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▶ **To cite this version:**

Adrien Barrasso, Francesco Russo. Backward Stochastic Differential Equations with no driving martingale, Markov processes and associated Pseudo Partial Differential Equations. Journal of Stochastic Analysis , 2022, 3 (1), 10.31390/josa.3.1.03 . hal-01431559v3

HAL Id: hal-01431559

https://hal.science/hal-01431559v3

Submitted on 24 Dec 2017

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Backward Stochastic Differential Equations with no driving martingale, Markov processes and associated Pseudo Partial Differential Equations.

Adrien BARRASSO * Francesco RUSSO†

December 2017

Abstract.
We discuss a class of Backward Stochastic Differential Equations (BSDEs) with no driving martingale. When the randomness of the driver depends on a general Markov process X, those BSDEs are denominated Markovian BSDEs and can be associated to a deterministic problem, called Pseudo-PDE which constitute the natural generalization of a parabolic semilinear PDE which naturally appears when the underlying filtration is Brownian. We consider two aspects of well-posedness for the Pseudo-PDEs: *classical* and *martingale* solutions.

MSC 2010 Classification. 60H30; 60H10; 35S05; 60J35; 60J60; 60J75.

KEY WORDS AND PHRASES. Martingale problem; pseudo-PDE; Markov processes; backward stochastic differential equation.

1 Introduction

This paper focuses on a new concept of Backward Stochastic Differential Equation (in short BSDE) with no driving martingale of the form

$$Y_t = \xi + \int_t^T \hat{f}\left(r, \cdot, Y_r, \sqrt{\frac{d\langle M\rangle}{dV}}(r)\right) dV_r - (M_T - M_t), \tag{1.1}$$

defined on a fixed stochastic basis fulfilling the usual conditions. V is a given bounded non-decreasing continuous adapted process, ξ (resp. $\hat{f}$) is a prescribed

*ENSTA ParisTech, Unité de Mathématiques appliquées, 828, boulevard des Maréchaux, F-91120 Palaiseau, France and Ecole Polytechnique, F-91128 Palaiseau, France. E-mail: adrien.barrasso@ensta-paristech.fr

†ENSTA ParisTech, Unité de Mathématiques appliquées, 828, boulevard des Maréchaux, F-91120 Palaiseau, France. E-mail: francesco.russo@ensta-paristech.fr

terminal condition (resp. driver). The unknown will be a couple of cadlag adapted processes (Y, M) where M is a martingale. When $V_t = t$ (1.1) is a particular case of the class of BSDEs introduced and studied by [26], for which bring a new light.

A special case of such BSDEs are the Markovian BSDEs of the form

$$Y_t^{s,x} = g(X_T) + \int_t^T f\left(r, X_r, Y_r^{s,x}, \sqrt{\frac{d\langle M^{s,x}\rangle}{dV}}(r)\right) dV_r - (M_T^{s,x} - M_t^{s,x}), \quad (1.2)$$

defined in a canonical space $\left(\Omega, \mathcal{F}^{s,x}, (X_t)_{t\in[0,T]}, (\mathcal{F}_t^{s,x})_{t\in[0,T]}, \mathrm{P}^{s,x}\right)$ where $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times E}$ corresponds to the laws (for different starting times s and starting points x) of an underlying forward Markov process with time index $[0, T]$, taking values in a Polish state space E. Indeed this Markov process is supposed to solve a *martingale problem* with respect to a given *deterministic* operator a, which is the natural generalization of stochastic differential equation in law. (1.2) will be naturally associated with a deterministic problem involving a, which will be called *Pseudo-PDE*, being of the type

$$\begin{cases} a(u)(t,x) + f\left(t, x, u(t,x), \sqrt{\Gamma(u,u)}(t,x)\right) &= 0 \quad \text{on } [0,T]\times E \\ u(T,\cdot) &= g, \end{cases} \quad (1.3)$$

where $\Gamma(u, u) = a(u^2) - 2ua(u)$ is a potential theory operator called the *carré du champs operator*. The Markovian BSDE (1.2) seems to be appropriated in the case when the forward underlying process X is a general Markov process which does not rely to a fixed reference process or random field as a Brownian motion or a Poisson measure.

The classical notion of Brownian BSDE was introduced in 1990 by E. Pardoux and S. Peng in [28], after an early work of J.M. Bismut in 1973 in [10]. It is a stochastic differential equation with prescribed terminal condition ξ and driver $\hat{f}$; the unknown is a couple (Y, Z) of adapted processes. Of particular interest is the case when the randomness of the driver is expressed through a forward diffusion process X and the terminal condition only depends on X_T. The solution, when it exists, is usually indexed by the starting time s and starting point x of the forward diffusion $X = X^{s,x}$, and it is expressed by

$$\begin{cases} X_t^{s,x} &= x + \int_s^t \mu(r, X_r^{s,x})dr + \int_s^t \sigma(r, X_r^{s,x})dB_r \\ Y_t^{s,x} &= g(X_T^{s,x}) + \int_t^T f\left(r, X_r^{s,x}, Y_r^{s,x}, Z_r^{s,x}\right) dr - \int_t^T Z_r^{s,x} dB_r, \end{cases} \quad (1.4)$$

where B is a Brownian motion. Existence and uniqueness of (1.4) (that we still indicate with BSDE) above was established first supposing essentially Lipschitz conditions on f with respect to the third and fourth variable. μ and σ were also supposed to be Lipschitz (with respect to x). In the sequel those conditions were considerably relaxed, see [30] and references therein.

In [31] and in [29] previous BSDE was linked to the semilinear PDE

$$\begin{cases} \partial_t u + \frac{1}{2} \sum_{i,j \leq d} (\sigma\sigma^\intercal)_{i,j} \partial^2_{x_i x_j} u + \sum_{i \leq d} \mu_i \partial_{x_i} u + f(\cdot, \cdot, u, \sigma \nabla u) = 0 \quad \text{on } [0, T[\times \mathrm{R}^d \\ u(T, \cdot) = g. \end{cases} \tag{1.5}$$

In particular, if (1.5) has a classical smooth solution u then $(Y^{s,x}, Z^{s,x}) := (u(\cdot, X^{s,x}_\cdot), \sigma \nabla u(\cdot, X^{s,x}_\cdot))$ solves the second line of (1.4). Conversely, only under the Lipschitz type conditions mentioned after (1.4), the solution of the BSDE can be expressed as a function of the forward process $(Y^{s,x}, Z^{s,x}) = (u(\cdot, X^{s,x}_\cdot), v(\cdot, X^{s,x}_\cdot))$, see [18]. When f and g are continuous, u is a viscosity solution of (1.5). Excepted in the case when u has some minimal differentiability properties, see e.g. [20], it is difficult to say something more on v. One major contribution of this paper consists in specifying v.

Since the pioneering work of [29], in the Brownian case, the relations between more general BSDEs and associated deterministic problems have been studied extensively, and innovations have been made in several directions.
In [5] the authors introduced a new kind of BSDE including a term with jumps generated by a Poisson measure, where an underlying forward process X solves a jump diffusion equation with Lipschitz type conditions. They associated with it an Integro-Partial Differential Equation (in short IPDE) in which some nonlocal operators are added to the classical partial differential maps, and proved that, under some continuity conditions on the coefficients, the BSDE provides a viscosity solution of the IPDE. In chapter 13 of [6], under some specific conditions on the coefficients of a Brownian BSDE, one produces a solution in the sense of distributions of the parabolic PDE. Later, the notion of mild solution of the PDE was used in [3] where the authors tackled diffusion operators generating symmetric Dirichlet forms and associated Markov processes thanks to the theory of Fukushima Dirichlet forms, see e.g. [21]. Those results were extended to the case of non symmetric Markov processes in [36]. Infinite dimensional setups were considered for example in [20] where an infinite dimensional BSDE could produce the mild solution of a PDE on a Hilbert space. Concerning the study of BSDEs driven by more general martingales than Brownian motion, we have already mentioned BSDEs driven by Poisson measures. In this respect, more recently, BSDEs driven by marked point processes were introduced in [13], see also [4]; in that case the underlying process does not contain any diffusion term. Brownian BSDEs involving a supplementary orthogonal term were studied in [18]. We can also mention the study of BSDEs driven by a general martingale in [11]. BSDEs of the same type, but with partial information have been investigated in [12]. A first approach to face deterministic problems for those equations appears in [25]; that paper also contains an application to financial hedging in incomplete markets. Finally, BSDEs in general filtered space were studied in [26] as we have already mentioned.

We come back to the motivations of the paper. Besides introducing and

studying the new class of BSDEs (1.1), (resp. Markovian BSDEs (1.2)), we study the corresponding Pseudo-PDE (1.3) and carefully explore their relations in the spirit of the existing links between (1.4) and (1.5). For the Pseudo-PDE, we analyze well-posedness at two different levels: *classical* solutions, which generalize the $C^{1,2}$-solutions of (1.5) and the so called *martingale solutions*. In the companion paper (see [8]), we also discuss other (analytical) solutions, that we denominate as *decoupled mild* solutions. The main contributions of the paper are essentially the following. In Section 3 we introduce the notion of BSDE with no driving martingale (1.1). Theorem 3.21 states existence and uniqueness of a solution for that BSDE, when the final condition ξ is square integrable and the driver $\hat{f}$ verifies some integrability and Lipschitz conditions. For technical reasons we have decided to provide an independent constructive proof from the one of [26]. Indeed we need that construction for the sequel of the paper. On the other hand, the particular form of our BSDE allows a simple and direct proof.

In Section 4, we consider an operator and its domain $(a, \mathcal{D}(a))$; V will be a continuous non-decreasing function. That section is devoted to the formulation of the martingale problem concerning our underlying process X. For each initial time s and initial point x the solution will be a probability $\mathrm{P}^{s,x}$ under which for any $\phi \in \mathcal{D}(a)$,

$$\phi(\cdot, X_\cdot) - \phi(s,x) - \int_s^\cdot a(\phi)(r, X_r) dV_r$$

is a local martingale starting in zero at time s. We will then assume that this martingale problem is well-posed and that its solution $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times E}$ defines a Markov process. In Proposition 4.10, we prove that, under each one of those probabilities, the angular bracket of every square integrable martingale is absolutely continuous with respect to dV. In Definition 4.14, we suitably define some extended domains for the operators a and Γ, using some locally convex topology. In Section 5 we introduce the Pseudo-PDE (1.3) to which we associate the Markovian BSDE (1.2), considered under every $\mathrm{P}^{s,x}$. We also introduce the notions of *classical solution* in Definition 5.3, and of *martingale solution* in Definition 5.18, which is fully probabilistic. Proposition 5.20 says the following. Classical solutions of (1.3) typically belong to the domain $\mathcal{D}(a)$ and are shown also to be essentially martingale solutions. Conversely a martingale solution belonging to $\mathcal{D}(a)$ is a classical solution, up to so called zero potential sets, see Definition 4.11. Proposition 5.9 asserts that, given a classical solution $u \in \mathcal{D}(a)$, then for any (s,x) the processes $Y^{s,x} = u(\cdot, X_\cdot)$ and $M^{s,x} = u(\cdot, X_\cdot) - u(s,x) - \int_s^\cdot f(\cdot, \cdot, u, \sqrt{\Gamma(u,u)})(r, X_r) dV_r$ solve (1.2) under the probability $\mathrm{P}^{s,x}$.

Theorems 5.21 and 5.22 state that the function u is the unique martingale solution of (1.3). Moreover v is also identified as a function of u through the an extesion of the carré du champs operator. This is the consequence of Theorem 5.15, which states that, without any assumptions of regularity, there exist Borel functions u and v such that for any $(s,x) \in [0,T] \times E$, the solution of (1.2)

verifies

$$\begin{cases} \forall t \geq s : Y_t^{s,x} = u(t, X_t) \quad \mathrm{P}^{s,x} \text{ a.s.} \\ \frac{d\langle M^{s,x}\rangle}{dV}(t) = v^2(t, X_t) \quad dV \otimes d\mathrm{P}^{s,x} \text{ a.e.} \end{cases}$$

In Section 6 we list some examples which are developed in [8]. These include Markov processes defined as weak solutions of Stochastic Differential Equations (in short SDEs) including possible jump terms, α-stable Lévy processes associated to fractional Laplace operators, solutions of SDEs with distributional drift and diffusions on compact manifolds.

2 Preliminaries

In the whole paper we will use the following notions, notations and vocabulary.

A topological space E will always be considered as a measurable space with its Borel σ-field which shall be denoted $\mathcal{B}(E)$ and if (F, d_F) is a metric space, $\mathcal{C}(E, F)$ (respectively $\mathcal{C}_b(E, F)$, $\mathcal{B}(E, F)$, $\mathcal{B}_b(E, F)$) will denote the set of functions from E to F which are continuous (respectively bounded continuous, Borel, bounded Borel).

On a fixed probability space $(\Omega, \mathcal{F}, \mathrm{P})$, for any $p \geq 1$, L^p will denote the set of random variables with finite p-th moment. A measurable space equipped with a right-continuous filtration $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathrm{T}})$ (where T is equal to R_+ or to $[0, T]$ for some $T \in \mathrm{R}_+^*$) will be called a **filtered space**. A probability space equipped with a right-continuous filtration $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathrm{T}}, \mathrm{P})$ will be called called a **stochastic basis** and will be said to **fulfill the usual conditions** if the probability space is complete and if $\mathcal{F}_0$ contains all the P-negligible sets. We introduce now some notations and vocabulary about spaces of stochastic processes, on a fixed stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathrm{T}}, \mathrm{P})$. Most of them are taken or adapted from [23] or [24]. A process $(X_t)_{t\in\mathrm{T}}$ is said to be **integrable** if X_t is an integrable r.v. for any t. We will denote $\mathcal{V}$ (resp $\mathcal{V}^+$) the set of adapted, bounded variation (resp non-decreasing) processes starting at 0; $\mathcal{V}^p$ (resp $\mathcal{V}^{p,+}$) the elements of $\mathcal{V}$ (resp $\mathcal{V}^+$) which are predictable, and $\mathcal{V}^c$ (resp $\mathcal{V}^{c,+}$) the elements of $\mathcal{V}$ (resp $\mathcal{V}^+$) which are continuous. If $A \in \mathcal{V}$, we will denote $Pos(A)$ and $Neg(A)$ the positive variation and negative variation parts of A, meaning the unique pair of elements $\mathcal{V}^+$ such that $A = Pos(A) - Neg(A)$ (see Proposition I.3.3 in [24] for their existence) and $Var(A) = Pos(A) + Neg(A)$ its total variation. $\mathcal{M}$ will be the space of cadlag martingales. For any $p \in [1, \infty]$ $\mathcal{H}^p$ will denote the Banach space of elements of $\mathcal{M}$ for which $\|M\|_{\mathcal{H}^p} := \mathrm{E}[|\sup_{t\in\mathrm{T}} M_t|^p]^{\frac{1}{p}} < \infty$ and in this set we identify indistinguishable elements. $\mathcal{H}_0^p$ will denote the Banach subspace of $\mathcal{H}^p$ of elements vanishing at zero.
If $\mathrm{T} = [0, T]$ for some $T \in \mathrm{R}_+^*$, a stopping time will take values in $[0, T] \cup \{+\infty\}$. We define a **localizing sequence of stopping times** as an a.s. increasing sequence of stopping times $(\tau_n)_{n\geq 0}$ such that there a.s. exists $N \in \mathrm{N}$ for which $\tau_N = +\infty$. Let Y be a process and τ a stopping time, we denote by Y^τ the

stopped process $t \mapsto Y_{t\wedge\tau}$. If $\mathcal{C}$ is a set of processes, we define its **localized class** $\mathcal{C}_{loc}$ as the set of processes Y such that there exists a localizing sequence $(\tau_n)_{n\geq 0}$ such that for every n, the stopped process Y^{τ_n} belongs to $\mathcal{C}$. In particular a process X is said to be locally integrable (resp. locally square integrable) if there is a localizing sequence $(\tau_n)_{n\geq 0}$ such that for every n, $X_t^{\tau_n}$ is integrable (resp. square integrable) for every t.
For any $M \in \mathcal{M}_{loc}$, we denote $[M]$ its **quadratic variation** and if moreover $M \in \mathcal{H}^2_{loc}$, $\langle M\rangle$ will denote its (predictable) **angular bracket**. $\mathcal{H}^2_0$ will be equipped with scalar product defined by $(M,N)_{\mathcal{H}^2_0} = \mathrm{E}[M_T N_T] = \mathrm{E}[\langle M,N\rangle_T]$ which makes it a Hilbert space. Two local martingales M, N will be said to be **strongly orthogonal** if MN is a local martingale starting in 0 at time 0. In $\mathcal{H}^2_{0,loc}$ this notion is equivalent to $\langle M,N\rangle = 0$.

3 BSDEs without driving martingale

In the whole present section we are given $T \in \mathrm{R}_+^*$, and a stochastic basis $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathrm{P}\big)$ fulfilling the usual conditions. Some proofs and intermediary results of the first part of this section are postponed to Appendix B.

Definition 3.1. *Let A and B be in $\mathcal{V}^+$. We will say that dB dominates dA* **in the sense of stochastic measures** *(written $dA \ll dB$) if for almost all ω, $dA(\omega) \ll dB(\omega)$ as Borel measures on $[0,T]$.*

We will say that dB and dA are mutually singular **in the sense of stochastic measures** *(written $dA \perp dB$) if for almost all ω, the Borel measures $dA(\omega)$ and $dB(\omega)$ are mutually singular.*

Let $B \in \mathcal{V}^+$. $dB \otimes d\mathrm{P}$ will denote the positive measure on $(\Omega \times [0,T], \mathcal{F}\otimes\mathcal{B}([0,T]))$ defined for any $F \in \mathcal{F}\otimes\mathcal{B}([0,T])$ by $dB\otimes d\mathrm{P}(F) = \mathrm{E}\left[\int_0^T \mathbb{1}_F(r,\omega)dB_r(\omega)\right]$. A property which holds true everywhere except on a null set for this measure will be said to be true $dB\otimes d\mathrm{P}$ almost everywhere (a.e).

The proof of Proposition below is in Appendix B.

Proposition 3.2. *For any A and B in $\mathcal{V}^{p,+}$, there exists a (non-negative $dB\otimes d\mathrm{P}$ a.e.) predictable process $\frac{dA}{dB}$ and a process in $\mathcal{V}^{p,+}$ $A^{\perp B}$ such that*

$$dA^{\perp B} \perp dB \text{ and } A = A^B + A^{\perp B} \text{ a.s.}$$

where $A^B = \int_0^\cdot \frac{dA}{dB}(r)dB_r$. The process $A^{\perp B}$ is unique and the process $\frac{dA}{dB}$ is unique $dB\otimes d\mathrm{P}$ a.e.
Moreover, there exists a predictable process K with values in $[0,1]$ (for every (ω,t)), such that $A^B = \int_0^\cdot \mathbb{1}_{\{K_r<1\}}dA_r$ and $A^{\perp B} = \int_0^\cdot \mathbb{1}_{\{K_r=1\}}dA_r$.

The predictable process $\frac{dA}{dB}$ appearing in the statement of Proposition 3.2 will beF the **Radon-Nikodym derivative** of A by B.

Remark 3.3. *Since for any $s < t$ $A_t - A_s = \int_s^t \frac{dA}{dB}(r)dB_r + A_t^{\perp B} - A_s^{\perp B}$ a.s. where $A^{\perp B}$ is increasing, it is clear that for any $s < t$,*
$\int_s^t \frac{dA}{dB}(r)dB_r \leq A_t - A_s$ a.s. and therefore that for any positive measurable process ϕ we have $\int_0^T \phi_r \frac{dA}{dB}(r)dB_r \leq \int_0^T \phi_r dA_r$ a.s.

If A is in $\mathcal{V}^p$, and $B \in \mathcal{V}^{p,+}$. We set $\frac{dA}{dB} := \frac{dPos(A)}{dB} - \frac{dPos(A)}{dB}$ and $A^{\perp B} := (Pos(A))^{\perp B} - (Neg(A))^{\perp B}$.

Proposition 3.4. *Let A_1 and A_2 be in $\mathcal{V}^p$, and $B \in \mathcal{V}^{p,+}$. Then,*
$\frac{d(A_1+A_2)}{dB} = \frac{dA_1}{dB} + \frac{dA_2}{dB}$ $dV \otimes d\mathrm{P}$ a.e. and $(A_1 + A_2)^{\perp B} = A_1^{\perp B} + A_2^{\perp B}$.

Proof. The proof is an immediate consequence of the uniqueness of the decomposition (3.2). □

Let $V \in \mathcal{V}^{p,+}$. We introduce two significant spaces related to V.
$\mathcal{H}^{2,V} := \{M \in \mathcal{H}_0^2 | d\langle M\rangle \ll dV\}$ and $\mathcal{H}^{2,\perp V} := \{M \in \mathcal{H}_0^2 | d\langle M\rangle \perp dV\}$.

The proof of the two propositions below are in Appendix B.

Proposition 3.5. *Let $M \in \mathcal{H}_0^2$, and let $V \in \mathcal{V}^{p,+}$. There exists a pair $(M^V, M^{\perp V})$ in $\mathcal{H}^{2,V} \times \mathcal{H}^{2,\perp V}$ such that $M = M^V + M^{\perp V}$ and $\langle M^V, M^{\perp V}\rangle = 0$.*

Moreover, we have $\langle M^V\rangle = \langle M\rangle^V = \int_0^\cdot \frac{d\langle M\rangle}{dV}(r)dV_r$ and $\langle M^{\perp V}\rangle = \langle M\rangle^{\perp V}$ and there exists a predictable process K with values in $[0, 1]$ such that $M^V = \int_0^\cdot \mathbb{1}_{\{K_r<1\}}dM_r$ and $M^{\perp V} = \int_0^\cdot \mathbb{1}_{\{K_r=1\}}dM_r$.

Proposition 3.6. *$\mathcal{H}^{2,V}$ and $\mathcal{H}^{2,\perp V}$ are orthogonal sub-Hilbert spaces of $\mathcal{H}_0^2$ and $\mathcal{H}_0^2 = \mathcal{H}^{2,V} \oplus^\perp \mathcal{H}^{2,\perp V}$. Moreover, any element of $\mathcal{H}_{loc}^{2,V}$ is strongly orthogonal to any element of $\mathcal{H}_{loc}^{2,\perp V}$.*

Remark 3.7. *All previous results extend when the filtration is indexed by R_+.*

We are going to introduce here our Backward Stochastic Differential Equation (BSDE) for which there is no need for having a particular martingale of reference.

We will denote $\mathcal{P}ro$ the σ-field generated by progressively measurable processes defined on $[0, T] \times \Omega$.
Given some $V \in \mathcal{V}^{c,+}$, we will indicate by $\mathcal{L}^2(dV \otimes d\mathrm{P})$ (resp. $\mathcal{L}^0(dV \otimes d\mathrm{P})$) the set of (up to indistinguishability) progressively measurable processes ϕ such that $\mathrm{E}[\int_0^T \phi_r^2 dV_r] < \infty$ (resp. $\int_0^T |\phi_r| dV_r < \infty$ P a.s.) and $L^2(dV \otimes d\mathrm{P})$ the quotient space of $\mathcal{L}^2(dV \otimes d\mathrm{P})$ with respect to the subspace of processes equal to zero $dV \otimes d\mathrm{P}$ a.e. More formally, $L^2(dV \otimes d\mathrm{P})$ corresponds to the classical L^2 space $L^2([0, T] \times \Omega, \mathcal{P}ro, dV \otimes d\mathrm{P})$ and is therefore complete for its usual norm.
$\mathcal{L}^{2,cadlag}(dV \otimes d\mathrm{P})$ (resp. $L^{2,cadlag}(dV \otimes d\mathrm{P})$) will denote the subspace of $\mathcal{L}^2(dV \otimes d\mathrm{P})$ (resp. $L^2(dV \otimes d\mathrm{P})$) of cadlag elements (resp. of elements having a cadlag representative). We emphasize that $L^{2,cadlag}(dV \otimes d\mathrm{P})$ is not a closed

subspace of $L^2(dV \otimes d\mathrm{P})$.
The application which associates to a process its corresponding class will be denoted $\phi \mapsto \dot{\phi}$.

The aforementioned BSDE will depend on a triple (V, ξ, f) of coefficients: V is an integrator process, ξ is the **final** condition, f is the **driver**.

Hypothesis 3.8. *1. V is bounded continuous non-decreasing adapted process;*

2. ξ is a square integrable $\mathcal{F}_T$-measurable r.v.

3. $\hat{f} : ([0,T] \times \Omega) \times \mathrm{R} \times \mathrm{R} \longrightarrow \mathrm{R}$, measurable with respect to $\mathcal{P}ro \otimes \mathcal{B}(\mathrm{R}) \otimes \mathcal{B}(\mathrm{R})$.

4. $\hat{f}(\cdot, \cdot, 0, 0) \in \mathcal{L}^2(dV \otimes d\mathrm{P})$.

5. There exist positive constants K^Y, K^Z such that, P a.s. we have for all t, y, y', z, z',

$$|\hat{f}(t, \cdot, y, z) - \hat{f}(t, \cdot, y', z')| \leq K^Y |y - y'| + K^Z |z - z'|. \tag{3.1}$$

We start with a lemma.

Lemma 3.9. *Let U_1 and U_2 be in $\mathcal{L}^2(dV \otimes d\mathrm{P})$ and such that $\dot{U}_1 = \dot{U}_2$. Let $F : [0,T] \times \Omega \times \mathrm{R} \longrightarrow \mathrm{R}$ be such that $F(\cdot, \cdot, U_1)$ and $F(\cdot, \cdot, U_2)$ are in $\mathcal{L}^0(dV \otimes d\mathrm{P})$, then the processes $\int_0^\cdot F(r, \cdot, U_r^1) dV_r$ and $\int_0^\cdot F(r, \cdot, U_r^2) dV_r$ are indistinguishable.*

Proof. There exists a P-null set $\mathcal{N}$ such that for any $\omega \in \mathcal{N}^c$, $U^1(\omega) = U^2(\omega)$ $dV(\omega)$ a.e. So for any $\omega \in \mathcal{N}^c$, $F(\cdot, \omega, U^1(\omega)) = F(\cdot, \omega, U^2(\omega))$ $dV(\omega)$ a.e. implying $\int_0^\cdot F(r, \omega, U_r^1(\omega)) dV_r(\omega) = \int_0^\cdot F(r, \omega, U_r^2(\omega)) dV_r(\omega)$. So $\int_0^\cdot F(r, \cdot, U_r^1) dV_r$ and $\int_0^\cdot F(r, \cdot, U_r^2) dV_r$ are indistinguishable processes. □

In some of the following proofs, we will have to work with classes of processes. According to Lemma 3.9, if $\dot{U}$ is an element of $L^2(dV \otimes d\mathrm{P})$, the integral $\int_0^\cdot F(r, \omega, U_r) dV_r$ will not depend on the representantive process U that we have chosen.

We will now give the formulation of our BSDE.

Definition 3.10. *We say that a couple $(Y, M) \in \mathcal{L}^{2,cadlag}(dV \otimes d\mathrm{P}) \times \mathcal{H}_0^2$ is a solution of $BSDE(\xi, \hat{f}, V)$ if it verifies*

$$Y = \xi + \int_\cdot^T \hat{f}\left(r, \cdot, Y_r, \sqrt{\frac{d\langle M \rangle}{dV}}(r)\right) dV_r - (M_T - M_\cdot) \tag{3.2}$$

in the sense of indistinguishability.

Proposition 3.11. *If (Y,M) solves $BSDE(\xi,\hat{f},V)$, and if we denote $\hat{f}\left(r,\cdot,Y_r,\sqrt{\frac{d\langle M\rangle}{dV}}(r)\right)$ by $\hat{f}_r$, then for any $t\in[0,T]$, a.s. we have*

$$\left\{\begin{array}{rcl} Y_t &=& \mathrm{E}\left[\xi+\int_t^T \hat{f}_r dV_r\middle|\mathcal{F}_t\right] \\ M_t &=& \mathrm{E}\left[\xi+\int_0^T \hat{f}_r dV_r\middle|\mathcal{F}_t\right]-\mathrm{E}\left[\xi+\int_0^T \hat{f}_r dV_r\middle|\mathcal{F}_0\right]. \end{array}\right. \tag{3.3}$$

Proof. Since $Y_t=\xi+\int_t^T \hat{f}_r dV_r-(M_T-M_t)$ a.s., Y being an adapted process and M a martingale, taking the expectation in (3.2) at time t, we directly get $Y_t=\mathrm{E}\left[\xi+\int_t^T \hat{f}_r dV_r\middle|\mathcal{F}_t\right]$ and in particular that $Y_0=\mathrm{E}\left[\xi+\int_0^T \hat{f}_r dV_r\middle|\mathcal{F}_0\right]$. Since $M_0=0$, looking at the BSDE at time 0 we get $M_T=\xi+\int_0^T \hat{f}_r dV_r-Y_0=\xi+\int_0^T \hat{f}_r dV_r-\mathrm{E}\left[\xi+\int_0^T \hat{f}_r dV_r\middle|\mathcal{F}_0\right]$. Taking the expectation with respect to $\mathcal{F}_t$ in the above inequality gives the second line of (3.3).

□

We will proceed showing that $BSDE(\xi,\hat{f},V)$ has a unique solution. At this point we introduce a significant map Φ which will map $L^2(dV\otimes d\mathrm{P})\times\mathcal{H}_0^2$ into its subspace $L^{2,cadlag}(dV\otimes d\mathrm{P})\times\mathcal{H}_0^2$. From now on, until Notation 3.15, we fix a couple $(\dot{U},N)\in L^2(dV\otimes d\mathrm{P})\times\mathcal{H}_0^2$ to which we will associate $(\dot{Y},M)$ which, as we will show, will belong to $L^{2,cadlag}(dV\otimes d\mathrm{P})\times\mathcal{H}_0^2$. We will show that $(\dot{U},N)\mapsto(\dot{Y},M)$ is a contraction for a certain norm. In all the proofs below, $\dot{U}$ will only appear in integrals driven by dV through a representative U.

Proposition 3.12. *For any $t\in[0,T]$, $\int_t^T \hat{f}^2\left(r,\cdot,U_r,\sqrt{\frac{d\langle N\rangle}{dV}}(r)\right)dV_r$ is in L^1 and $\left(\int_t^T \hat{f}\left(r,\cdot,U_r,\sqrt{\frac{d\langle N\rangle}{dV}}(r)\right)dV_r\right)$ is in L^2.*

Proof. By Cauchy-Schwarz inequality and thanks to the boundedness of V together the Lipschitz conditions on f in Hypothesis 3.8, there exist a positive constant C such that, for any $t\in[0,T]$, we have

$$\begin{aligned}&\left(\int_t^T \hat{f}\left(r,\cdot,U_r,\sqrt{\frac{d\langle N\rangle}{dV}}(r)\right)dV_r\right)^2\leq V_T^2\int_t^T \hat{f}^2\left(r,\cdot,U_r,\sqrt{\frac{d\langle N\rangle}{dV}}(r)\right)dV_r\\ &\leq C\left(\int_t^T \hat{f}^2\left(r,\cdot,0,0\right)dV_r+\int_t^T U_r^2 dV_r+\int_t^T \frac{d\langle N\rangle}{dV}(r)dV_r\right).\end{aligned} \tag{3.4}$$

The three terms on the right are in L^1. Indeed, by Remark 3.3 $\int_t^T \frac{d\langle N\rangle}{dV}(r)dV_r\leq(\langle N\rangle_T-\langle N\rangle_t)$ which belongs to L^1 since N is taken in $\mathcal{H}^2$. By Hypothesis 3.8, $f(\cdot,\cdot,0,0)$ is in $\mathcal{L}^2(dV\otimes d\mathrm{P})$, and $\dot{U}$ was also taken in $L^2(dV\otimes d\mathrm{P})$. This concludes the proof. □

We can therefore state the following definition.

Definition 3.13. *Setting* $\hat{f}_r = \hat{f}\left(r, \cdot, U_r, \sqrt{\frac{d\langle N\rangle}{dV}}(r)\right)$*, we define* M *as the cadlag version of the martingale* $t \mapsto \mathrm{E}\left[\xi + \int_0^T \hat{f}_r dV_r \middle| \mathcal{F}_t\right] - \mathrm{E}\left[\xi + \int_0^T \hat{f}_r dV_r \middle| \mathcal{F}_0\right]$.

It admits a cadlag version taking into account Theorem 4 in Chapter IV of [15], since the stochastic basis fulfills the usual conditions. We denote by Y *the cadlag process defined by* $Y_t = \xi + \int_t^T \hat{f}\left(r, \cdot, U_r, \sqrt{\frac{d\langle N\rangle}{dV}}(r)\right) dV_r - (M_T - M_t)$*. This will be called the* **cadlag reference process** *and we will often omit its dependence to* $(\dot{U}, N)$*.*

According to previous definition, it is not clear whether Y is adapted, however, we have the almost sure equalities

$$\begin{aligned} Y_t &= \xi + \int_t^T \hat{f}_r dV_r - (M_T - M_t) \\ &= \xi + \int_t^T \hat{f}_r dV_r - \left(\xi + \int_0^T \hat{f}_r dV_r - \mathrm{E}\left[\xi + \int_0^T \hat{f}_r dV_r \middle| \mathcal{F}_t\right]\right) \\ &= \mathrm{E}\left[\xi + \int_0^T \hat{f}_r dV_r \middle| \mathcal{F}_t\right] - \int_0^t \hat{f}_r dV_r \\ &= \mathrm{E}\left[\xi + \int_t^T \hat{f}_r dV_r \middle| \mathcal{F}_t\right]. \end{aligned} \tag{3.5}$$

Since Y is cadlag and adapted, by Theorem 15 Chapter IV of [14], it is progressively measurable.

Proposition 3.14. *M belongs to $\mathcal{H}_0^2$ and* $\sup_{t\in[0,T]} |Y_t| \in L^2$*.*

Proof. M is square integrable and vanishes at 0 by Definition 3.13 and Proposition 3.12. A consequence of Definition 3.13, of Cauchy-Schwarz inequality and of the boundedness of V is the existence of some $C, C' > 0$ such that, a.s.,

$$\begin{aligned} \sup_{t\in[0,T]} Y_t^2 &\leq C\left(\xi^2 + \sup_{t\in[0,T]}\left(\int_t^T \hat{f}_r dV_r\right)^2 + \sup_{t\in[0,T]}(M_T - M_t)^2\right) \\ &\leq C'\left(\xi^2 + \int_0^T \hat{f}_r^2 dV_r + \sup_{t\in[0,T]} M_t^2\right) \end{aligned} \tag{3.6}$$

which belongs to L^1 by Proposition 3.12 and the fact that ξ and M are square integrable. □

Since Y is cadlag progressively measurable, $\sup_{t\in[0,T]} |Y_t| \in L^2$ and since V is bounded, it is clear that $Y \in \mathcal{L}^{2,cadlag}(dV \otimes d\mathrm{P})$ and the corresponding class $\dot{Y}$ belongs to $L^{2,cadlag}(dV \otimes d\mathrm{P})$.

Notation 3.15. *We denote by* Φ *the operator which associates to a couple* $(\dot{U}, N)$ *the couple* $(\dot{Y}, M)$*.*

$$\Phi : \begin{array}{rcl} L^2(dV \otimes d\mathrm{P}) \times \mathcal{H}_0^2 & \longrightarrow & L^{2,cadlag}(dV \otimes d\mathrm{P}) \times \mathcal{H}_0^2 \\ (\dot{U}, N) & \longmapsto & (\dot{Y}, M). \end{array}$$

Proposition 3.16. *The mapping $(Y,M) \longmapsto (\dot{Y},M)$ induces a bijection between the set of solutions of $BSDE(\xi,\hat{f},V)$ and the set of fixed points of Φ.*

Proof. First, let (U,N) be a solution of $BSDE(\xi,\hat{f},V)$, let $(\dot{Y},M) := \Phi(\dot{U},N)$ and let Y be the reference cadlag process associated to U as in Definition 3.13. By this same definition, M is the cadlag version of
$t \mapsto \mathrm{E}\left[\xi + \int_0^T \hat{f}\left(r,\cdot,U_r,\sqrt{\frac{d\langle N\rangle}{dV}}(r)\right) dV_r \middle| \mathcal{F}_t\right] - \mathrm{E}\left[\xi + \int_0^T \hat{f}\left(r,\cdot,U_r,\sqrt{\frac{d\langle N\rangle}{dV}}(r)\right) dV_r \middle| \mathcal{F}_0\right]$,
but by Proposition 3.11, so is N, meaning $M = N$. Again by Definition 3.13, $Y = \xi + \int_\cdot^T \hat{f}\left(r,\cdot,U_r,\sqrt{\frac{d\langle N\rangle}{dV}}(r)\right) dV_r - (N_T - N_\cdot)$ which is equal to U thanks to (3.2), so $Y = U$ in the sense of indistinguishability, and in particular, $\dot{U} = \dot{Y}$, implying $(\dot{U},N) = (\dot{Y},M) = \Phi(\dot{U},N)$. The mapping $(Y,M) \longmapsto (\dot{Y},M)$ therefore does indeed map the set of solutions of $BSDE(\xi,\hat{f},V)$ into the set of fix points of Φ.

The map is surjective. Indeed let $(\dot{U},N)$ be a fixed point of Φ, the couple (Y,M) of Definition 3.13 verifies $Y = \xi + \int_\cdot^T \hat{f}\left(r,\cdot,U_r,\sqrt{\frac{d\langle N\rangle}{dV}}(r)\right) dV_r - (M_T - M_\cdot)$ in the sense of indistinguishability, and $(\dot{Y},M) = \Phi(\dot{U},N) = (\dot{U},N)$, so by Lemma 3.9, $\int_\cdot^T \hat{f}\left(r,\cdot,Y_r,\sqrt{\frac{d\langle M\rangle}{dV}}(r)\right) dV_r$ and $\int_\cdot^T \hat{f}\left(r,\cdot,U_r,\sqrt{\frac{d\langle N\rangle}{dV}}(r)\right) dV_r$ are indistinguishable and $Y = \xi + \int_\cdot^T \hat{f}\left(r,\cdot,Y_r,\sqrt{\frac{d\langle M\rangle}{dV}}(r)\right) dV_r - (M_T - M_\cdot)$, meaning that (Y,M) solves $BSDE(\xi,\hat{f},V)$.

We finally show that it is injective. Let us consider two solutions (Y^1,M) and (Y^2,M) of $BSDE(\xi,\hat{f},V)$ with $\dot{Y^1} = \dot{Y^2}$. By Lemma 3.9, the processes $\int_\cdot^T \hat{f}\left(r,\cdot,Y^1_r,\sqrt{\frac{d\langle M\rangle}{dV}}(r)\right) dV_r$ and $\int_\cdot^T \hat{f}\left(r,\cdot,Y^2_r,\sqrt{\frac{d\langle M\rangle}{dV}}(r)\right) dV_r$ are indistinguishable, so taking (3.2) into account, we have $Y^1 = Y^2$.

□

From now on, if $(\dot{Y},M)$ is the image by Φ of a couple $(\dot{U},N) \in L^2(dV \otimes d\mathrm{P}) \times \mathcal{H}^2_0$, by default, we will always refer to the cadlag reference process Y of $\dot{Y}$ defined in Definition 3.13.

Lemma 3.17. *Let Y be a cadlag adapted process satisfying $\mathrm{E}\left[\sup_{t\in[0,T]} Y_t^2\right] < \infty$ and M be a square integrable martingale. Then there exists a constant $C > 0$ such that for any $\epsilon > 0$ we have*

$$\mathrm{E}\left[\sup_{t\in[0,T]} \left|\int_0^t Y_{r-}\, dM_r\right|\right] \leq C\left(\frac{\epsilon}{2}\mathrm{E}\left[\sup_{t\in[0,T]} Y_t^2\right] + \frac{1}{2\epsilon}\mathrm{E}\left[[M]_T\right]\right).$$

In particular, $\int_0^\cdot Y_{r-}\, dM_r$ is a uniformly integrable martingale.

Proof. By Burkholder-Davis-Gundy and Cauchy-Schwarz inequalities, there exists $C > 0$ such that

$$\begin{aligned} & \mathrm{E}\left[\sup_{t\in[0,T]}\left|\int_0^t Y_{r^-}dM_r\right|\right] & \leq & \; C\mathrm{E}\left[\sqrt{\int_0^T Y_{r^-}^2 d[M]_r}\right] \\ \leq & \; C\mathrm{E}\left[\sqrt{\sup_{t\in[0,T]} Y_t^2 [M]_T}\right] & \leq & \; C\sqrt{\mathrm{E}\left[\sup_{t\in[0,T]} Y_t^2\right]\mathrm{E}[[M]]_T} \\ \leq & \; C\left(\tfrac{\epsilon}{2}\mathrm{E}\left[\sup_{t\in[0,T]} Y_t^2\right] + \tfrac{1}{2\epsilon}\mathrm{E}\left[[M]_T\right]\right) & < & \; +\infty. \end{aligned}$$

So $\int_0^\cdot Y_{r^-}dM_r$ is a uniformly integrable local martingale, and therefore a martingale. □

Lemma 3.18. *Let Y be a cadlag adapted process and $M \in \mathcal{H}^2$. Assume the existence of a constant $C > 0$ and an L^1-random variable Z such that for any $t \in [0,T]$, $Y_t^2 \leq C\left(Z + \left|\int_0^t Y_{r^-}dM_r\right|\right)$. Then $\sup_{t\in[0,T]} |Y_t| \in L^2$.*

Proof. For any stopping time τ we have

$$\sup_{t\in[0,\tau]} Y_t^2 \leq C\left(Z + \sup_{t\in[0,\tau]}\left|\int_0^t Y_{r^-}dM_r\right|\right). \tag{3.7}$$

Since Y_{t^-} is caglad and therefore locally bounded, (see Definition p164 in [32]) we define $\tau_n = \inf\{t > 0 : Y_{t^-} \geq n\}$. It yields $\int_0^{\cdot\wedge\tau_n} Y_{r^-}dM_r$ is in $\mathcal{H}^2$ since its angular bracket is equal to $\int_0^{\cdot\wedge\tau_n} Y_{r^-}^2 d\langle M\rangle_r$ which is inferior to $n^2\langle M\rangle_T \in L^1$. By Doob's inequality we know that $\sup_{t\in[0,\tau_n]}\left|\int_0^t Y_{r^-}dM_r\right|$ is L^2 and using (3.7), we get that $\sup_{t\in[0,\tau_n]} Y_t^2$ is L^1. By (3.7) applied with τ_n and taking expectation, we get $\mathrm{E}\left[\sup_{t\in[0,\tau_n]} Y_t^2\right] \leq C'\left(1 + \mathrm{E}\left[\sup_{t\in[0,\tau_n]}\left|\int_0^t Y_{r^-}dM_r\right|\right]\right)$, for some C' which does not depend on n. By Lemma 3.17 applied to (Y^{τ_n}, M) there exists $C'' > 0$ such that for any $n \in \mathrm{N}^*$ and $\epsilon > 0$,
$\mathrm{E}\left[\sup_{t\in[0,\tau_n]} Y_t^2\right] \leq C''\left(1 + \frac{\epsilon}{2}\mathrm{E}\left[\sup_{t\in[0,\tau_n]} Y_t^2\right] + \frac{1}{2\epsilon}\mathrm{E}\left[[M]_T\right]\right)$. Choosing $\epsilon = \frac{1}{C''}$, it follows that there exists $C_3 > 0$ such that for any $n > 0$,
$\frac{1}{2}\mathrm{E}\left[\sup_{t\in[0,\tau_n]} Y_t^2\right] \leq C_3\left(1 + \mathrm{E}\left[[M]_T\right]\right) < \infty$. By monotone convergence theorem, taking the limit in n we get the result. □

Proposition 3.19. *Let $\lambda \in \mathrm{R}$, let $(\dot{U}, N)$, $(\dot{U}', N')$ be in $L^2(dV \otimes d\mathrm{P}) \times \mathcal{H}_0^2$, let $(\dot{Y}, M)$, $(\dot{Y}', M')$ be their images by Φ and let Y, Y' be the cadlag representatives of $\dot{Y}$, $\dot{Y}'$ introduced in Definition 3.13. Then*

$\int_0^\cdot e^{\lambda V_r} Y_{r-} dM_r$, $\int_0^\cdot e^{\lambda V_r} Y'_{r-} dM'_r$, $\int_0^\cdot e^{\lambda V_r} Y_{r-} dM'_r$ *and* $\int_0^\cdot e^{\lambda V_r} Y'_{r-} dM_r$ *are martingales.*

Proof. Thanks to Proposition 3.14 we know that $\sup_{t\in[0,T]} |Y_t|$ and $\sup_{t\in[0,T]} |Y'_t|$ are L^2. Moreover since M and M' are square integrable, the statement yields therefore as a consequence of Lemma 3.17 and the fact that V is bounded. □

We will now show that Φ is a contraction for a certain norm. This will imply that it has a unique fixed point in $L^2(dV \otimes d\mathrm{P}) \times \mathcal{H}_0^2$ since this space is complete and therefore that $BSDE(\xi, \hat{f}, V)$ has a unique solution thanks to Proposition 3.16.
For any $\lambda > 0$, on $L^2(dV \otimes d\mathrm{P}) \times \mathcal{H}_0^2$ we define the norm $\|(\dot{Y}, M)\|_\lambda^2 := \mathrm{E}\left[\int_0^T e^{\lambda V_r} Y_r^2 dV_r\right] + \mathrm{E}\left[\int_0^T e^{\lambda V_r} d\langle M\rangle_r\right]$. Since V is bounded, these norms are all equivalent to the usual one of this space, which corresponds to $\lambda = 0$.

Proposition 3.20. *There exists $\lambda > 0$ such that for any $(\dot{U}, N) \in L^2(dV \otimes d\mathrm{P}) \times \mathcal{H}_0^2$, $\left\|\Phi(\dot{U}, N)\right\|_\lambda^2 \leq \frac{1}{2}\left\|(\dot{U}, N)\right\|_\lambda^2$. In particular, Φ is a contraction in $L^2(dV \otimes d\mathrm{P}) \times \mathcal{H}_0^2$ for the norm $\|\cdot\|_\lambda$.*

Proof. Let $(\dot{U}, N)$ and $(\dot{U}', N')$ be two couples of $L^2(dV \otimes d\mathrm{P}) \times \mathcal{H}_0^2$, let $(\dot{Y}, M)$ and $(\dot{Y}', M')$ be their images via Φ and let Y, Y' be the cadlag reference process of $\dot{Y}$, $\dot{Y}'$ introduced in Definition 3.13. We will write $\bar{Y}$ for $Y - Y'$ and we adopt a similar notation for other processes. We will also write
$\bar{f}_t := \hat{f}\left(t, \cdot, U_t, \sqrt{\frac{d\langle N\rangle}{dV}}(t)\right) - \hat{f}\left(t, \cdot, U'_t, \sqrt{\frac{d\langle N'\rangle}{dV}}(t)\right)$.

By additivity, we have $d\bar{Y}_t = -\bar{f}_t dV_t + d\bar{M}_t$. Since $\bar{Y}_T = \xi - \xi = 0$, applying the integration by parts formula to $\bar{Y}_t^2 e^{\lambda V_t}$ between 0 and T we get

$$\bar{Y}_0^2 - 2\int_0^T e^{\lambda V_r} \bar{Y}_r \bar{f}_r dV_r + 2\int_0^T e^{\lambda V_r} \bar{Y}_{r-} d\bar{M}_r + \int_0^T e^{\lambda V_r} d[\bar{M}]_r + \lambda \int_0^T e^{\lambda V_r} \bar{Y}_r^2 dV_r = 0.$$

Since, by Proposition 3.19, the stochastic integral with respect to $\bar{M}$ is a real martingale, by taking the expectations we get

$$\mathrm{E}\left[\bar{Y}_0^2\right] - 2\mathrm{E}\left[\int_0^T e^{\lambda V_r} \bar{Y}_r \bar{f}_r dV_r\right] + \mathrm{E}\left[\int_0^T e^{\lambda V_r} d\langle \bar{M}\rangle_r\right] + \lambda \mathrm{E}\left[\int_0^T e^{\lambda V_r} \bar{Y}_r^2 dV_r\right] = 0.$$

So by re-arranging and by using the Lipschitz condition on f stated in Hypothesis 3.8, we get

$$
\begin{aligned}
& \lambda \mathrm{E}\left[\int_0^T e^{\lambda V_r} \bar{Y}_r^2 dV_r\right] + \mathrm{E}\left[\int_0^T e^{\lambda V_r} d\langle \bar{M}\rangle_r\right] \\
\leq\ & 2K^Y \mathrm{E}\left[\int_0^T e^{\lambda V_r} |\bar{Y}_r||\bar{U}_r| dV_r\right] \\
& + 2K^Z \mathrm{E}\left[\int_0^T e^{\lambda V_r} |\bar{Y}_r| \left|\sqrt{\tfrac{d\langle N\rangle}{dV}}(r) - \sqrt{\tfrac{d\langle N'\rangle}{dV}}(r)\right| dV_r\right] \\
\leq\ & (K^Y\alpha + K^Z\beta)\mathrm{E}\left[\int_0^T e^{\lambda V_r} |\bar{Y}_r|^2 dV_r\right] + \tfrac{K^Y}{\alpha}\mathrm{E}\left[\int_0^T e^{\lambda V_r} |\bar{U}_r|^2 dV_r\right] \\
& + \tfrac{K^Z}{\beta}\mathrm{E}\left[\int_0^T e^{\lambda V_r} \left|\sqrt{\tfrac{d\langle N\rangle}{dV}}(r) - \sqrt{\tfrac{d\langle N'\rangle}{dV}}(r)\right|^2 dV_r\right],
\end{aligned}
$$

for any positive α and β. Then we pick $\alpha = 2K^Y$ and $\beta = 2K^Z$, which gives us

$$
\begin{aligned}
& \lambda \mathrm{E}\left[\int_0^T e^{\lambda V_r} \bar{Y}_r^2 dV_r\right] + \mathrm{E}\left[\int_0^T e^{\lambda V_r} d\langle \bar{M}\rangle_r\right] \\
\leq\ & 2((K^Y)^2 + (K^Z)^2)\mathrm{E}\left[\int_0^T e^{\lambda V_r} |\bar{Y}_r|^2 dV_r\right] \\
+\ & \tfrac{1}{2}\mathrm{E}\left[\int_0^T e^{\lambda V_r} |\bar{U}_r|^2 dV_r\right] + \tfrac{1}{2}\mathrm{E}\left[\int_0^T e^{\lambda V_r} \left|\sqrt{\tfrac{d\langle N\rangle}{dV}}(r) - \sqrt{\tfrac{d\langle N'\rangle}{dV}}(r)\right|^2 dV_r\right].
\end{aligned}
$$

We choose now $\lambda = 1 + 2((K^Y)^2 + (K^Z)^2)$ and we get

$$
\begin{aligned}
& \mathrm{E}\left[\int_0^T e^{\lambda V_r} \bar{Y}_r^2 dV_r\right] + \mathrm{E}\left[\int_0^T e^{\lambda V_r} d\langle \bar{M}\rangle_r\right] \\
\leq\ & \tfrac{1}{2}\mathrm{E}\left[\int_0^T e^{\lambda V_r} |\bar{U}_r|^2 dV_r\right] + \tfrac{1}{2}\mathrm{E}\left[\int_0^T e^{\lambda V_r} \left|\sqrt{\tfrac{d\langle N\rangle}{dV}}(r) - \sqrt{\tfrac{d\langle N'\rangle}{dV}}(r)\right|^2 dV_r\right].
\end{aligned} \tag{3.8}
$$

On the other hand, since by Proposition B.1 we know that $\frac{d\langle N\rangle}{dV}\frac{d\langle N'\rangle}{dV} - \left(\frac{d\langle N,N'\rangle}{dV}\right)^2$ is a positive process, we have

$$
\begin{aligned}
\left|\sqrt{\tfrac{d\langle N\rangle}{dV}} - \sqrt{\tfrac{d\langle N'\rangle}{dV}}\right|^2 &= \tfrac{d\langle N\rangle}{dV} - 2\sqrt{\tfrac{d\langle N\rangle}{dV}}\sqrt{\tfrac{d\langle N'\rangle}{dV}} + \tfrac{d\langle N'\rangle}{dV} \\
&\leq \tfrac{d\langle N\rangle}{dV} - 2\tfrac{d\langle N,N'\rangle}{dV} + \tfrac{d\langle N'\rangle}{dV} \\
&= \tfrac{d\langle \bar{N}\rangle}{dV}\ dV \otimes d\mathrm{P} \text{ a.e.}
\end{aligned} \tag{3.9}
$$

Therefore, since by Remark 3.3 we have $\int_0^\cdot e^{\lambda V_r}\frac{d\langle \bar{N}\rangle}{dV}(r)dV_r \leq \int_0^\cdot e^{\lambda V_r} d\langle \bar{N}\rangle_r$, then expression (3.8) implies
$\mathrm{E}\left[\int_0^T e^{\lambda V_r}\bar{Y}_r^2 dV_r + \int_0^T e^{\lambda V_r} d\langle \bar{M}\rangle_r\right] \leq \frac{1}{2}\mathrm{E}\left[\int_0^T e^{\lambda V_r}|\bar{U}_r|^2 dV_r + \int_0^T e^{\lambda V_r} d\langle \bar{N}\rangle_r\right]$, which proves the contraction for the norm $\|\cdot\|_\lambda$. □

Theorem 3.21. *If $(\xi, \hat{f})$ verifies Hypothesis 3.8 then $BSDE(\xi, \hat{f}, V)$ has a unique solution.*

Proof. The space $L^2(dV \otimes d\mathrm{P}) \times \mathcal{H}_0^2$ is complete and Φ defines on it a contraction for the norm $\|(\cdot,\cdot)\|_\lambda$ for some $\lambda > 0$, so Φ has a unique fixed point in $L^2(dV \otimes d\mathrm{P}) \times \mathcal{H}_0^2$. Then by Proposition 3.16, $BSDE(\xi, \hat{f}, V)$ has a unique solution. □

Remark 3.22. *Let (Y, M) be the solution of $BSDE(\xi, \hat{f}, V)$ and $\dot{Y}$ the class of Y in $L^2(dV \otimes d\mathrm{P})$. Thanks to Proposition 3.16, we know that $(\dot{Y}, M) = \Phi(\dot{Y}, M)$ and therefore by Propositions 3.14 and 3.19 that $\sup_{t\in[0,T]} |Y_t|$ is L^2 and that $\int_0^{\cdot} Y_{r-} dM_r$ is a real martingale.*

Remark 3.23. *Let $(\xi, \hat{f}, V)$ satisfying Hypothesis 3.8. Until now we have considered the related BSDE on the interval $[0, T]$. Without restriction of generality we can consider a BSDE on a restricted interval $[s, T]$ for some $s \in [0, T[$. The results and comments of this section immediately extend to this case. In particular there exists a unique couple of processes (Y^s, M^s), indexed by $[s, T]$ such that Y^s is adapted, cadlag and verifies $\mathrm{E}[\int_s^T (Y_r^s)^2 dV_r] < \infty$, such that M^s is a martingale starting at 0 in s and such that $Y_{\cdot}^s = \xi + \int_{\cdot}^T \hat{f}\left(r, \cdot, Y_r^s, \sqrt{\frac{d\langle M\rangle}{dV}}(r)\right) dV_r - (M_T^s - M_{\cdot}^s)$ in the sense of indistinguishability on $[s, T]$.*
Moreover, if (Y, M) denotes the solution of $BSDE(\xi, \hat{f}, V)$ then $(Y, M_{\cdot} - M_s)$ and (Y^s, M^s) coincide on $[s, T]$. This follows by the uniqueness argument for the restricted BSDE to $[s, T]$.

The lemma below shows that, in order to verify that a couple (Y, M) is the solution of $BSDE(\xi, \hat{f}, V)$, it is not necessary to verify the square integrability of Y since it will be automatically fulfilled.

Lemma 3.24. *Let $(\xi, \hat{f}, V)$ verify Hypothesis 3.8 and consider $BSDE(\xi, \hat{f}, V)$ defined in Definition 3.10. Assume that there exists a cadlag adapted process Y with $Y_0 \in L^2$, and $M \in \mathcal{H}_0^2$ such that*

$$Y = \xi + \int_{\cdot}^T \hat{f}\left(r, \cdot, Y_r, \sqrt{\frac{d\langle M\rangle}{dV}}(r)\right) dV_r - (M_T - M_{\cdot}), \tag{3.10}$$

in the sense of indistinguishability. Then $\sup_{t\in[0,T]} |Y_t| \in L^2$. In particular, $Y \in \mathcal{L}^2(dV \otimes d\mathrm{P})$ and (Y, M) is the unique solution of $BSDE(\xi, \hat{f}, V)$.

On the other hand if (Y, M) verifies (3.10) on $[s, T]$ with $s < T$, if $Y_s \in L^2$, $M_s = 0$ and if we denote (U, N) the unique solution of $BSDE(\xi, \hat{f}, V)$, then (Y, M) and $(U, N_{\cdot} - N_s)$ are indistinguishable on $[s, T]$.

Proof. Let $\lambda > 0$ and $t \in [0, T]$. By integration by parts formula applied to $Y^2 e^{-\lambda V}$ between 0 and t we get

$$\begin{aligned} Y_t^2 e^{-\lambda V_t} - Y_0^2 &= -2\int_0^t e^{-\lambda V_r} Y_r \hat{f}\left(r, \cdot, Y_r, \sqrt{\tfrac{d\langle M\rangle}{dV}}(r)\right) dV_r + 2\int_0^t e^{-\lambda V_r} Y_{r-} dM_r \\ &\quad + \int_0^t e^{-\lambda V_r} d[M]_r - \lambda \int_0^t e^{-\lambda V_r} Y_r^2 dM_r. \end{aligned}$$

By re-arranging the terms and using the Lipschitz conditions in Hypothesis

3.8, we get

$$\begin{aligned}
& Y_t^2 e^{-\lambda V_t} + \lambda \int_0^t e^{-\lambda V_r} Y_r^2 dV_r \\
\leq\ & Y_0^2 + 2\int_0^t e^{-\lambda V_r} |Y_r||\hat{f}| \left(r, \cdot, Y_r, \sqrt{\tfrac{d\langle M\rangle}{dV}}(r)\right) dV_r + 2\left|\int_0^t e^{-\lambda V_r} Y_{r^-} dM_r\right| \\
& + \int_0^t e^{-\lambda V_r} d[M]_r \\
\leq\ & Y_0^2 + \int_0^t e^{-\lambda V_r} |\hat{f}|^2(r,\cdot,0,0) dV_r + (2K^Y + 1 + K^Z)\int_0^t e^{-\lambda V_r}|Y_r|^2 dV_r \\
& +2\left|\int_0^t e^{-\lambda V_r} Y_{r^-} dM_r\right| + \int_0^t e^{-\lambda V_r} d[M]_r.
\end{aligned}$$

Choosing $\lambda = 2K^Y + 1 + K^Z$ this gives

$$\begin{aligned}
Y_t^2 e^{-\lambda V_t} \leq\ & Y_0^2 + \int_0^t e^{-\lambda V_r} |\hat{f}|^2(r,\cdot,0,0) dV_r + K^Z \int_0^t e^{-\lambda V_r} \tfrac{d\langle M\rangle}{dV}(r) dV_r \\
& +2\left|\int_0^t e^{-\lambda V_r} Y_{r^-} dM_r\right| + \int_0^t e^{-\lambda V_r} d[M]_r.
\end{aligned}$$

Since V is bounded, there is a constant $C > 0$, such that for any $t \in [0,T]$

$$Y_t^2 \leq C\left(Y_0^2 + \int_0^T |\hat{f}|^2(r,\cdot,0,0) dV_r + \int_0^T \frac{d\langle M\rangle}{dV}(r) dV_r + [M]_T + \left|\int_0^t Y_{r^-} dM_r\right|\right).$$

By Hypothesis 3.8 and since we assumed $Y_0 \in L^2$ and $M \in \mathcal{H}^2$, the first four terms on the right hand side are integrable and we can conclude by Lemma 3.18.

An analogous proof also holds on the interval $[s,T]$ taking into account Remark 3.23. □

If the underlying filtration is Brownian and $V_t = t$, we can identify the solution of the BSDE with no driving martingale to the solution of a Brownian BSDE.

Let B be a 1-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Let $T \in \mathrm{R}_+^*$ and for any $t \in [0,T]$, let $\mathcal{F}_t^B$ denote the σ-field $\sigma(B_r | r \in [0,t])$ augmented with the P-negligible sets.
In the stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}^B, \mathrm{P})$, let $V_t = t$ and $(\xi, \hat{f})$ satisfy Hypothesis 3.8. Let (Y, M) be the unique solution of $BSDE(\xi, \hat{f}, V)$, see Theorem 3.21.

Proposition 3.25. *We have $Y = U$, $M = \int_0^\cdot Z_r dB_r$, where (U, Z) is the unique solution of the Brownian BSDE*

$$U = \xi + \int_\cdot^T \hat{f}\left(r, \cdot, U_r, |Z_r|\right) dr - \int_\cdot^T Z_r dB_r. \tag{3.11}$$

Proof. By Theorem 1.2 in [27], (3.11) admits a unique solution (U, Z) of progressively measurable processes such that $Z \in L^2(dt \otimes d\mathrm{P})$. It is known that $\sup_{t\in[0,T]} |U_t| \in L^2$ and therefore that $U \in \mathcal{L}^2(dt \otimes d\mathrm{P})$, see Proposition 1.1 in [27] for instance. We define $N = \int_0^\cdot Z_r dB_r$. The couple (U, N) belongs to $\mathcal{L}^2(dt \otimes d\mathrm{P}) \times \mathcal{H}_0^2$. N verifies $\frac{d\langle N\rangle_r}{dr} = Z_r^2$ $dt \otimes d\mathrm{P}$ a.e. So by (3.11), the couple

(U,N) verifies $U = \xi + \int_\cdot^T \hat{f}\left(r,\cdot,U_r,\sqrt{\frac{d\langle N\rangle_r}{dr}}\right)dr - (N_T - N_\cdot)$ in the sense of indistinguishability. It therefore solves $BSDE(\xi,\hat{f},V)$ and the assertion yields by uniqueness of the solution. □

4 Martingale Problem and Markov classes

In this section, we introduce the Markov process which will later be the forward process which will be coupled to a BSDE in order to constitute Markovian BSDEs with no driving martingales. For details about the exact mathematical background that we use to define our Markov process, one can consult the Section A of the Appendix. We also introduce the martingale problem related to this Markov process.

Let E be a Polish space and $T \in \mathrm{R}_+^*$ be a fixed horizon. From now on, $\big(\Omega,\mathcal{F},(X_t)_{t\in[0,T]},(\mathcal{F}_t)_{t\in[0,T]}\big)$ denotes the canonical space defined in Definition A.1. We consider a canonical Markov class $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times E}$ associated to a transition function measurable in time as defined in Definitions A.5 and A.4, and for any $(s,x)\in[0,T]\times E$, $\big(\Omega,\mathcal{F}^{s,x},(\mathcal{F}_t^{s,x})_{t\in[0,T]},\mathrm{P}^{s,x}\big)$ will denote the stochastic basis introduced in Definition A.9 and which fulfills the usual conditions.

The following notion of Martingale Problem comes from [23] Chapter XI.

Definition 4.1. *Let χ be a family of stochastic processes defined on a filtered space $(\tilde{\Omega},\tilde{\mathcal{F}},(\tilde{\mathcal{F}}_t)_{t\in\mathrm{T}})$. We say that a probability measure P defined on $(\tilde{\Omega},\tilde{\mathcal{F}})$ solves the* ***martingale problem*** *associated to χ if under P all elements of χ are in $\mathcal{M}_{loc}$. We denote $\mathcal{MP}(\chi)$ the set of probability measures solving this martingale problem. P in $\mathcal{MP}(\chi)$ is said to be* ***extremal*** *if there can not exist distinct probability measures Q,Q' in $\mathcal{MP}(\chi)$ and $\alpha\in]0,1[$ such that $\mathrm{P} = \alpha\mathrm{Q} + (1-\alpha)\mathrm{Q}'$.*

We now introduce a Martingale problem associated to an operator, following closely the formalism of D.W. Stroock and S.R.S Varadhan in [35]. We will see in Remark 4.3 that both Definitions 4.1 and 4.2 are closely related.

Definition 4.2. *Let us consider a domain $\mathcal{D}(a)\subset\mathcal{B}([0,T]\times E,\mathrm{R})$ which is a linear algebra; a linear operator $a:\mathcal{D}(a)\longrightarrow\mathcal{B}([0,T]\times E,\mathrm{R})$ and a non-decreasing continuous function $V:[0,T]\to\mathrm{R}_+$ starting at 0.*
We say that a set of probability measures $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times E}$ defined on $(\Omega,\mathcal{F})$ solves the **martingale problem associated to** *$(\mathcal{D}(a),a,V)$ if, for any $(s,x)\in[0,T]\times E$, $\mathrm{P}^{s,x}$ verifies*

(a) $\mathrm{P}^{s,x}(\forall t\in[0,s],X_t = x) = 1$;

(b) for every $\phi\in\mathcal{D}(a)$, $\Big(t\longmapsto\phi(t,X_t)-\phi(s,x)-\int_s^t a(\phi)(r,X_r)dV_r\Big)$, $t\in[s,T]$, is a cadlag $(\mathrm{P}^{s,x},(\mathcal{F}_t)_{t\in[s,T]})$-local martingale.

We say that the Martingale Problem is ***well-posed*** *if for any $(s,x)\in[0,T]\times E$, $\mathrm{P}^{s,x}$ is the only probability measure satisfying those two properties.*

Remark 4.3. *In other words,* $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times E}$ *solves the martingale problem associated to* $(\mathcal{D}(a), a, V)$ *if and only if, for any* $(s,x)\in[0,T]\times E$, $\mathrm{P}^{s,x}\in\mathcal{MP}(\chi^{s,x})$ *(see Definition 4.1), where* $\chi^{s,x}$ *is the family of processes* $\left\{t\mapsto \mathbb{1}_{[s,T]}(t)\left(\phi(t,X_t)-\phi(s,x)-\int_s^t a(\phi)(r,X_r)dV_r\right)\middle|\phi\in\mathcal{D}(a)\right\}$, *together with processes* $\left\{t\mapsto \mathbb{1}_{\{r\}}(t)(X_t-x)\middle|r\in[0,s]\right\}$.

Notation 4.4. *For every* $(s,x)\in[0,T]\times E$ *and* $\phi\in\mathcal{D}(a)$, *the process* $t\mapsto \mathbb{1}_{[s,T]}(t)\left(\phi(t,X_t)-\phi(s,x)-\int_s^t a(\phi)(r,X_r)dV_r\right)$ *will be denoted* $M[\phi]^{s,x}$.

$M[\phi]^{s,x}$ is a cadlag $(\mathrm{P}^{s,x},(\mathcal{F}_t)_{t\in[0,T]})$-local martingale which is equal to 0 on $[0,s]$, and by Proposition A.10, it is also a $(\mathrm{P}^{s,x},(\mathcal{F}_t^{s,x})_{t\in[0,T]})$-local martingale.

The following Hypothesis 4.5 is assumed for the rest of this section.

Hypothesis 4.5. *The Markov class* $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times E}$ *solves a well-posed Martingale Problem associated to a triplet* $(\mathcal{D}(a), a, V)$ *in the sense of Definition 4.2.*

The bilinear operator below was introduced (in the case of time-homogeneous operators) by J.P. Roth in potential analysis (see Chapter III in [33]), and popularized by P.A. Meyer in the study of homogeneous Markov processes, see e.g. [16] Chapter XV Comment 23 or [23] Remark 13.46. It has finally become a fundamental tool in the study of Markov processes and semi-groups, see for instance [2]. It will be central in our work.

Definition 4.6. *We set*

$$\Gamma:\begin{array}{rcl}\mathcal{D}(a)\times\mathcal{D}(a) & \to & \mathcal{B}([0,T]\times E)\\ (\phi,\psi) & \mapsto & a(\phi\psi)-\phi a(\psi)-\psi a(\phi).\end{array}\tag{4.1}$$

The operator Γ *is called the* ***carré du champs operator****.*

This operator will appear in the expression of the angular bracket of the local martingales that we have defined.

Proposition 4.7. *For any* $\phi\in\mathcal{D}(a)$ *and* $(s,x)\in[0,T]\times E$, $M[\phi]^{s,x}$ *belongs to* $\mathcal{H}^2_{0,loc}$. *Moreover, for any* $(\phi,\psi)\in\mathcal{D}(a)\times\mathcal{D}(a)$ *and* $(s,x)\in[0,T]\times E$, *we have*

$$\langle M[\phi]^{s,x}, M[\psi]^{s,x}\rangle=\int_s^{\cdot}\Gamma(\phi,\psi)(r,X_r)dV_r,$$

on the interval $[s,T]$, *in the stochastic basis* $(\Omega,\mathcal{F}^{s,x},(\mathcal{F}_t^{s,x})_{t\in[0,T]},\mathrm{P}^{s,x})$.

Proof. We fix some $(s,x)\in[0,T]\times E$ and the associated probability $\mathrm{P}^{s,x}$. For any ϕ,ψ in $\mathcal{D}(a)$, by integration by parts on $[s,T]$ we have

$$\begin{array}{rl}& M[\phi]^{s,x}M[\psi]^{s,x}\\ = & \int_s^{\cdot} M[\phi]^{s,x}_{r^-}dM[\psi]^{s,x}_r+\int_s^{\cdot} M[\psi]^{s,x}_{r^-}dM[\phi]^{s,x}_r+[M[\phi]^{s,x},M[\psi]^{s,x}]\\ = & \int_s^{\cdot} M[\phi]^{s,x}_{r^-}dM[\psi]^{s,x}_r+\int_s^{\cdot} M[\psi]^{s,x}_{r^-}dM[\phi]^{s,x}_r+[\phi(\cdot,X_\cdot),\psi(\cdot,X_\cdot)]\\ = & \int_s^{\cdot} M[\phi]^{s,x}_{r^-}dM[\psi]^{s,x}_r+\int_s^{\cdot} M[\psi]^{s,x}_{r^-}dM[\phi]^{s,x}_r+\phi\psi(\cdot,X_\cdot)\\ & -\phi\psi(s,x)-\int_s^{\cdot}\phi(r^-,X_{r^-})d\psi(r,X_r)-\int_s^{\cdot}\psi(r^-,X_{r^-})d\phi(r,X_r).\end{array}$$

Since $\phi\psi$ belongs to $\mathcal{D}(a)$, we can use the decomposition of $\phi\psi(\cdot, X_\cdot)$ given by (b) in Definition 4.2 and

$$\begin{aligned}
& M[\phi]^{s,x}M[\psi]^{s,x} \\
= \; & \int_s^\cdot M[\phi]^{s,x}_{r^-}dM[\psi]^{s,x}_r + \int_s^\cdot M[\psi]^{s,x}_{r^-}dM[\phi]^{s,x}_r + \int_s^\cdot a(\phi\psi)(r, X_r)dV_r \\
& + M^{s,x}[\phi\psi] - \int_s^\cdot \phi a(\psi)(r, X_r)dV_r - \int_s^\cdot \psi a(\phi)(r, X_r)dV_r \\
& - \int_s^\cdot \phi(r^-, X_{r^-})dM^{s,x}[\psi]_r - \int_s^\cdot \psi(r^-, X_{r^-})dM^{s,x}[\phi]_r \\
= \; & \int_s^\cdot \Gamma(\phi, \psi)(r, X_r)dV_r + \int_s^\cdot M[\phi]^{s,x}_{r^-}dM[\psi]^{s,x}_r + \int_s^\cdot M[\psi]^{s,x}_{r^-}dM[\phi]^{s,x}_r \\
& + M^{s,x}[\phi\psi] - \int_s^\cdot \phi(r^-, X_{r^-})dM^{s,x}[\psi]_r - \int_s^\cdot \psi(r^-, X_{r^-})dM^{s,x}[\phi]_r.
\end{aligned} \tag{4.2}$$

Since V is continuous, this implies that $M[\phi]^{s,x}M[\psi]^{s,x}$ is a special semi-martingale with bounded variation predictable part $\int_s^\cdot \Gamma(\phi, \psi)(r, X_r)dV_r$. In particular taking $\phi = \psi$, we have on $[s, T]$ that $(M[\phi]^{s,x})^2 = \int_s^\cdot \Gamma(\phi, \phi)(r, X_r)dV_r + N^{s,x}$, where $N^{s,x}$ is some local martingale. The first element in previous sum is locally bounded since it is a continuous process. The second one is locally integrable as every local martingale. Finally $(M[\phi]^{s,x})^2$ is locally integrable, implying that $M[\phi]^{s,x}$ is in $\mathcal{H}^2_{0,loc}$.

Let us come back to two given $\phi, \psi \in \mathcal{D}(a)$. Since we know that $M[\phi]^{s,x}$, $M[\psi]^{s,x}$ belong to $\mathcal{H}^2_{0,loc}$ we can consider $\langle M[\phi]^{s,x}, M[\psi]^{s,x}\rangle$ which, by definition, is the unique predictable process with bounded variation such that $M[\phi]^{s,x}M[\psi]^{s,x} - \langle M[\phi]^{s,x}, M[\psi]^{s,x}\rangle$ is a local martingale. So necessarily, taking (4.2) into account, $\langle M[\phi]^{s,x}, M[\psi]^{s,x}\rangle = \int_s^\cdot \Gamma(\phi, \psi)(r, X_r)dV_r$. □

Taking $\phi = \psi$ in Proposition 4.7, yields the following.

Corollary 4.8. *For any $(s, x) \in [0, T] \times E$ and $\phi \in \mathcal{D}(a)$, $M[\phi]^{s,x} \in \mathcal{H}^{2,V}_{loc}$.*

We now show that in our setup, $\mathcal{H}^2_0$ is always equal to $\mathcal{H}^{2,V}$. This can be seen as a generalization of Theorem 13.43 in [23].

Proposition 4.9. *Let $(s, x) \in [0, T] \times E$ and $\mathrm{P}^{s,x}$ be fixed. If $N \in \mathcal{H}^\infty_{0,loc}$ is strongly orthogonal to $M[\phi]^{s,x}$ for all $\phi \in \mathcal{D}(a)$ then it is necessarily equal to 0.*

Proof. In Hypothesis 4.5, for any $(s, x) \in [0, T] \times E$ we have assumed that $\mathrm{P}^{s,x}$ was the unique element of $\mathcal{MP}(\chi^{s,x})$, where $\chi^{s,x}$ was introduced in Remark 4.3. Therefore $\mathrm{P}^{s,x}$ is extremal in $\mathcal{MP}(\chi^{s,x})$. So thanks to the Jacod-Yor Theorem (see e.g. Theorem 11.2 in [23]), we know that if an element N of $\mathcal{H}^\infty_{0,loc}$ is strongly orthogonal to all the $M[\phi]^{s,x}$ then it is equal to zero. □

Proposition 4.10. *If Hypothesis 4.5 is verified then for any $(s, x) \in [0, T] \times E$, in the stochastic basis $\left(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}^{s,x}_t)_{t\in[0,T]}, \mathrm{P}^{s,x}\right)$, we have $\mathcal{H}^2_0 = \mathcal{H}^{2,V}$.*

Proof. We fix $(s, x) \in [0, T] \times E$. It is enough to show the inclusion $\mathcal{H}^2_0 \subset \mathcal{H}^{2,V}$. We start considering a bounded martingale $N \in \mathcal{H}^\infty_0$ and showing that it belongs to $\mathcal{H}^{2,V}$. Since N belongs to $\mathcal{H}^2_0$, we can consider the corresponding $N^V, N^{\perp V}$ in $\mathcal{H}^2_0$, appearing in the statement of Proposition 3.5. We show below that N^V and $N^{\perp V}$ are locally bounded, which will permit us to use Jacod-Yor theorem

on $N^{\perp V}$.

Indeed, by Proposition 3.5 there exists a predictable process K such that $N^V = \int_s^{\cdot} \mathbb{1}_{\{K_r<1\}} dN_r$ and $N^{\perp V} = \int_s^{\cdot} \mathbb{1}_{\{K_r=1\}} dN_r$. So if N is bounded then it has bounded jumps; by previous way of characterizing N^V and $N^{\perp V}$, their jumps can be expressed $(\Delta N^V)_t = \mathbb{1}_{\{K_t<1\}}\Delta N_t$ and $(\Delta N^{\perp V})_t = \mathbb{1}_{\{K_t=1\}}\Delta N_t$ (see Theorem 8 Chapter IV.3 in [32]), so they also have bounded jumps which implies that they are locally bounded, see (2.4) in [23].

So $N^{\perp V}$ is in $\mathcal{H}^{\infty}_{0,loc}$ and by construction it belongs to $\mathcal{H}^{2,\perp V}$. Since by Corollary 4.8, all the $M[\phi]^{s,x}$ belong to $\mathcal{H}^{2,V}_{loc}$, then, by Proposition 3.6, $N^{\perp V}$ is strongly orthogonal to all the $M[\phi]^{s,x}$. Consequently, by Proposition 4.9, $N^{\perp V}$ is equal to zero. This shows that $N = N^V$ which by construction belongs to $\mathcal{H}^{2,V}$, and consequently that $\mathcal{H}^{\infty}_0 \subset \mathcal{H}^{2,V}$, which concludes the proof when N is a bounded martingale.

We can conclude by density arguments as follows. Let $M \in \mathcal{H}^2_0$. For any integer $n \in \mathrm{N}^*$, we denote by M^n the martingale in $\mathcal{H}^{\infty}_0$ defined as the cadlag version of $t \mapsto \mathrm{E}^{s,x}[((-n)\vee M_T \wedge n)|\mathcal{F}_t]$. Now $(M^n_T - M_T)^2 \underset{n\to\infty}{\longrightarrow} 0$ a.s. and this sequence is bounded by $4M_T^2$ which is an integrable r.v. So by the dominated convergence theorem $\mathrm{E}^{s,x}\left[(M^n_T - M_T)^2\right] \underset{n\to\infty}{\longrightarrow} 0$. Then by Doob's inequality, $\sup_{t\in[0,T]} (M^n_t - M_t) \underset{n\to\infty}{\overset{L^2}{\longrightarrow}} 0$ meaning that $M^n \underset{n\to\infty}{\overset{\mathcal{H}^2}{\longrightarrow}} M$. Since $\mathcal{H}^{\infty}_0 \subset \mathcal{H}^{2,V}$, then M^n belongs to $\mathcal{H}^{2,V}$ for any $n \geq 0$. Moreover $\mathcal{H}^{2,V}$ is closed in $\mathcal{H}^2$, since by Proposition 3.6, it is a sub-Hilbert space. Finally we have shown that $M \in \mathcal{H}^{2,V}$. $\square$

Since V is continuous, it follows in particular that every $(\mathrm{P}^{s,x}, (\mathcal{F}^{s,x}_t)_{t\in[0,T]})$-square integrable martingale has a continuous angular bracket. By localization, the same assertion holds for local square integrable martingales.

We will now be interested in extending the domain $\mathcal{D}(a)$.

For any $(s,x) \in [0,T]\times E$ we define the positive bounded **potential measure** $U(s,x,\cdot)$ on $([0,T]\times E, \mathcal{B}([0,T])\otimes\mathcal{B}(E))$ by

$$U(s,x,\cdot) : \begin{array}{rcl} \mathcal{B}([0,T])\otimes\mathcal{B}(E) & \longrightarrow & [0,V_T] \\ A & \longmapsto & \mathrm{E}^{s,x}\left[\int_s^T \mathbb{1}_{\{(t,X_t)\in A\}} dV_t\right]. \end{array}$$

Definition 4.11. *A Borel set $A \subset [0,T]\times E$ will be said to be* **of zero potential** *if, for any $(s,x) \in [0,T]\times E$ we have $U(s,x,A) = 0$.*

Notation 4.12. *Let $p > 0$. We introduce*
$\mathcal{L}^p_{s,x} := \mathcal{L}^p(U(s,x,\cdot)) = \left\{ f \in \mathcal{B}([0,T]\times E, \mathrm{R}) : \mathrm{E}^{s,x}\left[\int_s^T |f|^p(r,X_r)dV_r\right] < \infty \right\}$.
That classical $\mathcal{L}^p$-space is equipped with the seminorm
$\|\cdot\|_{p,s,x} : f \mapsto \left(\mathrm{E}^{s,x}\left[\int_s^T |f(r,X_r)|^p dV_r\right]\right)^{\frac{1}{p}}$. *We also introduce*

$\mathcal{L}^0_{s,x} := \mathcal{L}^0(U(s,x,\cdot)) = \left\{ f \in \mathcal{B}([0,T]\times E, \mathrm{R}) : \int_s^T |f|(r,X_r)dV_r < \infty \quad \mathrm{P}^{s,x} \text{ a.s.} \right\}$.
We then denote for any $p \in \mathrm{N}$

$$\mathcal{L}^p_X = \bigcap_{(s,x)\in[0,T]\times E} \mathcal{L}^p_{s,x}. \tag{4.3}$$

Let $\mathcal{N}$ *be the linear sub-space of* $\mathcal{B}([0,T]\times E, \mathrm{R})$ *containing all functions which are equal to 0,* $U(s,x,\cdot)$ *a.e. for every* (s,x).
For any $p \in \mathrm{N}$*, we define the quotient space* $L^p_X = \mathcal{L}^p_X/\mathcal{N}$.
If $p \in \mathrm{N}^*$*,* L^p_X *can be equipped with the topology generated by the family of semi-norms* $(\|\cdot\|_{p,s,x})_{(s,x)\in[0,T]\times E}$ *which makes it a separate locally convex topological vector space, see Theorem 5.76 in [1].*

Proposition 4.13. *Let* f *and* g *be in* $\mathcal{L}^0_X$*. Then* f *and* g *are equal up to a set of zero potential if and only if for any* $(s,x) \in [0,T]\times E$*, the processes* $\int_s^\cdot f(r,X_r)dV_r$ *and* $\int_s^\cdot g(r,X_r)dV_r$ *are indistinguishable under* $\mathrm{P}^{s,x}$*. Of course in this case* f *and* g *correspond to the same element of* L^0_X.

Proof. Let $\mathrm{P}^{s,x}$ be fixed. Evaluating the total variation of $\int_s^\cdot (f-g)(r,X_r)dV_r$ yields that $\int_s^\cdot f(r,X_r)dV_r$ and $\int_s^\cdot g(r,X_r)dV_r$ are indistinguishable if and only if $\int_s^T |f-g|(r,X_r)dV_r = 0$ a.s. Since that r.v. is non-negative, this is true if and only if $\mathrm{E}^{s,x}\left[\int_s^T |f-g|(r,X_r)dV_r\right] = 0$ and therefore if and only if $U(s,x,N) = 0$, where N is the Borel subset of $[0,T]\times E$, defined by $\{(t,y) : f(t,y) \neq g(t,y)\}$. This concludes the proof of Proposition 4.13.

□

We can now define our notion of **extended generator**.

Definition 4.14. *We first define the* ***extended domain*** $\mathcal{D}(\mathfrak{a})$ *as the set functions* $\phi \in \mathcal{B}([0,T]\times E, \mathrm{R})$ *for which there exists*
$\psi \in \mathcal{B}([0,T]\times E, \mathrm{R})$ *such that under any* $\mathrm{P}^{s,x}$ *the process*

$$\mathbb{1}_{[s,T]}\left(\phi(\cdot,X.) - \phi(s,x) - \int_s^\cdot \psi(r,X_r)dV_r\right) \tag{4.4}$$

(which is not necessarily cadlag) has a cadlag modification in $\mathcal{H}^2_0$.

Proposition 4.15. *Let* $\phi \in \mathcal{B}([0,T]\times E, \mathrm{R})$*. There is at most one (up to zero potential sets)* $\psi \in \mathcal{B}([0,T]\times E, \mathrm{R})$ *such that under any* $\mathrm{P}^{s,x}$*, the process defined in* (4.4) *has a modification which belongs to* $\mathcal{M}_{loc}$.
If moreover $\phi \in \mathcal{D}(a)$*, then* $a(\phi) = \psi$ *up to zero potential sets. In this case, according to Notation 4.4, for every* $(s,x) \in [0,T]\times E$*,* $M[\phi]^{s,x}$ *is the* $\mathrm{P}^{s,x}$ *cadlag modification in* $\mathcal{H}^2_0$ *of* $\mathbb{1}_{[s,T]}\left(\phi(\cdot,X.) - \phi(s,x) - \int_s^\cdot \psi(r,X_r)dV_r\right)$.

Proof. Let ψ^1 and ψ^2 be two functions such that for any $\mathrm{P}^{s,x}$,
$\mathbb{1}_{[s,T]}\left(\phi(\cdot,X.) - \phi(s,x) - \int_s^\cdot \psi^i(r,X_r)dV_r\right), \quad i = 1,2$, admits a cadlag modification which is a local martingale. Then, under a fixed $\mathrm{P}^{s,x}$, $\phi(\cdot,X.)$ has two

cadlag modifications which are therefore indistinguishable, and by uniqueness of the decomposition of special semi-martingales, $\int_s^\cdot \psi^1(r,X_r)dV_r$ and $\int_s^\cdot \psi^2(r,X_r)dV_r$ are indistinguishable on $[s,T]$. Since this is true under any $\mathrm{P}^{s,x}$, the two functions are equal up to a zero-potential set because of Proposition 4.13.
Concerning the second part of the statement, let $\phi \in \mathcal{D}(a) \cap \mathcal{D}(\mathfrak{a})$. The result follows by Definition 4.2 and the uniqueness of the function ϕ established just before. □

Definition 4.16. *Let $\phi \in \mathcal{D}(\mathfrak{a})$ as in Definition 4.14. We denote again by $M[\phi]^{s,x}$, the unique cadlag version of the process* (4.4) *in $\mathcal{H}_0^2$. Taking Proposition 4.13 into account, this will not generate any ambiguity with respect to Notation 4.4. Proposition 4.13, also permits to define without ambiguity the operator*

$$\mathfrak{a}: \begin{array}{ccc} \mathcal{D}(\mathfrak{a}) & \longrightarrow & L_X^0 \\ \phi & \longmapsto & \psi. \end{array}$$

*$\mathfrak{a}$ will be called the **extended generator**.*

We now want to extend the carré du champs operator $\Gamma(\cdot,\cdot)$ to $\mathcal{D}(\mathfrak{a}) \times \mathcal{D}(\mathfrak{a})$.

Proposition 4.17. *Let ϕ_1, ϕ_2 be in $\mathcal{D}(\mathfrak{a})$. There exists a (unique up to zero-potential sets) function in $\mathcal{B}([0,T] \times E, \mathrm{R})$ which we will denote $\mathfrak{G}(\phi_1,\phi_2)$ such that under any $\mathrm{P}^{s,x}$, $\langle M[\phi_1]^{s,x}, M[\phi_2]^{s,x}\rangle = \int_s^\cdot \mathfrak{G}(\phi_1,\phi_2)(r,X_r)dV_r$ on $[s,T]$, up to indistinguishability.*
If moreover ϕ_1 and ϕ_2 belong to $\mathcal{D}(a)$, then $\Gamma(\phi_1,\phi_2) = \mathfrak{G}(\phi_1,\phi_2)$ up to zero potential sets.

Proof. Let $\phi_1, \phi_2 \in \mathcal{D}(\mathfrak{a})$ according to Definition 4.16. We take some representative of the classes $\mathfrak{a}(\phi_i)$ for $i = 1,2$, still denoted by the same symbol and define the square integrable MAFs (see Definition A.11) $M[\phi_i]$ by

$$M[\phi_i]_u^t(\omega) = \begin{cases} \phi_i(u,X_u(\omega)) - \phi_i(t,X_t(\omega)) - \int_t^u \mathfrak{a}(\phi_i)(r,X_r(\omega))dV_r \\ \qquad \text{if } \int_t^u |\mathfrak{a}(\phi_i)|(r,X_r(\omega))dV_r < +\infty \\ 0 \text{ otherwise.} \end{cases} \tag{4.5}$$

Indeed, for every $(s,x) \in [0,T] \times E$, $M[\phi_i]^{s,x}$ is the cadlag version under $\mathrm{P}^{s,x}$. The existence of $\mathfrak{G}(\phi_1,\phi_2)$ therefore derives from Proposition A.12. By Proposition 4.13 that function is determined up to a zero-potential set.
The second statement holds because of Proposition 4.7. □

Definition 4.18. *The bilinear operator $\mathfrak{G}: \mathcal{D}(\mathfrak{a}) \times \mathcal{D}(\mathfrak{a}) \longmapsto L_X^0$ will be called the **extended carré du champs operator**.*

According to Definition 4.14, we do not have necessarily $\mathcal{D}(a) \subset \mathcal{D}(\mathfrak{a})$, however we have the following.

Corollary 4.19. *If $\phi \in \mathcal{D}(a)$ and $\Gamma(\phi,\phi) \in \mathcal{L}_X^1$, then $\phi \in \mathcal{D}(\mathfrak{a})$ and $(a(\phi),\Gamma(\phi,\phi)) = (\mathfrak{a}(\phi),\mathfrak{G}(\phi,\phi))$ up to zero potential sets.*

Proof. Given some $\phi \in \mathcal{D}(a)$, by Definition 4.14, if for every $(s,x) \in [0,T] \times E$, $M[\phi]^{s,x}$ is square integrable, then $\phi \in \mathcal{D}(\mathfrak{a})$. By Proposition 4.7, for every $(s,x) \in [0,T] \times E$ $M[\phi]^{s,x}$ is a $\mathrm{P}^{s,x}$ square integrable if and only if $\Gamma(\phi,\phi) \in \mathcal{L}^1_X$. So the statement holds because of Propositions 4.15 and 4.17. □

5 Pseudo-PDEs and associated Markovian BSDEs with no driving martingale

In this section, we still consider $T \in \mathrm{R}_+^*$, a Polish space E and the associated canonical space $\big(\Omega, \mathcal{F}, (X_t)_{t\in[0,T]}, (\mathcal{F}_t)_{t\in[0,T]}\big)$, see Definition A.1. We also consider a canonical Markov class $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times E}$ and assume the following for the rest of the Section.

Hypothesis 5.1. *The transition function of* $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times E}$ *is measurable in time (see Definitions A.5 and A.4) and* $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times E}$ *solves a well-posed martingale problem associated to a triplet* $(\mathcal{D}(a), a, V)$*, see Definition 4.2 and Hypothesis 4.5.*

We will investigate here a specific type of BSDE with no driving martingale $BSDE(\xi, \hat{f}, V)$ which we will call **of Markovian type**, or **Markovian** BSDE, in the following sense. The process V will be the (deterministic) function V introduced in Definition 4.2, the final condition ξ will only depend on the final value of the canonical process X_T and the randomness of the driver $\hat{f}$ at time t will only appear via the value at time t of the forward process X. Given a function
$f : [0,T] \times E \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, we will set $\hat{f}(t,\omega,y,z) = f(t, X_t(\omega), y, z)$ for $t \in [0,T], \omega \in \Omega, y, z \in \mathbb{R}$.

That BSDE will be connected with the deterministic problem below.

Notation 5.2. *From now on, we fix some* $g \in \mathcal{B}(E, \mathrm{R})$ *and*
$f \in \mathcal{B}([0,T] \times E \times \mathrm{R} \times \mathrm{R}, \mathrm{R})$.

Definition 5.3. *We will call* ***Pseudo-Partial Differential Equation*** *(in short Pseudo-PDE) the following equation with final condition:*

$$\begin{cases} a(u)(t,x) + f\left(t, x, u(t,x), \sqrt{\Gamma(u,u)(t,x)}\right) &= 0 \quad \text{on } [0,T] \times E \\ u(T,\cdot) &= g. \end{cases} \tag{5.1}$$

We will say that u *is a* ***classical solution*** *of the Pseudo-PDE if it belongs to* $\mathcal{D}(a)$ *and verifies* (5.1).

Notation 5.4. *Equation* (5.1) *will be denoted* $Pseudo-PDE(f,g)$.

For the rest of this section, we will also assume that f, g verify the following.

Hypothesis 5.5. • *$g(X_T)$ is L^2 under $\mathbb{P}^{s,x}$ for every $(s,x) \in [0,T] \times E$;*

- *$t \longmapsto f(t, X_t, 0, 0) \in \mathcal{L}^2_X$;*
- *there exist $K^Y, K^Z > 0$ such that for all (t, x, y, y', z, z'),*

$$|f(t,x,y,z) - f(t,x,y',z')| \leq K^Y |y - y'| + K^Z |z - z'|. \tag{5.2}$$

With the equation $Pseudo - PDE(f,g)$, we will associate the family of BSDEs with no driving martingale indexed by $(s,x) \in [0,T] \times E$ and defined on the interval $[0,T]$ and in the stochastic basis $\left(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}^{s,x}_t)_{t\in[0,T]}, \mathrm{P}^{s,x}\right)$, given by

$$Y^{s,x}_t = g(X_T) + \int_t^T f\left(r, X_r, Y^{s,x}_r, \sqrt{\frac{d\langle M^{s,x}\rangle}{dV}}(r)\right) dV_r - (M^{s,x}_T - M^{s,x}_t). \tag{5.3}$$

Notation 5.6. *Equation* (5.3) *will be denoted $BSDE^{s,x}(f,g)$. It corresponds to $BSDE(g(X_T), \hat{f}, V)$ with $\mathrm{P} := \mathrm{P}^{s,x}$.*

Remark 5.7. .

1. *If Hypothesis 5.5 is verified then Hypothesis 3.8 is verified for* (5.3)*. By Theorem 3.21, for any (s,x), $BSDE^{s,x}(f,g)$ has a unique solution, in the sense of Definition 3.10.*
2. *Even if the underlying process X admits no (even generalized) moments, given a couple (f,g) such that $f(\cdot,\cdot,0,0)$ and g are bounded, the considerations of this section still apply. In particular the connection between the $BSDE^{s,x}(f,g)$ and the corresponding $Pseudo - PDE(f,g)$ still exists.*

Notation 5.8. *From now on, $(Y^{s,x}, M^{s,x})$ will always denote the (unique) solution of $BSDE^{s,x}(f,g)$.*

The goal of our work is to understand if and how the solutions of equations $BSDE^{s,x}(f,g)$ produce a solution of $Pseudo - PDE(f,g)$ and reciprocally.

We will start by showing that if $Pseudo - PDE(f,g)$ has a classical solution, then this one provides solutions to the associated $BSDE^{s,x}(f,g)$.

Proposition 5.9. *Let u be a classical solution of $Pseudo - PDE(f,g)$ verifying $\Gamma(u,u) \in \mathcal{L}^1_X$. Then, for any $(s,x) \in [0,T] \times E$, if $M[u]^{s,x}$ is as in Notation 4.4, we have that $(u(\cdot, X.), M[u]^{s,x})$ and $(Y^{s,x}, M^{s,x}_\cdot - M^{s,x}_s)$ are $\mathbb{P}^{s,x}$-indistinguishable on $[s,T]$.*

Proof. Let (s,x) be fixed. Since $u \in \mathcal{D}(a)$, the martingale problem in the sense of Definition 4.2 and (5.1) imply that, on $[s,T]$, under $\mathrm{P}^{s,x}$

$$\begin{aligned}
& u(\cdot, X.) \\
= \;& u(T, X_T) - \textstyle\int_\cdot^T a(u)(r, X_r) dV_r - (M[u]^{s,x}_T - M[u]^{s,x}_\cdot) \\
= \;& g(X_T) + \textstyle\int_\cdot^T f\left(r, X_r, u(r, X_r), \sqrt{\Gamma(u,u)(r, X_r)}\right) - (M[u]^{s,x}_T - M[u]^{s,x}_\cdot) \\
= \;& g(X_T) + \textstyle\int_\cdot^T f\left(r, X_r, Y_r, \sqrt{\frac{d\langle M[u]^{s,x}\rangle}{dV}}(r)\right) dV_r - (M[u]^{s,x}_T - M[u]^{s,x}_\cdot),
\end{aligned}$$

where the latter equality comes from Proposition 4.7. Since $\Gamma(u,u) \in \mathcal{L}^1_X$ it follows that $\mathrm{E}^{s,x}\left[\langle M[u]^{s,x}\rangle_T\right] = \mathrm{E}^{s,x}\left[\int_s^T \Gamma(u,u)(r,X_r)dV_r\right] < \infty$. This means that $M[u]^{s,x} \in \mathcal{H}^2_0$, so by Lemma 3.24 $(u(\cdot,X.), M[u]^{s,x}_\cdot)$ and $(Y^{s,x}, M^{s,x}_\cdot - M^{s,x}_s)$ are indistinguishable on $[s,T]$. □

We will now adopt the converse point of view, starting with the solutions of the equations $BSDE^{s,x}(f,g)$. Below we will show that there exist Borel functions u and $v \geq 0$ such that for any $(s,x) \in [0,T] \times E$, for all $t \in [s,T]$, $Y^{s,x}_t = u(t,X_t)$ $\mathrm{P}^{s,x}$-a.s., and $\frac{d\langle M^{s,x}\rangle}{dV} = v^2(\cdot,X.)$ $\quad dV \otimes d\mathrm{P}^{s,x}$ a.e. on $[s,T]$. This will be the object of Theorem 5.15, whose an analogous formulation exists in the Brownian framework, see e.g. Theorem 4.1 in [18]. We start with a lemma.

Lemma 5.10. *Let $\tilde{f} \in \mathcal{L}^2_X$. Let, for any $(s,x) \in [0,T] \times E$, $(\tilde{Y}^{s,x}, \tilde{M}^{s,x})$ be the (unique by Theorem 3.21 and Remark 3.23) solution of*

$$\tilde{Y}^{s,x}_t = g(X_T) + \int_t^T \tilde{f}\left(r,X_r\right) dV_r - (\tilde{M}^{s,x}_T - \tilde{M}^{s,x}_t), \quad t \in [s,T],$$

in $\left(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}^{s,x}_t)_{t\in[0,T]}, \mathrm{P}^{s,x}\right)$. Then there exist two functions u and $v \geq 0$ in $\mathcal{B}([0,T]\times E, \mathrm{R})$ such that for any $(s,x) \in [0,T] \times E$

$$\begin{cases} \forall t \in [s,T] : \tilde{Y}^{s,x}_t = u(t,X_t) \quad \mathrm{P}^{s,x} a.s. \\ \frac{d\langle \tilde{M}^{s,x}\rangle}{dV} = v^2(\cdot,X.) \quad dV \otimes d\mathrm{P}^{s,x} \ a.e. \ on \ [s,T]. \end{cases}$$

Proof. We set $u : (s,x) \mapsto \mathrm{E}^{s,x}\left[g(X_T) + \int_s^T \tilde{f}\left(r,X_r\right) dV_r\right]$ which is Borel by Proposition A.7 and Lemma A.8. Therefore by (A.3) in Remark A.6, for a fixed $t \in [s,T]$ $\mathbb{P}^{s,x}$- a.s. we have

$$\begin{aligned} u(t,X_t) &= \mathrm{E}^{t,X_t}\left[g(X_T) + \int_t^T \tilde{f}\left(r,X_r\right) dV_r\right] \\ &= \mathrm{E}^{s,x}\left[g(X_T) + \int_t^T \tilde{f}\left(r,X_r\right) dV_r \Big| \mathcal{F}_t\right] \\ &= \mathrm{E}^{s,x}\left[\tilde{Y}^{s,x}_t + (\tilde{M}^{s,x}_T - \tilde{M}^{s,x}_t)|\mathcal{F}_t\right] \\ &= \tilde{Y}^{s,x}_t, \end{aligned}$$

since $\tilde{M}^{s,x}$ is a martingale and $\tilde{Y}^{s,x}$ is adapted. Then the square integrable AF (see Definition A.11) defined by

$$M^t_{t'}(\omega) = \begin{cases} u(t',X_{t'}) - u(t,X_t) + \int_t^{t'} \tilde{f}(r,X_r(\omega))dV_r \\ \qquad \text{if } \int_t^{t'} |\tilde{f}|(r,X_r(\omega))dV_r < +\infty \\ 0 \text{ otherwise} \end{cases} \tag{5.4}$$

is a MAF whose cadlag version under $\mathrm{P}^{s,x}$ is $\tilde{M}^{s,x}$. The existence of the function v follows setting $v = \sqrt{k}$ in Proposition A.12.

□

We now define the Picard iterations associated to the contraction defining the solution of the BSDE associated with $BSDE^{s,x}(f,g)$.

Notation 5.11. *For a fixed $(s,x) \in [0,T] \times E$, $\Phi^{s,x}$ will denote the contraction $\Phi : L^2(dV \otimes d\mathrm{P}^{s,x}) \times \mathcal{H}_0^2$ introduced in Notation 3.15 with respect to the BSDE associated with $BSDE^{s,x}(f,g)$, see Notation 5.8 In the sequel we will not distinguish between a couple $(\dot{Y}, M)$ in $L^2(dV \otimes d\mathrm{P}^{s,x}) \times \mathcal{H}_0^2$ and (Y, M), where Y is the reference cadlag process of $\dot{Y}$, according to Definition 3.13. We then convene the following.*

1. $(Y^{0,s,x}, M^{0,s,x}) := (0,0)$;
2. $\forall k \in \mathrm{N}^* : (Y^{k,s,x}, M^{k,s,x}) := \Phi^{s,x}(Y^{k-1,s,x}, M^{k-1,s,x})$,

meaning that for $k \in \mathrm{N}^*$, $(Y^{k,s,x}, M^{k,s,x})$ is the solution of the BSDE

$$Y^{k,s,x} = g(X_T) + \int_{\cdot}^{T} f\left(r, X_r, Y^{k-1,s,x}, \sqrt{\frac{d\langle M^{k-1,s,x}\rangle}{dV}}(r)\right) dV_r - (M_T^{k,s,x} - M_{\cdot}^{k,s,x}). \tag{5.5}$$

The processes $(Y^{k,s,x}, M^{k,s,x})$ will be called the **Picard iterations** of $BSDE^{s,x}(f,g)$

Proposition 5.12. *For each $k \in \mathrm{N}$, there exist functions u_k and $v_k \geq 0$ in $\mathcal{B}([0,T] \times E, \mathrm{R})$ such that for every $(s,x) \in [0,T] \times E$*

$$\begin{cases} \forall t \in [s,T] : Y_t^{k,s,x} = u_k(t, X_t) \quad \mathrm{P}^{s,x} a.s. \\ \frac{d\langle M^{k,s,x}\rangle}{dV} = v_k^2(\cdot, X_\cdot) \quad dV \otimes d\mathrm{P}^{s,x} \ a.e. \ on \ [s,T]. \end{cases} \tag{5.6}$$

Lemma 5.13. *Let $(s,x) \in [0,T] \times E$ be fixed and let ϕ, ψ be two measurable processes. If ϕ and ψ are $\mathrm{P}^{s,x}$-modifications of each other, then they are equal $dV \otimes d\mathrm{P}^{s,x}$ a.e.*

Proof. Since for any $t \in [0,T]$, $\phi_t = \psi_t$ $\mathrm{P}^{s,x}$ a.s. we can write by Fubini's theorem $\mathrm{E}^{s,x}\left[\int_0^T \mathbb{1}_{\phi_t \neq \psi_t} dV_t\right] = \int_0^T \mathrm{P}^{s,x}(\phi_t \neq \psi_t) dV_t = 0$. □

Proof of Proposition 5.12.
We proceed by induction on k. It is clear that $(u_0, v_0) = (0,0)$ verifies the assertion for $k = 0$.
Now let us assume that functions u_{k-1}, v_{k-1} exist, for some integer $k \geq 1$, verifying (5.6) for k replaced with $k-1$.
We fix $(s,x) \in [0,T] \times E$. By Lemma 5.13, $(Y^{k-1,s,x}, Z^{k-1,s,x}) = (u_{k-1}, v_{k-1})(\cdot, X_\cdot)$ $dV \otimes \mathrm{P}^{s,x}$ a.e. on [s,T]. Therefore by (5.5), on $[s,T]$
$Y^{k,s,x} = g(X_T) + \int_{\cdot}^{T} f\left(r, X_r, u_{k-1}(r, X_r), v_{k-1}(r, X_r)\right) dV_r - (M_T^{k,s,x} - M_{\cdot}^{k,s,x})$.
Since $\Phi^{s,x}$ maps $L^2(dV \otimes d\mathrm{P}^{s,x}) \times \mathcal{H}_0^2$ into itself (see Definition 3.15), obviously all the Picard iterations belong to $L^2(dV \otimes d\mathrm{P}^{s,x}) \times \mathcal{H}_0^2$. In particular, $Y^{k-1,s,x}$ and $\sqrt{\frac{d\langle M^{k-1,s,x}\rangle}{dV}}$ belong to $\mathcal{L}^2(dV \otimes d\mathrm{P}^{s,x})$. So, by recurrence assumption on u_{k-1} and v_{k-1}, it follows that u_{k-1} and v_{k-1} belong to $\mathcal{L}_X^2$. Therefore, using

the assumptions f in Hypothesis 5.5, $f(\cdot,\cdot,u_{k-1},v_{k-1}) \in \mathcal{L}^2_X$. The existence of u_k and v_k now comes from Lemma 5.10 applied to $\tilde{f} := f(\cdot,\cdot,u_{k-1},v_{k-1})$. This establishes the induction step for a general k and allows to conclude the proof. □

Now we intend to pass to the limit in k. For any $(s,x) \in [0,T] \times E$, we have seen in Proposition 3.20 that $\Phi^{s,x}$ is a contraction in $\left(L^2(dV \otimes d\mathrm{P}^{s,x}) \times \mathcal{H}^2_0, \|\cdot\|_\lambda\right)$ for some $\lambda > 0$, so we know that the sequence $(Y^{k,s,x}, M^{k,s,x})$ converges to $(Y^{s,x}, M^{s,x})$ in this topology.
The proposition below also shows an a.e. corresponding convergence, adapting the techniques of Corollary 2.1 in [18].

Proposition 5.14. *For any $(s,x) \in [0,T] \times E$, $Y^{k,s,x} \underset{k\to\infty}{\longrightarrow} Y^{s,x}$ $dV \otimes d\mathrm{P}^{s,x}$ a.e. and $\sqrt{\frac{d\langle M^{k,s,x}\rangle}{dV}} \underset{k\to\infty}{\longrightarrow} \sqrt{\frac{d\langle M^{s,x}\rangle}{dV}}$ $dV \otimes d\mathrm{P}^{s,x}$ a.e.*

Proof. We fix (s,x) and the associated probability. In this proof, all superscripts s,x are dropped. We set $Z^k = \sqrt{\frac{d\langle M^k\rangle}{dV}}$ and $Z = \sqrt{\frac{d\langle M\rangle}{dV}}$. By Proposition 3.20, there exists $\lambda > 0$ such that for any $k \in \mathrm{N}^*$

$$\begin{aligned} & \mathrm{E}\left[\int_0^T e^{-\lambda V_r}|Y^{k+1}_r - Y^k_r|^2 dV_r + \int_0^T e^{-\lambda V_r} d\langle M^{k+1} - M^k\rangle_r\right] \\ \leq\ & \tfrac{1}{2}\mathrm{E}\left[\int_0^T e^{-\lambda V_r}|Y^k_r - Y^{k-1}_r|^2 dV_r + \int_0^T e^{-\lambda V_r} d\langle M^k - M^{k-1}\rangle_r\right], \end{aligned}$$

therefore

$$\begin{aligned} & \sum_{k\geq 0} \mathrm{E}\left[\int_0^T e^{-\lambda V_r}|Y^{k+1}_r - Y^k_r|^2 dV_r\right] + \mathrm{E}\left[\int_0^T e^{-\lambda V_r} d\langle M^{k+1} - M^k\rangle_r\right] \\ \leq\ & \sum_{k\geq 0} \tfrac{1}{2^k}\left(\mathrm{E}\left[\int_0^T e^{-\lambda V_r}|Y^1_r|^2 dV_r\right] + \mathrm{E}\left[\int_0^T e^{-\lambda V_r} d\langle M^1\rangle_r\right]\right) \\ <\ & \infty. \end{aligned} \tag{5.7}$$

Thanks to (3.9) and (5.7) we have
$\sum_{k\geq 0}\left(\mathrm{E}\left[\int_0^T e^{-\lambda V_r}|Y^{k+1}_r - Y^k_r|^2 dV_r\right] + \mathrm{E}\left[\int_0^T e^{-\lambda V_r}|Z^{k+1}_r - Z^k_r|^2 dV_r\right]\right) < \infty.$
So by Fubini's theorem we have
$\mathrm{E}\left[\int_0^T e^{-\lambda V_r}\left(\sum_{k\geq 0}(|Y^{k+1}_r - Y^k_r|^2 + |Z^{k+1}_r - Z^k_r|^2)\right) dV_r\right] < \infty.$

Consequently the sum $\sum_{k\geq 0}\left(|Y^{k+1}_r(\omega) - Y^k_r(\omega)|^2 + |Z^{k+1}_r(\omega) - Z^k_r(\omega)|^2\right)$ is finite on a set of full $dV \otimes d\mathrm{P}$ measure. So on this set of full measure, the sequence $(Y^{k+1}_t(\omega), Z^{k+1}_t(\omega))$ converges, and the limit is necessarily equal to $(Y_t(\omega), Z_t(\omega))$ $dV \otimes d\mathrm{P}$ a.e. because of the $L^2(dV \otimes d\mathrm{P})$ convergence that we have mentioned in the lines before the statement of the present Proposition 5.14. □

Theorem 5.15. *There exist two functions u and $v \geq 0$ in $\mathcal{B}([0,T] \times E, \mathrm{R})$ such that for every $(s,x) \in [0,T] \times E$,*

$$\begin{cases} \forall t \in [s,T] : Y^{s,x}_t = u(t, X_t) \quad \mathrm{P}^{s,x} \text{ a.s.} \\ \frac{d\langle M^{s,x}\rangle}{dV} = v^2(\cdot, X_\cdot) \quad dV \otimes d\mathrm{P}^{s,x} \text{ a.e. on } [s,T]. \end{cases} \tag{5.8}$$

Proof. We set $\bar{u} := \limsup_{k\in\mathbb{N}} u_k$, in the sense that for any $(s,x) \in [0,T] \times E$, $\bar{u}(s,x) = \limsup_{k\in\mathbb{N}} u_k(s,x)$ and $v := \limsup_{k\in\mathbb{N}} v_k$. $\bar{u}$ and v are Borel functions. We know by Propositions 5.12, 5.14 and Lemma 5.13 that for every $(s,x) \in [0,T]\times E$

$$\begin{cases} u_k(\cdot, X_\cdot) \underset{k\to\infty}{\longrightarrow} Y^{s,x} & dV \otimes d\mathrm{P}^{s,x} \text{ a.e. on } [s,T] \\ v_k(\cdot, X_\cdot) \underset{k\to\infty}{\longrightarrow} Z^{s,x} & dV \otimes d\mathrm{P}^{s,x} \text{ a.e. on } [s,T], \end{cases}$$

where $Z^{s,x} := \sqrt{\frac{d\langle M^{s,x}\rangle}{dV}}$. Therefore, for some fixed $(s,x) \in [0,T] \times E$ and on the set of full $dV \otimes d\mathrm{P}^{s,x}$ measure on which these convergences hold we have

$$\begin{cases} \bar{u}(t, X_t(\omega)) = \limsup_{k\in\mathbb{N}} u_k(t, X_t(\omega)) = \lim_{k\in\mathbb{N}} u_k(t, X_t(\omega)) = Y_t^{s,x}(\omega) \\ v(t, X_t(\omega)) = \limsup_{k\in\mathbb{N}} v_k(t, X_t(\omega)) = \lim_{k\in\mathbb{N}} v_k(t, X_t(\omega)) = Z_t^{s,x}(\omega). \end{cases} \tag{5.9}$$

This shows in particular the existence of v and the validity of the second line of (5.8).

It remains to show the existence of u so that the first line of (5.8) holds. Thanks to the $dV \otimes d\mathrm{P}^{s,x}$ equalities concerning v and $\bar{u}$ stated in (5.9), under every $\mathrm{P}^{s,x}$ we actually have

$$Y^{s,x} = g(X_T) + \int_\cdot^T f\left(r, X_r, \bar{u}(r, X_r), v(r, X_r)\right) dV_r - (M_T^{s,x} - M_\cdot^{s,x}). \tag{5.10}$$

Now (5.10) can be considered as a BSDE where the driver does not depend on y and z. For any $(s,x) \in [0,T] \times E$, $Y^{s,x}$ and $Z^{s,x}$ belong to $\mathcal{L}^2(dV \otimes d\mathrm{P}^{s,x})$, then by (5.9), so do $\bar{u}(\cdot, X_\cdot)\mathbb{1}_{[s,T]}$ and $v(\cdot, X_\cdot)\mathbb{1}_{[s,T]}$, meaning that $\bar{u}$ and v belong to $\mathcal{L}^2_X$. Using the two assumptions made on f in Hypothesis 5.5, $f(\cdot,\cdot,\bar{u},v)$ also belongs to $\mathcal{L}^2_X$. We can therefore apply Lemma 5.10 to $\tilde{f} = f(\cdot,\cdot,\bar{u},v)$, and conclude to the existence of a Borel function u such that for every $(s,x) \in [0,T] \times E$, $Y^{s,x}$ is on $[s,T]$ a $\mathrm{P}^{s,x}$-version of $u(\cdot, X_\cdot)$.

□

Remark 5.16. *Since $\bar{u}(\cdot, X_\cdot) = Y^{s,x} = u(\cdot, X_\cdot)$ $dV \otimes d\mathrm{P}^{s,x}$ a.e. for every $(s,x) \in [0,T] \times E$, one can remark that $u = \bar{u}$ up to a zero potential set, and in particular that $u \in \mathcal{L}^2_X$ since $\bar{u}$ does.*
Moreover, for any $(s,x) \in [0,T] \times E$, the stochastic convergence $(Y^{k,s,x}, M^{k,s,x}) \xrightarrow[k\to\infty]{L^2(dV\otimes d\mathrm{P}^{s,x})\times\mathcal{H}^2} (Y^{s,x}, M^{s,x})$ now has the functional counterpart $\begin{cases} u_k \xrightarrow{\|\cdot\|_{2,s,x}} u \\ v_k \xrightarrow{\|\cdot\|_{2,s,x}} v, \end{cases}$ which yields $\begin{cases} u_k \xrightarrow{L^2_X} u \\ v_k \xrightarrow{L^2_X} v, \end{cases}$ where we recall that the locally convex topological space L^2_X was introduced in Notation 4.12.

Corollary 5.17. *For any $(s,x) \in [0,T] \times E$ and for any $t \in [s,T]$, the couple of functions (u,v) obtained in Theorem 5.15 verifies $\mathrm{P}^{s,x}$ a.s.*

$$u(t,X_t) = g(X_T) + \int_t^T f\left(r, X_r, u(r,X_r), v(r,X_r)\right) dV_r - (M_T^{s,x} - M_t^{s,x}),$$

where $M^{s,x}$ denotes the martingale part of the unique solution of $BSDE^{s,x}(f,g)$.

Proof. The corollary follows from Theorem 5.15 and Lemma 5.13. □

We now introduce now a probabilistic notion of solution for $Pseudo{-}PDE(f,g)$.

Definition 5.18. *A function $u : [0,T] \times E \to \mathrm{R}$ will be said to be a* **martingale solution** *of $Pseudo-PDE(f,g)$ if $u \in \mathcal{D}(\mathfrak{a})$ and*

$$\left\{ \begin{array}{rcl} \mathfrak{a}(u) & = & -f(\cdot,\cdot,u,\sqrt{\mathfrak{G}(u,u)}) \\ u(T,\cdot) & = & g. \end{array} \right. \tag{5.11}$$

Remark 5.19. *The first equation of (5.11) holds in L^0_X, hence up to a zero potential set. The second one is a pointwise equality.*

Proposition 5.20. *A classical solution u of $Pseudo-PDE(f,g)$ such that $\Gamma(u,u) \in \mathcal{L}^1_X$, is also a martingale solution.*
Conversely, if u is a martingale solution of $Pseudo-PDE(f,g)$ belonging to $\mathcal{D}(a)$, then u is a classical solution of $Pseudo{-}PDE(f,g)$ up to a zero-potential set, meaning that the first equality of (5.1) holds up to a set of zero potential.

Proof. Let u be a classical solution of $Pseudo-PDE(f,g)$ verifying $\Gamma(u,u) \in \mathcal{L}^1_X$, Definition 5.3 and Corollary 4.19 imply that $u \in \mathcal{D}(\mathfrak{a})$, $u(T,\cdot) = g$, and the equalities up to zero potential sets

$$\mathfrak{a}(u) = a(u) = -f(\cdot,\cdot,u,\Gamma(u,u)) = -f(\cdot,\cdot,u,\mathfrak{G}(u,u)), \tag{5.12}$$

which shows that u is a martingale solution. Similarly, the second statement follows by Definition 5.18 and again Corollary 4.19.

□

Theorem 5.21. *Assume Hypothesis 5.1 and 5.5 and let (u,v) be the functions defined in Theorem 5.15.*

Then $u \in \mathcal{D}(\mathfrak{a})$, $v^2 = \mathfrak{G}(u,u)$ and u is a martingale solution of $Pseudo-PDE(f,g)$.

Proof. For any $(s,x) \in [0,T] \times E$, by Corollary 5.17, for $t \in [s,T]$, we have $u(t,X_t) - u(s,x) = -\int_s^t f(r,X_r,u(r,X_r),v(r,X_r))dV_r + (M_t^{s,x} - M_s^{s,x})$ $\mathrm{P}^{s,x}$ a.s. so by Definition 4.16, $u \in \mathcal{D}(\mathfrak{a})$, $\mathfrak{a}(u) = -f(\cdot,\cdot,u,v)$ and $M[u]^{s,x} = M_\cdot^{s,x} - M_s^{s,x}$.
Moreover by Theorem 5.15 we have $\frac{d\langle M^{s,x}\rangle}{dV} = v^2(\cdot,X_\cdot)\, dV \otimes d\mathrm{P}^{s,x}$ a.e. on $[s,T]$, so by Proposition 4.17 it follows $v^2 = \mathfrak{G}(u,u)$ and therefore, the L^2_X equality $\mathfrak{a}(u) = -f(\cdot,\cdot,u,\sqrt{\mathfrak{G}(u,u)})$, which establishes the first line of (5.11).
Concerning the second line, we have for any $x \in E$,
$u(T,x) = u(T,X_T) = g(X_T) = g(x)$ $\mathrm{P}^{T,x}$ a.s. so $u(T,\cdot) = g$ (in the deterministic pointwise sense). □

We conclude the section with Theorem 5.22 which states that the previously constructed martingale solution of $Pseudo-PDE(f,g)$ is unique.

Theorem 5.22. *Under Hypothesis 5.1 and 5.5, $Pseudo-PDE(f,g)$ admits a unique martingale solution.*

Proof. Existence has been the object of Theorem 5.21.

Let u and u' be two elements of $\mathcal{D}(\mathfrak{a})$ solving (5.11) and let $(s,x)\in[0,T]\times E$ be fixed. By Definition 4.14 and Remark 3.23, the process $u(\cdot,X.)$ (respectively $u'(\cdot,X.)$) under $\mathrm{P}^{s,x}$ admits a cadlag modification $U^{s,x}$ (respectively $U'^{s,x}$) on $[s,T]$, which is a special semi-martingale with decomposition

$$\begin{aligned} U^{s,x} &= u(s,x)+\int_s^{\cdot}\mathfrak{a}(u)(r,X_r)dV_r+M[u]^{s,x}\\ &= u(s,x)-\int_s^{\cdot}f\left(r,X_r,u(r,X_r),\sqrt{\mathfrak{G}(u,u)}(r,X_r)\right)dV_r+M[u]^{s,x}\\ &= u(s,x)-\int_s^{\cdot}f\left(r,X_r,U^{s,x},\sqrt{\mathfrak{G}(u,u)}(r,X_r)\right)dV_r+M[u]^{s,x}, \end{aligned} \tag{5.13}$$

where the third equality of (5.13) comes from Lemma 5.13. Similarly we have $U'^{s,x}=u'(s,x)-\int_s^{\cdot}f\left(r,X_r,U'^{s,x},\sqrt{\mathfrak{G}(u',u')}(r,X_r)\right)dV_r+M[u']^{s,x})$.
The processes $M[u]^{s,x}$ and $M[u']^{s,x}$ (introduced at Definition 4.16) belong to $\mathcal{H}_0^2$; by Proposition 4.17, $\langle M[u]^{s,x}\rangle=\int_s^{\cdot}\mathfrak{G}(u,u)(r,X_r)dV_r$ (respectively $\langle M[u']^{s,x}\rangle=\int_s^{\cdot}\mathfrak{G}(u',u')(r,X_r)dV_r$). Moreover since $u(T,\cdot)=u'(T,\cdot)=g$, then $u(T,X_T)=u'(T,X_T)=g(X_T)$ a.s. then the couples $(U^{s,x},M[u]^{s,x})$ and $(U'^{s,x},M[u']^{s,x})$ both verify the equation (with respect to $\mathrm{P}^{s,x}$).

$$Y.=g(X_T)+\int_{.}^{T}f\left(r,X_r,Y_r,\sqrt{\frac{d\langle M\rangle}{dV}}(r)\right)dV_r-(M_T-M.) \tag{5.14}$$

on $[s,T]$.
Even though we do not have a priori information on the square integrability of $U^{s,x}$ and $U'^{s,x}$, we know that $M[u]^{s,x}$ and $M[u']^{s,x}$ are in $\mathcal{H}^2$ and equal to zero at time s, and that $U_s^{s,x}$ and $U_s'^{s,x}$ are deterministic so L^2. By Lemma 3.24 and the fact that $(U^{s,x},M[u]^{s,x})$ and $(U'^{s,x},M[u']^{s,x})$ solve the BSDE in the weaker sense (5.14), it is sufficient to conclude that both solve $BSDE^{s,x}(f,g)$ on $[s,T]$. By Theorem 3.21 and Remark 3.23 the two couples are $\mathbb{P}^{s,x}$-indistinguishable. This implies that $u(\cdot,X.)$ and $u'(\cdot,X.)$ are $\mathbb{P}^{s,x}$-modifications one of the other on $[s,T]$. In particular, considering their values at time s, we have $u(s,x)=u'(s,x)$. We therefore have $u'=u$.

□

Corollary 5.23. *There is at most one classical solution u of $Pseudo-PDE(f,g)$ such that $\Gamma(u,u)\in\mathcal{L}_X^1$.*

Proof. The proof follows from Proposition 5.20 and Theorem 5.22. □

6 Applications

In [8] which is the continuation of the present paper, several examples are studied. The examples below fit in the framework of Section 4.
A first typical example is the setup of jump diffusions as in the formalism D.W. Stroock in [34]. These are Markov processes which solve a Martingale problem associated to an operator of the type

$$\begin{aligned} a(\phi) &= \partial_t\phi + \tfrac{1}{2}\sum_{i,j\leq d}(\sigma\sigma^{\intercal})_{i,j}\partial^2_{x_ix_j}\phi + \sum_{i\leq d}\mu_i\partial_{x_i}\phi \\ &\quad + \int\left(\phi(\cdot,\cdot+y)-\phi(\cdot,y)-\tfrac{1}{1+\|y\|^2}\sum_{i\leq d}y_i\partial_{x_i}\phi\right)K(\cdot,\cdot,dy), \end{aligned}$$

where μ is a bounded Borel function with values in R^d and σ is a continuous-Borel function with values in $GL_d(\mathrm{R})$, the set of invertible matrices of size d. K is a Lévy kernel.

We also study Markov processes associated to a large class of pseudo-differential operators with the formalism of N. Jacob in [22]. A typical example of equation considered is

$$\begin{cases} \partial_t u - (-\Delta)^{\frac{\alpha}{2}}u = f(\cdot,\cdot,u,\sqrt{\Gamma^\alpha(u,u)}) \text{ on } [0,T]\times\mathrm{R}^d \\ u(T,\cdot) = g. \end{cases} \tag{6.1}$$

Here, the fractional Laplace operator $(-\Delta)^{\frac{\alpha}{2}}$ is given for some $\alpha\in]0,2[$ by $\phi\longmapsto c_\alpha PV\int_{\mathrm{R}^d}\frac{(\phi(\cdot+y)-\phi)}{\|y\|^{d+\alpha}}dy$ where c_α is some positive constant and PV denotes the principal value operator.

$$\Gamma^\alpha(\phi,\phi) = c_\alpha PV\int_{\mathrm{R}^d}\frac{(\phi(\cdot,\cdot+y)-\phi)^2}{\|y\|^{d+\alpha}}dy \tag{6.2}$$

is the corresponding Carré du champ. The forward process of the corresponding BSDEs is the α-stable Levy process.

An other example of application is given by solutions of SDEs with distributional drift, which are studied in [19]. These permit to tackle semilinear parabolic PDEs with distributional drift of type

$$\begin{cases} \partial_t u + \frac{1}{2}\sigma^2\partial^2_x u + b'\partial_x u + f(\cdot,\cdot,u,\sigma|\partial_x u|) = 0 \quad \text{on } [0,T]\times\mathrm{R} \\ u(T,\cdot) = g, \end{cases} \tag{6.3}$$

where b is only a continuous function, hence b' is a distribution.

Finally, examples in non Euclidean state spaces are given with the study of diffusions in a compact differential manifold M. A typical example is the Brownian motion in a Riemannian manifold. The equation considered is then of type

$$\begin{cases} \partial_t u + \Delta_M u + f(\cdot,\cdot,u,\|\nabla_M u\|_2) = 0 \quad \text{on } [0,T]\times M \\ u(T,\cdot) = g, \end{cases} \tag{6.4}$$

where Δ_M is the Laplace-Beltrami operator and ∇_M is the gradient in local coordinates. More general equations are considered in [8].

Appendices

A Markov classes

We recall in this Appendix some basic definitions and results concerning Markov processes. For a complete study of homogeneous Markov processes, one may consult [16], concerning non-homogeneous Markov classes, our reference was chapter VI of [17]. Some results are only stated, but the advised reader may consult [7] and [9] in which all announced results are carefully proven.

The first definition refers to the canonical space that one can find in [23], see paragraph 12.63.

Notation A.1. *In the whole section E will be a fixed Polish space (a separable completely metrizable topological space), and $\mathcal{B}(E)$ its Borel σ-field. E will be called the* ***state space****.*

We consider $T \in \mathrm{R}_+^*$. We denote $\Omega := \mathrm{D}(E)$ the Skorokhod space of functions from $[0,T]$ to E right-continuous with left limits and continuous at time T (e.g. cadlag). For any $t \in [0,T]$ we denote the coordinate mapping $X_t : \omega \mapsto \omega(t)$, and we introduce on Ω the σ-field $\mathcal{F} := \sigma(X_r | r \in [0,T])$.

On the measurable space $(\Omega, \mathcal{F})$, we introduce the measurable ***canonical process***

$$X : \begin{array}{rcl} (t,\omega) & \longmapsto & \omega(t) \\ ([0,T] \times \Omega, \mathcal{B}([0,T]) \otimes \mathcal{F}) & \longrightarrow & (E, \mathcal{B}(E)), \end{array}$$

and the right-continuous filtration $(\mathcal{F}_t)_{t\in[0,T]}$ where $\mathcal{F}_t := \bigcap_{s\in]t,T]} \sigma(X_r | r \leq s)$ if $t < T$, and $\mathcal{F}_T := \sigma(X_r | r \in [0,T]) = \mathcal{F}$.

$\big(\Omega, \mathcal{F}, (X_t)_{t\in[0,T]}, (\mathcal{F}_t)_{t\in[0,T]}\big)$ will be called the ***canonical space*** *(associated to T and E).*

For any $t \in [0,T]$ we denote $\mathcal{F}_{t,T} := \sigma(X_r | r \geq t)$, and for any $0 \leq t \leq u < T$ we will denote $\mathcal{F}_{t,u} := \bigcap_{n\geq 0} \sigma(X_r | r \in [t, u + \frac{1}{n}])$.

Remark A.2. *Previous definitions and all the notions of this Appendix, extend to a time interval equal to R_+ or replacing the Skorokhod space with the Wiener space of continuous functions from $[0,T]$ (or R_+) to E.*

Definition A.3. *The function*

$$p : \begin{array}{rcl} (s,x,t,A) & \longmapsto & p(s,x,t,A) \\ [0,T] \times E \times [0,T] \times \mathcal{B}(E) & \longrightarrow & [0,1], \end{array}$$

will be called ***transition function*** *if, for any s, t in $[0,T]$, $x \in E$, $A \in \mathcal{B}(E)$, it verifies*

1. *$x \mapsto p(s,x,t,A)$ is Borel,*
2. *$B \mapsto p(s,x,t,B)$ is a probability measure on $(E, \mathcal{B}(E))$,*
3. *if $t \leq s$ then $p(s,x,t,A) = \mathbb{1}_A(x)$,*
4. *if $s < t$, for any $u > t$, $\int_E p(s,x,t,dy)p(t,y,u,A) = p(s,x,u,A)$.*

The latter statement is the well-known **Chapman-Kolmogorov equation**.

Definition A.4. *A transition function p for which the first item is reinforced supposing that $(s,x) \longmapsto p(s,x,t,A)$ is Borel for any t, A, will be said* ***measurable in time****.*

Definition A.5. *A* ***canonical Markov class*** *associated to a transition function p is a set of probability measures $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times E}$ defined on the measurable space $(\Omega, \mathcal{F})$ and verifying for any $t \in [0,T]$ and $A \in \mathcal{B}(E)$*

$$\mathrm{P}^{s,x}(X_t \in A) = p(s,x,t,A), \tag{A.1}$$

and for any $s \leq t \leq u$

$$\mathrm{P}^{s,x}(X_u \in A|\mathcal{F}_t) = p(t, X_t, u, A) \quad \mathrm{P}^{s,x} \ a.s. \tag{A.2}$$

Remark A.6. *Formula 1.7 in Chapter 6 of [17] states that for any $F \in \mathcal{F}_{t,T}$ yields*

$$\mathrm{P}^{s,x}(F|\mathcal{F}_t) = \mathrm{P}^{t,X_t}(F) = \mathrm{P}^{s,x}(F|X_t) \ \ \mathrm{P}^{s,x} a.s. \tag{A.3}$$

Property (A.3) *will be called* ***Markov property****.*

For the rest of this section, we are given a canonical Markov class $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times E}$ which transition function is measurable in time.

Proposition A.7. *For any event $F \in \mathcal{F}$, $(s,x) \longmapsto \mathrm{P}^{s,x}(F)$ is Borel. For any random variable Z, if the function $(s,x) \longmapsto \mathrm{E}^{s,x}[Z]$ is well-defined (with possible values in $[-\infty, \infty]$), then it is Borel.*

Lemma A.8. *Let V be a continuous non-decreasing function on $[0,T]$ and $f \in \mathcal{B}([0,T] \times E)$ be such that for every (s,x), $\mathrm{E}^{s,x}[\int_s^T |f(r,X_r)|dV_r] < \infty$, then $(s,x) \longmapsto \mathrm{E}^{s,x}[\int_s^T f(r,X_r)dV_r]$ is Borel.*

Definition A.9. *For any $(s,x) \in [0,T]\times E$ we will consider the (s,x)-****completion*** *$\left(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}_t^{s,x})_{t\in[0,T]}, \mathrm{P}^{s,x}\right)$ of the stochastic basis $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathrm{P}^{s,x}\right)$ by defining $\mathcal{F}^{s,x}$ as the $\mathrm{P}^{s,x}$-completion of $\mathcal{F}$, by extending $\mathrm{P}^{s,x}$ to $\mathcal{F}^{s,x}$ and finally by defining $\mathcal{F}_t^{s,x}$ as the $\mathrm{P}^{s,x}$-closure of $\mathcal{F}_t$ for every $t \in [0,T]$.*

We remark that, for any $(s,x) \in [0,T] \times E$, $\left(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}_t^{s,x})_{t\in[0,T]}, \mathrm{P}^{s,x}\right)$ is a stochastic basis fulfilling the usual conditions.

Proposition 3.13 in [9] states the following.

Proposition A.10. *Let $(s,x) \in [0,T] \times E$ be fixed, Z be a random variable and $t \in [s,T]$, then $\mathrm{E}^{s,x}[Z|\mathcal{F}_t] = \mathrm{E}^{s,x}[Z|\mathcal{F}_t^{s,x}]$ $\mathrm{P}^{s,x}$ a.s.*

We recall here Definition 4.1 in [9].

Definition A.11. *We denote $\Delta := \{(t,u) \in [0,T]^2 | t \leq u\}$. On $(\Omega, \mathcal{F})$, we define a **non-homogeneous Additive Functional** (shortened AF) as a random-field indexed by Δ $A := (A_u^t)_{(t,u)\in\Delta}$, with values in R, verifying the two following conditions.*

1. *For any $(t,u) \in \Delta$, A_u^t is $\mathcal{F}_{t,u}$-measurable;*
2. *for any $(s,x) \in [0,T] \times E$, there exists a real cadlag $\mathcal{F}^{s,x}$-adapted process $A^{s,x}$ (taken equal to zero on $[0,s]$ by convention) such that for any $x \in E$ and $s \leq t \leq u$, $A_u^t = A_u^{s,x} - A_t^{s,x}$ $\mathrm{P}^{s,x}$ a.s.*

*$A^{s,x}$ will be called the **cadlag version of** A **under** $\mathrm{P}^{s,x}$.*

*An AF will be called a **non-homogeneous square integrable Martingale Additive Functional** (shortened square integrable MAF) if under any $\mathrm{P}^{s,x}$ its cadlag version is a square integrable martingale.*

A immediate consequence of Proposition 4.17 in [9] is the following.

Proposition A.12. *Given an increasing continuous function V, if in every stochastic basis $\left(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}_t^{s,x})_{t\in[0,T]}, \mathrm{P}^{s,x}\right)$, we have $\mathcal{H}_0^2 = \mathcal{H}^{2,V}$, then we can state the following.*
Let M and M' be two square integrable MAFs and let $M^{s,x}$ (respectively $M'^{s,x}$) be the cadlag version of M (respectively M') under a fixed $\mathrm{P}^{s,x}$. There exists a Borel function $k \in \mathcal{B}([0,T] \times E, \mathrm{R})$ such that for any $(s,x) \in [0,T] \times E$, $\langle M^{s,x}, M'^{s,x} \rangle = \int_s^\cdot k(r, X_r) dV_r$.
In particular if M is a square integrable MAF and $M^{s,x}$ its cadlag version under a fixed $\mathrm{P}^{s,x}$, there exists a Borel function $k \in \mathcal{B}([0,T] \times E, \mathrm{R})$ (which can be taken positive) such that for any $(s,x) \in [0,T] \times E$, $\langle M^{s,x} \rangle = \int_s^\cdot k(r, X_r) dV_r$.

B Technicalities related to Section 3

Proof of Proposition 3.2. Since we have $dA \ll dA + dB$ in the sense of stochastic measures with A, B predictable, there exists a predictable positive process K such that $A = \int_0^\cdot K_s dA_s + \int_0^\cdot K_s dB_s$ up to indistinguishability, see Proposition I.3.13 in [24]. Now there exists a P-null set $\mathcal{N}$ such that for any $\omega \in \mathcal{N}^c$ we have $0 \leq \int_0^\cdot K_s(\omega) dB_s(\omega) = \int_0^\cdot (1 - K_s(\omega)) dA_s(\omega)$, so $K(\omega) \leq 1$ $dA(\omega)$ a.e. on $\mathcal{N}^c$. Therefore if we set $E(\omega) = \{t : K_t(\omega) = 1\}$ and $F(\omega) = \{t : K_t(\omega) < 1\}$ then $E(\omega)$ and $F(\omega)$ are disjoint Borel sets and $dA(\omega)$ has all its mass in $E(\omega) \cup F(\omega)$ so we can decompose $dA(\omega)$ within these two sets.

We therefore define the processes $A^{\perp B} = \int_0^\cdot \mathbb{1}_{\{K_s=1\}} dA_s$ and ; $A^B = \int_0^\cdot \mathbb{1}_{\{K_s<1\}} dA_s$.

$A^{\perp B}$ and A^B are both in $\mathcal{V}^{p,+}$, and $A = A^{\perp B} + A^B$. In particular the (stochastic) measures $dA^{\perp B}$ and dA^B fulfill $dA^{\perp B}(\omega)(G) = dA(\omega)(E(\omega) \cap G)$ and $dA^B(\omega)(G) = dA(\omega)(F(\omega) \cap G)$.
We remark $dA^{\perp B} \perp dB$ in the sense of stochastic measures. Indeed, fixing $\omega \in \mathcal{N}^c$, for $t \in E(\omega)$, $K_t(\omega) = 1$, so $\int_{E(\omega)} dA(\omega) = \int_{E(\omega)} dA(\omega) + \int_{E(\omega)} dB(\omega)$ implying that $\int_{E(\omega)} dB(\omega) = 0$. Since for any $\omega \in \mathcal{N}^c$, $dB(\omega)\,(E(\omega)) = 0$ while $dA^{\perp B}(\omega)$ has all its mass in $E(\omega)$, which gives this first result.

Now let us prove $dA^B \ll dB$ in the sense of stochastic measure.
Let $\omega \in \mathcal{N}^c$, and let $G \in \mathcal{B}([0,T])$, such that $\int_G dB(\omega) = 0$. Then

$$\begin{array}{rcl} \int_G dA^B(\omega) & = & \int_{G\cap F(\omega)} dA(\omega) \\ & = & \int_{G\cap F(\omega)} K(\omega) dA(\omega) + \int_{G\cap F(\omega)} K(\omega) dB(\omega) \\ & = & \int_{G\cap F(\omega)} K(\omega) dA(\omega). \end{array}$$

So $\int_{G\cap F(\omega)} (1 - K(\omega)) dA(\omega) = 0$, but $(1 - K(\omega)) > 0$ on $F(\omega)$.
So $dA^B(\omega)(G) = 0$. Consequently for every $\omega \in \mathcal{N}^c$, $dA^B(\omega) \ll dB(\omega)$ and so that $dA^B \ll dB$.
Now, since K is positive and $K(\omega) \leq 1$ $dA(\omega)$ a.e. for almost all ω, we can replace K by $K \wedge 1$ which is still positive predictable, without changing the associated stochastic measures $dA^B, dA^{\perp B}$; therefore we can consider that $K_t(\omega) \in [0,1]$ for all (ω, t).
We remark that for P almost all ω the decomposition $A^{\perp B}$ and A^B is unique because of the corresponding uniqueness of the decomposition in the Lebesgue-Radon-Nikodym theorem for each fixed $\omega \in \mathcal{N}^c$.

Since $dA^B \ll dB$, again by Proposition I.3.13 in [24], there exists a predictable positive process that we will call $\frac{dA}{dB}$ such that $A^B = \int_0^{\cdot} \frac{dA}{dB} dB$ and which is only unique up to $dB \otimes d\mathrm{P}$ null sets. □

Proposition B.1. *Let M and M' be two local martingales in $\mathcal{H}^2_{loc}$ and let $V \in \mathcal{V}^{p,+}$. We have $\frac{d\langle M\rangle}{dV}\frac{d\langle M'\rangle}{dV} - \left(\frac{d\langle M,M'\rangle}{dV}\right)^2 \geq 0 \quad dV \otimes d\mathrm{P}$ a.e.*

Proof. Let $x \in \mathrm{Q}$. Since $\langle M + xM'\rangle$ is an increasing process starting at zero, then by Proposition 3.2, we have $\frac{d\langle M+xM'\rangle}{dV} \geq 0 \quad dV \otimes d\mathrm{P}$ a.e.
By the linearity property stated in Proposition 3.4, we have
$0 \leq \frac{d\langle M+xM'\rangle}{dV} = \frac{d\langle M\rangle}{dV} + 2x\frac{d\langle M,M'\rangle}{dV} + x^2\frac{d\langle M'\rangle}{dV}$ $dV \otimes d\mathrm{P}$ a.e. Since Q is countable, there exists a $dV \otimes d\mathrm{P}$-null set $\mathcal{N}$ such that for $(\omega, t) \notin \mathcal{N}$ and $x \in \mathrm{Q}$,
$\frac{d\langle M\rangle}{dV}(\omega,t) + 2x\frac{d\langle M,M'\rangle}{dV}(\omega,t) + x^2\frac{d\langle M'\rangle}{dV}(\omega,t) \geq 0$. By continuity of polynomes, this holds for any $x \in \mathrm{R}$. Expressing the discriminant of this polynome, we deduce that $4\left(\frac{d\langle M,M'\rangle}{dV}(\omega,t)\right)^2 - 4\frac{d\langle M\rangle}{dV}(\omega,t)\frac{d\langle M'\rangle}{dV}(\omega,t) \leq 0$ for all $(\omega,t) \notin \mathcal{N}$. □

Proof of Proposition 3.5. Since the angular bracket $\langle M\rangle$ of a square integrable martingale M always belongs to $\mathcal{V}^{p,+}$, by Proposition 3.2, we can

consider the processes $\langle M\rangle^V$ and $\langle M\rangle^{\perp V}$; in particular there exists a predictable process K with values in $[0,1]$ such that $\langle M\rangle^V = \int_0^\cdot \mathbb{1}_{\{K_s<1\}} d\langle M\rangle_s$ and $\langle M\rangle^{\perp V} = \int_0^\cdot \mathbb{1}_{\{K_s=1\}} d\langle M\rangle_s$.

We can then set $M^V = \int_0^\cdot \mathbb{1}_{\{K_s<1\}} dM_s$ and $M^{\perp V} = \int_0^\cdot \mathbb{1}_{\{K_s=1\}} dM_s$ which are well-defined because K is predictable, and therefore $\mathbb{1}_{\{K_t<1\}}$ and $\mathbb{1}_{\{K_t=1\}}$ are also predictable. $M^V, M^{\perp V}$ belong to $\mathcal{H}_0^2$ because their angular brackets are both bounded by $\langle M\rangle_T \in L^1$. Since K takes values in $[0,1]$, we have
$M^V + M^{\perp V} = \int_0^\cdot \mathbb{1}_{\{K_s<1\}} dM_s + \int_0^\cdot \mathbb{1}_{\{K_s=1\}} dM_s = M$;
$\langle M^V\rangle = \int_0^\cdot \mathbb{1}_{\{K_s<1\}} d\langle M\rangle_s = \langle M\rangle^V$; $\langle M^{\perp V}\rangle = \int_0^\cdot \mathbb{1}_{\{K_s=1\}} d\langle M\rangle_s = \langle M\rangle^{\perp V}$
and $\langle M^V, M^{\perp V}\rangle = \int_0^\cdot \mathbb{1}_{\{K_s<1\}}\mathbb{1}_{\{K_s=1\}} d\langle M\rangle_s = 0$. □

Proof of Proposition 3.6. We start by remarking that for any M_1, M_2 in $\mathcal{H}_0^2$, a consequence of Kunita-Watanabe's decomposition (see Theorem 4.27 in [24]) is that $dVar(\langle M_1, M_2\rangle) \ll d\langle M_1\rangle$ and $dVar(\langle M_1, M_2\rangle) \ll d\langle M_2\rangle$.

Now, let M_1 and M_2 be in $\mathcal{H}^{2,V}$. We have $dVar(\langle M_1, M_2\rangle) \ll d\langle M_1\rangle \ll dV$. So since $\langle M_1 + M_2\rangle = \langle M_1\rangle + 2\langle M_1, M_2\rangle + \langle M_2\rangle$, then $d\langle M_1 + M_2\rangle \ll dV$ which shows that $\mathcal{H}^{2,V}$ is a vector space.

If M_1 and M_2 are in $\mathcal{H}^{2,\perp V}$, then since $dVar(\langle M_1, M_2\rangle) \ll d\langle M_1\rangle$ we can write $Var(\langle M_1, M_2\rangle) = \int_0^\cdot \frac{dVar(\langle M_1,M_2\rangle)}{d\langle M_1\rangle} d\langle M_1\rangle$ which is almost surely singular with respect to dV since M_1 belongs to $\mathcal{H}^{2,\perp V}$. So, by the bilinearity of the angular bracket $\mathcal{H}^{2,\perp V}$ is also a vector space.

Finally if $M_1 \in \mathcal{H}^{2,V}$ and $M_2 \in \mathcal{H}^{2,\perp V}$ then $dVar(\langle M_1, M_2\rangle) \ll d\langle M_1\rangle \ll dV$ but we also have seen that if $d\langle M_2\rangle$ is singular to dV then so is $dVar(\langle M_1, M_2\rangle) \ll d\langle M_2\rangle$.
For fixed ω, a measure being simultaneously dominated and singular with respect to to $dV(\omega)$ is necessarily the null measure, so $dVar(\langle M_1, M_2\rangle) = 0$ as a stochastic measure. Therefore M_1 and M_2 are strongly orthogonal, which implies in particular that M_1 and M_2 are orthogonal in $\mathcal{H}_0^2$.
So we have shown that $\mathcal{H}^{2,V}$ and $\mathcal{H}^{2,\perp V}$ are orthogonal sublinear-spaces of $\mathcal{H}_0^2$ but we also know that $\mathcal{H}_0^2 = \mathcal{H}^{2,V} + \mathcal{H}^{2,\perp V}$ thanks to Proposition 3.5, so
$\mathcal{H}_0^2 = \mathcal{H}^{2,V} \oplus^\perp \mathcal{H}^{2,\perp V}$. This implies that $\mathcal{H}^{2,V} = (\mathcal{H}^{2,\perp V})^\perp$ and
$\mathcal{H}^{2,\perp V} = (\mathcal{H}^{2,V})^\perp$ and therefore that these spaces are closed. So they are sub-Hilbert spaces. We also have shown that they were strongly orthogonal spaces, in the sense that any $M^1 \in \mathcal{H}^{2,V}$, $M^2 \in \mathcal{H}^{2,\perp V}$ are strongly orthogonal. By localization the strong orthogonality property also extends to $M^1 \in \mathcal{H}_{loc}^{2,V}$, $M^2 \in \mathcal{H}_{loc}^{2,\perp V}$. □

ACKNOWLEDGEMENTS. The authors are grateful to Andrea Cosso for stimulating discussions. The research of the first named author was provided by a PhD fellowship (AMX) of the Ecole Polytechnique.

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