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Description of the the symmetric convex random closed sets as zonotopes from their Feret’s diameters

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Abstract

In this paper, the 2-D random closed sets (RACS) are studied by means of the Feret’s diameter, also known as the caliper diameter. More specifically, it is shown that a 2-D symmetric convex RACS can be approximated as precisely as we want by some random zonotopes (polytopes formed by the Minkowski sum of line segments) in terms of the Hausdorff’s distance. Such an approximation is fully defined from the Feret’s diameter of the 2-D convex RACS. Particularly, the moments of the random vector representing the face lengths of the zonotope approximation are related to the moments of the Feret’s diameter random process of the RACS.

Keywords : Zonotopes, Random Closed Set, Feret’s diameter, Polygonal approximation.

MSC[2010]: 60D-XX, 52A22.

1 Introduction

1.1 Context and objectives

The geometrical characterization of granular media (grains, pores, fibers...) is an important issue in materials and process sciences. Indeed, several granular media can be modeled as random sets where the heterogeneity of the particles is studied with a probabilistic approach [12, 24]. In this context, the random closed sets (RACS) have been particularly studied [29, 21, 8, 2] to get geometrical characteristics of such granular media. A RACS denotes a random variable defined on a probability space $(\Omega, \mathfrak{A}, P)$ valued in $(\mathbb{F}, \mathfrak{F})$ the family of closed subsets of \mathbb{R}^d provided with the σ -algebra $\mathfrak{F} := \sigma\{\{F \in \mathbb{F} \mid F \cap X \neq \emptyset\} \mid X \in \mathfrak{K}\}$ where \mathfrak{K} denotes the class of compact subsets on \mathbb{R}^d . In a probabilistic point of view, the distribution of a convex RACS is uniquely determined from the Choquet capacity functional [23, 16]. However, such a description is not suitable for explicitly determining the geometrical shape of the RACS. An alternative way is to describe a RACS by the probability distribution of real valued geometrical characteristics (area, perimeter, diameters...).

1.2 Original contribution

The aim of this paper is to show how such characteristics can be used to describe the geometrical shape of a convex random closed set in \mathbb{R}^2 . It has already been shown [25] that the moments of the Feret's diameter of a convex random closed set in \mathbb{R}^2 can be obtained by the area measures on morphological transforms of it. A Feret's diameter (also known as caliper diameter) is a measure of a set size along a specified direction. It can be defined as the distance between the two parallel planes restricting the set perpendicular to that direction.

A set $X \in \mathbb{R}^2$ is said to be central symmetric or more simply symmetric if it is equal to the set $\check{X} := -X$. Note that the Feret's diameter is not sensitive to such a central symmetrization [22]. Indeed, for a non empty compact convex set $X \subset \mathbb{R}^2$, its symmetrized set $\frac{1}{2}(X \oplus \check{X})$ (see [21, 17]) has the same Feret's diameter as X . Consequently, the Feret's diameter of a convex set X is not enough to fully reconstruct X (but only its symmetrized set). However, the Feret's diameter is still useful to describe the shape of convex sets for two reasons. Firstly, a convex set X and its symmetrized set $\frac{1}{2}(X \oplus \check{X})$ share a lot of common geometrical descriptors (perimeter, eccentricity...). Secondly, there is many applications in which symmetric convex particles are considered. In this way, the reported work is focused on the symmetric convex sets (i.e $X = \frac{1}{2}(X \oplus \check{X})$). By abusing the notation, the conditions non-empty and compact will be often omitted in this paper. In other words, without explicit mention of the contrary, a *convex set* will refer to a non-empty compact convex set.

In this paper, it will be shown that the Feret's diameter of a random symmetric convex set can be used to define some approximations of it as random zonotopes. The polygonal approximation of a deterministic convex set have already been studied several times [18, 14, 4, 7]. However, in most cases the approximation is made by using the support function, which is not available in most of the geometric stochastic models. Random polygons have already been studied several time [10, 3, 20]. However, they are defined in different ways and for other objectives, and they are not characterized from their Feret's diameters. In our point of view, a zonotope (which is a Minkowski sum of line segments) is described by its faces (direction and length) and can be characterized by its Feret's diameter. It will be shown that the Feret's diameter of a symmetric convex set evaluated on a finite number of directions $N > 1$ can be used to define some approximations of it as zonotope. Such zonotope approximations will be generalized to the random symmetric convex sets. Therefore, a random symmetric convex set will be approximated by a random zonotope, and such approximations will be characterized from the Feret's diameter of the random symmetric convex set. The considered random zonotope will be uniquely determined by the lengths of its faces, their directions will be assumed to be known. The approximations considered are consistent as $N \rightarrow \infty$ with respect to the Hausdorff's distance.

This work is a preliminary study in order to describe the geometrical characteristics of a population of convex particles in the context of image analysis. Indeed, such images of population of convex particles can be modelled by stochastic geometric models. In such a model, the particle's projection is rep-

resented by a random convex set. Consequently, this work can be used to get information on such convex particles. In addition, when the particles are supposed to be symmetric, they have a symmetric 2-D projection that can be fully characterized by the Feret's diameter. Such a symmetric hypothesis is suitable in several industrial applications in chemical engineering (gas absorption, distillation, liquid-liquid extraction, petroleum processes, crystallization, etc.).

An area of application is the gas-liquid reactions. Indeed in a such process, the gas bubbles can be modelled as ellipsoid which 2-D projections are ellipses (see [32, 30, 5]). The main area of application is crystals manufacturing. Indeed, many crystals are 3-D zonohedrons and their 2-D projections are zonotopes. For example, the crystals of oxalate ammonium [26, 1], the crystals of calcium oxalate dihydrate [31] or the (L)-glutamic acid [6]. In such applications, the considered approximation coincide with the real data.

1.3 Paper outline

The paper is organized in the following way. The first part is devoted to the case of a deterministic symmetric convex set X . Some properties of Feret's diameter are firstly recalled, then for any integer $N > 1$ an approximation $X_0^{(N)}$ of X as a zonotope [11] is described. It is shown that this approximation is consistent as $N \rightarrow \infty$ with respect to the Hausdorff's distance [27]. A more accurate zonotope approximation $\tilde{X}_0^{(N)}$ of X which is invariant up to a rotation is also defined where the consistency is also satisfied. This approximation is particularly interesting to describe the geometrical shape of X .

The second part is devoted to the characterization of the random zonotopes. Firstly, some properties of the random process associated to the Feret's diameter will be explored. Then the random zonotopes will be studied. Some classes of them will be defined and their descriptions by their faces will be discussed. Finally, the characterization of some random zonotopes from their Feret's diameters random process will be studied.

In the last part, the random symmetric convex set X is studied. It is shown that it can still be described as precisely as we want by a random zonotope $X_0^{(N)}$ and up to a rotation by a random zonotope $X_\infty^{(N)}$ with respect to the Hausdorff's distance. The properties of these approximates are given and it is shown that they can be characterized from the Feret's diameter random process of X . In particular, the mean and auto-covariance of the Feret's diameter random process of X can be used to get the mean and the variances of the random vectors composed by the face lengths of their zonotope approximations.

2 Description of a symmetric convex set as a zonotope from its Feret's diameter

The aim of this section is to discuss how a convex set X can be described as a zonotope. It will be shown that X can always be approximated as precisely as we want by zonotopes, and how such zonotopes can be characterized from the Feret's diameter of X . Firstly, there is a need to recall the definition of the

Feret's diameter and some of its properties.

2.1 Feret's diameter and the support function

Definition 2.1 (Support function). Let $X \subset \mathbb{R}^2$ be a convex set, the support function of X is defined as:

$$f_X : \begin{cases} \mathbb{R}^2 & \longrightarrow \mathbb{R} \\ x & \longmapsto \sup_{s \in X} \langle x, s \rangle = \max_{s \in X} \langle x, s \rangle \end{cases}$$

where \langle, \rangle denote the Euclidean dot product.

The support function allows to fully characterize a convex set. Indeed, any positive homogeneous, convex, real-valued function on \mathbb{R}^2 is the support function of a convex set [27]. In the following, some important properties of the support function are given. The proofs are omitted since they can be found in the literature [13, 27].

Proposition 2.1 (Properties of the support function). Let $X \subset \mathbb{R}^2$ be a convex set, its support function satisfies the following properties:

- i. Positive homogeneity: $\forall r \geq 0, f_X(rx) = rf_X(x)$
- ii. Sub-additivity: $f_X(x+y) \leq f_X(x) + f_X(y)$
- iii. $f_{X \oplus Y} = f_X + f_Y$, where \oplus denotes the Minkowski's addition.
- iv. Let s be a vectorial similarity and $b \in \mathbb{R}^2$ then $f_{s(X)+b}(x) = f_X(s(x)) + \langle x, b \rangle$.
- v. Reconstruction:

$$X = \bigcap_{x \in \mathbb{R}^2} \{y \in \mathbb{R}^2 \mid \langle y, x \rangle \leq f_X(x)\} \quad (1)$$

- vi. If in addition $0 \in X$ then $f_X \geq 0$.
- vii. $d_H(X, Y) = \|f_X - f_Y\|_\infty$ where d_H denote the Hausdorff's distance and $\|\cdot\|_\infty$ the uniform norm on the unit sphere.

The items i and ii relate the convexity of the support function and the expression (1) allows the reconstruction of a convex set from its support function. Note that the positive homogeneity of the support function involves that it can be completely determined on the Euclidean unit sphere. The following representation for the support function of X is adopted in this paper:

$$h_X : \begin{cases} \mathbb{R} & \longrightarrow \mathbb{R} \\ \theta & \longmapsto h_X(\theta) = f_X(t(-\sin(\theta), \cos(\theta))) \end{cases}$$

h_X is a continuous and 2π -periodic function.

Note that the Feret's diameter of a convex set X , denoted H_X , can be expressed by the support function as:

$$\forall \theta \in \mathbb{R}, H_X(\theta) = h_X(\theta) + h_{\bar{X}}(\theta) \quad (2)$$

where \check{X} is the usual notation for the symmetric set $-X$. It is easy to see that the Feret's diameter of X coincides with the support function of $X \oplus \check{X}$, where \oplus denotes the Minkowski sum. Therefore, the functional H_X is enough to fully characterize the symmetrized body $\frac{1}{2}(X \oplus \check{X})$. Note that if X is already symmetric then H_X fully characterizes X . Some important properties of the Feret's diameter are recalled in the following.

Proposition 2.2 (Properties of the Feret's diameter). *Let X be a convex set then its Feret's diameter H_X satisfies the following properties:*

- i. Let X and Y be two convex sets, then $H_{X \oplus Y} = H_X + H_Y$
- ii. $\forall r \in \mathbb{R}$, $H_{rX} = |r|H_X$
- iii. Let R_η be a rotation and $b \in \mathbb{R}^2$, then $\forall \theta \in \mathbb{R}$, $H_{R_\eta(X)+b}(\theta) = H_X(\theta + \eta)$
- iv. π -periodicity: $\forall \theta \in \mathbb{R}$, $H_X(\theta + \pi) = H_X(\theta)$
- v. Let X and Y be two symmetric bodies, then $H_X \leq H_Y \Leftrightarrow X \subseteq Y$
- vi. For any $\theta, \beta \in [0, 2\pi]$,

$$H_X(\theta + \beta) \leq H_X(\theta) + 2 \left| \sin\left(\frac{\beta}{2}\right) \right| H_X\left(\theta + \frac{\beta + \pi}{2}\right) \quad (3)$$

Proof.

- 1,2,3. According to the equation (2), the first three points come directly from proposition [Proposition 2.1](#).
4. $\forall \theta \in \mathbb{R}$, $h_{\check{X}}(\theta) = h_X(\theta + \pi)$ then it follows the π -periodicity.
5. Because of the symmetry of X and Y , if $H_X \leq H_Y$ then $h_X \leq h_Y$. Therefore for any $x \in \mathbb{R}^2$ $f_X(x) \leq f_Y(x)$, so $\{y \in \mathbb{R}^2 \mid \langle y, x \rangle \leq f_X(x)\} \subseteq \{y \in \mathbb{R}^2 \mid \langle y, x \rangle \leq f_Y(x)\}$ then $X \subset Y$ according to the proposition [2.1.v](#).
Suppose that $X \subset Y$ then $\forall x \in \mathbb{R}^2$, $\{\langle s, x \rangle \mid s \in X\} \subset \{\langle s, x \rangle \mid s \in Y\} \Rightarrow f_X(x) \leq f_Y(x) \Rightarrow h_X \leq h_Y \Rightarrow H_X \leq H_Y$.
6. For any $(\theta, \beta) \in \mathbb{R}^2$, let $\alpha = \beta + \pi$, $x = {}^t(-\sin(\theta), \cos(\theta))$, $z = {}^t(-\sin(\theta + \alpha), \cos(\theta + \alpha))$ and $y = z + x$, so:

$$\begin{aligned} f_X(y - x) &\leq f_X(-x) + f_X(y) \\ h_X(\theta + \alpha) &\leq h_X(\theta + \pi) + f_X(y) \end{aligned}$$

and :

$$\begin{aligned} \|y\| &= \sqrt{(\sin(\theta) + \sin(\theta + \alpha))^2 + (\cos(\theta) + \cos(\theta + \alpha))^2} \\ &= \sqrt{2 + 2(\sin(\theta)\sin(\theta + \alpha) + \cos(\theta)\cos(\theta + \alpha))} \\ &= \sqrt{2}\sqrt{1 + \cos(\alpha)} \\ &= \sqrt{2}\sqrt{2\cos^2\left(\frac{\alpha}{2}\right)} \end{aligned}$$

$$\begin{aligned}
&= 2|\cos(\frac{\alpha}{2})| \\
&= 2|\sin(\frac{\beta}{2})|
\end{aligned}$$

By using the Euler's formula:

$$\begin{aligned}
\sin(\theta) + \sin(\theta + \alpha) &= 2\sin(\theta + \frac{\alpha}{2})\cos(\frac{\alpha}{2}) \\
\cos(\theta) + \cos(\theta + \alpha) &= 2\cos(\theta + \frac{\alpha}{2})\cos(\frac{\alpha}{2})
\end{aligned}$$

and by taking $\eta \in \mathbb{R}$ such that $y = \|y\|^t (-\sin(\eta), \cos(\eta))$, it follows:

$$\begin{aligned}
\sin(\eta) &= \frac{2\sin(\theta + \frac{\alpha}{2})\cos(\frac{\alpha}{2})}{\|y\|} \\
\cos(\eta) &= \frac{2\cos(\theta + \frac{\alpha}{2})\cos(\frac{\alpha}{2})}{\|y\|}
\end{aligned}$$

Let s be the sign of $\cos(\frac{\alpha}{2})$ then $\sin(\eta) = s\sin(\theta + \frac{\alpha}{2})$ and $\cos(\eta) = s\cos(\theta + \frac{\alpha}{2})$.

Finally, $\eta \in \{\theta + \frac{\beta+\pi}{2}, \theta + \frac{\beta+\pi}{2} + \pi\}$ and it can be expressed as:

$$h_X(\theta + \beta - \pi) \leq h_X(\theta + \pi) + 2|\sin(\frac{\beta}{2})|h_X(\eta)$$

This result is true for any convex set X , in particular for $Y = \frac{1}{2}(X \oplus \check{X})$. However, $h_y = H_X$ then by using the π -periodicity of the Feret's diameter:

$$\forall \theta, \beta \in [0, 2\pi], H_X(\theta + \beta) \leq H_X(\theta) + 2|\sin(\frac{\beta}{2})|H_X(\theta + \frac{\beta + \pi}{2})$$

□

The Feret's diameter can also be related to the mixed area [27] by using a line segment as structural element. Indeed, by using the Steiner's formula [27] with the two convex sets X and Y :

$$A(X \oplus Y) = A(X) + 2W(X, Y) + A(Y)$$

where $W(X, Y)$ denote the mixed area between X and Y . The mixed area functional $W(., .)$ is a symmetric mapping which is also homogeneous in its two variables (see [19, 27] for details). It is often used to describe some morphological characteristics of a convex X by using different structuring elements. For instance, if X is a bounded convex set and B the unit disk, then $W(X, B) = \frac{1}{2}U(X)$, where $U(X)$ denotes the perimeter of X . Let X be a bounded convex set, and S_θ a unit line segment directed by θ , then:

$$W(X, S_\theta) = \frac{1}{2}H_X(\theta) \tag{4}$$

The proof is omitted since it consists in a simple drawing and can be found in the literature [25, 19].

Remark 2.1. *This relation is very important because it involves an interpretation of the mixed area of a convex set with the Minkowski addition of line segments from its Feret's diameter. Indeed, for any $\theta_1, \theta_2 \in [0, \pi]$ and $\alpha_1, \alpha_2 \in \mathbb{R}_+$:*

$$\begin{aligned} A(X \oplus \alpha_1 S_{\theta_1} \oplus \alpha_2 S_{\theta_2}) &= A(X \oplus \alpha_1 S_{\theta_1}) + 2W(X \oplus \alpha_1 S_{\theta_1}, \alpha_2 S_{\theta_2}) \\ &= A(X) + \alpha_1 H_X(\theta_1) + \alpha_2 H_{X \oplus \alpha_1 S_{\theta_1}}(\theta_2) \\ &= A(X) + \alpha_1 H_X(\theta_1) + \alpha_2 H_X(\theta_2) + \alpha_2 H_{\alpha_1 S_{\theta_1}}(\theta_2) \end{aligned}$$

However, $\alpha_2 H_{\alpha_1 S_{\theta_1}}(\theta_2) = W(\alpha_1 S_{\theta_1}, S_{\theta_2}) = A(\alpha_1 S_{\theta_1} \oplus \alpha_2 S_{\theta_2})$. Then,

$$W(X, \alpha_1 S_{\theta_1} \oplus \alpha_2 S_{\theta_2}) = \frac{1}{2}(\alpha_1 H_X(\theta_1) + \alpha_2 H_X(\theta_2))$$

This result can be easily generalized by induction to any Minkowski sum of line segments, $\forall n \geq 1, \forall i = 1, \dots, n, \alpha_i \in \mathbb{R}_+, \theta_i \in \mathbb{R}$:

$$W(X, \bigoplus_{i=1}^n \alpha_i S_{\theta_i}) = \frac{1}{2} \sum_{i=1}^n \alpha_i H_X(\theta_i) \quad (5)$$

The relation (5) has an important kind of linearity. Indeed, it implies formulae for the computation of the mixed area between a convex set and a symmetric body from their Feret's diameter (see remark [Remark 2.3](#)).

2.2 Approximation of a symmetric convex set by a 0-regular zonotope

It has been given some properties of the Feret's diameter of a convex set and its connection with the mixed area. Here the zonotope will be defined and particularly the class of the 0-regular zonotopes. It will be discussed some properties of the zonotopes. In particular, it will be shown how a symmetric convex set can be approximated by a 0-regular zonotope as precisely as we want.

Let \mathcal{C} denote the class of all symmetric convex sets of \mathbb{R}^2 , where the symmetry is given in the sense of Minkowski: $X = \frac{1}{2}(X \oplus \check{X})$. Let S_0 be the unit line segment $[-\frac{1}{2}, \frac{1}{2}]$ and S_t its rotation by the angle $t \in [0, \pi[$. Consider now the convex set \check{X} such that:

$$X = \bigoplus_{i=1}^n \alpha_i S_{\theta_i}, \quad n \in \mathbb{N}^*, \quad \forall i = 1, \dots, n, \quad \alpha_i \in \mathbb{R}_+, \quad \theta_i \in [0, \pi[\quad (6)$$

Note that X is a compact convex symmetric polygon with at most $2n$ faces, where $\forall i = 1, \dots, n, \alpha_i$ is the length of the two faces of X oriented by θ_i . It is easy to see that every compact convex symmetric polygon has an even number of faces and can be represented as (6) up to a translation. Furthermore, note that X has a non-empty interior if and only if $n > 1$.

Definition 2.2 (Zonotopes). *Any compact convex symmetric polygon such as (6) is called a zonotope. Let $N \in \mathbb{N}^*$, $\mathcal{C}^{(N)}$ denotes the set of all zonotopes with at most $2N$ faces:*

$$\mathcal{C}^{(N)} = \left\{ \bigoplus_{i=1}^N \alpha_i S_{\theta_i} \mid \alpha \in \mathbb{R}_+^N, \theta \in [0, \pi[^N \right\}$$

where $\alpha = {}^t(\alpha_1, \dots, \alpha_N)$ and $\theta = {}^t(\theta_1, \dots, \theta_N)$.

Several geometric characteristics and properties of zonotopes can be easily expressed from the representation (6).

Proposition 2.3 (Geometrical characterization of zonotopes). *Let $N \in \mathbb{N}^*$ and $X = \bigoplus_{i=1}^N \alpha_i S_{\theta_i}$ be an element of $\mathcal{C}^{(N)}$. Let H_X be its Feret's diameter function, $U(X)$ its perimeter, and $A(X)$ its area. Then:*

$$\forall \eta \in \mathbb{R}, H_X(\eta) = \sum_{i=1}^N \alpha_i |\sin(\eta - \theta_i)| \quad (7)$$

$$U(X) = 2 \sum_{i=1}^N \alpha_i \quad (8)$$

$$A(X) = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j |\sin(\theta_i - \theta_j)| \quad (9)$$

Proof.

- (6). For any $(\beta, \eta) \in \mathbb{R}^2$, the support function of the line segment S_β in the direction η is:

$$\begin{aligned} h_{S_\beta}(\eta) &= \max_{t \in [-\frac{1}{2}, \frac{1}{2}]} \{t(-\cos(\beta) \sin(\eta) + \sin(\beta) \cos(\eta))\} \\ &= \max_{t \in [-\frac{1}{2}, \frac{1}{2}]} \{t \sin(\beta - \eta)\} \\ &= \frac{1}{2} |\sin(\beta - \eta)| \\ &\Rightarrow H_{S_\beta}(\eta) = |\sin(\beta - \eta)| \end{aligned}$$

Then from propositions 2.2.i and 2.2.ii, it follows the relation (7).

- (7). Considering that X is a polygon of $2N$ faces of length $\alpha_i, i = 1, \dots, N$, the perimeter can be obtained by adding up the face lengths.

- (8). For the area, the result (9) is proved by induction on N : for $N = 1$, $X = S_{\theta_1}$ and $A(X) = 0$ then (9) is verified. Suppose that (9) is true for $n \leq N$ and let us show that it is true for $N + 1$. $X = (\bigoplus_{i=1}^N \alpha_i S_{\theta_i}) \oplus \alpha_{N+1} S_{\theta_{N+1}}$ then from Steiner formula:

$$A(X) = A\left(\bigoplus_{i=1}^N \alpha_i S_{\theta_i}\right) + 2W\left(\bigoplus_{i=1}^N \alpha_i S_{\theta_i}, \alpha_{N+1} S_{\theta_{N+1}}\right)$$

then according to (4):

$$2W\left(\bigoplus_{i=1}^N \alpha_i S_{\theta_i}, \alpha_{N+1} S_{\theta_{N+1}}\right) = \alpha_{N+1} H_{\bigoplus_{i=1}^N \alpha_i S_{\theta_i}}(\theta_{N+1})$$

and finally according to the heredity assumption and (7):

$$\begin{aligned}
A(X) &= \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j |\sin(\theta_i - \theta_j)| + \alpha_{N+1} H_{\bigoplus_{i=1}^N \alpha_i S_{\theta_i}}(\theta_{N+1}) \\
&= \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j |\sin(\theta_i - \theta_j)| + \alpha_{N+1} \sum_{i=1}^N \alpha_i |\sin(\theta_{N+1} - \theta_i)| \\
&= \frac{1}{2} \sum_{i=1}^{N+1} \sum_{j=1}^{N+1} \alpha_i \alpha_j |\sin(\theta_i - \theta_j)|
\end{aligned}$$

which proves (9). □

In the following, a regular subdivision θ is used. It will be shown that if the subdivision step is sufficiently small, any symmetric convex set can be approximated by a zonotope as precisely as we want.

Definition 2.3 (0-regular zonotopes). Let $N \in \mathbb{N}^*$, $\mathcal{C}_0^{(N)}$ denotes the class of all zonotopes with at most $2N$ faces oriented by the regular subdivision of $[0, \pi[$ by N elements:

$$\mathcal{C}_0^{(N)} = \left\{ \bigoplus_{i=1}^N \alpha_i S_{\theta_i} \mid \alpha \in \mathbb{R}_+^N \text{ with } \theta_i = \frac{(i-1)\pi}{N}, i = 1, \dots, N \right\}.$$

Such zonotopes are called 0-regular zonotopes.

One can remark that $\mathcal{C}_0^{(N)} \subset \mathcal{C}^{(N)}$ and $\mathcal{C}_0^{(N_1)} \subset \mathcal{C}_0^{(N_2)}$ if and only if N_1 is a splitter of N_2 . In addition $\mathcal{C}_0^{(N)}$ can be identified to \mathbb{R}_+^N by the application $\alpha \rightarrow X = (\bigoplus_{i=1}^N \alpha_i S_{\theta_i}) \in \mathcal{C}$ which is an isomorphism between the semi-groups $(\mathbb{R}_+^N, +)$ and $(\mathcal{C}_0^{(N)}, \oplus)$. That is to say, this application is a bijection and:

$$\forall (\alpha, \alpha') \in \mathbb{R}_+^N \times \mathbb{R}_+^N, \left(\bigoplus_{i=1}^N (\alpha_i + \alpha'_i) S_{\theta_i} \right) = \left(\bigoplus_{i=1}^N \alpha_i S_{\theta_i} \right) \oplus \left(\bigoplus_{i=1}^N \alpha'_i S_{\theta_i} \right)$$

Theorem 2.1 (Approximation in $\mathcal{C}_0^{(N)}$). Let $X \in \mathcal{C}$.

- i. For all $N > 1$, let $F^{(N)}$ denote the squared matrix $(|\sin(\theta_i - \theta_j)|)_{1 \leq i, j \leq N}$ and $H_X^{(N)} = {}^t (H_X(\theta_1), \dots, H_X(\theta_N))$, then:

$$X_0^{(N)} = \bigoplus_{i=1}^N (F^{(N)-1} H_X^{(N)})_i S_{\theta_i} \tag{10}$$

belongs to $\mathcal{C}_0^{(N)}$ and satisfies:

$$\forall N > 1, d_H(X, X_0^{(N)}) \leq (6 + 2\sqrt{2}) \sin\left(\frac{\pi}{2N}\right) \text{diam}(X) \tag{11}$$

where $\text{diam}(X) = \sup_{s \in \mathbb{R}} (H_X(s))$ denotes the maximal diameter of X and d_H the Hausdorff's distance.

Consequently, the sequence of 0-regular zonotopes $(X_0^{(N)})_{N>1}$ approximates X in the following sense:

$$d_H(X, X_0^{(N)}) \longrightarrow 0 \text{ as } N \longrightarrow \infty \quad (12)$$

$X_0^{(N)}$ will be called the $\mathcal{C}_0^{(N)}$ -approximation of X .

ii. In addition, for any $N > 1$ the set $X_0^{(N)}$ is the unique element of $\mathcal{C}_0^{(N)}$ satisfying:

$$H_{X_0^{(N)}}(\theta_i) = H_X(\theta_i), \quad i = 1, \dots, N \quad (13)$$

iii. Furthermore $X_0^{(N)}$ contains X and it can be expressed as:

$$X_0^{(N)} = \bigcap_{i=1}^N \{x \in \mathbb{R}^2, | \langle x, {}^t(-\sin(\theta_i), \cos(\theta_i)) \rangle | \leq \frac{1}{2} H_X(\theta_i)\} \quad (14)$$

Proof.

2. Let $N > 1$ be an integer, it is easy to see that the matrix $F^{(N)}$ is invertible since $F^{(N)}$ is a circulant matrix [15] and its eigenvalues are exactly the coefficients of the discrete Fourier transform [28] of the signal $|\sin(\cdot)|$ (these coefficients are all strictly positive). Let $\alpha = F^{(N)-1} H_X^{(N)}$ such that:

$$X_0^{(N)} = \bigoplus_{i=1}^N \alpha_i S_{\theta_i}$$

Let us show that $X_0^{(N)}$ is the unique element of $\mathcal{C}_0^{(N)}$ satisfying $H_{X_0^{(N)}}(\theta_i) = H_X(\theta_i)$, $i = 1, \dots, N$. Suppose that there exists $X' \in \mathcal{C}_0^{(N)}$ satisfying $H_{X'}(\theta_i) = H_X(\theta_i)$, $i = 1, \dots, N$ then X' can be written as $X' = \bigoplus_{i=1}^N \alpha'_i S_{\theta_i}$ and then $H_{X'}^{(N)} = F^{(N)} \alpha'$. The invertibility of $F^{(N)}$ implies $\alpha = \alpha'$ which means $X_0^{(N)} = X'$.

1. Let us find an upper bound for the Hausdorff distance.

For all $\eta \in \mathbb{R}$ there exists $i \in \{1, \dots, N\}$ such that $\eta = \theta_i + \delta$ with $|\delta| \leq \frac{\pi}{2N}$.

By using the inequality (3) with $\theta = \theta_i, \beta = \delta$ for $X_0^{(N)}$:

$$H_{X_0^{(N)}}(\eta) \leq H_{X_0^{(N)}}(\theta_i) + 2 \left| \sin\left(\frac{\delta}{2}\right) \right| H_{X_0^{(N)}}\left(\theta_i + \frac{\delta + \pi}{2}\right)$$

By using the inequality (3) with $\theta = \eta, \beta = -\delta$ for X :

$$\begin{aligned} H_X(\theta_i) &\leq H_X(\eta) + 2 \left| \sin\left(\frac{-\delta}{2}\right) \right| H_X\left(\theta_i + \frac{\delta + \pi}{2}\right) \\ \Rightarrow -H_X(\eta) &\leq -H_X(\theta_i) + 2 \left| \sin\left(\frac{\delta}{2}\right) \right| H_X\left(\theta_i + \frac{\delta + \pi}{2}\right) \end{aligned}$$

Considering the equality $H_{X_0^{(N)}}(\theta_i) = H_X(\theta_i)$, it follows from the two previous inequalities:

$$H_{X_0^{(N)}}(\eta) - H_X(\eta) \leq 2 \left| \sin\left(\frac{\delta}{2}\right) \right| \left(H_X\left(\theta_i + \frac{\delta + \pi}{2}\right) + H_{X_0^{(N)}}\left(\theta_i + \frac{\delta + \pi}{2}\right) \right)$$

In the same manner, by using (3) with $\theta = \theta_i, \beta = \delta$ for X , and with $\theta = \eta, \beta = -\delta$ for $X_0^{(N)}$:

$$H_X(\eta) - H_{X_0^{(N)}}(\eta) \leq 2 \left| \sin\left(\frac{\delta}{2}\right) \right| \left(H_X\left(\theta_i + \frac{\delta + \pi}{2}\right) + H_{X_0^{(N)}}\left(\theta_i + \frac{\delta + \pi}{2}\right) \right)$$

Therefore by denoting $diam(X) = \sup_{\theta} \{H_X(\theta)\}$ and $diam(X_0^{(N)}) = \sup_{\theta} \{H_{X_0^{(N)}}(\theta)\}$, it follows:

$$|H_X(\eta) - H_{X_0^{(N)}}(\eta)| \leq 2 \sin\left(\frac{\pi}{2N}\right) (diam(X) + diam(X_0^{(N)})) \quad (15)$$

Furthermore,

$$\begin{aligned} H_{X_0^{(N)}}(\eta) &= \sum_{j=1}^N \alpha_j |\sin(\theta_i + \delta - \theta_j)| \\ &= \sum_{j=1}^N \alpha_j |\sin(\theta_i - \theta_j) \cos(\delta) - \cos(\theta_i - \theta_j) \sin(\delta)| \\ &\leq |\cos(\delta)| \sum_{j=1}^N \alpha_j |\sin(\theta_i - \theta_j)| + |\sin(\delta)| \sum_{j=1}^N |\sin(\theta_i - \theta_j + \frac{\pi}{2})| \\ &\leq |\cos(\delta)| H_{X_0^{(N)}}(\theta_i) + |\sin(\delta)| H_{X_0^{(N)}}\left(\frac{\pi}{2}\right) \\ &\leq |\cos(\delta)| H_X(\theta_i) + |\sin(\delta)| diam(X_0^{(N)}) \\ &\leq |\cos(\delta)| diam(X) + |\sin(\delta)| diam(X_0^{(N)}) \\ &\leq diam(X) + \sin\left(\frac{\pi}{2N}\right) diam(X_0^{(N)}) \\ &\Rightarrow diam(X_0^{(N)}) (1 - \sin\left(\frac{\pi}{2N}\right)) \leq diam(X) \\ N \geq 2 &\Rightarrow diam(X_0^{(N)}) \leq \frac{\sqrt{2}}{\sqrt{2}-1} diam(X) \end{aligned}$$

Then from (15):

$$\begin{aligned} |H_X(\eta) - H_{X_0^{(N)}}(\eta)| &\leq 2 \sin\left(\frac{\pi}{2N}\right) \left(1 + \frac{\sqrt{2}}{\sqrt{2}-1}\right) diam(X) \\ \Rightarrow \sup_{\eta} |(H_X(\eta) - H_{X_0^{(N)}}(\eta))| &= d_H(X, X_0^{(N)}) \\ &\leq (6 + 2\sqrt{2}) \sin\left(\frac{\pi}{2N}\right) diam(X) \end{aligned}$$

Consequently $d_H(X, X_0^{(N)}) \rightarrow 0$ as $N \rightarrow \infty$.

3. Let us note $Y_N = \bigcap_{i=1}^N \{x \in \mathbb{R}^2, | \langle x, {}^t(-\sin(\theta_i), \cos(\theta_i)) \rangle | \leq \frac{1}{2}H_X(\theta_i)\}$. Note that $Y_N \in \mathcal{C}_0^{(N)}$. Indeed each set of the intersection is the space between two lines oriented by one of the θ_i thus Y_N is a polygon with faces directed by the θ_i and therefore it belongs to $\mathcal{C}_0^{(N)}$. Because of the symmetry of X it is easy to see that $X = \bigcap_{s \in [0, \pi]} \{x \in \mathbb{R}^2, | \langle x, {}^t(-\sin(s), \cos(s)) \rangle | \leq \frac{1}{2}H_X(s)\}$ then $X \subset Y_N$ and consequently $H_X \leq H_{Y_N}$. Furthermore because of the expression of Y_N for any $i = 1, \dots, N$, $H_X(\theta_i) \geq H_{Y_N}(\theta_i)$ it follows the equality on the θ_i , and according to the foregoing $Y_N = X_0^{(N)}$. □

This theorem shows that a symmetric body can be always approximated by a 0-regular zonotope as close as we want. Note that the choice of the sequence $X_0^{(N)}$ is not the best one. Indeed, by taking $\frac{\text{diam}(X)}{\text{diam}(X_0^{(N)})} X_0^{(N)}$ there is a finer approximation with respect to the Hausdorff's distance. However, the sequence $X_0^{(N)}$ presents some important advantages: it always contains X , the approximation of a Minkowski sum is the Minkowski sum of the approximations, and its face length vector is expressed only from a linear combination of the Feret's diameter of X . Furthermore, if $\exists M > 1$, $X \in \mathcal{C}_0^{(M)}$ then $X_0^{(M)} = X$ and X is an adhesion value of the sequence $X_0^{(N)}$.

Remark 2.2 (Equivalence between perimeter and maximal diameter).
Notice that the $\text{diam}(X)$ can be replaced by $\frac{1}{2}U(X)$ in the relation (11). In fact, for any convex set X there is the relation:

$$2\text{diam}(X) \leq U(X) \leq 4\text{diam}(X) \quad (16)$$

Indeed, according to the definition of $\text{diam}(X)$ there exists a line segment $S \subseteq X$ which has a length greater than $\text{diam}(X)$ then $U(X) \geq U(S) \geq 2\text{diam}(X)$. The second inequality comes by considering that there is a square of side $\text{diam}(X)$ containing X .

Remark 2.3 (Expression of the mixed area from the Feret's diameter).
An interpretation of the mixed area between a convex set and a symmetric convex set can be given from the theorem iii. Indeed, let $N > 1$, Y be a convex set (not necessary symmetric), X be a symmetric convex set and $X_0^{(N)} = \bigoplus_{i=1}^N \alpha_i S_{\theta_i}$ be its $\mathcal{C}_0^{(M)}$ -approximation. Then, according to the continuity of the area and the Minkowski addition there is

$$W(Y, X_0^{(N)}) \rightarrow W(Y, X) \text{ as } N \rightarrow \infty$$

Furthermore, according to the theorem iii, $W(Y, X_0^{(N)})$ can be expressed as:

$$W(Y, X_0^{(N)}) = \sum_{i=1}^N H_Y(\theta_i) \sum_{j=1}^N F_{ij}^{(N)-1} H_X(\theta_j)$$

Then, the mixed area $W(Y, X)$ can be computed as:

$$W(Y, X) = \lim_{N \rightarrow \infty} \sum_{i=1}^N \sum_{j=1}^N F_{ij}^{(N)-1} H_Y(\theta_i) H_X(\theta_j)$$

Notice that a continuous version of the expression above can be written in terms of convolution. However, that's not our objective.

Of course the $\mathcal{C}_0^{(N)}$ -approximation is sensitive to rotations (see Figure 1). Obviously, it can be problematic to describe the geometry of sets. Let us consider the following example of an ellipse.

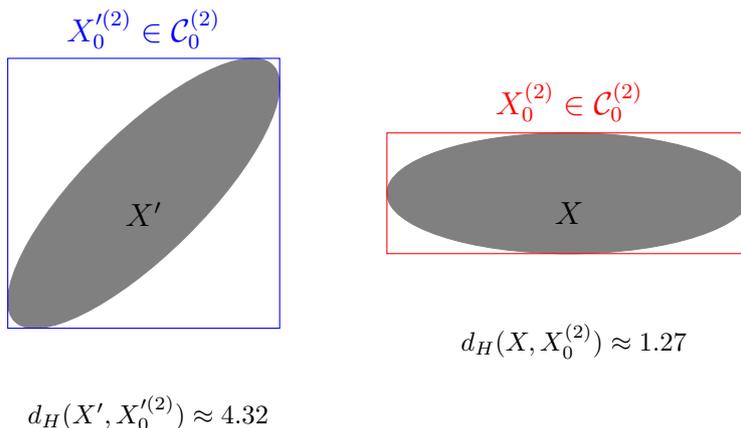


Figure 1: The $\mathcal{C}_0^{(N)}$ -approximations of an ellipse X and its rotation X' with respect to the angle $\frac{\pi}{4}$.

Example 2.1. Let X be an ellipse with semi-axis $a = 1$ and $b = 3$, and suppose that the major semi-axis b is horizontally oriented. Firstly consider the case $N = 2$, and let us note $X' := R_{\frac{\pi}{4}}(X)$, the following Figure 1 shows that the $\mathcal{C}_0^{(N)}$ -approximation of X is better than the one of X' (in terms of the Hausdorff's distance). Indeed, $d_H(X, X_0^{(2)}) \ll d_H(X', X_0'^{(2)})$. Furthermore, the $\mathcal{C}_0^{(2)}$ -approximation of the rotation is not the rotation of the $\mathcal{C}_0^{(2)}$ -approximation. Therefore, it can be problematic to use the $\mathcal{C}_0^{(2)}$ -approximation to describe the shape of X . Note that for the ellipse X of Figure 1, the orientations 0 and $\frac{\pi}{4}$ are respectively the better and the worst case for the $\mathcal{C}_0^{(2)}$ -approximation.

Let us consider now the more general case of the approximation of the rotations of X for different values of N . For each $N = 1, \dots, 20$, the $\mathcal{C}_0^{(N)}$ -approximations of all of the rotations $R_\eta(X)$ of X has been computed. Among these approximations, the better η_b and the worst η_w angles (in terms of the Hausdorff's distance) have been retain. The corresponding Hausdorff's distances are represented in Figure 2. Consequently, whatever the orientation of the ellipse, the Hausdorff's distance is inside the gray region. It can be noticed, that for small values of N , the difference between the worst and the better case is more important.

For the reasons mentioned above, it can be interesting to have an isometric invariant approximation. Fortunately, for a symmetric convex set X , the better $\mathcal{C}_0^{(N)}$ -approximation (in terms of the Hausdorff's distance) of the family of rotations of X can be used to define such an isometric invariant approximation.

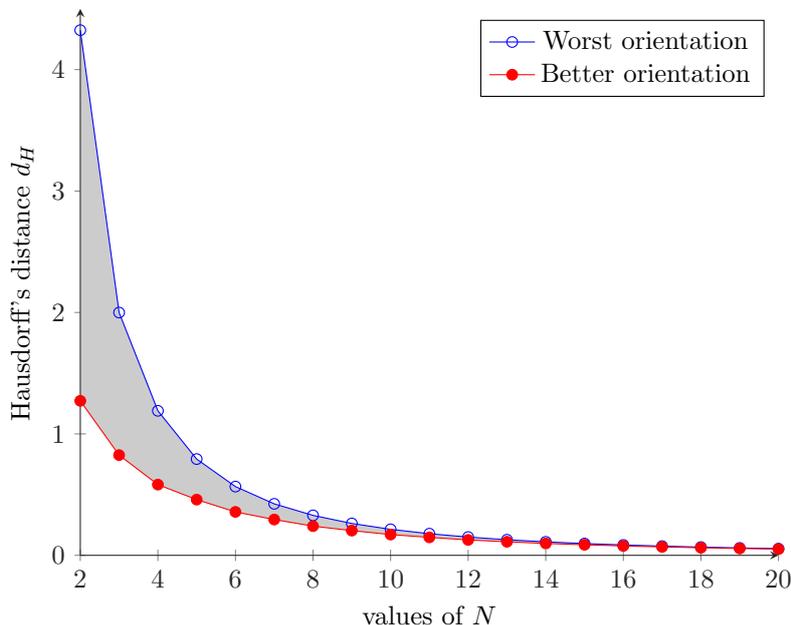


Figure 2: The Hausdorff's distance between an ellipse of semi-axis $(1, 3)$ and its $\mathcal{C}_0^{(N)}$ -approximation for several values of N . The case of the better direction in red and worst in blue. The gray region, represent the possible values for this distance.

2.3 Approximation of a symmetric convex set by a regular zonotope

It has been shown previously how a symmetric convex set X can be approximated in the class of the 0-regular zonotope. Such approximation is sensitive to the rotations. However, in order to study convex sets, there is sometime a need to have isometric invariant tools. Therefore, it will be defined here an approximation which is invariant up to a rotation. To meet this goal there is a need to perform the approximation on a class larger than $\mathcal{C}_0^{(N)}$, namely the class of the regular zonotopes.

Definition 2.4 (t -regular and regular zonotopes). Let $t \in \mathbb{R}$ and $N > 1$ be an integer, $\mathcal{C}_t^{(N)}$ denotes the class of the rotated elements of $\mathcal{C}_0^{(N)}$ with respect to the angle t :

$$\mathcal{C}_t^{(N)} = \{R_t(X) | X \in \mathcal{C}_0^{(N)}\}$$

Any element of $\mathcal{C}_t^{(N)}$ is called a t -regular zonotope with $2N$ faces.

Furthermore, $\mathcal{C}_\infty^{(N)} = \bigcup_{t \in \mathbb{R}} \mathcal{C}_t^{(N)}$ denotes the set of the regular zonotopes with $2N$ faces.

All the properties of $\mathcal{C}_0^{(N)}$ cited above are also true for $\mathcal{C}_t^{(N)}$, $t \in \mathbb{R}$. Therefore, it will be defined an approximation in $\mathcal{C}_\infty^{(N)}$.

Theorem 2.2 (Approximation in $\mathcal{C}_\infty^{(N)}$). Let $X \in \mathcal{C}$ and let us note $X_0^N(t)$ the $\mathcal{C}_0^{(N)}$ -approximation of $R_{-t}(X)$.

i. There exists $\tau \in [0, \pi[$ satisfying:

$$d_H(R_\tau(X_0^N(\tau)), X) = d_H(X_0^N(\tau), R_{-\tau}(X)) = \min_{t \in \mathbb{R}} d_H(X_0^N(t), R_{-t}(X)) \quad (17)$$

$X_0^N(\tau)$, also denoted \tilde{X}_0^N , will be named the $\mathcal{C}_0^{(N)}$ -rotational approximation of X .

ii. The $\mathcal{C}_0^{(N)}$ -rotational approximation of X is invariant under rotations of X .

The set $R_\tau(X_0^N(\tau))$ will be called a \mathcal{C}_∞^N -approximation of X in $\mathcal{C}_\infty^{(N)}$ and will be denoted by $X_\infty^{(N)}$.

Proof.

1. First of all, because of the symmetry of the 0-regular zonotopes:

$$\forall t \in \mathbb{R}, \mathcal{C}_t^{(N)} = \mathcal{C}_{t+\pi}^{(N)} \Rightarrow \min_{t \in \mathbb{R}} d_H(X_0^N(t), R_{-t}(X)) = \min_{t \in [0, \pi]} d_H(X_0^N(t), R_{-t}(X)).$$

For any $t \in \mathbb{R}$, let us note $\alpha(t)$ the face length vector of $X_0^N(t)$ then for any $h \in \mathbb{R}$:

$$\begin{aligned} \|\alpha(t) - \alpha(t+h)\|_1 &= \|F^{(N)-1}(H_{R_{-t}(X)}^{(N)} - H_{R_{-t-h}(X)}^{(N)})\|_1 \\ &\Rightarrow \|\alpha(t) - \alpha(t+h)\|_1 \leq \|F^{(N)-1}\|_1 \|H_{R_{-t}(X)}^{(N)} - H_{R_{-t-h}(X)}^{(N)}\|_1 \end{aligned}$$

However, $\forall \eta \in \mathbb{R}, H_{R_{-t-h}(X)}(\eta) = H_{R_{-t}(X)}(\eta+h)$. Because of the continuity of the Feret's diameter $\|H_{R_{-t}(X)}^{(N)} - H_{R_{-t-h}(X)}^{(N)}\|_1 \rightarrow 0$ as $h \rightarrow 0$ then $\|\alpha(t) - \alpha(t+h)\|_1 \rightarrow 0$ as $h \rightarrow 0$.

Therefore from the expression(7) about the Feret's diameter of a zonotope, $\forall \eta \in \mathbb{R},$:

$$\begin{aligned} |H_{X_0^N(t+h)}(\eta) - H_{X_0^N(t)}(\eta)| &= \left| \sum_{i=1}^N (\alpha_i(t) - \alpha_i(t+h)) |\sin(\eta - \theta_i)| \right| \\ &\leq N \max_{i=1, \dots, N} \{(\alpha_i(t) - \alpha_i(t+h))\} \end{aligned}$$

then $|H_{X_0^N(t+h)}(\eta) - H_{X_0^N(t)}(\eta)| \rightarrow 0$ as $h \rightarrow 0$ and finally $d_H(X_0^N(t), X_0^N(t+h)) \rightarrow 0$ as $h \rightarrow 0$. Consequently, the map $t \mapsto X_0^N(t)$ is continuous with respect to the Hausdorff's distance.

Note that $\forall x \in \mathbb{R}, H_{R_t(X_0^N(t))}(x) = H_{X_0^N(t)}(x-t)$ and $H_X(x) = H_{R_{-t}(X)}(x-t)$ then:

$$\begin{aligned} H_{R_t(X_0^N(t))}(x) - H_X(x) &= H_{X_0^N(t)}(x-t) - H_{R_{-t}(X)}(x-t) \\ &\Rightarrow d_H(R_t(X_0^N(t)), X) = d_H(X_0^N(t), R_{-t}(X)) \\ &\Rightarrow \min_{t \in \mathbb{R}} d_H(R_t(X_0^N(t)), X) = \min_{t \in \mathbb{R}} d_H(X_0^N(t), R_{-t}(X)) \end{aligned}$$

Furthermore for any $x, h \in \mathbb{R}$,

$$|H_{R_t(X_0^N(t))}(x) - H_{R_{t+h}(X_0^N(t+h))}(x)| = |H_{X_0^N(t)}(x-t) - \dots$$

$$\begin{aligned}
& \dots |H_{X_0^N(t+h)}(x-t) + H_{X_0^N(t+h)}(x-t) - H_{X_0^N(t+h)}(x-t-h)| \\
& \leq |H_{X_0^N(t)}(x-t) - H_{X_0^N(t+h)}(x-t)| \dots \\
& \dots + |H_{X_0^N(t+h)}(x-t) - H_{X_0^N(t+h)}(x-t-h)|
\end{aligned}$$

then from the continuity of the Feret's diameter and of the map $t \mapsto X_N(t)$, it follows the continuity of $t \mapsto R_t(X_0^N(t))$. As a consequence the map $t \mapsto d_H(X_0^N(t), X)$ is also continuous and the minimum $\min_{t \in [0, \pi]} d_H(R_t(X_0^N(t)), X)$ is achieved. Then there is $a \in [0, \pi]$ such that $d_H(R_\tau(X_0^N(\tau)), X) = \min_{t \in \mathbb{R}} d_H(X_0^N(t), R_{-t}(X))$.

- Let us prove the invariance by rotations. Let $\eta \in [0, \pi]$ and $Y = R_\eta(X)$, then $Y_0^N(t)$ is the $\mathcal{C}_0^{(N)}$ -approximation of $R_{-(t-\eta)}(X)$ and $Y_0^N(t) = X_0^N(t-\eta)$. Furthermore,

$$\begin{aligned}
\min_{t \in \mathbb{R}} d_H(Y_0^N(t), R_{-t}(Y)) &= \min_{t \in \mathbb{R}} d_H(X_0^N(t-\eta), R_{-(t-\eta)}(X)) \\
&= \min_{t \in \mathbb{R}} d_H(X_0^N(t), R_{-t}(X)) \\
&= d_H(X_0^N(\tau), R_{-\tau}(X))
\end{aligned}$$

Then $X_0^N(\tau)$ is a $\mathcal{C}_0^{(N)}$ -rotational approximation of Y and the $\mathcal{C}_\infty^{(N)}$ -approximation associated is $R_\eta(R_\tau(X_0^N(\tau)))$ (indeed $Y_0^N(\tau+\eta) = X_0^N(\tau)$).

□

The proposition above gives important informations. The $\mathcal{C}_\infty^{(N)}$ -approximation of a symmetric convex set X is the best regular zonotope with at most $2N$ faces containing X . It is always a better approximation than the $\mathcal{C}_0^{(N)}$ -approximation. This approximation can be used for not so large N . For example, for $N = 2$, the $\mathcal{C}_0^{(2)}$ -approximation of an ellipse depends on the orientation of the ellipse, but its $\mathcal{C}_\infty^{(2)}$ -approximation is the best way to put the ellipse inside a rectangle (see Figure 3). An illustration of the approximations of that ellipse for higher value of N is represented Figure 4.

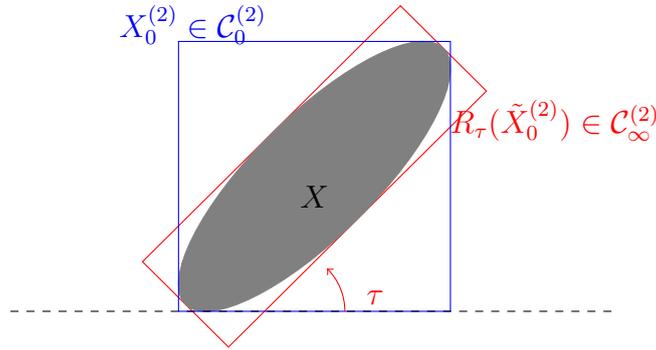


Figure 3: An ellipse and its approximations: $X_2 \in \mathcal{C}_0^{(2)}$ in blue and $R_\tau(\tilde{X}_2) \in \mathcal{C}_\infty^{(2)}$ in red.

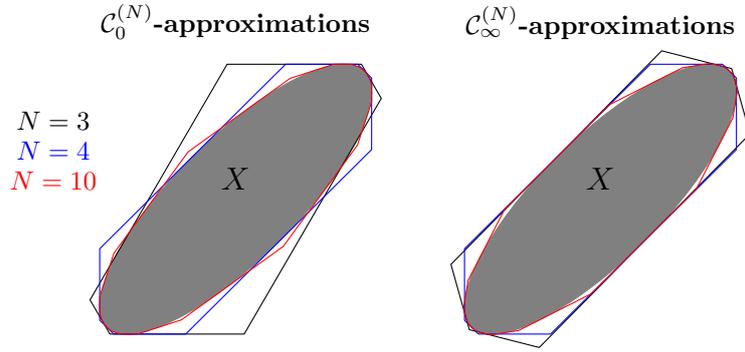


Figure 4: The $\mathcal{C}_0^{(N)}$ -approximations(left) and $\mathcal{C}_\infty^{(N)}$ -approximations(right) of an ellipse of semi-axis $(3, 1)$ for different values of $N(= 3, 4, 10)$.

The accuracy of the $\mathcal{C}_0^{(N)}$ -approximation has been presented in Figure 2 and one can remark that the best orientation corresponds to the $\mathcal{C}_\infty^{(N)}$ -approximation. Then, for the considered ellipse, the accuracy of the $\mathcal{C}_\infty^{(N)}$ -approximation in function of the number of faces N has already been represented in Figure 2. However, the accuracy of the $\mathcal{C}_\infty^{(N)}$ -approximation depends both on the shape and the size of the symmetric convex X .

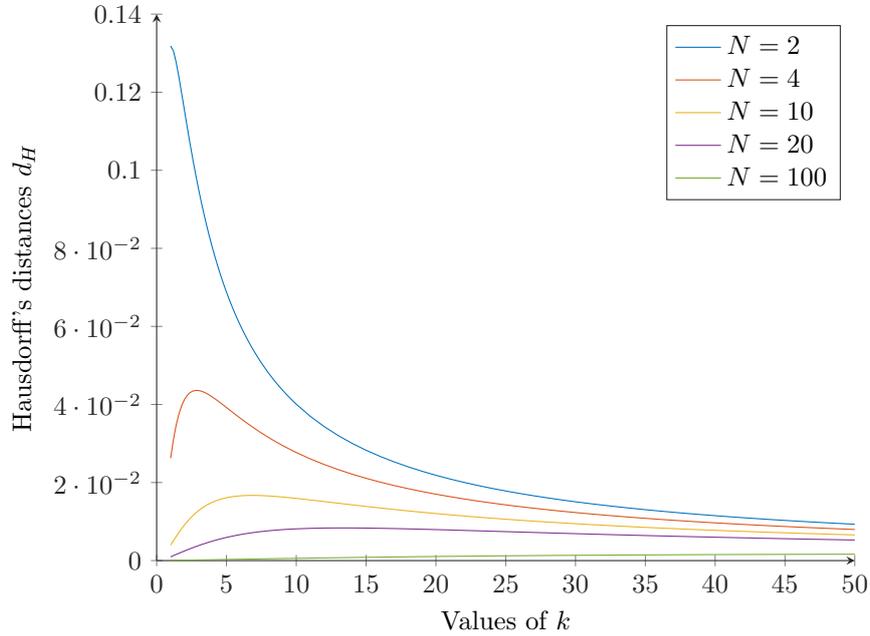


Figure 5: The Hausdorff's distance between an ellipse of unit perimeter and its $\mathcal{C}_\infty^{(N)}$ -approximations for several values of N in function of its axis ratio k .

Remark 2.4 (Accuracy of the $\mathcal{C}_\infty^{(N)}$ -approximation). *The size dependence of the accuracy is easy to understand: the accuracy decreases proportionally to the size factor. Indeed, for $Y := kX$, $k \in \mathbb{R}_+$ we have $d_H(Y_\infty^{(N)}, Y) = kd_H(X_\infty^{(N)}, X)$ (because of the homogeneity of the Feret's diameter). In order to study the impact of the shape (independently of its size) on the approximation accuracy, there is a need to use an homothetic invariant descriptor. In order to do this, the Feret's diameter of a symmetric convex set X will be normalized by its perimeter. According to Cauchy's formula [27] the perimeter is equal to the Feret's diameter total mass $\int_0^\pi H_X(\theta)d\theta$. Then, according to the homogeneity of the Feret's diameter, an involved distance can be defined as $\forall X, Y \in \mathcal{C}$, $\tilde{d}_H(X, Y) := d_H(\frac{X}{U(X)}, \frac{Y}{U(Y)})$. Such a distance can be used to study the approximation accuracy. Notice that it is equivalent to work with sets of unit perimeters and using the usual Hausdorff's distance. Such a consideration will be done in the following example.*

Let us consider an ellipse X with unit perimeter and an axis ratio $k \in [1, +\infty[$, the case $k = 1$ referring to the disk. The accuracy of the $\mathcal{C}_\infty^{(N)}$ -approximation as a function of N and k is shown. More specifically, on the Figure 5 it can be seen that the behaviour of the curves are very different for different values of N . Indeed, the worst shape for $N = 2$ is the disk. However, as it can be seen that is not the case for others values of N . It can be noticed that when the ratio k increases, the importance of N for the approximation decreases. This suggests that when an object X is elongated, one can choose a small value of N .

It has been studied two different approximations of a symmetric convex set X . The first one is an approximation of X as 0-regular zonotope, and the second as a regular zonotope. These approximations have been characterized from the Feret's diameter of X . The next objective is to study these approximations when X become a random symmetric body, and then how they can be characterized from the Feret's diameter of X . In order to do this, there is a need to study some properties of the random zonotopes, which lead us to the following section.

3 The random zonotopes

The aim of this section is to investigate how a random zonotope can be described by a random vector representing its faces and how such random vector can be characterized from the Feret's diameter of the random zonotope. Firstly, the properties of the random process corresponding to the Feret's diameter of a random set, will be investigated. In a second time, the description of a random zonotope by its faces will be explored. Finally, the characterization of some random zonotopes from their Feret's diameter random process will be given.

3.1 Feret's diameter process and isotropic random set

Let X be a random convex set, i.e a random closed set which is almost surely a convex set. In this subsection, some properties of the random process [9] corresponding to the Feret's diameter of X are stated.

Definition 3.1 (Feret's diameter random process). *Let X be a random convex set of \mathbb{R}^2 . For all $\omega \in \Omega$ a.s $X(\omega)$ is a convex set. Then, for any $t \in \mathbb{R}$, the positive random variable $H_X(t) : \omega \mapsto H_{X(\omega)}(t)$ is almost surely defined. The random process $\{H_X(t), t \in \mathbb{R}\}$ will be named the Feret's diameter random process of X .*

The trajectories of H_X are the Feret's diameter of the realizations of X , then the properties of the proposition 2.2 are also true for these trajectories, especially the continuity and the π -periodicity. It can be also noticed that the Feret's diameter random process characterizes the symmetric convex sets.

Definition 3.2 (Isotropised set of a random symmetric body). *Let X' be a symmetric random convex set and let η be a random uniform variable on $[0, \pi]$ independent of X' , the set*

$$X := R_\eta(X')$$

is isotropic (a random compact is said to be isotropic if and only if its distribution is isometric invariant [8]) and will be called an isotropised set of X' .

Let X' be a random symmetric body and X be an isotropised set of it, then X and X' have the same shape distribution and they also have the same zonotope rotational approximations (see theorem 2.2).

In the following, it will be shown that the Feret's diameter random process $H_{X'}$ of X' can be expressed from the one of X . This property will be used to show that a random symmetric convex set can be described up to a rotation by an isotropic random zonotope.

Let us recall that the Feret's diameter random process $H_{X'}$ of X' is enough to characterize X' , then for any $\theta \in \mathbb{R}$ the Feret's diameter $H_{X'}$ of X' can be expressed as:

$$H_X(\theta) = H_{X'}(\theta - \eta)$$

Let B be a Borel subset of \mathbb{R} . Because of the uniformity of η and its independence with X' , it follows:

$$\begin{aligned} \mathbb{P}(H_X(\theta) \in B) &= \mathbb{P}(H_{X'}(\theta - \eta) \in B) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \mathbb{P}(H_{X'}(\theta - t) \in B) dt \end{aligned}$$

Furthermore, by using the π -periodicity of the Feret's diameter, the distribution of $H_X(\theta)$ can be expressed as:

$$\mathbb{P}(H_X(\theta) \in B) = \frac{1}{\pi} \int_0^\pi \mathbb{P}(H_{X'}(\theta - t) \in B) dt \quad (18)$$

Consequently, the moments of the Feret's diameter process of the set X' and the isotropised set X are related. Of course there is a need to ensure their existence, but it will be treated after.

Proposition 3.1 (Moments of the Feret's diameter process of the isotropised set). *Let X' be a random convex set and X the isotropised set of X' . Suppose that the first and second order moments of the Feret's diameter random process $H_{X'}$ of X' exists, then those of X exists and they can be expressed as:*

$$\forall \theta \in [0, 2\pi], \mathbb{E}[H_X(\theta)] = \frac{1}{\pi} \int_0^\pi \mathbb{E}[H_{X'}(\theta)] d\theta$$

$$\forall (s, t) \in [0, 2\pi]^2, \mathbb{E}[H_X(s)H_X(t)] = \frac{1}{\pi} \int_0^\pi \mathbb{E}[H_{X'}(\theta)H_{X'}(\theta + s - t)] d\theta$$

Proof. Let X' be a random convex set and $X = R_\eta(X')$ an isotropised set of it and suppose that first and second order moments of $H_{X'}$ exist. Let us recall that $\forall \theta \in \mathbb{R}$, $H_X(\theta) = H_{X'}(\theta - \eta)$ and that η is independent of X' thus the result comes by integrating with respect to the uniform distribution of η . \square

Proposition 3.2 (Feret's diameter process of an isotropic random convex set). *Let X' be a random convex set.*

- i. If X' is isotropic then the random variables $H_{X'}(\theta)$, $\theta \in [0, \pi]$ are identically distributed (i.e the random process $H_{X'}$ is stationary).*
- ii. Furthermore, if X' is symmetric the reciprocal is true.*

Proof.

1. Let η be a uniform random variable on $[0, \pi]$ independent of X' and let note $X = R_\eta(X')$. If X' is isotropic then X and X' have the same distribution then H_X and $H_{X'}$ have also the same distribution. Consequently, according to (18), for any $\theta \in [0, \pi]$ and for any borel set B ,

$$\mathbb{P}(H_{X'}(\theta) \in B) = \mathbb{P}(H_X(\theta) \in B) = \frac{1}{\pi} \int_0^\pi \mathbb{P}(H_{X'}(\theta - t) \in B) dt$$

Because of the π -periodicity of the Feret's diameter, the integral is undependable of θ and thus the random variables $H_{X'}(\theta)$, $\theta \in [0, \pi]$ are identically distributed.

2. Suppose that X' is symmetric and $H_{X'}(\theta)$, $\theta \in [0, \pi]$ are identically distributed. Then the random process $H_{X'}$ is stationary, that is to say: for any $x \in \mathbb{R}$, the random process $(H_{X'}(\theta))_{\theta \in \mathbb{R}}$ and the translated process $(\tilde{H}_{X'}(\theta) = H_{X'}(\theta + x))_{\theta \in \mathbb{R}}$ have the same distribution. However $\tilde{H}_{X'}$ is exactly the random process corresponding to the Feret's diameter of $R_x(X')$. It has been already established that the Feret's diameter characterizes the symmetric bodies, then for any $x \in \mathbb{R}$, $R_x(X')$ and X' have the same distribution, so X' is isotropic. \square

It has been shown some properties of the Feret's diameter random process. Let us discuss now about the random zonotopes, that is to say the random sets almost surely valued in $\mathcal{C}^{(N)}$.

3.2 Description of the random zonotopes from their faces

Here it will be defined some classes of random zonotopes. Particularly, the class of the random zonotopes almost surely valued in $\mathcal{C}_0^{(N)}$ and the class of those almost surely valued in $\mathcal{C}_\infty^{(N)}$. Several properties of the random zonotopes will be studied. In particular, it will be shown how a random zonotope can be described by a random vector corresponding to its faces.

Definition 3.3 (Random zonotopes). *Let $N > 1$ be an integer, a random closed set X which has almost surely its realizations in $\mathcal{C}^{(N)}$ will be called a random zonotope with at most $2N$ faces or, in a more concise way, a random zonotope when there is no possible confusion.*

Such a random set can be described almost surely as:

$$\forall \omega \in \Omega \text{ a.s.}, X(\omega) = \bigoplus_{i=1}^N \alpha_i(\omega) S_{\beta_i(\omega)}$$

The distribution of the random vector (α, β) characterizes X . The random vector α will be named a *face length vector* of X .

According to proposition 2.3, for any face length vector α of X , some geometrical characteristics (Feret's diameter, perimeter, area) of X can be expressed as:

$$\forall \omega \in \Omega \text{ a.s.}, \forall t \in \mathbb{R}, H_X(t) = \sum_{i=1}^N \alpha_i |\sin(t - \beta_i)| \quad (19)$$

$$\forall \omega \in \Omega \text{ a.s.}, U(X) = 2 \sum_{i=1}^N \alpha_i \quad (20)$$

$$\forall \omega \in \Omega \text{ a.s.}, A(X) = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j |\sin(\beta_i - \beta_j)| \quad (21)$$

Proposition 3.3 (Existence conditions for the auto-covariance of the Feret's diameter process). *Let X be a random zonotope with $2N$ faces and α its face length vector, then the following properties are equivalent:*

$$\mathbb{E}[U(X)^2] < \infty \quad (22)$$

$$\alpha \in L^2(\mathbb{R}_+^N) \quad (23)$$

Furthermore, if one of these conditions is satisfied then $\mathbb{E}[A(X)] < \infty$ and $\forall (s, t) \in [0, \pi]^2$, $\mathbb{E}[H_X(s)H_X(t)] < \infty$.

Proof. According to (20), $U(X)^2 = (2 \sum_{i=1}^N \alpha_i)^2$, the first equivalence is trivial (because of the positivity of α).

The proposition 2.3 also shows that: $\forall (s, t) \in [0, \pi]^2$,

$$\begin{aligned} H_X(s)H_X(t) &= \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j |\sin(s - \beta_i) \sin(t - \beta_j)| \\ &\leq \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j \end{aligned}$$

$$\leq \frac{1}{4}U(X)^2$$

Then the expectation $\mathbb{E}[H_X(s)H_X(t)]$ exists and the existence of $\mathbb{E}[A(X)]$ comes from the isoperimetric inequality. \square

Definition 3.4 (0-regular random zonotopes). *Let $N > 1$ be an integer, a random closed set X which has almost surely its realizations in $\mathcal{C}_0^{(N)}$ will be called a 0-regular random zonotope with at most $2N$ faces or, in a more concise way, a 0-regular random zonotope when there is no possible confusion.*

A 0-regular random zonotope X can be almost surely expressed as:

$$\forall \omega \in \Omega \text{ a.s.}, X(\omega) = \bigoplus_{i=1}^N \alpha_i(\omega) S_{\theta_i}$$

where $\theta_i, i = 1, \dots, N$ denotes the regular subdivision on $[0, \pi]$.

The distribution of the face length vector α characterizes the distribution of X . In addition, this relation is bijective; in other word, the α 's distribution is uniquely defined, it will be named the face length distribution.

Of course the 0-regular random zonotopes can be used to approximate the random symmetric convex sets as $N \rightarrow \infty$ (see section 4.1). However, it is not the best way to model a random symmetric convex set. Indeed, it can be noticed that a 0-regular random zonotope cannot be isotropic. For instance, there is a need to use a large N in order to describe a random set built as an isotropic random square (see example 4.1). That is the reason for using a larger class of random zonotopes.

Definition 3.5 (Regular random zonotopes). *Let $N > 1$ be an integer, any random compact set taking its value almost surely in $\mathcal{C}_\infty^{(N)}$ will be named a regular random zonotope and can be expressed as:*

$$X = R_x \left(\bigoplus_{i=1}^N \alpha_i S_{\theta_i} \right)$$

where x is a random variable on $[0, \pi]$ and α a random vector taking values in \mathbb{R}_+^N . The random vector α will be named a random face length vector of X .

Proposition 3.4 (Isotropic regular random zonotope). *Let $X = R_x \left(\bigoplus_{i=1}^N \alpha_i S_{\theta_i} \right)$ be an isotropic regular random zonotope, then X has the same distribution of the following random set:*

$$X \stackrel{\text{a.s.}}{=} R_\eta \left(\bigoplus_{i=1}^N \alpha_i S_{\theta_i} \right) \tag{24}$$

where η is a uniform random variable on $[0, \pi]$ independent of α .

Proof. Let $X = R_x \left(\bigoplus_{i=1}^N \alpha_i S_{\theta_i} \right)$ be an isotropic regular random zonotope, and η' be a uniform random variable independent of α . Because of the isotropy of X , the random set $R_{\eta'}(X)$ has the same distribution as X . Let $\eta = x + \eta'[\pi]$ then

the random set $R_{\eta'}(X)$ can be expressed as $R_{\eta}(\bigoplus_{i=1}^N \alpha_i S_{\theta_i})$. Consequently, $R_{\eta}(\bigoplus_{i=1}^N \alpha_i S_{\theta_i})$ has the same distribution as X .

Let us show that η is a uniform variable independent of α . Let B be a Borel set of \mathbb{R}^N and for all $t \in [0, \pi]$, E denotes the event $E = \{\eta \in [0, t]\} \cap \{\alpha \in B\}$ then:

$$\begin{aligned} E &= \{\alpha \in B\} \cap \left(\bigcup_{z \in [0, \pi]} \{x = z\} \cap \{\eta' + z[\pi] \leq t\} \right) \\ &= \bigcup_{z \in [0, \pi]} \{\alpha \in B\} \{x = z\} \cap \{\eta' + z[\pi] \leq t\} \end{aligned}$$

Note that this union is disjointed, then because of the independence of η' :

$$\mathbb{P}(E) = \int_0^\pi \mathbb{P}(\{\alpha \in B\} \{x = z\}) \mathbb{P}(\{\eta' + z[\pi] \leq t\}) dz$$

The quantity $\mathbb{P}(\{\eta' + z[\pi] \leq t\})$ is independent of the value of z and it can be easily computed as $\mathbb{P}(\{\eta' + z[\pi] \leq t\}) = \frac{t}{\pi}$. Consequently:

$$\begin{aligned} \mathbb{P}(E) &= \frac{t}{\pi} \int_0^\pi \mathbb{P}(\{\alpha \in B\} \{x = z\}) \mathbb{P}(\{\eta' + z[\pi] \leq t\}) dz \\ \mathbb{P}(E) &= \frac{t}{\pi} \mathbb{P}(\{\alpha \in B\}) \end{aligned}$$

then η is a uniform random variable on $[0, \pi]$ independent of α . \square

The proposition above shows that an isotropic regular random zonotope can always be described as (24). Such a zonotope is consequently defined by its random face length vector α . However, different distributions of α can lead to the same distribution of X as mentioned in the following proposition.

Proposition 3.5 (Family of the random face length vectors). *Let α be a random face length vector of the isotropic regular random zonotope X . The following family of random face length vectors, denoted $\mathcal{F}_N(X)$, provides the same distribution of the random set X :*

$$\mathcal{F}_N(X) = \{\alpha' \stackrel{a.s.}{=} J^n \alpha \mid \forall \omega \in \Omega \text{ a.s.}, n(\omega) \in \{0, \dots, N-1\}\} \quad (25)$$

where J is the circulant matrix $J = \text{Circ}(0, 1, 0, \dots, 0)$.

Proof. First of all, it is easy to see that $\mathcal{F}_N(X)$ is not empty by construction of X . Let α, α' be two representative random vectors of X then there exists two uniform random variables η and η' satisfying $\eta \perp \alpha$ and $\eta' \perp \alpha'$ such that :

$$\begin{aligned} \forall \omega \in \Omega \text{ a.s.}, \bigoplus_{i=1}^N \alpha_i(\omega) S_{\theta_i + \eta(\omega)} &= \bigoplus_{i=1}^M \alpha'_i(\omega) S_{\theta_i + \eta'(\omega)} \\ \Rightarrow \forall \omega \in \Omega \text{ a.s.}, R_{-\eta'(\omega)} \left(\bigoplus_{i=1}^N \alpha_i(\omega) S_{\theta_i + \eta(\omega)} \right) &= R_{-\eta'(\omega)} \left(\bigoplus_{i=1}^N \alpha'_i(\omega) S_{\theta_i + \eta'(\omega)} \right) \\ \Rightarrow \forall \omega \in \Omega \text{ a.s.}, \bigoplus_{i=1}^N \alpha'_i(\omega) S_{\theta_i} &= \bigoplus_{i=1}^N \alpha_i(\omega) S_{\theta_i + \eta(\omega) - \eta'(\omega)} \end{aligned}$$

Then, because of the uniqueness of the face length vector in $\mathcal{C}_0^{(N)}$, for any $\omega \in \Omega$ a.s there is $j(\omega) \in \{1, \dots, N\}$ such that:

$$\begin{aligned}\theta_1 &= (\theta_{j(\omega)} + \eta(\omega) - \eta'(\omega))[\pi] \text{ and } \alpha'_1(\omega) = \alpha_j(\omega) \\ \Rightarrow \theta_{j(\omega)} &= (\eta'(\omega) - \eta(\omega))[\pi] \text{ and } \alpha'_1(\omega) = \alpha_j(\omega) \\ \Rightarrow \alpha'_i(\omega) &= \alpha_{i+j-1[M]}(\omega) \\ \Rightarrow \alpha'(\omega) &= J^{j(\omega)-1}\alpha(\omega)\end{aligned}$$

By taking $\forall \omega \in \Omega$ a.s , $n(\omega) = j(\omega) - 1[N]$ it follows $\alpha' = J^n \alpha$ and consequently $\mathcal{F}_N(X) \subset \{\alpha' = J^n \alpha | \forall \omega \in \Omega \text{ a.s } , n(\omega) \in \{0 \dots, N-1\}\}$.

The other inclusion can be proved by taking η' such that $\forall \omega \in \Omega$ a.s , $\eta'(\omega) = \beta_{n(\omega)+1} + \eta[\pi]$. For such an η' it follows $\forall \omega \in \Omega$ a.s , $X(\omega) = \bigoplus_{i=1}^N \alpha'_i(\omega) S_{\theta_i + \eta'(\omega)}$. \square

Definition 3.6 (Central random face length vector). *Let $\alpha \in \mathcal{F}_N(X)$ and n be a uniform random variable on $\{0 \dots, M-1\}$ independent of α , then the random face length vector $\alpha' = J^n \alpha$ will be called a central random face length vector of X .*

Notice that a central random face length vector has all of these components identically distributed. Furthermore, its distribution has many interesting properties.

Proposition 3.6 (Uniqueness of the central face length distribution). *There is a unique distribution for any central random face length vectors. In other words, let $\tilde{\alpha}', \alpha'$ be two central random face length vectors of X , then they have same distribution. Such a distribution will be named the central face length distribution of X .*

Proof. Let $\tilde{\alpha}'$ and α' be two central representations of X . Then there exists a random face length vector $\tilde{\alpha}$ and an independent uniform variable \tilde{n} on $\{0 \dots, N-1\}$ such that $\tilde{\alpha}' = J^{\tilde{n}} \tilde{\alpha}$. In addition, $\tilde{\alpha} \in \mathcal{F}_N(X)$ so there exists n such that $\tilde{\alpha} = J^n \alpha'$. Consequently $\tilde{\alpha}' = J^{\tilde{n}+n} \alpha'$. Let $n' = \tilde{n} + n[N]$, it is easy to see that $J^{\tilde{n}+n} = J^{n'}$, thus:

$$\tilde{\alpha}' = J^{n'} \alpha'$$

Let us prove that n' is a uniform variable on $\{0 \dots, M-1\}$ independent of α' .

For any $k \in \{0 \dots, N-1\}$,

$$\begin{aligned}\mathbb{P}(\{n' = k\}) &= \mathbb{P}\left(\bigcup_{i=0}^{N-1} \{\tilde{n} = k - i[N]\} \cap \{n = i\}\right) \\ &= \sum_{i=0}^{N-1} \mathbb{P}(\{\tilde{n} = k - i[N]\}) \mathbb{P}(\{n = i\}) \\ &= \frac{1}{N}\end{aligned}$$

then n' is a uniform variable on $\{0 \dots, N-1\}$. Furthermore for any Borel set B and any $k \in \{0 \dots, N-1\}$:

$$\mathbb{P}(\{n' = k\} \cap \{\alpha' \in B\}) = \mathbb{P}\left(\bigcup_{i=0}^{N-1} \{\tilde{n} = k - i[N]\} \cap \{n = i\} \cap \{\alpha' \in B\}\right)$$

$$\begin{aligned}
&= \sum_{i=0}^{N-1} \mathbb{P}(\{\tilde{n} = k - i[N]\} \cap \{n = i\} \cap \{\alpha' \in B\}) \\
&= \sum_{i=0}^{N-1} \mathbb{P}(\{\tilde{n} = k - i[N]\}) \mathbb{P}(\{n = i\} \cap \{\alpha' \in B\}) \\
&= \frac{1}{N} \sum_{i=0}^{N-1} \mathbb{P}(\{n = i\} \cap \{\alpha' \in B\}) \\
&= \frac{1}{N} \mathbb{P}(\{\alpha' \in B\}) \\
&= \mathbb{P}(\{n' = k\}) \mathbb{P}(\{\alpha' \in B\})
\end{aligned}$$

Now let us prove that α' and $\tilde{\alpha}'$ have the same distribution. Let $B = B_0 \times \cdots \times B_{N-1}$ a product of Borel sets of \mathbb{R} . Firstly, note that $\forall k \in \{0 \cdots, N-1\}$, $\mathbb{P}(J^k \alpha' \in B) = \mathbb{P}(\alpha' \in B)$. Indeed, by definition, α' can be written as $\alpha' = J^n \alpha$ with α a representative of X and n an independent uniform random variable on $\{0 \cdots, N-1\}$. Therefore,

$$\begin{aligned}
\mathbb{P}(\{\alpha' \in B\}) &= \mathbb{P}\left(\bigcup_{i=0}^{N-1} \{J^i \alpha \in B\} \cap \{n = i\}\right) \\
&= \frac{1}{N} \sum_{i=0}^{N-1} \mathbb{P}(\{J^i \alpha \in B\}) \\
&= \frac{1}{N} \sum_{i=0}^{N-1} \mathbb{P}(\{\alpha \in B_i \times \cdots \times B_0 \cdots \times B_{N-1-i}\})
\end{aligned}$$

In the same manner,

$$\begin{aligned}
\mathbb{P}(\{J^k \alpha' \in B\}) &= \frac{1}{N} \sum_{i=0}^{N-1} \mathbb{P}(\{J^{i+k} \alpha \in B\}) \\
&= \frac{1}{N} \sum_{i=0}^{N-1} \mathbb{P}(\{\alpha \in B_{i+k} \times \cdots \times B_0 \cdots \times B_{N-1-i-k}\}) \\
&= \frac{1}{N} \sum_{i=0}^{N-1} \mathbb{P}(\{\alpha \in B_i \times \cdots \times B_0 \cdots \times B_{N-1-i}\}) \\
&= \mathbb{P}(\{\alpha' \in B\})
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathbb{P}(\{\tilde{\alpha}' \in B\}) &= \mathbb{P}\left(\bigcup_{k=0}^{N-1} \{J^k \alpha' \in B\} \cap \{n' = k\}\right) \\
&= \frac{1}{N} \sum_{k=0}^{N-1} \mathbb{P}(\{J^k \alpha' \in B\}) \\
&= \mathbb{P}(\{\alpha' \in B\})
\end{aligned}$$

Finally $\tilde{\alpha}'$ and α' have same distribution. □

Proposition 3.7. [*Properties of the central face length distribution*]

Let α be a central random face length vector of X , then the first and second order moments of its distribution have the following properties:

i. first order moment:

$$\forall i = 1, \dots, N, \mathbb{E}[\alpha_i] = \frac{U(X)}{2N} \quad (26)$$

ii. second order moment:

The matrix $C[\alpha] = (\mathbb{E}[\alpha_i \alpha_j])_{1 \leq i, j \leq N}$ is a circulant matrix defined by the first column $V[\alpha] = {}^t(\mathbb{E}[\alpha_1 \alpha_1], \dots, \mathbb{E}[\alpha_1 \alpha_N])$: $C[\alpha] = \text{Circ}(V[\alpha])$. Furthermore this matrix is symmetric, it depends only on $(\lfloor \frac{N}{2} \rfloor + 1)$ values, where $\lfloor \frac{N}{2} \rfloor$ denotes the floor of $\frac{N}{2}$. Let us note $m = \lfloor \frac{N}{2} \rfloor$ and $v = {}^t(\mathbb{E}[\alpha_1 \alpha_1], \dots, \mathbb{E}[\alpha_1 \alpha_{m+1}])$ therefore if N is an even integer $V = {}^t(v_0, \dots, v_{m-1}, v_m, v_{m-1}, \dots, v_1)$ and if N is an odd integer $V = {}^t(v_0, \dots, v_m, v_m, \dots, v_1)$.

Proof.

1. The first point is trivial. Indeed the marginals of α are identically distributed then $\forall i, j \mathbb{E}[\alpha_i] = \mathbb{E}[\alpha_j]$ and $U(X) = 2 \sum_{i=1}^N \alpha_i \Rightarrow \forall i = 1, \dots, N, \mathbb{E}[\alpha_i] = \frac{U(X)}{2N}$.
2. It has been shown that for any $k \in \{0 \dots, N-1\}$, the random variables α and $J^k \alpha$ have same distribution, then they have the same covariance matrix. Therefore $\forall 1 \leq i, j \leq N$:

$$\forall k \in \{0 \dots, N-1\}, \mathbb{E}[\alpha_i \alpha_j] = \mathbb{E}[\alpha_{i+k[N]+1} \alpha_{j+k[N]+1}]$$

so $\mathbb{E}[\alpha_i \alpha_j]$ is a circulant matrix, it depends only on $i - j[N]$ and because of its symmetry also only on $j - i[N]$. Let $1 \leq i \leq j \leq N$ then there is two possible case, first suppose that $N = 2m$ is an even integer, then $\forall 0 \leq k \leq m-1$:

$$\mathbb{E}[\alpha_1 \alpha_{1+m+k}] = \mathbb{E}[\alpha_{1+m} \alpha_{1+k}] = \mathbb{E}[\alpha_{1+m+N-k} \alpha_{1+N}] = \mathbb{E}[\alpha_1 \alpha_{1+m-k}]$$

then by noting $V = {}^t(\mathbb{E}[\alpha_1 \alpha_1], \dots, \mathbb{E}[\alpha_1 \alpha_N])$ and $v = {}^t(\mathbb{E}[\alpha_1 \alpha_1], \dots, \mathbb{E}[\alpha_1 \alpha_{m+1}])$ there is $V = {}^t(v_0, \dots, v_{m-1}, v_m, v_{m-1}, \dots, v_1)$.

If N is an odd integer, then $N = 2m + 1$ and for any $0 \leq k \leq m$:

$$\mathbb{E}[\alpha_1 \alpha_{1+m+k}] = \mathbb{E}[\alpha_{1+m+1} \alpha_{1+k}] = \mathbb{E}[\alpha_{2+m+N-k} \alpha_{1+N}] = \mathbb{E}[\alpha_1 \alpha_{2+m-k}]$$

then by noting $V[\alpha] = {}^t(\mathbb{E}[\alpha_1 \alpha_1], \dots, \mathbb{E}[\alpha_1 \alpha_N])$ and $v = {}^t(\mathbb{E}[\alpha_1 \alpha_1], \dots, \mathbb{E}[\alpha_1 \alpha_{m+1}])$ there is $V = {}^t(v_0, \dots, v_m, v_m, \dots, v_1)$. Finally $C[\alpha]$ is a symmetric circulant matrix.

□

Example 3.1. In order to illustrate the properties of the face length vector distributions, let us discuss about the case $N = 2$. Then, $X = R_\eta(\alpha_1 S_0 \oplus \alpha_2 S_{\frac{\pi}{2}})$ with η a uniform random variable on $[0, \pi]$ independent of α .

Therefore X is an isotropic random rectangle described by its sides (α_1, α_2) . However, that is not the unique way to describe it. Indeed, even for a deterministic rectangle of sides (a, b) it can also be said that its sides is (b, a) . This simple fact involves a lot of different distributions for the face length vectors of an isotropic random rectangle.

Let us take this simple example: suppose Y is equiprobably the rectangle of sides $(1, 2)$ or the rectangle of sides $(3, 4)$. Then, there is at least the following four possible descriptions for the sides of Y 's realization:

- $(1, 2)$ or $(3, 4)$
- $(2, 1)$ or $(3, 4)$
- $(2, 1)$ or $(4, 3)$
- $(1, 2)$ or $(4, 3)$

Therefore, there are four corresponding face length distributions $\frac{1}{2}\Delta_{(1,2)} + \frac{1}{2}\Delta_{(3,4)}$, $\frac{1}{2}\Delta_{(2,1)} + \frac{1}{2}\Delta_{(3,4)}$, ... where $\Delta_{(a,b)}$ denotes the Dirac measure in (a, b) . However, there are not the only possibilities. Indeed, many other can be built from the previous distributions, such as the distribution $\frac{1}{4}\Delta_{(1,2)} + \frac{1}{4}\Delta_{(2,1)} + \frac{1}{2}\Delta_{(3,4)}$. Notice that the central distribution of Y is $\frac{1}{4}\Delta_{(1,2)} + \frac{1}{4}\Delta_{(2,1)} + \frac{1}{4}\Delta_{(3,4)} + \frac{1}{4}\Delta_{(4,3)}$.

Let us return now to the general case of the isotropic random rectangle X with a face length vector α . According to the foregoing, it is easy to see that any another face length vector α' of X can be built as :

$$\alpha' = \begin{pmatrix} 1 - \delta & \delta \\ \delta & 1 - \delta \end{pmatrix} \alpha \quad (27)$$

where δ is any Bernoulli variable (i.e valued in $\{0, 1\}$) eventually correlated to α .

Indeed, it should be noticed that $\begin{pmatrix} 1 - \delta & \delta \\ \delta & 1 - \delta \end{pmatrix} = J^\delta$, therefore by taking $\eta' = \eta + \delta \frac{\pi}{2} [\pi]$,

$$X = R_\eta(\alpha_1 S_0 \oplus \alpha_2 S_{\frac{\pi}{2}}) = R_{\eta'}(\alpha'_1 S_0 \oplus \alpha'_2 S_{\frac{\pi}{2}})$$

it can be easily proved that η' is a uniform random variable on $[0, \pi]$ independent of α' (see proof of proposition 3.6), then α' is a face length vector of X .

Let us consider now the central face length distribution, so let δ be a Bernoulli variable of parameter $\frac{1}{2}$ (i.e a uniform variable on $\{0, 1\}$) independ of α , and let $\alpha' = J^\delta \alpha$ be a central face length vector, then according to (27),

$$\begin{aligned} \alpha'_1 &= (1 - \delta)\alpha_1 + \delta\alpha_2 \\ \alpha'_2 &= \delta\alpha_1 + (1 - \delta)\alpha_2 \end{aligned}$$

Consequently, the first and second order moments of the face length distribution can be computed as:

$$\mathbb{E}[\alpha'_1] = \mathbb{E}[\alpha'_2] = \frac{1}{2}\mathbb{E}[\alpha_1 + \alpha_2]$$

$$\begin{aligned}\mathbb{E}[\alpha_1'^2] &= \mathbb{E}[\alpha_2'^2] = \frac{1}{2}\mathbb{E}[\alpha_1^2 + \alpha_2^2] \\ \mathbb{E}[\alpha_1'\alpha_2'] &= \mathbb{E}[\alpha_1\alpha_2]\end{aligned}$$

Notice that the property 3.7 is well verified,. Indeed, $\mathbb{E}[\alpha_1'] = \mathbb{E}[\alpha_2'] = \frac{1}{4}\mathbb{E}[U(X)]$ and the matrix $C[\alpha]$ is a circulant matrix depending on two parameters.

3.3 Characterizing an isotropic regular random zonotope from its Feret's diameter random process

It has been shown that the distribution of an isotropic random zonotope X can be described by its central face length distribution. The properties of such distributions have been studied. Here it will be shown how its characteristics can be connected to the geometrical characteristics of the random zonotope. In particular, it will be given formulae which allows to connect the first and second order moments of Feret's diameter of X to those of the central face length distribution.

Let X be an isotropic random zonotope represented by its face length vector α . Let us recall that X can be almost surely expressed as:

$$X = R_\eta\left(\bigoplus_{i=1}^N \alpha_i S_{\theta_i}\right)$$

where η is a uniform random variable independent of $\alpha \geq 0$. Suppose that the condition $\mathbb{E}(U(X)^2) < \infty$ is satisfied, then according to the proposition 3.3, $\alpha \in L^2(\mathbb{R}_+^N)$ and the mean and the auto-covariance of H_X exist.

According to the proposition 2.3, for any representative α of X , some geometrical characteristics of X can be expressed as:

$$\forall t \in \mathbb{R}, H_X(t) = \sum_{i=1}^N \alpha_i |\sin(t - \eta - \theta_i)|$$

$$U(X) = 2 \sum_{i=1}^N \alpha_i$$

$$A(X) = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j |\sin(\theta_i - \theta_j)|$$

Therefore, by considering $\alpha \in L^2(\mathbb{R}_+^N)$ and the independence of α and η , their expectation can be computed by integration with respect to the uniform distribution of η :

$$\forall t \in \mathbb{R}, \mathbb{E}[H_X(t)] = \frac{2}{\pi} \sum_{i=1}^N \mathbb{E}[\alpha_i] \tag{28}$$

$$\mathbb{E}[U(X)] = 2 \sum_{i=1}^N \mathbb{E}[\alpha_i] \tag{29}$$

$$A(X) = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \mathbb{E}[\alpha_i \alpha_j] |\sin(\theta_i - \theta_j)| \quad (30)$$

$$\forall t, t' \in \mathbb{R}, \mathbb{E}[H_X(t)H_X(t+t')] = \sum_{i=1}^N \sum_{j=1}^N \mathbb{E}[\alpha_i \alpha_j] k_S(t' + \theta_i - \theta_j) \quad (31)$$

$$\text{where } \forall t \in \mathbb{R}, k_S(t) = \frac{1}{\pi} \int_0^\pi |\sin(t+z)\sin(z)| dz \quad (32)$$

Note that k_S is a π -periodic function and it can be expressed on $[0, \pi]$ as:

$$k_S(t) = \frac{1}{2\pi} (2\sin^3(t) + \cos(t)(\pi - 2t + \sin(2t))) \quad (33)$$

By using the equation (31) and the stationarity of H_X :

$$\forall t, t' \in \mathbb{R}, \mathbb{E}[H_X(t)H_X(t+t')] = \mathbb{E}[H_X(t)H_X(t-t')] \quad (34)$$

$$\forall t, t' \in \mathbb{R}, \sum_{i=1}^N \sum_{j=1}^N \mathbb{E}[\alpha_i \alpha_j] k_S(t' + \theta_i - \theta_j) = \sum_{j=1}^N \sum_{i=1}^N \mathbb{E}[\alpha_i \alpha_j] k_S(t' + \theta_j - \theta_i) \quad (35)$$

and by introducing the following functional:

$$\forall t \in \mathbb{R}, \forall 1 \leq i, j \leq N, K_{ij}(t) = k_S(t + \theta_i - \theta_j) \quad (36)$$

it follows:

$$\forall t, t' \in \mathbb{R}, \mathbb{E}[H_X(t)H_X(t+t')] = \sum_{i=1}^N \sum_{j=1}^N \mathbb{E}[\alpha_i \alpha_j] K_{ij}(t') \quad (37)$$

Proposition 3.8. *For any real t , $K(t)$ is a circulant matrix. Furthermore, by denoting $((k_1(t), \dots, k_N(t)))$ the first line of $K(t)$, we have $K(t) = \text{Circ}((k_1(t), \dots, k_N(t)))$ and $K_{ij}(t) = k_j(\theta_i + t)$ for $1 \leq i, j \leq N$.*

Proof. Let us show that $K(t)$ is a circulant matrix. For any real t , $t + \theta_i - \theta_j$ depends only on $i - j$ then $K(t)$ is a Toeplitz matrix. Furthermore for $1 \leq i \leq N - 1$ and $1 \leq j \leq N$, $K_{(i+1)j}(t) = k_S(t + \theta_i - (\theta_j - \frac{\pi}{N}))$ but $(\theta_j - \frac{\pi}{N}) = \theta_{\sigma(j)}$ where $\sigma(j) = (j - 2[N]) + 1$, then $K_{(i+1)j}(t) = K_{i\sigma(j)}$. Therefore, the line index $i + 1$ of $K(t)$ is a cyclic permutation of the line index i of $K(t)$ so $K(t)$ is a circulant matrix. Furthermore $k_j(\theta_i + t) = k_S(t + \theta_i - \theta_j) = K_{ij}(t)$. \square

Suppose now that α is a central representative of X , it will be shown that the first and second order moments of the central distribution can be easily expressed from the Feret's diameter process.

Theorem 3.1 (Moments of the central face length distribution). *Let X be an isotropic random zonotope represented by a central face length vector α , then:*

$$\forall x \in \mathbb{R}, \forall i = 1, \dots, N, \mathbb{E}[\alpha_i] = \frac{\pi}{2N} \mathbb{E}[H_X(x)] \quad (38)$$

$$V[\alpha] = \frac{1}{N} K(0)^{-1} V[H_X^{(N)}] \quad (39)$$

where $V[x]$ denotes the vector ${}^t(\mathbb{E}[x_1 x_1], \dots, \mathbb{E}[x_1 x_N])$.

Proof. Suppose that α is a central representative of X , then according to the propositions 3.7 and the equation (28) the first order moment of the central distribution can be expressed as:

$$\forall x \in \mathbb{R}, \forall i = 1, \dots, N, \mathbb{E}[\alpha_i] = \frac{\pi}{2N} \mathbb{E}[H_X(x)] \quad (40)$$

Considering the propositions 3.8 and 3.7, it comes $\mathbb{E}[\alpha_i \alpha_j] = V[\alpha]_{j-i[N]+1}$ and $K_{ij}(t) = k_{j-i[N]+1}$ then for all $t \in \mathbb{R}$,

$$\begin{aligned} \mathbb{E}[H_X(0)H_X(t)] &= \sum_{i=1}^N \sum_{j=1}^N \mathbb{E}[\alpha_i \alpha_j] K_{ij}(t) \\ &= \sum_{i=1}^N \sum_{j=1}^N V[\alpha]_{j-i[N]+1} k_{j-i[N]+1}(t) \\ &= \sum_{i=1}^N \sum_{j=1}^i V[\alpha]_{j-i[N]+1} k_{j-i[N]+1}(t) + \sum_{i=1}^N \sum_{j=i}^N V[\alpha]_{j-i[N]+1} k_{j-i[N]+1}(t) \\ &\quad - \sum_{i=1}^N V[\alpha]_1 k_1(t) \\ &= \sum_{i=1}^N \sum_{s=0}^{i-1} V[\alpha]_{s+1} k_{s+1}(t) + \sum_{i=1}^N \sum_{s=0}^{N-i} V[\alpha]_{s+1} k_{N-s[N]+1}(t) - NV[\alpha]_1 k_1(t) \\ &= \sum_{i=1}^N \sum_{s=0}^{i-1} V[\alpha]_{s+1} k_{s+1}(t) + \sum_{i=1}^N \sum_{z=N}^i V[\alpha]_{N-z+1} k_{z[N]+1}(t) - NV[\alpha]_1 k_1(t) \\ &= \sum_{i=1}^N \sum_{s=0}^{i-1} V[\alpha]_{s+1} k_{s+1}(t) + \sum_{i=1}^N \sum_{z=N}^i V[\alpha]_{z[N]+1} k_{z[N]+1}(t) - NV[\alpha]_1 k_1(t) \\ &= \sum_{i=1}^N \sum_{s=0}^{i-1} V[\alpha]_{s+1} k_{s+1}(t) + \sum_{i=1}^N \sum_{s=i}^N V[\alpha]_{s[N]+1} k_{s[N]+1}(t) - NV[\alpha]_1 k_1(t) \\ &= \sum_{i=1}^N \sum_{s=0}^{i-1} V[\alpha]_{s+1} k_{s+1}(t) + \sum_{i=1}^N \sum_{s=i}^N V[\alpha]_{s[N]+1} k_{s[N]+1}(t) - NV[\alpha]_1 k_1(t) \\ &= \sum_{i=1}^N \sum_{s=0}^N V[\alpha]_{s[N]+1} k_{s[N]+1}(t) - NV[\alpha]_1 k_1(t) \\ &= \sum_{i=1}^N \sum_{s=0}^{N-1} V[\alpha]_{s[N]+1} k_{s[N]+1}(t) + \sum_{i=1}^N V[\alpha]_1 k_1(t) - NV[\alpha]_1 k_1(t) \\ &= \sum_{i=1}^N \sum_{s=0}^{N-1} V[\alpha]_{s+1} k_{s+1}(t) \\ &= \sum_{i=1}^N \sum_{s=1}^N V[\alpha]_s k_s(t) \end{aligned}$$

$$\Rightarrow \mathbb{E}[H_X(0)H_X(t)] = N \sum_{s=1}^N V[\alpha]_s k_s(t) \quad (41)$$

Let us note $V[H_X^{(N)}] = {}^t(\mathbb{E}[H_X(0)H_X(\theta_1)], \dots, \mathbb{E}[H_X(0)H_X(\theta_N)])$. For $1 \leq i \leq N$, $V[H_X^{(N)}]_i = N \sum_{s=1}^N V[\alpha]_s k_s(\theta_i) = N \sum_{s=1}^N V[\alpha]_s K_{is}(0)$ then,

$$V[H_X^{(N)}] = NK(0)V[\alpha] \quad (42)$$

It is easy to see that $K(0)$ is a symmetric positive definite matrix. Indeed, for $1 \leq i, j \leq N$, $K_{ij}(0) = k_S(\theta_i - \theta_j) = k_S(\theta_j - \theta_i) = K_{ji}(0)$ then $K(0)$ is a symmetric matrix, furthermore $\forall x \in \mathbb{R}^N$,

$$\begin{aligned} {}^t x K(0) x &= \sum_{i=1}^N \sum_{j=1}^N x_i x_j K_{ij}(0) \\ &= \sum_{i=1}^N \sum_{j=1}^N x_i x_j \frac{1}{\pi} \int_0^\pi |\sin(\theta_i - \theta_j + z) \sin(z)| dz \\ &= \frac{1}{\pi} \int_0^\pi \sum_{i=1}^N \sum_{j=1}^N x_i x_j |\sin(z - \theta_i) \sin(z - \theta_j)| dz \\ &= \frac{1}{\pi} \int_0^\pi \left(\sum_{i=1}^N x_i |\sin(z - \theta_i)| \right)^2 dz \end{aligned}$$

By denoting Y the following real valued random variable $Y = \sum_{i=1}^N x_i |\sin(z - \theta_i)|$ where z is a uniform random variable on $[0, \pi]$, then ${}^t x K(0) x = \mathbb{E}[Y^2]$ so ${}^t x K(0) x \geq 0$. Furthermore ${}^t x K(0) x = 0$ if and only if $Y = 0$ almost surely, $Y = 0$ a.s. $\Rightarrow \forall z \in [0, \pi]$, $\sum_{i=1}^N x_i |\sin(z - \theta_i)| = 0 \Rightarrow x = 0$. Finally $K(0)$ is a symmetric positive definite matrix, then it is invertible, and it follows:

$$V[\alpha] = \frac{1}{N} K(0)^{-1} V[H_X^{(N)}]$$

□

This theorem gives the 1st and 2nd order moments of the central face length distribution from those of the Feret's diameter. Note that $V[\alpha]$ and $V[H_X^{(N)}]$ satisfy the properties of symmetry. Indeed, by denoting $m = \lfloor \frac{N}{2} \rfloor$, it has been shown that if N is an even integer $V[\alpha] = {}^t(v_0, \dots, v_{m-1}, v_m, v_{m-1}, \dots, v_1)$ and if N is an odd integer $V[\alpha] = {}^t(v_0, \dots, v_m, v_m, \dots, v_1)$, where $v_k = \mathbb{E}[\alpha_1 \alpha_{1+k}]$, $k = 0, \dots, m$. The vector $V[H_X^{(N)}]$ can be expressed in the same way, for $i = 1, \dots, N$, $\pi - \theta_i = \frac{N-i+2-1}{N} \pi = \theta_{N-i+1[N]+1}$

$$\begin{aligned} V[H_X^{(N)}]_i &= \mathbb{E}[H_X(0)H_X(\theta_i)] \\ &= \mathbb{E}[H_X(0)H_X(\pi - \theta_i)] \\ &= V[H_X^{(N)}]_{N-i+1[N]+1} \end{aligned}$$

Therefore, if N is an even integer, $V[H_X^{(N)}] = {}^t(c_0, \dots, c_{m-1}, c_m, c_{m-1}, \dots, v_1)$, and if N is an odd integer, $V[H_X^{(N)}] = {}^t(c_0, \dots, c_m, c_m, \dots, c_1)$, where $c_k =$

$V[H_X^{(N)}]_{k+1}$, $k = 0, \dots, m$. In practice the vector $V[\alpha]$ can be computed by the knowledge of the $m + 1$ first components of $V[H_X^{(N)}]$ and the linear problem (42) can be rewritten and solved as a linear problem of size $m + 1$.

Remark 3.1. *In practice, the estimation of $\mathbb{E}[H_X(0)H_X(t)]$ for $t \in [0, \pi]$ will be often noised. Then, it is a better choice to find $V[\alpha]$ in the least squares sense. Let $N' \geq m + 1$ and let $0 = t_1 \leq \dots \leq t_{N'} = \frac{\pi}{2}$ be a subdivision of $[0, \pi]$ containing $\{\theta_1, \dots, \theta_{m+1}\}$, the $(t_i)_{1 \leq i \leq N'}$ are observation points. Let us recall that $\forall t \in [0, \frac{\pi}{2}]$, $\mathbb{E}[H_X(0)H_X(t)] = \mathbb{E}[H_X(0)H_X(\pi - t)]$ then it can be considered that there exist $2(N' - 1)$ points of observation such that: $z_i = t_i$ for $i = 1, \dots, N'$, and $z_i = t_{2N'-i}$ for $i = N' + 1, \dots, 2N' - 2$. Let $Q_{ij} = k_j(z_i)$, $V[H_X^{(2(N'-1))}] = {}^t(\mathbb{E}[H_X(0)H_X(z_1)], \dots, \mathbb{E}[H_X(0)H_X(z_{2(N'-1)})])$ then from (41) :*

$$V[H_X^{(2(N'-1))}] = QV[\alpha]$$

Finally, if $\hat{V}[H_X^{(2(N'-1))}]$ is a noisy estimation of $V[H_X^{(2(N'-1))}]$, the following least square estimator of $V[\alpha]$ is better than the one provided by (39),

$$\tilde{V}[\alpha] = \arg \min_{V \in \mathbb{R}_+^N} \|\hat{V}[H_X^{(2(N'-1))}] - QV\|^2 \quad (43)$$

It has been discussed some properties of the random zonotopes. The 0-regular random zonotopes and the regular random zonotope have been defined and studied. It has been shown that a 0-regular random zonotope can be describe by a unique face length distribution. Such distribution can be easily relate to the Feret's diameter of the 0-regular random zonotope with the relations established in the first section.

The different face length distributions of a regular random zonotope has been studied. It has been show that among them, one can be identified, the central face length distribution. Finally, it has been given some formulae which allows to compute the first and second order moments of the central face length distribution from whose of the Feret's diameter of the regular random zonotope. The following section is devoted to the description of a random symmetric convex set as a 0-regular random zonotope and as a regular random zonotope.

4 Description of a random symmetric convex set as a random zonotope from its Feret's diameter

In the first section, it has been defined some approximations of a symmetric convex set as zonotopes. In the second section, the regular and 0-regular random zonotopes have been characterized from their Feret's diameters random process. The aim of this section is to generalize the previous approximations to a random symmetric convex set X .

Firstly, it will be shown that the 0-regular random zonotope corresponding to the $\mathcal{C}_0^{(N)}$ -approximation of X can be characterized from the Feret's diameter random process of X . In a second time, it will be shown that the isotropic regular

random zonotope corresponding to the $\mathcal{C}_\infty^{(N)}$ -approximation of an isotropised set of X can be estimated from the Feret's diameter random process of X .

4.1 Approximation of a random symmetric convex set by a 0-regular random zonotope

The approximation of a random symmetric convex set X by a 0-regular random zonotope will be investigated. It will be shown that the random set $X_0^{(N)}$ valued in $\mathcal{C}_0^{(N)}$ which is defined as the $\mathcal{C}_0^{(N)}$ -approximation of X 's realizations can be characterize from the Feret's diameter of X . Finally, some formulas which allow to compute the moments of the random vector of the faces of $X_0^{(N)}$ will be given.

Proposition 4.1 (Approximation by a 0-regular random zonotope).

Let X be a random convex set. For any $\omega \in \Omega$ a.s , let $X_0^{(N)}(\omega)$ be the $\mathcal{C}_0^{(N)}$ -approximation of $X(\omega)$. The 0-regular random zonotope $X_0^{(N)}$ will be called the $\mathcal{C}_0^{(N)}$ -approximation of the random set X .

For any $N > 1$, an interval of confidence for the Hausdorff's distance can be built. Indeed, for any $a > 0$ there is the relation:

$$\mathbb{P}(d_H(X, X_0^{(N)}) > a) \leq \frac{(6 + 2\sqrt{2})}{a} \sin\left(\frac{\pi}{2N}\right) \mathbb{E}[\text{diam}(X)] \quad (44)$$

If $\epsilon \in [0, 1]$ is a confidence level, $a(\epsilon, N) = \frac{(6+2\sqrt{2})}{\epsilon} \sin\left(\frac{\pi}{2N}\right) \mathbb{E}[\text{diam}(X)]$ can be considered as an upper bound for $d_H(X, X_0^{(N)})$ with confidence $1 - \epsilon$.

Consequently, such an approximation is consistent as $N \rightarrow \infty$.

Proof. Let X be a random convex set. For any $\omega \in \Omega$ a.s , let $X_0^{(N)}(\omega)$ be the $\mathcal{C}_0^{(N)}$ -approximation of $X(\omega)$ in $\mathcal{C}_0^{(N)}$. According to the theorem [iii](#), for any real $a > 0$:

$$\begin{aligned} \forall \omega \in \Omega \text{ a.s , } d_H(X, X_0^{(N)}) &\leq (6 + 2\sqrt{2}) \sin\left(\frac{\pi}{2N}\right) \text{diam}(X) \\ \Rightarrow \mathbb{P}(d_H(X, X_0^{(N)}) > a) &\leq \mathbb{P}\left((6 + 2\sqrt{2}) \sin\left(\frac{\pi}{2N}\right) \text{diam}(X) > a\right) \end{aligned}$$

By using the Markov inequality [\[9\]](#), it follows:

$$\mathbb{P}(d_H(X, X_0^{(N)}) > a) \leq \frac{(6 + 2\sqrt{2})}{a} \sin\left(\frac{\pi}{2N}\right) \mathbb{E}[\text{diam}(X)]$$

The consistence of the approximation as $N \rightarrow \infty$ comes directly from this relation. \square

According to to the relation [16](#), $\mathbb{E}[\text{diam}(X)]$ can be replaced by $\frac{1}{2}\mathbb{E}[U(X)]$. Let X be a random symmetric convex set and $X_0^{(N)}$ be its $\mathcal{C}_0^{(N)}$ -approximation, in the same way as the deterministic case, the face length distribution can be related to the Feret's diameter of X .

Proposition 4.2 (Characterization of the $\mathcal{C}_0^{(N)}$ -approximation from the Feret's diameter process). *Let $N > 1$ be an integer and X a random symmetric convex set. Let $X_0^{(N)}$ be the $\mathcal{C}_0^{(N)}$ -approximation of X . Its face length vector α can be characterized from the Feret's diameter process:*

$$\forall \omega \in \Omega \text{ a.s } \alpha(\omega) = F^{(N)-1} H_X^{(N)}(\omega) \quad (45)$$

$$\mathbb{E}[\alpha] = F^{(N)-1} \mathbb{E}[H_X^{(N)}] \quad (46)$$

$$C[\alpha] = F^{(N)-1} C[H_X^{(N)}] F^{(N)-1} \quad (47)$$

where $H_X^{(N)} = {}^t(H_X(\theta_1), \dots, H_X(\theta_1))$ is the random vector composed by the Feret's diameter evaluated on the regular subdivision. The matrix $F^{(N)}$ is still defined as $(|\sin(\theta_i - \theta_j)|)_{1 \leq i, j \leq N}$ and for a vector x the notation $C[x]$ denotes its second order moments $\mathbb{E}[x^t x]$.

Proof. According to the theorem iii, the matrix $F^{(N)}$ is invertible then by definition of the approximation it follows the relation (45). Note that $\alpha^t \alpha = F^{(N)-1} H_X^{(N)t} H_X^{(N)} F^{(N)-1}$, then the relation (46) and (47) come from the linearity of the expectation. \square

Remark 4.1. The $\mathcal{C}_0^{(N)}$ -approximation of a random symmetric convex set X is a consistent approximation as $N \rightarrow \infty$. Furthermore, if X is already a 0-regular random zonotope in $\mathcal{C}_0^{(N)}$, its M^{th} approximation $X_0^{(M)}$ coincides with X if and only if N is a divider of M .

Such an approximation is sensitive to a rotation of X . Indeed, if $R_\eta(X)$ is the rotation of X by the random angle η , the random set X and $R_\eta(X)$ have different approximations. This property can be seen as an advantage or a disadvantage. Indeed, if the objective is to describe the direction of some random set it is an advantage, but there is a need to use a large N . However, when the objective is to describe the shape of a random set with a small N without taking into consideration its direction, it can be a great disadvantage, see the following example.

Example 4.1. Let $N = 2$ and $\theta_1 = 0$, $\theta_2 = \frac{\pi}{2}$ the regular subdivision. Let us consider the random symmetric convex set X as a deterministic square of side 1, i.e. $X = S_{\theta_1} \oplus S_{\theta_2}$. Its $\mathcal{C}_0^{(2)}$ -approximation coincides with X : $X_0^{(2)} = X$. The matrix $F^{(N)}$ is defined as:

$$F^{(N)} = F^{(N)-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and consequently:

$$\mathbb{E}[\alpha_X] = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, C[\alpha_X] = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } Cov(\alpha_X) = 0$$

Consider now the random symmetric convex set $Y = R_\eta(X)$ where η is a uniform random variable on $[0, \pi]$, then the mean and the covariance of its Feret's diameter can be computed (see (31) to (33)) as:

$$\mathbb{E}[H_Y^{(N)}] = \begin{pmatrix} \frac{4}{\pi} \\ \frac{4}{\pi} \end{pmatrix} \text{ and } C[H_Y^{(N)}] = \left(1 + \frac{2}{\pi}\right) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

So

$$\mathbb{E}[\alpha_Y] = \frac{4}{\pi} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, C[\alpha_Y] = \left(\frac{\pi + 2}{\pi}\right) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and

$$\text{Cov}(\alpha_Y) = \frac{\pi^2 + 2\pi - 16}{\pi^2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

The random set Y is approximated by a random rectangle which has a shape that can varies ($\text{Cov}(\alpha_Y) \neq 0$). However, Y have the same geometrical shape as X . This example shows that the $\mathcal{C}_0^{(N)}$ -approximation can not be used to describe the shape of a random symmetric convex set for small N .

In order to describe the shape of a random symmetric convex set as a zonotope with a small number of faces, there is a need to have an approximation insensitive to the rotations. Which lead us to the following approximation.

4.2 Approximation of a random symmetric convex set by an isotropic random zonotope.

Previously, it has been shown that a random symmetric convex set can be approximated as a random 0-regular zonotope. However, it has been also shown that such approximate can be problematic for small values of N . The aim of this section is to define and characterize an approximation in $\mathcal{C}_\infty^{(N)}$ which is invariant up to a rotation and which can be used for not so large N . For this objective, the approximation is given for an isotropised set of X instead of X . It will be shown that a random symmetric convex set can be approximated up to a rotation by an isotropic random regular zonotope.

Let us note $Y = R_z(X)$ the isotropised set of X with z an independent uniform variable on $[0, \pi]$. Let $X_\infty^{(N)}$ be a $\mathcal{C}_\infty^{(N)}$ -approximation of X , then,

$$\forall \omega \in \Omega \text{ a.s.}, X_\infty^{(N)}(\omega) = R_{\tau(\omega)}(\tilde{X}_0^{(N)}(\omega))$$

According to the definition of the $\mathcal{C}_\infty^{(N)}$ -approximation, the random set $Y_\infty^{(N)} = R_z(X_\infty^{(N)})$ is a $\mathcal{C}_\infty^{(N)}$ -approximation of Y . Consequently, $Y_\infty^{(N)} = R_{z+\tau}(\tilde{X}_0^{(N)})$. Because of the independence of η and X and according to the property of the addition modulo π , the random variable $\eta = z + \tau$ is a uniform random variable on $[0, \pi]$ independent of X , then $Y_\infty^{(N)}$ is an isotropic regular zonotope. Such a random regular zonotope will be used as the approximation of X up to a rotation.

Definition 4.1 ($\mathcal{C}_\infty^{(N)}$ -isotropic approximation). Let X be a random symmetric convex set and $Y = R_z(X)$ its isotropised set, the isotropic random regular zonotope $Y_\infty^{(N)} = R_z(X_\infty^{(N)})$ will be named the $\mathcal{C}_\infty^{(N)}$ -isotropic approximation of X and denoted by $\tilde{X}_\infty^{(N)}$.

Proposition 4.3 (Properties of the $\mathcal{C}_\infty^{(N)}$ -isotropic approximation). Let X be a random symmetric convex set, and $\tilde{X}_\infty^{(N)}$ be its $\mathcal{C}_\infty^{(N)}$ -isotropic approximation.

i. $\tilde{X}_\infty^{(N)}$ is an isotropic random regular zonotope.

ii. $\forall \omega \in \Omega \text{ a.s.}, \exists t(\omega) \in [0, \pi], \forall i = 1, \dots, N, H_X(t + \theta_i) = H_{\tilde{X}_\infty^{(N)}}(t + \theta_i)$

iii. $\forall \omega \in \Omega$ a.s , $d_P(X, Y_\infty^{(N)}) \rightarrow 0$ as $N \rightarrow \infty$

iv. The $\mathcal{C}_\infty^{(N)}$ -isotropic approximation is invariant up to a rotation of X .

v. if X is a random regular zonotope, any face length vector of $\tilde{X}_\infty^{(N)}$ is a face length vector of X .

Proof.

1. It is easy to see that $\tilde{X}_\infty^{(N)}$ is an isotropised set of $X_\infty^{(N)}$. Consequently it is an isotropic random regular zonotope.

2-4. These properties are direct consequence of the theorem 2.2.

5. Suppose that X is a random regular zonotope then X and $\tilde{X}_\infty^{(N)}$ coincide up to a random rotation, then any face length vector of the one is a face length vector of the other one.

□

In order to describe the shape of X , the best way would be to characterize the central face length distribution of $\tilde{X}_\infty^{(N)}$ from information available on X . Unfortunately, there is no way to compute the characteristics of the random process $H_{\tilde{X}_\infty^{(N)}}$ from those of H_X . However, the approximation of the first and second order moments of $H_{\tilde{X}_\infty^{(N)}}$ can be estimated from those of the Feret's diameters of an isotropised set of X (i.e H_Y , where Y is an isotropised set of X).

Proposition 4.4 (Approximation of the moments of the central face length distribution). *Let X be a symmetric random convex set, Y its isotropised set, $\tilde{X}_\infty^{(N)}$ the $\mathcal{C}_\infty^{(N)}$ -isotropic approximation of X and α the central face length vector of $\tilde{X}_\infty^{(N)}$.*

i. *An approximation of the first and second order moments of α is given by*

$$\forall x \in [0, \pi], \forall i = 1, \dots, N, \hat{\mathbb{E}}[\alpha] = \frac{\pi}{2N} \mathbb{E}[H_Y^{(N)}] \quad (48)$$

$$\hat{V}[\alpha] = \frac{1}{N} K(0)^{-1} V[H_Y^{(N)}] \quad (49)$$

Such an approximation is consistent as $N \rightarrow \infty$: $\hat{\mathbb{E}}[\alpha] - \mathbb{E}[\alpha] \rightarrow 0$ and $\hat{V}[\alpha] - V[\alpha] \rightarrow 0$ as $N \rightarrow \infty$.

ii. *If $\hat{\alpha}$ is a positive random vector satisfying $V[\hat{\alpha}] = \hat{V}[\alpha]$, $\mathbb{E}[\hat{\alpha}] = \hat{\mathbb{E}}[\alpha]$ and η an independent uniform variable on $[0, \pi]$, then the random set \hat{X} defined as:*

$$\hat{X} = R_\eta \left(\bigoplus_{i=1}^N \hat{\alpha}_i S_{\theta_i} \right) \quad (50)$$

satisfies $\mathbb{E}[U(X)] = \mathbb{E}[U(\hat{X})]$.

Proof.

1. The consistence of the estimate is trivial regarding that $\mathbb{E}[H_Y^{(N)}] \rightarrow \mathbb{E}[H_{\tilde{X}_\infty^{(N)}}^{(N)}]$ and $V[H_Y^{(N)}] \rightarrow V[H_{\tilde{X}_\infty^{(N)}}^{(N)}]$ as $N \rightarrow \infty$.

2. Let $\hat{\alpha}$ be a positive random vector satisfying $V[\hat{\alpha}] = \hat{V}[\alpha]$, $\mathbb{E}[\hat{\alpha}] = \hat{\mathbb{E}}[\alpha]$ and η an independent uniform variable on $[0, \pi]$. Because of the isotropy of Y , the vector $\mathbb{E}[H_Y^{(N)}]$ has all of his components equal to $\frac{1}{\pi}\mathbb{E}[(U(Y))]$, then the random set $\hat{X} = R_\eta(\bigoplus_{i=1}^N \hat{\alpha}_i S_{\theta_i})$ satisfies:

$$\begin{aligned}\mathbb{E}[U(\hat{X})] &= 2 \sum_{i=1}^N \mathbb{E}[\hat{\alpha}_i] \\ &= \mathbb{E}[U(Y)] \\ &= \mathbb{E}[U(X)]\end{aligned}$$

□

Remark 4.2. *Firstly, note that the quantities $\mathbb{E}[H_Y^{(N)}]$ and $V[H_Y^{(N)}]$ are easily obtained from the mean and auto-covariance of H_X by using the property 3.1. The approximations $\hat{\mathbb{E}}[\alpha]$ and $\hat{V}[\alpha]$ can be regarded as the characteristics of the central face length vector of an isotropic random regular zonotope \hat{X} , which has the same Feret's diameter on the θ_i than an isotropised set of X . In particular, such a zonotope has the same mean perimeter as X .*

Furthermore if X is a N^{th} -random regular zonotope, such quantities coincide with the ones of a face length vector of X . Consequently it will be more interesting to use the $\mathcal{C}_\infty^{(N)}$ -isotropic approximation when X is assumed to be a N^{th} -random regular zonotope.

5 Conclusions and prospects

In this paper, different approximations of a symmetric convex set as a zonotope have been proposed. These approximations have been further generalized to random symmetric convex sets. It has been shown that a random convex set can be approximated as precisely as we want as a random zonotope in terms of the Hausdorff's distance. More specifically, for a random symmetric convex set X , the first and second order moments of the face length vector of its zonotope approximation can be computed from the first and second order moments of the Feret's diameter process of X .

This work involves several perspectives. The first one would be to get higher moments of the central face length distribution, and to generalize this work in higher dimension. One potential application of this work would be to describe the primary grain of the germ-grain model. Indeed, in a large class of such models, there exists estimators for the moments of the Feret's diameter of the primary grain [25]. In particular, we prospect to apply this to the images of oxalate ammonium crystals modelled by the Boolean model (see [25, 26]). However, there is a need to study the estimators involved by the zonotope approximation in those germ-grains models.

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