of radiuses $(R_n^k)_{n \geq k \geq 1}$, $l \in \mathbb{N}^$, $(x_n^i)_{1 \leq i \leq l} \in (\mathbb{T}_n^2)^l$, $M_k \rightarrow \infty$ such that :*

- $\forall n \geq 1, 1 \leq i \leq l, |\rho_n(x_n^i) - 1| \geq \delta$.
- For any $n \geq k$, $\inf_{i \neq j} d(x_n^i, x_n^j) \geq 10R_n^k$ and $A_n^\delta \subset \bigsqcup_{i=1}^l B(x_n^i, R_n^k)$.
- There exists C independent of δ, n, k such that for any $n \geq k, 1 \leq i \leq l$

$$\left| \int_{(\cup B(x_n^i, R_n^k))^c} \widetilde{e}(\rho_n, \phi_n) - c_n p(\rho_n, \phi_n) dx \right| \leq C \left(\delta \int_{(\cup B(x_n^i, R_n^k))^c} \widetilde{e}(\rho_n, \phi_n) dx + \frac{\widetilde{E}_n(\rho_n, \phi_n)}{M_k} \right). \quad (4.6)$$

- $\forall k \geq 1, R_n^k \rightarrow_n R^k < \infty, R^k \rightarrow_k \infty$.

Moreover, if $c < 1$, for δ small enough we can replace (4.6) by

$$\left| \int_{(\cup B(x_n^i, R_n^k))^c} \widetilde{e}(\rho_n, \phi_n) dx \right| \leq C \frac{\widetilde{E}_n(\rho_n, \phi_n)}{(1-c)M_k}. \quad (4.7)$$

Remark 6. Basically, the lemma states that there are two areas: several balls far from each other on which non trivial profiles persist as $n \rightarrow \infty$, and a rest where there may be some “spreading” contribution to the total energy, but which is almost equal to the spreading part of the momentum. If $c < 1$ there is no spreading.

Note also that $l \in \mathbb{N}^*$ means that “pure spreading” does not occur.

The better estimate available if $c < 1$ makes this case quite simpler. Actually a consequence of (1.15) is that $c \geq 1$ does not occur, unfortunately, the existence of minimizers is a prerequisite to this estimate.

An interesting alternative approach would have been to prove directly that there exists no solution to (1.12) if $c \geq 1$, as was done in [21] for the Gross-Pitaevskii case.

The case $c \geq 1$

The dichotomy scenario In this paragraph, we show that the sequence of minimizers can not split in several profiles.

Proposition 4.8. *In lemma 4.7, we have for δ small enough $l = 1$.*

Proof. First we note that l as a function of δ is nonincreasing, as the existence of l points such that $d(x_n^i, x_n^j) \rightarrow \infty$ and $|\rho(x_n^i) - 1| \geq \varepsilon$ prevents from covering A_n^δ for $\delta < \varepsilon$ by less than l balls of radius bounded in n . We assume by contradiction that there exists $\varepsilon > 0$, such that $l(\varepsilon) \geq 2$. This implies the existence of $(y_n^1, y_n^2) \in \mathbb{T}_n^2$ such that $|\rho(y_n^i) - 1| \geq \varepsilon$, $d(y_n^1, y_n^2) \xrightarrow{n} \infty$. For $0 < \delta \leq \varepsilon$, we can assume up to a reindexation

$$\forall n \geq k, y_n^1 \in B(x_n^1, R_n^k), y_n^2 \in B(x_n^2, R_n^k). \quad (4.8)$$

Since $\|\nabla \rho_n\|_\infty$ is bounded uniformly in n , we also remark

$$\exists q > 0 \text{ independent of } \delta, \forall n \geq k \geq 1, \forall i = 1, 2, \int_{B(x_n^i, R_n^k)} \tilde{G}(\rho_n) \geq q. \quad (4.9)$$

We apply proposition 4.5 to $(\rho_n, \phi_n)(\cdot - x_n^i)$: up to an extraction there exists $(\rho^i, \nabla \phi^i) \in \cap_{j \geq 0} H^j(\mathbb{R}^2)$, solutions of (1.12) with speed c and

$$\forall K \text{ compact}, \forall j \geq 1, \|\rho_n(\cdot - x_n^i) - \rho^i\|_{H^j(K)} + \|\nabla \phi_n(\cdot - x_n^i) - \nabla \phi^i\|_{H^{j-1}(K)} \rightarrow 0.$$

We can assume that $\widetilde{E}_n(\rho^n, \phi^n)$ converges and thus

$$\forall k \geq 1, \exists (\mu_k, \nu_k) \in \mathbb{R}^+ \times \mathbb{R} : \begin{cases} \lim_n \int_{\mathbb{T}_n^2} \tilde{e}(\rho_n, \phi_n) dx = \sum_{i=1}^l \int_{B(0, R^k)} \tilde{e}(\rho^i, \phi^i) dx + \mu_k, \\ \lim_n \int_{\mathbb{T}_n^2} p(\rho_n, \phi_n) dx = \sum_{i=1}^l \int_{B(0, R^k)} p(\rho^i, \phi^i) dx + \nu_k, \\ |\mu_k - c\nu_k| \leq C \left(\delta \mu_k + \frac{\lim_n \widetilde{E}_n(\rho_n, \phi_n)}{M_k} \right), \end{cases} \quad (4.10)$$

where $B(0, R^k)$ is now the usual ball of \mathbb{R}^2 . Letting $k \rightarrow \infty$ implies

$$\lim_n \widetilde{E}_n(\rho^n, \phi^n) = \sum_{i=1}^l \widetilde{E}(\rho^i, \phi^i) + \mu, \quad \lim_n P_n(\rho_n, \phi_n) = \sum_{i=1}^l P(\rho^i, \phi^i) + \nu, \quad |\mu - c\nu| \leq C\delta\mu.$$

Since C is an absolute constant, we can assume $C\delta < 1$, thus $\nu \geq 0$. Let us set $p_i := P(\rho^i, \phi^i)$. Since $(\rho_i, \nabla\phi_i)$ is a solution of (1.12), identity (5.3) is true, namely

$$cp_i = 2 \int_{\mathbb{R}^2} \widetilde{G}(\rho_i) dx > 0,$$

thus $p_i > 0$ and from (4.9) $p_1 \geq 2q/M$, $p_2 \geq 2q/M$. On the other hand, we know that $\forall n$, $P_n(\rho_n, \phi_n) = p$, so that by subadditivity and proposition 4.4

$$\begin{aligned} \sum_{i=1}^l \widetilde{E}(\rho^i, \phi^i) + \mu &= \lim_n \widetilde{E}_n(\rho_n, \phi_n) \leq \widetilde{E}_{\min}(p) = \widetilde{E}_{\min}\left(\sum_{i=1}^l p_i + \nu\right) \\ \Rightarrow \widetilde{E}(\rho^1, \phi^1) + \widetilde{E}(\rho^2, \phi^2) + \sum_{i=3}^l \widetilde{E}_{\min}(p_i) &\leq \widetilde{E}_{\min}\left(\sum_{i=1}^l p_i\right) + \widetilde{E}_{\min}(\nu) - \mu \end{aligned}$$

Next we use proposition 3.3: $\widetilde{E}_{\min}(\nu) \leq \nu \leq c\nu$, subadditivity and $|\mu - c\nu| \leq C\delta\mu$

$$\widetilde{E}_{\min}(p_1) + \widetilde{E}_{\min}(p_2) \leq \widetilde{E}(\rho^1, \phi^1) + \widetilde{E}(\rho^2, \phi^2) \leq \widetilde{E}_{\min}(p_1 + p_2) + C\delta\mu$$

But from proposition 3.4, $\widetilde{E}_{\min}(p_1) + \widetilde{E}_{\min}(p_2) - \widetilde{E}_{\min}(p_1 + p_2) \geq D(2q/M, 2q/M) > 0$, while letting $\delta \rightarrow 0$ we find

$$D(2q/M, 2q/M) \leq 0, \tag{4.11}$$

which is a contradiction. □

The spreading scenario Ruling out this scenario follows the same scheme, but simpler. Since $l = 1$, from the same computations as in the previous paragraph for any $\delta > 0$ there exists (ρ, ϕ) , $\mu \geq 0$, $p_1 > 0$ such that $P(\rho, \phi) = p_1$ and

$$\lim_n (\widetilde{E}_n)_{\min}(\rho_n, \phi_n) = \widetilde{E}(\rho, \phi) + \mu, \quad \lim_n P_n(\rho_n, \phi_n) = p_1 + \nu, \quad |c\nu - \mu| \leq C\delta\mu.$$

We use $\widetilde{E}_{\min}(\nu) \leq \nu \leq c\nu \leq \mu + C\delta\mu$ so that

$$E_{\min}(p_1) + E_{\min}(\nu) \leq E_{\min}(p_1) + \mu + C\delta\mu = \lim_n E_n(\rho_n, \phi_n) + C\delta\mu \leq E_{\min}(p_1 + \nu) + C\delta\mu.$$

proposition 3.4 with $q = \min(\nu, p_1)$ implies $0 \leq -D(q, q) + C\delta\mu$, letting $\delta \rightarrow 0$ we get $q = 0$, thus $\mu = \nu = 0$.

Conclusion We have obtained that there exists $(\rho, \nabla\phi) \in (\cap_{j \geq 0} H^j)^2$ such that

$$\begin{aligned} \forall K \text{ compact, } \|\rho_n - \rho\|_{H^j(K)} + \|\nabla\phi_n - \nabla\phi\|_{H^j(K)} &\longrightarrow_n 0, \\ \lim_n \widetilde{E}_n(\rho_n, \phi_n) = E(\rho, \phi), \quad p = \lim_n P_n(\rho_n, \phi_n) &= P(\rho, \phi). \end{aligned}$$

this ends the proof of 4.6 in the case $c \geq 1$.

The case $c < 1$ With the same notations as in the case $c \geq 1$ we have the existence of $(\rho^i, \nabla\phi^i)_{1 \leq i \leq l}$ such that $\|\rho_n(\cdot - x_n^i) - \rho^i\|_{H^j(K)} \longrightarrow 0$, $\|\nabla\phi_n^i(\cdot - x_n^i) - \nabla\phi^i\|_{H^j(K)} \longrightarrow 0$. Let us fix δ small enough so that inequality (4.7) is true. Thanks to the pointwise inequality $|(\rho - 1)\partial_1\phi| \lesssim C\tilde{e}(\rho, \phi)$ we get the following identities

$$\begin{aligned} \lim_n \int \widetilde{E}_n(\rho_n, \phi_n) &= \lim_k \sum_{i=1}^l \int_{B(0, R^k)} \tilde{e}(\rho^i, \phi^i) dx + O(1/M_k) = \sum_{i=1}^l \widetilde{E}(\rho^i, \phi^i), \\ p &= \lim_n \int P_n(\rho_n, \phi_n) dx = \sum_{i=1}^l P(\rho^i, \phi^i). \end{aligned}$$

For $1 \leq i \leq l$, set $p_i = P(\rho_i, \phi_i)$. If for some $\delta > 0$, $l \geq 2$, then we have as for the case $c \geq 1$

$$\widetilde{E}_{\min}(p_1) + \widetilde{E}_{\min}(p_2) + \sum_{i=3}^l \widetilde{E}_{\min}(p_i) \leq \widetilde{E}_{\min}(p_1 + p_2) + \sum_{i=1}^3 \widetilde{E}_{\min}(p_i),$$

which leads to the absurd inequality $0 \leq -D(p_1, p_2)$. Thus $l = 1$, the conclusion is the same as for $c \geq 1$.

5 Pohozaev type identities and applications

In this section we complete the proof of theorem 1.1 with the sharp estimates on the energy near $p = 0$.

The first proposition does not rely on the fact that the dimension d is 2, therefore we state it in general settings. Since the solutions to (1.12) that we constructed in the previous section are smooth we state our identities for smooth functions, but they are true under much weaker assumptions.

For conciseness we write K for $K(\chi(\rho))$, $K' = d(K \circ \chi)/d\rho$.

Proposition 5.1. *Let (ρ, ϕ) be a smooth finite energy solution of (1.12). If $(\rho - 1, \phi) \in (H^2)^2$,*

then it satisfies the Pohozaev identities

$$\tilde{E}(\rho, \phi) = \int_{\mathbb{R}^d} \chi |\partial_1 \phi|^2 + K |\partial_1 \rho|^2 dx, \quad (5.1)$$

$$\forall 2 \leq j \leq d, \quad \tilde{E}(\rho, \phi) = \int_{\mathbb{R}^d} \chi |\partial_j \phi|^2 + K |\partial_j \rho|^2 dx + cP(\rho, \phi), \quad (5.2)$$

$$\frac{d-2}{2} \int_{\mathbb{R}^d} \chi |\nabla \phi|^2 + K |\nabla \rho|^2 dx = -d \int_{\mathbb{R}^d} \tilde{G}(\rho) dx + (d-1)cP(\rho, \phi). \quad (5.3)$$

Moreover we have

$$cP(\rho, \phi) = \int_{\mathbb{R}^d} \chi |\nabla \phi|^2 dx \quad (5.4)$$

Proof. Multiply the first equation of (1.12) by $x_1 \partial_1 \phi$ and integrate (note that the integrals are not clearly convergent, for a rigorous argument see e.g. proposition 5 in [21]):

$$\begin{aligned} \int_{\mathbb{R}^d} -cx_1 \partial_1 \rho \partial_1 \phi - \chi \nabla \phi \cdot \nabla (x_1 \partial_1 \phi) dx &= \int_{\mathbb{R}^d} -cx_1 \partial_1 \rho \partial_1 \phi - \chi |\partial_1 \phi|^2 - \chi x_1 \partial_1 \frac{|\nabla \phi|^2}{2} dx \\ &= \int_{\mathbb{R}^d} -cx_1 \partial_1 \rho \partial_1 \phi - \chi |\partial_1 \phi|^2 + \frac{\chi |\nabla \phi|^2}{2} \\ &\quad + \frac{x_1 \partial_1 \chi |\nabla \phi|^2}{2} dx \quad (5.5) \\ &= 0 \end{aligned}$$

Now the multiplication of the second equation of (1.12) by $x_1 \partial_1 \rho$ and integration gives

$$\begin{aligned} 0 &= \int_{\mathbb{R}^d} -cx_1 \partial_1 \phi \partial_1 \rho + \frac{x_1 \chi' \partial_1 \rho |\nabla \phi|^2}{2} - \left(K \Delta \rho + \frac{1}{2} K' |\nabla \rho|^2 \right) x_1 \partial_1 \rho + \tilde{g}(\rho) x_1 \partial_1 \rho dx, \\ \text{with } \int_{\mathbb{R}^d} -K \Delta \rho x_1 \partial_1 \rho dx &= \int_{\mathbb{R}^d} K |\partial_1 \rho|^2 + x_1 K' |\nabla \rho|^2 \partial_1 \rho + x_1 K \partial_1 \left(\frac{|\nabla \rho|^2}{2} \right) dx \\ &= \int_{\mathbb{R}^d} K |\partial_1 \rho|^2 + \frac{x_1 K' |\nabla \rho|^2 \partial_1 \rho}{2} - \frac{K |\nabla \rho|^2}{2} dx, \\ \text{and } \int_{\mathbb{R}^d} x_1 \tilde{g}(\rho) \partial_1 \rho &= \int_{\mathbb{R}^d} -\tilde{G}(\rho) dx, \end{aligned}$$

so that

$$0 = \int_{\mathbb{R}^d} -cx_1 \partial_1 \phi \partial_1 \rho + \frac{x_1 \chi' \partial_1 \rho |\nabla \phi|^2}{2} + K |\partial_1 \rho|^2 - \frac{K |\nabla \rho|^2}{2} - \tilde{G}(\rho) dx. \quad (5.6)$$

Finally, if we add (5.6) to (5.5) we obtain (5.1)

$$0 = \int_{\mathbb{R}^d} \chi \frac{|\nabla \phi|^2}{2} + K \frac{|\nabla \rho|^2}{2} + \tilde{G}(\rho) - \chi |\partial_1 \phi|^2 - K |\partial_1 \rho|^2.$$

The same computations with multipliers $x_j \partial_j \rho$ and $x_j \partial_j \phi$, $j \geq 2$ lead to

$$0 = \int \frac{\chi |\nabla \phi|^2 + K |\nabla \rho|^2}{2} + \tilde{G}(\rho) - \chi |\partial_j \phi|^2 - K |\partial_j \rho|^2 - c(\partial_1 \rho x_j \partial_j \phi - c \partial_j \rho x_j \partial_1 \phi) dx.$$

This gives (5.2), indeed an integration by part shows

$$\begin{aligned} \int \partial_1 \rho x_j \partial_j \phi - c \partial_j \rho x_j \partial_1 \phi dx &= \int -x_j (\rho - 1) \partial_1 \partial_j \phi + x_j (\rho - 1) \partial_1 \partial_j \phi + (\rho - 1) \partial_1 \phi dx \\ &= P(\rho, \phi). \end{aligned}$$

The third identity is obtained by summing the previous ones. The last identity is obtained by multiplying the first equation in (1.12) by ϕ and integration. \square

Proposition 5.2. *Let $p_0 > 0$ given by prop 3.3 p_0 , α, β positive such that for any (smooth) minimizer of speed c and momentum $p \leq p_0$,*

$$\alpha p^2 \leq 1 - c \leq \beta p^2.$$

Proof. For $p \leq p_0$, let (ρ, ϕ) be such a minimiser. From (5.3), (5.4) and proposition 3.3

$$cp = \frac{1}{2} \int 2\tilde{G} + \chi |\nabla \phi|^2 dx \leq \tilde{E}_{\min}(p) \leq p - \alpha p^3$$

which gives $1 - c \geq \alpha p^2$. The other inequality follows an idea from [10]: applying ∂_1 to the first equation in (1.12) gives

$$-c \partial_1^2 \rho + \partial_1 \Delta \phi + \partial_1 \operatorname{div}((\chi - 1) \nabla \phi) = 0.$$

Next, multiply the momentum equation by $K(1)/K$ and apply Δ :

$$-c \Delta \partial_1 \phi - K(1) \Delta^2 \rho + \Delta \rho + \Delta \left(c \left(1 - \frac{K(1)}{K} \right) \partial_1 \phi + \frac{K(1)g}{K} - \rho + \frac{K(1)\chi'}{2K} |\nabla \phi|^2 - \frac{K(1)K'}{2K} |\nabla \rho|^2 \right) = 0.$$

if we add these equalities we obtain

$$\begin{aligned} (K(1) \Delta^2 - \Delta + c \partial_1^2) \rho &= \Delta \left(c \left(\frac{K(1)}{K} - 1 \right) \partial_1 \phi + \frac{K(1)g}{K} - \rho + \frac{K(1)\chi'}{2K} |\nabla \phi|^2 \right. \\ &\quad \left. - \frac{K(1)K'}{2K} |\nabla \rho|^2 \right) + c \partial_1 \operatorname{div}((\chi - 1) \nabla \phi) \\ &:= \Delta A + c \partial_1 \operatorname{div} B. \end{aligned}$$

As $\chi(\rho)$ is bounded, $\tilde{g}'(1) = 1$ and $K(\chi(1)) = K(1)$, it is easy to see

$$\|A\|_{L^1} + \|B\|_{L^1} \lesssim \tilde{E}(\rho, \phi) = \tilde{E}_{\min}(p).$$

Since the Fourier transform maps continuously L^1 to L^∞ , we deduce

$$\begin{aligned} \|\rho - 1\|_2 = 2\pi\|\widehat{\rho - 1}\|_2 &\leq C(\|A\|_{L^1} + \|B\|_{L^1}) \left\| \frac{|\xi|^2 + |\xi_1||\xi|}{K(1)|\xi|^4 + |\xi|^2 - c|\xi_1|^2} \right\|_2 \\ &\leq C\tilde{E}_{\min}(p) \left\| \frac{|\xi|^2 + |\xi_1||\xi|}{K(1)|\xi|^4 + |\xi|^2 - c|\xi_1|^2} \right\|_2. \end{aligned}$$

As $c \leq 1 - \alpha p^2 < 1$ the L^2 norm on the right hand side is finite, and an elementary explicit computation (see [10] claim 2.59) gives

$$\|\rho - 1\|_2^2 \lesssim \frac{\tilde{E}_{\min}^2}{\sqrt{1-c}}.$$

On the other hand we have from proposition 5.1 the (1.9) $\int(\rho - 1)^2 dx \sim 2 \int \tilde{G}(\rho) dx = cp = \int \chi(\partial_1 \phi)^2 dx$, we deduce

$$p = \int_{\mathbb{R}^2} (\rho - 1) \partial_1 \phi \lesssim \int_{\mathbb{R}^2} \tilde{G}(\rho) + \frac{\chi \partial_1 \phi^2}{2} dx = cp,$$

so that $c \gtrsim 1$. Next $\int(\rho - 1)^2 dx \sim cp \geq c\tilde{E}_{\min}(p) \gtrsim \tilde{E}_{\min}(p)$ and we can conclude

$$\sqrt{1-c} \leq C\sqrt{2}\tilde{E}_{\min}(p) \leq C\sqrt{2}p \Rightarrow c - 1 \gtrsim -p^2.$$

□

Corollary 5.3. *There exists $p_0 > 0$, such that for $p \leq p_0$, if there exists a minimiser of momentum p ,*

$$p - \beta p^3 \leq \tilde{E}_{\min}(p) \leq p - \alpha p^3. \quad (5.7)$$

with the same α, β as in proposition 5.2.

Proof. The inequality $\tilde{E}_{\min}(p) \leq p - \alpha p^3$ is proposition 3.3. Conversely thanks to propositions 5.1 and 5.2

$$\tilde{E}(\rho, \phi) \geq \int \frac{\chi |\nabla \phi|^2}{2} + \tilde{G}(\rho) dx = cp \geq p - \beta p^3.$$

□

Remark 7. Corollary 5.3 is rather natural with the following heuristic : consider the formal relation $\delta \tilde{E} = c \delta P \Rightarrow \frac{dE_{\min}}{dp} = c$. If this was true corollary 5.3 would merely be a consequence of the integration in p of the estimates on c .

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A Proof of the existence of the profile decomposition

This section is devoted to the proof of lemma 4.7. First, we recall that c_n is bounded, up to an extraction we assume $c_n \rightarrow c > 0$ (for the sign of c , see prop 4.5).

According to proposition 4.4 and proposition 3.3, $\underline{\lim}(\tilde{E}_n)_{\min}(p) \leq \tilde{E}_{\min}(p) \leq 1 - \alpha p^2$. Therefore for n large enough, $\tilde{E}_n)_{\min}(p) \leq 1 - \alpha p^2/2$, and a straightforward modification of proposition 3.3 implies $\|\rho_n - 1\|_{L^\infty(\mathbb{T}_n^2)} \gtrsim p^2$. This ensures that $A_n^\delta = \{|\rho_n - 1| \geq \delta\}$ is not empty at least for $\delta \lesssim p^2$ and n large enough. Next for any $n \geq 0$, the set A_n^δ is compact, thus there exists a finite covering $\cup_{i=1}^{l(n)} B(x_n^i, 1/3) \supset A_n^\delta$ such that $|\rho(x_n^i) - 1| \geq \delta$. Using Vitali's lemma, there is a subset $J_n \subset \{1, \dots, l\}$ such that for $i, j \in J_n$, $B(x_n^i, 1/3) \cap B(x_n^j, 1/3) = \emptyset$ and $\bigcup_{i \in J_n} B(x_n^i, 1) \supset A_n^\delta$. From lemma 4.2 $\|\rho_n - 1\|_{W^{1,\infty}}$ is bounded uniformly in n , then

$$\frac{|J_n|\delta^2}{\|\rho_n - 1\|_{W^{1,\infty}}} \lesssim \sum \int_{B(x_n^i, 1/3)} (\rho_n - 1)^2 dx \lesssim \tilde{E}_n(\rho_n, \phi_n),$$

so $|J_n|$ must be bounded uniformly too. Up to an extraction, we can assume that $|J_n|$ is a constant l .

There are two key lemmas. The first one is a kind of improved Vitali's lemma, stating that the ball can be chosen very far away from each other.

Lemma A.1. *Given a collection $\bigcup_{i=1}^l B(x_i, R) \subset \mathbb{T}_n^2$, for any $M \geq 2$, there exists a subset $J \subset \{1, \dots, l\}$ and $R \leq R' \leq (2M)^l R$ such that $\sqcup_{j \in J} B(x_j, R') \supset \cup_{i=1}^l B(x_i, R)$ and for any $(j, k) \in J^2$, $d(x_j, x_k) \geq MR'$.*

For the proof, we refer to [10] lemma 4.12.

The second lemma looks a lot like proposition 4.2 from [10]. We include a proof since there is a few non trivial differences. Let us first fix some notations : for fixed n , $R \geq 1$, $M \gg 1$, we apply lemma A.1 to $\cup B(x_n^i, R)$. Up to reindexing, there exists $l_n \leq l$, $(x_n^i)_{1 \leq i \leq l_n} \sqcup_{i=1}^{l_n} B(x_n^i, R'_n) \supset \cup_{i=1}^l B(x_n^i, R) \supset A_n^\delta$, $d(x_n^i, x_n^j) \geq R'_n M$.

Lemma A.2. *If $\cup_{i=1}^{l_n} B(x_n^i, R'_n) \supset A_n^\delta$ is as in lemma A.1, and $k \in \mathbb{N}^*$ such that $2^k < M/2$, there exists $1 \leq m \leq k$, C an absolute constant such that setting*

$$S_n^k := (\cup_{i=1}^{l_n} B(x_n^i, 2^m R'_n))^c,$$

then

$$\left| \int_{S_n^k} c_n p(\rho_n, \phi_n) - \tilde{e}(\rho_n, \phi_n) dx \right| \leq C \left(\delta \int_{S_n^k} \tilde{e}(\rho_n, \phi_n) dx + \frac{\tilde{E}_n(\rho_n, \phi_n)}{k} \right). \quad (\text{A.1})$$

Proof. We first remark that since the balls $B(x_n^i, 2^k R')$ are disjoint,

$$\int_{\bigcup_{i=1}^{l_n} B(x_n^i, 2^k R') \setminus B(x_n^i, R')} \tilde{e} dx = \sum_{p=1}^k \int_{\bigcup_{i=1}^{l_n} B(x_n^i, 2^p R') \setminus B(x_n^i, 2^{p-1} R')} \tilde{e} dx$$

In particular, there exists $1 \leq m \leq k$ such that

$$\int_{\bigcup_{i=1}^{l_n} B(x_n^i, 2^m R') \setminus B(x_n^i, 2^{m-1} R')} \tilde{e} dx \leq \frac{\int_{\bigcup_{i=1}^{l_n} B(x_n^i, 2^k R') \setminus B(x_n^i, R')} \tilde{e} dx}{k} \leq \frac{\widetilde{E}_n(\rho_n, \phi_n)}{k}. \quad (\text{A.2})$$

Now let $\psi \in C_c^\infty(\mathbb{R}^+)$ such that $\psi|_{[0,1]} = 1$, $\text{supp}(\psi) \subset [0, 2]$, we define $\check{\phi}_n$ by

$$\begin{aligned} \check{\phi}_n &= \phi_n, \quad x \in (\cup B(x_n^i, 2^m R'))^c, \\ \check{\phi}_n &= \psi\left(\frac{|x - x_n^i|}{2^{m-1} R'}\right) \int_{B(x_n^i, 2^m R') \setminus B(x_n^i, 2^{m-1} R')} \phi_n dx + (1 - \psi) \phi_n, \quad x \in B(x_n^i, 2^m R'). \end{aligned}$$

We recall the Poincaré-Wirtinger inequality

$$\int_{B(0,2) \setminus B(0,1)} |f - \int f|^2 dx \leq C \|\nabla f\|_{L^2(B(0,2) \setminus B(0,1))}^2,$$

so that from a scaling argument

$$\int_{B(x_n^i, 2^m R') \setminus B(x_n^i, 2^{m-1} R')} |\nabla \check{\phi}_n|^2 dx \lesssim \|\nabla \phi_n\|_{L^2(B(x_n^i, 2^m R') \setminus B(x_n^i, 2^{m-1} R'))}^2. \quad (\text{A.3})$$

If we multiply the first equation of (4.2) by $\check{\phi}_n$ and integrate over \mathbb{T}_n^2 we obtain

$$\begin{aligned} \int_{S_n^k} c_n p(\rho_n, \phi_n) dx - \chi(\rho_n) |\nabla \phi_n|^2 dx + \sum_{i=1}^{l(n)} \int_{B(x_n^i, 2^m R') \setminus B(x_n^i, 2^{m-1} R')} c_n (\rho_n - 1) \partial_1 \check{\phi}_n \\ - \chi(\rho_n) \nabla \phi_n \nabla \check{\phi}_n dx = 0. \end{aligned}$$

Using Cauchy-Schwarz's inequality, (A.2) and (A.3) we can bound the second term

$$\begin{aligned} \left| \sum_{i=1}^{l(n)} \int_{B(x_n^i, 2^m R') \setminus B(x_n^i, 2^{m-1} R')} c_n (\rho_n - 1) \partial_1 \check{\phi}_n - \chi \nabla \phi_n \nabla \check{\phi}_n dx \right| &\lesssim \int_{\cup B(x_n^i, 2^m R') \setminus B(x_n^i, 2^{m-1} R')} \tilde{e}(\rho_n, \phi_n) dx \\ &\leq \frac{\widetilde{E}_n(\rho_n, \phi_n)}{k}. \end{aligned}$$

We have obtained

$$\left| \int_{S_n^k} c_n p(\rho_n, \phi_n) - \rho_n |\nabla \phi_n|^2 dx \right| \leq C \frac{\widetilde{E}_n(\rho_n, \phi_n)}{k}. \quad (\text{A.4})$$

We turn to symmetric computations on the second equation of (4.2). We set

$$\check{\rho}_n = \begin{cases} \rho_n, & x \in (\cup B(x_n^i, 2^m R'))^c, \\ \check{\rho}_n = \psi\left(\frac{|x - x_n^i|}{2^{m-1}R'}\right) + (1 - \psi)\rho_n, & x \in B(x_n^i, 2^m R'). \end{cases}$$

In this case, since $(\rho_n - 1)^2 \lesssim \tilde{e}_n(\rho_n, \phi_n)$ and $|\check{\rho}_n - 1| \leq |\rho_n - 1|$ we will not need the Poincaré-Wirtinger inequality. As in the previous section, we denote K for $(K \circ \chi)(\rho_n)$, $K' = dK \circ \chi/d\rho$. Multiplying the second equation of (4.2) by $\check{\rho}_n - 1$ and integrating on \mathbb{T}_n^2 gives

$$\int_{\mathbb{T}_n^2} -c_n(\check{\rho}_n - 1)\partial_1\phi_n + \frac{(\check{\rho}_n - 1)|\nabla\phi_n|^2}{2} + K\nabla\rho_n\nabla\check{\rho}_n + \frac{1}{2}K'|\nabla\rho_n|^2(\check{\rho}_n - 1) + \tilde{g}(\rho_n)(\check{\rho}_n - 1)dx = 0$$

We point out that $\check{\rho}_n - 1 = 0$ on $\cup B(x_n^i, 2^{m-1}R')$, thus

$$\begin{aligned} & \int_{\cup B(x_n^i, 2^m R') \setminus B(x_n^i, 2^{m-1} R')} -c_n(\check{\rho}_n - 1)\partial_1\phi_n + \frac{(\check{\rho}_n - 1)|\nabla\phi_n|^2}{2} + K\nabla\rho_n\nabla\check{\rho}_n \\ & \quad + \frac{1}{2}K'|\nabla\rho_n|^2(\check{\rho}_n - 1) + \tilde{g}(\rho_n)(\check{\rho}_n - 1)dx \\ & = - \int_{S_n^k} -c_n p(\rho_n, \phi_n) + K|\nabla\rho_n|^2 + \tilde{g}(\rho_n)(\rho_n - 1) \\ & \quad + \frac{(\rho_n - 1)|\nabla\phi_n|^2}{2} + \frac{1}{2}K'|\nabla\rho_n|^2(\rho_n - 1)dx \end{aligned} \quad (\text{A.5})$$

To estimate the left hand side, we observe that on $(\cup_i B(x_n^i, R'))^c$, $|\check{\rho}_n - 1| \leq \min(|\rho_n - 1|, \delta)$, therefore

$$\begin{aligned} & \int_{\cup_i B(x_n^i, 2^m R') \setminus B(x_n^i, 2^{m-1} R')} -c_n(\check{\rho}_n - 1)\partial_1\phi_n + \frac{(\check{\rho}_n - 1)|\nabla\phi_n|^2}{2} \\ & \quad + \frac{1}{2}K'|\nabla\rho_n|^2(\check{\rho}_n - 1) + g(\rho_n)(\check{\rho}_n - 1)dx \lesssim \frac{\widetilde{E}_n(\rho_n, \phi_n)}{k}, \end{aligned}$$

Moreover $|\nabla\check{\rho}_n| \lesssim |\nabla\rho_n| + |\rho_n - 1|$, therefore

$$\int_{B(x_n^i, 2^p R') \setminus B(x_n^i, 2^{p-1} R')} |\nabla\check{\rho}_n|^2 dx \lesssim \int_{B(x_n^i, 2^p R') \setminus B(x_n^i, 2^{p-1} R')} \tilde{e}(\rho_n, \phi_n) dx,$$

so that the left hand side in (A.5) is bounded by E_n/k . This estimate, combined with $|\rho_n - 1||\nabla\rho_n|^2 \leq \delta|\nabla\rho_n|^2$ on S_n^k implies

$$\begin{aligned} & \left| \int_{S_n^k} -c_n(\rho_n - 1)\partial_1\phi_n + K(\rho_n)|\nabla\rho_n|^2 + g(\rho_n)(\rho_n - 1) dx \right| \\ & \leq C \left(\delta \int_{S_n^k} \tilde{e}_n dx + \int_{\cup B(x_n^i, 2^p R') \setminus B(x_n^i, 2^{p-1} R')} \tilde{e}_n dx \right). \end{aligned}$$

To conclude, we remark that near $\rho = 1$, $\tilde{g}(\rho) \sim \rho - 1$, $\tilde{G}(\rho) \sim (\rho - 1)^2/2$, so that $|\tilde{g}(\rho_n)(\rho_n - 1) - 2\tilde{G}(\rho_n)| \lesssim \delta\tilde{G}(\rho_n) \leq \delta\tilde{e}$. As a consequence

$$\left| \int_{\mathcal{S}_n^k} -c_n p(\rho_n, \phi_n) + K|\nabla\rho_n|^2 + 2\tilde{G}(\rho_n) dx \right| \leq C \left(\delta \int_{\mathcal{S}_n^k} \tilde{e}_n dx + \frac{\tilde{E}_n(\rho_n, \phi_n)}{k} \right) \quad (\text{A.6})$$

Putting together (A.4) and (A.6), we find the expected result

$$\left| \int_{\mathcal{S}_n^k} -c_n p(\rho_n, \phi_n) + \frac{K(\rho_n)|\nabla\rho_n|^2 + \chi(\rho_n)|\nabla\phi_n|^2}{2} + \tilde{G}(\rho_n) dx \right| \leq C \left(\delta \int_{\mathcal{S}_n^k} \tilde{e}_n dx + \frac{\tilde{E}_n}{k} \right).$$

□

Now we combine these two lemmas to construct the sequence R_n^k through a diagonal extraction.

A.1 The case $c \geq 1$

Construction of R_n^1 We recall that for any $n \geq 0$, $A_n^\delta \subset \cup_{i=1}^l B(x_n^i, 1)$. We apply lemma A.1 with $M = 10$, this gives for any n a subset $J_n^1 \subset \{1, \dots, l\}$ and $1 \leq R_n^1 \leq (20)^l$ such that

$$\sqcup_{j \in J_n^1} B(x_n^j, R_n^1) \supset \cup_{i=1}^l B(x_n^i, 1), \text{ and for any } (i, j) \in J_n^1, d(x_n^j, x_n^i) \geq 10R_n^1.$$

We apply lemma A.2 with $k = 1^5$, then (A.1) is true on $\mathcal{S}_n^1 = (\sqcup_{j \in J_n^1} B(x_n^j, 2R_n^1))^c$. Since $(2R_n^1)_n$ and $|J_n^1|$ are bounded, there is an extraction $\psi_1(n)$ such that $2R_{\psi_1(n)}^1$ converges to some $\mathcal{R}^1 \geq 2$ and $J_{\psi_1(n)}^1 = J^1$ does not depend of n .

Construction of R_n^2 We apply once more lemma A.1 to $\cup_{i=1}^l B(x_{\psi_1(n)}^i, 2)$ with $M = 3 \cdot 10$. For any n there is a subset $J_{\psi_1(n)}^2 \subset \{1, \dots, l\}$, $2 \leq R_{\psi_1(n)}^2 \leq 2(60)^l$ such that

$$\begin{aligned} \sqcup_{i \in J_{\psi_1(n)}^2} B(x_{\psi_1(n)}^i, R_{\psi_1(n)}^2) &\supset \cup_{i=1}^l B(x_{\psi_1(n)}^i, 1), \\ \text{and for any } (i, j) \in J_{\psi_1(n)}^2, &d(x_{\psi_1(n)}^j, x_{\psi_1(n)}^i) \geq 30R_{\psi_1(n)}^2. \end{aligned}$$

From lemma A.2 with $k = 2$, for any n there exists $1 \leq m_{\psi_1(n)}^2 \leq 2$ such that (A.1) is true on $\mathcal{S}_{\psi_1(n)}^2 = (\sqcup_{j \in J_{\psi_1(n)}^2} B(x_{\psi_1(n)}^j, 2^{m_{\psi_1(n)}^2} R_{\psi_1(n)}^2))^c$. Since $(2^{m_{\psi_1(n)}^2} R_{\psi_1(n)}^2)_n$ and $|J_{\psi_1(n)}^2|$ are bounded, there is a sub-extraction $\psi_2(n)$ such that $2^{m_{\psi_2(n)}^2} R_{\psi_2(n)}^2 \rightarrow_n \mathcal{R}^2 \geq 4$ and $J_{\psi_2(n)}^2 = J^2$.

The generic argument at step k to construct of R_n^k is the following :

⁵In this case obviously $p = 1$, but this will not be the case in the rest of the induction argument.

Construction of R_n^k At step k , we have an extraction $\psi_{k-1}(n)$, we apply lemma A.2 to $\cup_{i=1}^l B(x_{\psi_{k-1}(n)}^i, 2^k)$ with $M = 10 \cdot 3^{k-1}$, which gives again $2^k \leq R_{\psi_{k-1}(n)}^k \leq 2^k (20 \cdot 3^{k-1})^l$, $J_{\psi_{k-1}(n)}^k \subset \{1, \dots, l\}$ as before, then lemma A.2 provides $1 \leq m_{\psi_{k-1}(n)}^k \leq 2^k$ such that (A.1) is true on $\mathcal{S}_{\psi_{k-1}(n)}^k = (\cup_{J_{\psi_{k-1}(n)}^k} B(x_{\psi_{k-1}(n)}^j, 2^{m_{\psi_{k-1}(n)}^k} R_{\psi_{k-1}(n)}^k))$. The union is disjoint since

$$d(x_{\psi_{k-1}(n)}^j, x_{\psi_{k-1}(n)}^i) \geq 10 \cdot 3^{k-1} R_{\psi_{k-1}(n)} \geq 5 \cdot (3/2)^{k-1} (2^{m_{\psi_{k-1}(n)}^k} R_{\psi_{k-1}(n)}^k).$$

Since $(2^{m_{\psi_{k-1}(n)}^k} R_{\psi_{k-1}(n)}^k)_n, (|J_{\psi_{k-1}(n)}^k|)_n$ are bounded in n , there is an extraction ψ_k such that

$$2^{m_{\psi_k(n)}^k} R_{\psi_k(n)}^k \rightarrow_n \mathcal{R}^k \geq 2^{k+1}, \quad J_{\psi_k(n)}^k = J^k.$$

Conclusion Since $2 \leq |J^k| \leq l$, there exists an extraction σ such that $J^{\sigma(k)} = J$ does not depend on k and $|J| \geq 2$. We consider the diagonal extraction $\psi_{\sigma(n)}(\sigma(n)) = \Psi(n)$ and set for $n \geq k$, $\mathcal{R}_n^k := 2^{m_{\Psi(n)}^{\sigma(k)}} R_{\psi(n)}^{\sigma(k)}$, $(X_n^j)_{j \in J} := (x_{\Psi(n)}^j)_{j \in J}$. By construction,

$$d(X_n^i, X_n^j) \geq 5 \cdot (3/2)^{k-1} \mathcal{R}_n^k, \quad \mathcal{R}_n^k \rightarrow_n \mathcal{R}^k \geq 2^{\sigma(k)} \rightarrow_k +\infty,$$

and for any $n \geq k$, according to lemma A.2

$$\left| \int_{(\cup_{j \in J} B(X_n^j, \mathcal{R}_n^k))^c} (c_{\Psi(n)} p - \tilde{e})(\rho_{\Psi(n)}, \phi_{\Psi(n)}) dx \right| \leq C \left(\delta \int_{(\cup_{j \in J} B(X_n^j, \mathcal{R}_n^k))^c} \tilde{e} dx + \frac{\widetilde{E_{\Psi(n)}}}{\sigma(k)} \right).$$

A.2 The case $c < 1$

In this case, for an arbitrary subset Ω we use the simple estimate :

$$\forall x \in \Omega, \quad |p| \leq \frac{(\rho - 1)^2 + \chi |\partial_1 \phi|^2}{2 \inf_{\Omega} \sqrt{\chi}}$$

Combining this with $G(\rho) = (1 - \rho)^2/2 + O((1 - \rho)^3)$, this implies for δ small

$$\exists C > 0 : \forall x \in A_n^\delta, \quad |p(\rho_n(x), \phi_n(x))| \leq \frac{\tilde{e}(\rho_n(x), \phi_n(x))}{1 - C\delta}.$$

For any set $\mathcal{S} \subset A_n^\delta$, provided n is large enough, δ small enough, we get

$$\left| \int_{\mathcal{S}} \tilde{e} - c_n p dx \right| \geq \left(1 - \frac{c_n}{1 - C\delta} \right) \int_{\mathcal{S}} \tilde{e} dx \geq \frac{1 - c}{2} \int_{\mathcal{S}} \tilde{e} dx.$$

Now lemma A.2 with $k \geq 1$, $R = 1$, $M = 10 \cdot 3^{k-1}$ provides $\mathcal{S}_n^k \subset \mathcal{A}_n^\delta$ on which equation (A.1) combined with the inequality above implies for δ small enough

$$\frac{1-c}{2} \int_{\mathcal{S}_n^k} \tilde{e} dx \leq C \left(\frac{\tilde{E}_n}{k} + \delta \int_{\mathcal{S}_n^k} \tilde{e} dx \right) \Rightarrow \int_{\mathcal{S}_n^k} \tilde{e} \lesssim \frac{\tilde{E}_n}{k}.$$

Therefore, arguing as for $c \geq 1$ we obtain extractions Ψ, σ such that for $n \geq k$

$$d(X_n^i, X_n^j) \geq 5 \cdot (3/2)^{k-1} \mathcal{R}_n^k, \quad \int_{\sqcup B(X_n^j, \mathcal{R}_n^k)} \tilde{e}(\rho_{\Psi(n)}, \phi_{\Psi(n)}) dx, \leq C \frac{\tilde{E}_n}{\sigma(k)}$$

with $\lim_n \mathcal{R}_n^k = \mathcal{R}^k \geq 2^{\sigma(k)}$.

B Remarks on the one dimensional case

The existence and stability of solitary waves for nonlinear Schrödinger type equations

$$i\partial_t \psi + \partial_x^2 \psi = g(|\psi|^2) \psi, \quad \text{with } g(\rho_0) = 0,$$

is now quite well understood. Existence follows from basic ODE technics since the corresponding equation is integrable, stability is a more delicate issue, but can nevertheless be tackled in several ways. The first approach is to consider the minimization problem $\inf\{E_{NLS}(\psi), P_{NLS}(\psi) = p\}$. Due to better Sobolev embeddings in dimension 1 it can be directly solved, the stability of minimizers then follows by the classical Cazenave-Lions [15] argument. This program has been carried at least in the Gross-Pitaevskii case $g(\rho) = \rho - 1$ in [8]. More recently D. Chiron studied extensively in [18] the stability and instability of traveling waves for very general $g(\rho)$. Among the variety of technics developed was an approach à la Grillakis-Shatah-Strauss which is very efficient in our case too. In this section, we want to underline that traveling waves of (1.1) and NLS share remarkable common features :

1. their speed is bounded by the sound speed $c_s = \sqrt{\rho_0 g'(\rho_0)}$ for (1.1), $\sqrt{2\rho_0 g'(\rho_0)}$ for NLS,
2. if there exists a traveling wave of speed $c_0 < c_s$, there exists a local branch of traveling waves parametrized by their speed as ψ_c or (ρ_c, ϕ_c) ,
3. the stability criterion is $dP_{NLS}(\psi_c)/dc < 0$, resp. $dP(\rho_c, \phi_c) < 0$.

The existence and conditional stability of solitary waves for (1.1) in dimension one was already obtained in [5] with a stability criterion that can be easily proved as equivalent to $dP/dc < 0$ (see remark 8). Nonlinear instability was left open, but using methods developed for Schrödinger type equations in [26], we will prove that $dP/dc > 0$ implies nonlinear instability. This is the only new result of this section, which is structured as follows : we rewrite the equations in a more convenient form, and show the existence of traveling waves that can

be parametrized by their speed (proposition B.1). Next we recall the stability criterion of [22] and show that its assumptions are satisfied. Finally, we prove in theorem B.4 that the failure of the stability criterion implies nonlinear instability.

Let us now turn to the equations under study. We take $\rho_\infty > 0$ and assume $g(\rho_\infty) = 0$, $g'(\rho_\infty) > 0$, we will study traveling waves with $\lim_{\pm\infty} \rho = \rho_\infty$. As in the rest of the article, we assume that g and K are smooth on $]0, +\infty[$ in order to avoid technical issues. G is the primitive of g that cancels at ρ_∞ . In order to avoid the peculiar space \dot{H}^1 , we will use a slight modification of the hamiltonian and momentum. Instead of

$$E(\rho, \phi) = \int_{\mathbb{R}} \frac{\rho|\nabla\phi|^2 + K|\nabla\rho^2|}{2} + G(\rho)dx,$$

defined for $(\rho, \phi) \in H^1 \times \dot{H}^1$, we consider

$$E(\rho, u) = \int_{\mathbb{R}} \frac{\rho|u|^2 + K|\nabla\rho^2|}{2} + G(\rho)dx, \quad P(\rho, u) = \int_{\mathbb{R}} (\rho - \rho_\infty)udx.$$

defined for $(\rho, u) \in (\rho_\infty + H^1) \times L^2$ with $\rho > 0$. For the variables (ρ, u) , the Euler-Korteweg system has the following hamiltonian structure

$$\partial_t \begin{pmatrix} \rho \\ u \end{pmatrix} = \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta E}{\delta \rho} \\ \frac{\delta E}{\delta u} \end{pmatrix} = J\delta E. \quad (\text{B.1})$$

Traveling waves of speed c can be seen as critical points of $E - cP$: if $\rho(x - ct)$, $u(x - ct)$ solves (B.1) with $\lim_{\pm\infty} \rho = \rho_\infty$, $\lim_{\pm\infty} u = 0$, then

$$\begin{cases} -c(\rho - \rho_\infty) + \rho u = 0 \\ -cu + u^2/2 + g(\rho) = K\rho'' + \frac{1}{2}K'(\rho')^2 \end{cases} \Leftrightarrow c \begin{pmatrix} \frac{\delta P}{\delta \rho} \\ \frac{\delta P}{\delta u} \end{pmatrix} = \begin{pmatrix} \frac{\delta E}{\delta \rho} \\ \frac{\delta E}{\delta u} \end{pmatrix}$$

Obviously if (ρ, u) is a traveling wave of speed c , $(\rho, -u)$ is a traveling wave of speed $-c$, therefore we focus on the case $c > 0$ (we choose not to consider the degenerate case $c = 0$). This ODE system can be elementarily integrated: from the first equation, $u = c(\rho - \rho_\infty)/\rho$, injecting this in the second equation, and multiplying it by ρ' , we obtain after integration

$$\frac{-c^2}{2\rho}(\rho - \rho_\infty)^2 + G(\rho) = \frac{1}{2}K(\rho')^2, \quad (\text{B.2})$$

Letting $x \rightarrow \infty$, we find

$$0 \leq \frac{1}{2}K(\rho')^2 = \frac{(\rho - \rho_\infty)^2}{2\rho_\infty}(\rho_\infty g'(\rho_\infty) - c^2) + O(\rho - \rho_\infty)^3.$$

We deduce the so-called subsonic condition

$$|c| \leq \sqrt{\rho_\infty g'(\rho_\infty)} := c_s.$$

Conversely, if $0 < c < \sqrt{\rho_\infty g'(\rho_\infty)}$ consider the application

$$F(\rho) = \frac{-c^2}{2\rho}(\rho - \rho_\infty)^2 + G(\rho).$$

On a neighbourhood of ρ_∞ , $F > 0$, and since $\lim_{0+} F_c(\rho) = -\infty$ we can define $\rho_m = \sup\{\rho < \rho_\infty : F(\rho) = 0\} > 0$. The set $\{(\rho, \rho') : \rho' = \pm\sqrt{F(\rho)}, \rho \in [\rho_m, \rho_\infty]\}$ forms a homoclinic orbit of the differential equation (B.2) under the (generically true) condition $F'(\rho_m) > 0$. If $F'(\rho_c) = 0$ the set corresponds to two heteroclinic profiles (of infinite energy). Symmetrically, if $\rho_M = \inf\{\rho > \rho_\infty : F(\rho) = 0\}$ is finite, the set $\{(\rho, \rho') : \rho' = \pm\sqrt{F(\rho)}, \rho \in [\rho_\infty, \rho_M]\}$ forms a homoclinic orbit if $F'(\rho_M) < 0$. We also point out the identity

$$P(\rho_c, u_c) = c \int_{\mathbb{R}} \frac{(\rho_c - \rho_\infty)^2}{\rho_c} dx, \quad (\text{B.3})$$

so that for any traveling wave with non zero speed, $P(\rho_c, u_c) \neq 0$.

Finally, consider F as a function of (ρ, c) . Given $0 < c_0 < c_s$, the condition $F(\rho_m, c_0) = 0$, $\partial_\rho F(\rho_m, c_0) \neq 0$ implies from the implicit function theorem there exists $\rho_I(c)$ smooth, defined on a neighbourhood of c_0 and a neighbourhood of (ρ_m, c) such that $F(\rho, c) = 0$ iff $\rho = \rho_I(c)$. Up to shrinking the neighbourhood of c , $\rho_I(c) = \sup\{\rho < \rho_m : F(\rho, c) = 0\}$, by continuity $\partial_\rho F(\rho_I(c), c) \neq 0$, and in particular this gives a small *branch* of solitary waves parametrized by c , that have for minimal value $\rho_I(c)$. These observations can be summarized with the following proposition.

Proposition B.1. *There exists no nontrivial traveling wave for $c > c_s$. For $0 < c < c_s$, there exists a nontrivial traveling wave if and only if at least one of the two cases is true*

- *There exists $\rho_m < \rho_\infty$ such that $F > 0$ on (ρ_m, ρ_∞) , $F(\rho_m) = 0$, $F'(\rho_m) > 0$. In this case, up to translation ρ is the solution of the Cauchy problem*

$$\begin{cases} \frac{1}{2}K'(\rho')^2 + K\rho'' = \frac{-c^2(\rho^2 - \rho_\infty)^2}{\rho^2}, \\ \rho(0) = \rho_m, \rho'(0) = 0. \end{cases}$$

It is even, decreasing on $] -\infty, 0]$.

- *There exists $\rho_M > \rho_\infty$ such that $F > 0$ on $]\rho_\infty, \rho_M[$, $F(\rho_M) = 0$, $F'(\rho_M) < 0$. In this case, up to translation ρ is the solution of the Cauchy problem*

$$\begin{cases} \frac{1}{2}K'(\rho')^2 + K\rho'' = \frac{-c^2(\rho^2 - \rho_\infty)^2}{\rho^2}, \\ \rho(0) = \rho_M, \rho'(0) = 0. \end{cases}$$

It is symmetric, increasing on $] -\infty, 0]$.

In both cases, $P(\rho, u) > 0$. Moreover, near any traveling wave of speed $c_0 < c_s$ there exists a branch of traveling waves that can be parametrized by $c \in (c_0 - \varepsilon, c_0 + \varepsilon)$ for ε small enough.

Given a branch of traveling waves defined on some interval of speeds I , we abusively denote $E(c)$, $P(c)$ the energy and momentum of the traveling wave of speed c in this branch, E' , P' their derivative with respect to c . Regarding stability, following the famous result of Grillakis-Shatah-Strauss [22], the *moment of instability* was defined in [6] as

$$m(c) = E(c) - cP(c).$$

Let us shortly summarize the framework from [22]: the Euler-Korteweg equations are seen as the hamiltonian system (B.1), it is invariant by translation, the conservation law associated to the translation invariance is the momentum $P(\rho, u)$. Since a traveling wave satisfies $\delta E - c\delta P = 0$, it is a critical point of $E - cP$.

We say that a traveling wave is conditionally orbitally stable if for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|(\rho_0, u_0) - (\rho_c, u_c)\|_{H^1 \times L^2} < \delta$ and the solution exists on $[0, T)$ then

$$\sup_{t \in [0, T)} \inf_{y \in \mathbb{R}} \|(\rho(t, \cdot + y), u(t, \cdot + y)) - (\rho_c, u_c)\|_{H^1 \times L^2} < \varepsilon.$$

Theorem B.2 ([22]). *Under the following assumptions:*

- $\delta^2 E - c\delta^2 P$ has only one negative simple eigenvalue
- its kernel is spanned by $\partial_x(\rho_c, u_c)$, the rest of its spectrum is positive bounded away from 0
- J is onto

then the traveling wave of speed c is conditionally orbitally stable if and only if $m''(c) > 0$. If J is not onto the “if” part remains true, but the “only if” part may fail.

Remark 8. An alternative version of $m''(c) > 0$ can be stated as follows: since any traveling wave of speed c is a critical point of the functional $(\rho, u) \mapsto E - cP$, we have for any $c \in I$, $E'(c) - cP'(c) = 0$, differentiating twice $E(c) - cP(c)$, we find

$$m''(c) = -P'(c),$$

so that $m'' > 0$ is equivalent to $P' < 0$. In this case the application $c \rightarrow P(c)$ is locally invertible and we may parametrize E by P . Since $dE/dP = E'/P' = c$, we have

$$\frac{d^2 E}{dP^2} = \frac{dc}{dP} < 0,$$

so that the stability condition implies the strict concavity of $E(P)$. We point out that in dimension 2 the curve $\tilde{E}_{\min}(P)$ is concave (proposition 3.2). This is an indication in favour of the stability of the traveling waves that we constructed.

Notations: (ρ_c, u_c) is a branch of traveling waves locally parametrized by their speed c . We denote $\langle \cdot, \cdot \rangle$ for both the L^2 and $(L^2)^2$ scalar product. We use the variable $r = \rho - \rho_\infty$, set $r_c := \rho_c - \rho_\infty$ and set with an abusive notation $P(r, u) := P(\rho, u)$, then

$$P(r, u) = \int ru \, dx, \quad \delta P(r, u) = \begin{pmatrix} u \\ r \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r \\ u \end{pmatrix}.$$

We denote $\mathcal{L} = \delta^2 E - c\delta^2 P$, $U_c = (r_c, u_c)$.

For any function depending on the speed f_c (and possibly on the x variable), we denote $f'_c := df_c/dc$. To avoid confusion we denote ∂_x the spatial derivative.

Spectral assumptions for $\mathcal{L} := \delta^2 E - c\delta^2 P$ They were obtained in [5] for the lagrangian formulation of the equations. The argument in the eulerian variable is slightly more involved, we include it for completeness:

$$\mathcal{L}(\rho_c, u_c) = \begin{pmatrix} \mathcal{M} & u_c - c \\ u_c - c & \rho_c \end{pmatrix}, \quad \mathcal{M}r = \left(G'' - \frac{K''(\partial_x \rho_c)^2}{2} - K' \partial_x^2 \rho_c \right) r - \partial_x (K \partial_x r). \quad (\text{B.4})$$

Due to the invariance by translation, we have $(\delta E - c\delta P)(\rho_c, u_c)(\cdot + x) = 0$, by differentiation in x we get $\mathcal{L}(\rho_c, u_c)\partial_x(\rho_c, u_c) = 0$. Conversely if $U = (U_1, U_2) \in \text{Ker}(\mathcal{L})$, we have $U_2 = \frac{c-u_c}{\rho_c} U_1$, and $U_1 \in \text{Ker}(\mathcal{M} - (u_c - c)^2/\rho_c)$. As $\mathcal{M} - (u_c - c)^2/\rho_c$ is a Sturm-Liouville type operator, its kernel is of dimension one and since $\partial_x \rho_c \in \text{Ker}(\mathcal{M} - (u_c - c)^2/\rho_c)$, there exists $\lambda \in \mathbb{R}$ such that $U_1 = \lambda \partial_x \rho_c$. Next using $u_c = c(1 - \rho_\infty/\rho_c)$

$$U_2 = \lambda \frac{c - u_c}{\rho_c} \partial_x \rho_c = \frac{\lambda c \rho_\infty}{\rho_c^2} \partial_x \rho_c = \lambda \partial_x u_c,$$

so $\partial_x(\rho_c, u_c)$ spans $\text{Ker}(\delta^2 E - c\delta^2 P)$.

Furthermore as $\partial_x \rho_c$ has exactly one zero, from Sturm Liouville's theory the operator $\mathcal{M} - \frac{(u_c - c)^2}{\rho_c}$ has exactly one negative eigenvalue. In particular, if r_- is an eigenvector associated to the negative eigenvalue and $U_- = (r_-, -(u_c - c)r_-/\rho_c)$, then

$$\langle \mathcal{L}U_-, U_- \rangle = \langle (\mathcal{M} - (u_c - c)^2/\rho_c)r_-, r_- \rangle < 0, \quad (\text{B.5})$$

so that $\delta^2 E - c\delta^2 P$ has at least one negative eigenvalue. Conversely, if $\lambda < 0$ is an eigenvalue of $\delta^2 E - c\delta^2 P$ with eigenvector (U_1, U_2) , from basic computations

$$\left(\mathcal{M} - \frac{(u_c - c)^2}{\rho_c} - \frac{\lambda(u_c - c)^2}{\rho_c(\lambda - \rho_c)} \right) U_1 = \lambda U_1,$$

so that λ is an eigenvalue of $\delta^2 E - c\delta^2 P$ if and only if it is an eigenvalue of

$$\mathcal{M}_\lambda = \mathcal{M} - \frac{(u_c - c)^2}{\rho_c} - \frac{\lambda(u_c - c)^2}{\rho_c(\lambda - \rho_c)}.$$

As the application $\lambda \in \mathbb{R}^- \rightarrow \lambda/(\lambda - \rho_c)$ is decreasing, the family \mathcal{M}_λ is decreasing too (in the sense of the scalar product). Let $\lambda_- < 0$ be the minimal eigenvalue of $\delta^2 E - c\delta^2 P$. Since $\mathcal{M}_0 = \mathcal{M} - (u_c - c)^2/\rho_c$, it has only one negative eigenvalue, and thus so does \mathcal{M}_λ for $\lambda_- \leq \lambda \leq 0$. If $\delta^2 E - c\delta^2 P$ had an other negative eigenvalue $\lambda_- < \lambda' < 0$, then λ' would be the only negative eigenvalue of $\mathcal{M}_{\lambda'}$. By monotony $\lambda' < \lambda_-$ which is absurd.

For the last condition, we have characterized the negative eigenvalue and the kernel. It suffices then to observe that thanks to the subsonic condition

$$\lim_{x \rightarrow \infty} G''(\rho_c) - \frac{K''(\rho_c)(\partial_x \rho_c)^2}{2} - K'(\rho_c)\partial_x^2 \rho_c - \frac{(u_c - c)^2}{\rho_c} = \frac{\rho_\infty g'(\rho_\infty) - c^2}{\rho_\infty} > 0,$$

thus the essential spectrum of $\mathcal{M} - \frac{(u_c - c)^2}{\rho_c}$ is positive bounded away from zero.

Theorem B.2 can now be applied :

Corollary B.3 (orbital stability, [5]). *If $-P'(c) = m''(c) > 0$, then (ρ_c, u_c) is conditionally orbitally stable.*

Remark 9. Unfortunately, the well-posedness theory from [7] only provides local existence for $(\rho(t=0), u(t=0)) \in (\rho_0 + H^{s+1}) \times H^s$, $s > 3/2$, therefore it is not clear if a smooth solution starting near a traveling wave exists for all times. At least in the case $K = 1/\rho$, one can combine the existence of global solutions to NLS that remain bounded away from 0 and use the Madelung transform to convert them into solutions of (1.1).

Remark 10. The condition $P'(c) < 0$ seems a bit easier to check than $m'' > 0$. For example for $\rho_m < \rho_\infty$ from (B.2)

$$P(\rho_c, u_c) = \int_{\mathbb{R}} (\rho_c - \rho_\infty) u_c dx = 2 \int_{\rho_m}^{\rho_\infty} \frac{c(\rho - \rho_\infty)^2}{\rho} \sqrt{2(G - \frac{c^2}{2\rho}(\rho - \rho_\infty)^2)} d\rho,$$

with ρ_m the first zero of $G - \frac{c^2}{2\rho}(\rho - \rho_\infty)^2$ below ρ_∞ .

Nonlinear instability is not a direct application of theorem B.2, indeed $\begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix}$ is not onto so the only if part can not be used. Of course there is no gain in adopting the formulation with $(\rho, \phi) \in H^1 \times \dot{H}^1$: in this case $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, but \dot{H}^1 is not a Hilbert space. Nevertheless this obstruction was overcome in various settings, in particular we shall follow the approach of Lin [26] (see also [12]) to prove the following result:

Theorem B.4. *Let (ρ_c, u_c) be a traveling wave of speed $c > 0$. If $\frac{dP}{dc} > 0$, then the traveling wave is unstable, i.e., there exists $\varepsilon > 0$ such that for any $\delta > 0$, there exists $(\rho_0, u_0) \in H^3 \times H^2$ such that $\|\rho_0 - \rho_c\|_{H^1} + \|u_0 - u_c\|_{L^2} < \delta$ and either the corresponding solution (ρ, u) blows up in finite time, or*

$$\sup_{t \in \mathbb{R}^+} \inf_{y \in \mathbb{R}} \|\rho(t, \cdot + y) - \rho_c\|_{H^1} + \|u(t, \cdot + y) - u_c\|_{L^2} \geq \varepsilon.$$

We recall the notation $U_c = (r_c, u_c) = (\rho_c - 1, u_c)$. The proof in the framework of [22] relies on the existence of a smooth curve $\psi(s) : (-\eta, \eta) \rightarrow H^1 \times L^2$ for some $\eta > 0$, with

$$\psi(0) = (r_c, u_c), \quad P(\psi(s)) = P(r_c, u_c), \quad \langle (\delta^2 E - c\delta^2 P)\psi'(0), \psi'(0) \rangle < 0.$$

It provides an “unstable direction” $y = d\psi/ds|_{s=0}$ such that

$$\langle \delta^2(E - cP)y, y \rangle < 0, \quad \langle \delta P(U_c), y \rangle = 0, \quad (\text{B.6})$$

and a Lyapunov function $A(U) = \langle -J^{-1}y, U(\cdot + x_{\min}(U)) \rangle$, where $x_{\min}(U)$ minimizes $\|(r_c, u_c) - U(\cdot + x)\|_{H^1 \times L^2}$ (see lemma B.5 below). For $0 < s \ll 1$, it is proved that the solution $(r(t), u(t))$ with Cauchy data $(r(0), u(0)) = \psi(s)$ is unstable due to some growth of $A(r(t), u(t))$. This approach raises two issues:

- $J^{-1}y$ does not exist a priori. The method in [26] is to construct $y_1 \in \text{range}(J)$ close to y , which still satisfies (B.6), and carry on the proof.
- All constructions are performed in the natural functional settings $(r, u) \in H^1 \times L^2$, but the best local well-posedness result requires $(r(0), u(0)) \in H^{s+1} \times H^s$, $s > 3/2$ (see [7]). We use a density argument to replace the unstable initial data $\psi(s) \in H^1 \times L^2$ by a regularized version.

This program requires a collection of lemmas that we prove only when there is a significant difference with [22].

Lemma B.5 (lemma 3.2 [22]). *Let*

$$V_\varepsilon = \{(r, u) \in H^1 \times L^2 : \inf_x (\|r(\cdot + x) - r_c\|_{H^1} + \|u(\cdot + x) - u_c\|_{L^2}) < \varepsilon\}.$$

For ε small enough, there exists a smooth map $x_{\min} : V_\varepsilon \rightarrow \mathbb{R}$ which realises the inf, namely :

$$\|U(\cdot + x_{\min}(U)) - U_c\|_{H^1 \times L^2} = \inf_x \|U(\cdot + x) - U_c\|_{H^1 \times L^2}.$$

Moreover $x_{\min}(U(\cdot + r)) = x_{\min}(U) - r$.

The following lemma is the only one where the lack of surjectivity of J requires some corrections.

Lemma B.6 (theorem 4.1 [22]). *There exists $y \in \text{Im}(J) \cap (H^1)^2$ such that*

$$\langle \delta P(U_c), y \rangle = 0, \quad \langle \mathcal{L}y, y \rangle < 0,$$

and a smooth curve $\psi : (-\eta, \eta) \rightarrow \{(U \in H^1 \times L^2 : P(U) = P(U_c))\}$ with

$$\frac{d\psi}{ds}(0) = y, \quad \frac{d^2\psi}{ds^2}(0) < 0.$$

In particular, $s = 0$ is a local maximum of $E(\psi(s))$.

Proof. Let U_- as in (B.5), $y_0 = \alpha U'_c + U_-$, $\alpha = -\langle \delta P(U_c), U_- \rangle / \langle \delta P(U_c), U'_c \rangle$. We have $\langle \delta P(U_c), y_0 \rangle = 0$, moreover $\delta E(U_c) - c\delta P(U_c) = 0$, by differentiation in c , $\mathcal{L}U'_c = \delta P(U_c)$. This implies

$$\begin{aligned} \langle \mathcal{L}y_0, y_0 \rangle &= \alpha \langle \mathcal{L}U'_c, y_0 \rangle + \langle \mathcal{L}U_-, y_0 \rangle = \alpha \langle \delta P(U_c), y_0 \rangle + \langle \mathcal{L}U_-, y_0 \rangle \\ &= \langle \mathcal{L}U_-, U_- \rangle + \alpha \langle \delta P(U_c), U_- \rangle \\ &= \langle \mathcal{L}U_-, U_- \rangle - \frac{(\langle \delta P(U_c), U_- \rangle)^2}{\langle \delta P(U_c), U'_c \rangle} \end{aligned}$$

From (B.5), $\langle \mathcal{L}U_-, U_- \rangle < 0$, $\langle \delta P(U_c), U'_c \rangle = P'(c) > 0$, thus $\langle \mathcal{L}y_0, y_0 \rangle < 0$.

We construct then $y \in \text{Im}(J)$ close to y_0 . From classical ODE arguments $U_c(x)$ and $U_-(x)$ converge exponentially fast to 0 at infinity, in particular $(1 + |x|)y_0(x) \in L^1$. According to [26], lemma 5.2, for any $\mu > 0$ there exists $d = (d_1, d_2) \in (H^1)^2$, $\|d\|_{H^1} < \mu$, such that

$$(1 + |x|)d \in L^1, \quad \int_{\mathbb{R}} y_0 + d \, dx = 0, \quad \int_{\mathbb{R}} d_1 u_c \, dx = \int_{\mathbb{R}} d_2 r_c \, dx = 0.$$

In particular $\langle \delta P(U_c), d \rangle = \int d_1 u_c + d_2 r_c \, dx = 0$. Let us set $y = y_0 + d$. Then by construction $\langle \delta P(U_c), y \rangle = 0$. For μ small enough

$$\langle \mathcal{L}y, y \rangle < 0, \quad \text{moreover} \quad \int_{-\infty}^x y(s) \, ds = O(1/(1 + |x|)),$$

so that $J^{-1}y$ is well defined and belongs to $(H^2)^2$. Now since $\delta P(U_c) \neq 0$, $E := \{U : \langle \delta P(U_c), U \rangle = 0\}$ is a closed hyperplane of $H^1 \times L^2$ with $y \in E$. By the implicit function theorem there exists a neighbourhood $\mathcal{U} \subset E$ and an application $F : \mathcal{U} \rightarrow H^1 \times L^2$ such that for $e \in \mathcal{U}$, $P(U_c + F(e)) = P(U_c)$, $\delta F(0) = I_d$. In particular, if we set for s small enough $\psi(s) = U_c + F(sy)$ we obtain $P(\psi(s)) = P(U_c)$, $\left. \frac{d\psi}{ds} \right|_0 = y$. Using $E(\psi(s)) = (E - cP)(\psi(s)) + cP(U_c)$ we have

$$\left. \frac{dE(\psi(s))}{ds} \right|_{s=0} = \langle (\delta E - c\delta P)(U_c), y \rangle = 0, \quad \left. \frac{d^2 E(\psi(s))}{ds^2} \right|_{s=0} = \langle \mathcal{L}y, y \rangle < 0.$$

□

The next lemmas correspond to [22] from lemma 4.2 to lemma 4.6. Let $y = (y_1, y_2)$ from lemma B.6 and define $Y := -\int_{-\infty}^x (y_2, y_1) \in (H^2)^2$ so that $JY = y$.

Lemma B.7. *The map $A : U \in V_\varepsilon \rightarrow A(U) = \langle -Y, U(\cdot + x_{\min}(U)) \rangle$ is C^1 and satisfies*

$$\forall U \in V_\varepsilon, \quad J\delta A(U) \in H^1 \times L^2, \quad J\delta A(U_c) = -y, \quad \langle \delta P(U), J\delta A(U) \rangle = 0.$$

Lemma B.8. *The differential equation $U'(\lambda) = -J\delta A(U(\lambda))$, $U(0) = U_0 \in V_\varepsilon$ defines a local flow in V_ε , denoted $R(\lambda, U_0)$. It satisfies*

$$R(\lambda, U_0)(\cdot + s) = R(\lambda, U_0(\cdot + s)), \quad (\text{B.7})$$

$$\frac{d}{d\lambda} P(R(\lambda, U_0)) = \langle \delta P(R(\lambda, U_0)), -J\delta A(R(\lambda, U_0)) \rangle = 0, \quad (\text{B.8})$$

$$\left. \frac{dR(\lambda, U_c)}{d\lambda} \right|_0 = -J\delta A(U_c) = y. \quad (\text{B.9})$$

Lemma B.9. *Let $M(U) := U(\cdot + x_{\min}(U))$, U_{-1} an eigenvector associated to the negative eigenvalue of $\delta^2 E - c\delta^2 P$. The solutions of $\langle M(R(\lambda, U)) - U_c, U_{-1} \rangle = 0$ can be parametrized as $(\Lambda(U), U)$, where Λ is a functional in $C^1(V_\varepsilon, \mathbb{R})$. For $U \in V_\varepsilon$ such that $P(U) = P(U_c)$,*

$$E(U_c) < E(U) + \Lambda(U)Q(U), \quad Q = \langle \delta E, -J\delta A \rangle. \quad (\text{B.10})$$

Lemma B.10. *For $\psi : (-\eta, \eta) \rightarrow V_\varepsilon$ given in lemma B.6, $Q(\psi(s))$ changes sign at 0.*

End of proof of theorem B.4 Since Q changes sign, there exists s such that $Q(\psi(s)) > 0$. Since $\lim_0 \psi(s) = U_c$, $\psi(s)$ can be chosen arbitrarily close to U_c . From lemma B.6 $E(\psi(s)) < E(U_c)$ and for s small enough using (B.3) we have $P(\psi(s)) > 0$. For $(\varphi_n)_{n \geq 0}$ a standard sequence of mollifiers,

$$\|\varphi_n * \psi(s) - \psi(s)\|_{H^1 \times L^2} \rightarrow 0, \quad \varphi_n * \psi(s) \in H^3 \times H^2.$$

For n large we can assume $P(\varphi_n * \psi(s)) \neq 0$, $E(\varphi_n * \psi(s)) < E(U_c)$ and we define

$$U_n = \sqrt{\frac{P(\psi(s))}{P(\varphi_n * \psi(s))}} \varphi_n * \psi(s),$$

As $P(\psi(s))/P(\varphi_n * \psi(s)) \rightarrow_n 1$, for n large enough $E(U_n) < E(U_c)$ and by construction $P(U_n) = P(U_c)$. Let $U = (r, u)(t)$ the solution of (B.1) with initial data U_n . By conservation of E and P (see [7]), and (B.10), as long as $U(t)$ remains in V_ε

$$E(U(t)) = E(U_n) < E(U_c), \quad E(U_c) < E(U(t)) + \Lambda(U(t))Q(U(t)).$$

This implies $\Lambda > 0$ and up to diminishing ε we can assume $\Lambda \leq 1$, so that $Q(U(t)) \geq E(U_c) - E(U_n) > 0$. Then if $U(t) \in V_\varepsilon$,

$$A(U(t)) \leq \|Y\|_2(\|U_c\|_2 + \varepsilon), \quad \frac{d}{dt} A(U(t)) = \langle J\delta E, \delta A \rangle = Q(U(t)) \geq E(U_c) - E(U_n),$$

which can only remain true for a finite time. Thus $U(t)$ must exit V_ε or blows up before.

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