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# Duality in nondominated discrete-time models for American options \*

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## Abstract

We aim to generalize the duality results of Bouchard & Nutz [10] to the case of American options. By introducing an enlarged canonical space, we reformulate the superhedging problem for American options as a problem for European options. Then in a discrete time market with finitely many liquid options, we show that the minimum superhedging cost of an American option equals to the supremum of the expectation of the payoff at all (weak) stopping times and under a suitable family of martingale measures. Moreover, by taking the limit on the number of liquid options, we obtain a new class of martingale optimal transport problems as well as a Kantorovich duality result.

**Key words.** Super-replication, American option, nondominated model, martingale optimal transport, Kantorovich duality.

**MSC (2010).** Primary: 60G40, 60G05; Secondary: 49M29.

## 1 Introduction

In a complete market, where every contingent claim can be perfectly replicated by a self-financing trading strategy, the option price is given by its replication cost, under the no-arbitrage assumption. In the classical dominated model, the no-arbitrage condition is proved to be equivalent to the existence of the equivalent martingale measures, by the so-called first fundamental theorem of asset pricing, see e.g. Delbaen & Schachermayer [13], Föllmer & Schied [18], etc.

In a nondominated model, where the market is incomplete, a safe way of pricing is to use the minimum super-replication cost of the derivative option. Using the duality result, the super-replication problem is related to a model-free pricing problem, i.e. the supremum of the expectations of the payoff under a suitable family of “martingale measures” models. For the continuous time model under “volatility

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uncertainty”, this duality result has been established under different formulations, see, among many others, Denis & Martini [14], Soner, Touzi & Zhang [32], Nutz & Neufeld [28], Possamai, Royer & Touzi [31], etc.

Another branch of literature studied the superhedging problem using dynamic strategy on the underlying risky asset as well as the static strategy on a given set (finite or infinite) of liquid options. This approach has been initiated by the seminal work of Hobson [22], with the optimal Skorokhod embedding problem (SEP) approach, i.e. finding the optimal stopping time on a Brownian motion under a marginal constraint on the stopped Brownian motion. This approach is justified by the fact that a continuous martingale can be considered as a time-changed Brownian motion, and moreover, one can recover the marginal distribution of the underlying martingale with (infinitely many) liquid call/put option prices of different strikes but at a fixed maturity (see e.g. Breeden & Litzenberg [9]). We would like refer to the survey papers [23, 30] for more details on this approach. More recently, this problem has been studied by the so-called martingale optimal transport (MOT) approach, starting from [5] and [19], etc. In particular, the superhedging duality becomes a Kantorovich type duality as in the classical optimal transport problem. We also refer to [1], [4], [16], [26], [21], [34], etc., among many others, for duality results in different contexts.

In a discrete-time nondominated model, with presence of finite number (could be 0) of liquid options, Bouchard & Nutz [10] introduce a notion of no-arbitrage in a quasi-sure sense, and relate it to the existence of the “dominated” martingale measures. In particular, a general duality result has been established between the minimum superhedging cost (in a quasi-sure sense) and the supremum over a suitable family of martingale measures, for European type derivative options. Similar techniques and results have then been extended to the continuous time case with continuous underlying process in Biagini, Bouchard, Kardaras and Nutz [7]. We also notice that a similar duality result has also been obtained by Burzoni, Frittelli & Maggis [11] in a slightly different discrete time setting.

The main objective of this paper is to extend the arguments and duality results in Bouchard & Nutz [10] to the case of American type options, under their “quasi-sure” no-arbitrage conditions. For American options, the superhedging strategy should super-replicate the exercise payoff value of the option regardless of its exercise time, and its natural dual formulation should be the supremum of the expected value of the payoff at all stopping times and under a suitable family of martingale measures. By introducing an enlarged canonical space, we reformulate the superhedging problem for American options as a problem for European options, and its natural dual problem turns to be the supremum over a family of “weak” stopping times. Then by adapting the arguments in [10], we obtained a general duality results. Moreover, restricting to the context where the liquid options are European call/put options and are numerous enough, so that one can recover the marginal distribution of the underlying risky assets, it leads naturally to a MOT problem. By the approximating arguments, we obtain a Kantorovich type duality for the MOT problem. We also discuss the equivalence between the “weak” and “strong” stopping time formulations for the supremum (dual) problem.

While most of the literature on the robust hedging problem focuses on the (exotic) European options, some works are devoted to the American options. For example, Cox & Hoeggerl [12] studies the necessary (and sufficient in some cases) conditions on the American Put prices for the absence of arbitrage. Dolinsky [15]

studied the Game options (including American options) in a nondominated discrete-time market, but without liquid options in the superhedging strategy. Neuberger [27] considered a discrete time, discrete space market with presence of liquid European vanilla options, and obtained a duality result by using a “weak” dual formulation. This approach has recently been exploited and presented with more fruitful results in Hobson & Neuberger [24]. Bayraktar, Huang & Zhou [2] studied the same superhedging problem as our paper, but they only considered the “strong” stopping times for the dual formulation, which leads to a duality gap in general cases. More recently, Bayraktar and Zhou [3] prove a duality result by considering “randomized” stopping times, under some regularity and integrability conditions on the payoff functions. Our “weak” formulation of the dual problem is more or less in the same spirit of [27, 24, 3], but our approach leads to a duality results in a more general setting, and/or under more general conditions (see more discussions in Section 2.3 and also [25]).

The rest of the paper is organized as follows. Section 2 formulates the main problem and provides the main results. The proofs are completed in Section 3.

## 2 Main results

### 2.1 Setting

**Notations** We first recall some notations used in Bouchard & Nutz [10]. Given a measurable space  $(\Omega, \mathcal{A})$ , we denote by  $\mathfrak{B}(\Omega)$  the set of all probability measures on  $\mathcal{A}$ . If  $\Omega$  is a topological space,  $\mathcal{B}(\Omega)$  denotes its Borel  $\sigma$ -field. If  $\Omega$  is a Polish space, a subset  $A \subseteq \Omega$  is analytic if it is the image of a Borel subset of another Polish space under a Borel measurable mapping. A function  $f : \Omega \rightarrow \overline{\mathbb{R}} := [-\infty, \infty]$  is upper semianalytic if  $\{\omega \in \Omega : f(\omega) > c\}$  is analytic for all  $c \in \mathbb{R}$ . Given a probability measure  $\mathbb{P} \in \mathfrak{B}(\Omega)$  and a measurable function  $f : \Omega \rightarrow \overline{\mathbb{R}}$ , we define the expectation

$$\mathbb{E}^{\mathbb{P}}[f] := \mathbb{E}^{\mathbb{P}}[f^+] - \mathbb{E}^{\mathbb{P}}[f^-], \quad \text{with the convention } \infty - \infty = -\infty.$$

For a family  $\mathcal{P} \subseteq \mathfrak{B}(\Omega)$  of probability measures, a subset  $A \subset \Omega$  is called  $\mathcal{P}$ -polar if  $A \subset A'$  for some  $A' \in \mathcal{A}$  satisfying  $\mathbb{P}[A'] = 0$  for all  $\mathbb{P} \in \mathcal{P}$ , and a property is said to hold  $\mathcal{P}$ -quasi surely or  $\mathcal{P}$ -q.s if it holds true outside a  $\mathcal{P}$ -polar set.

**A nondominated discrete-time model** Following [10], we will consider a discrete-time model (with slightly different notations). Let  $N \in \mathbb{N}$  be the time horizon,  $\Omega_0 = \{\omega_0\}$  be a singleton and  $\Omega_1$  a Polish space. For each  $k \in \{1, \dots, N\}$ , we define  $\Omega_k := \Omega_1^k$  as the  $k$ -fold Cartesian product. For each  $k$ , we denote by  $\mathcal{F}_k^0 := \mathcal{B}(\Omega_k)$  and by  $\mathcal{F}_k$  its universal completion. In particular, we notice that  $\mathcal{F}_0^0$  is trivial and we denote  $\Omega := \Omega_N$  and  $\mathcal{F} := \mathcal{F}_N$ . We shall often see  $\mathcal{F}_k$  as a sub- $\sigma$ -field of  $\mathcal{F}_N$ , and hence obtain a filtration  $\mathbb{F} = (\mathcal{F}_k)_{0 \leq k \leq N}$  on  $\Omega$ . Let  $k \in \{0, \dots, N-1\}$  and  $\omega \in \Omega_k$ , we are given a non-empty **convex** set  $\mathcal{P}_k(\omega) \subseteq \mathfrak{B}(\Omega_1)$  of probability measures, which represents the set of all possible models for the  $k+1$ -th period, given state  $\omega$  at time  $k$ . We assume that for each  $k$ ,

$$\text{graph}(\mathcal{P}_k) := \{(\omega, \mathbb{P}) : \omega \in \Omega_k, \mathbb{P} \in \mathcal{P}_k(\omega)\} \subseteq \Omega_k \times \mathcal{P}(\Omega_1) \text{ is analytic.} \quad (2.1)$$

Given such a kernel  $\mathbb{P}_k$  for each  $k \in \{0, 1, \dots, N-1\}$ , we define a probability measure  $\mathbb{P}$  on  $\Omega$  by Fubini's theorem:

$$\mathbb{P}(A) := \int_{\Omega_1} \cdots \int_{\Omega_1} \mathbf{1}_A(\omega_1, \omega_2, \dots, \omega_N) \mathbb{P}_{N-1}(\omega_1, \dots, \omega_{N-1}; d\omega_N) \cdots \mathbb{P}_0(d\omega_1)$$

We can then introduce the set  $\mathcal{P} \subseteq \mathfrak{B}(\Omega)$  of possible models for the multi-period market up to time  $N$ :

$$\mathcal{P} := \{ \mathbb{P}_0 \otimes \mathbb{P}_1 \otimes \cdots \otimes \mathbb{P}_{N-1} : \mathbb{P}_k(\cdot) \in \mathcal{P}_k(\cdot), k = 0, 1, \dots, N-1 \}. \quad (2.2)$$

Notice that the condition (2.1) ensures that  $\mathcal{P}_k$  has always a universally measurable selector:  $\mathbb{P}_k : \Omega_k \rightarrow \mathcal{P}(\Omega_1)$  such that  $\mathbb{P}_k(\omega) \in \mathcal{P}_k(\omega)$  for all  $\omega \in \Omega_k$ . Then the set  $\mathcal{P}$  defined in (2.2) is nonempty.

Let  $d \in \mathbb{N}$ , we equip an  $\mathbb{F}^0$ -adapted process  $S = (S_k)_{0 \leq k \leq N}$ , so that  $S_k = (S_k^1, \dots, S_k^d) : \Omega \rightarrow \mathbb{R}^d$  is Borel measurable. Equivalently, we can consider  $S_k$  as a Borel measurable random vector defined on  $\Omega_k$ , and it represents the (discounted) price of the traded stocks at time  $k \in \{0, 1, \dots, N\}$ . Let  $\mathcal{H}$  be the set of all  $\mathbb{F}$ -predictable  $\mathbb{R}^d$ -valued (dynamic strategy) processes, then given  $H \in \mathcal{H}$ , we denote

$$H \cdot S := ((H \cdot S)_k)_{k=0,1,\dots,N}, \quad (H \cdot S)_k := \sum_{i=1}^k H_i \Delta S_i,$$

where  $\Delta S_i := S_i - S_{i-1}$  and  $H_i \Delta S_i$  denotes the inner product  $\sum_{j=1}^d H_i^j \Delta S_i^j$  on  $\mathbb{R}^d$ . Further, let  $e \in \mathbb{N} \cup \{0\}$ , and  $g = (g^1, \dots, g^e) : \Omega \rightarrow \mathbb{R}^e$  be Borel measurable. For each  $i = 1, \dots, e$ ,  $g^i$  is seen as a liquid option which can be bought or sold at time  $k = 0$  with price  $g_0^i = 0$ . Suppose that the options  $g = (g^1, \dots, g^e)$  can only be traded statically, so that a semi-static hedging strategy is a pair  $(H, h) \in \mathcal{H} \times \mathbb{R}^e$ . We also denote by  $\mathcal{M}_k(\omega)$  the collection of all probability measures  $\mathbb{Q}$  on  $\Omega_1$ , such that  $\mathbb{Q} \ll \mathbb{P}$  for some  $\mathbb{P} \in \mathcal{P}_k(\omega)$  and under which  $\mathbb{E}^{\mathbb{Q}}[\Delta S_{k+1}] = 0$ .

The following notion of no-arbitrage condition ( $\text{NA}(\mathcal{P})$ ) has been introduced by Bouchard & Nutz [10].

**Definition 2.1.** *Condition  $\text{NA}(\mathcal{P})$  holds if for all  $(H, h) \in \mathcal{H} \times \mathbb{R}^e$*

$$(H \cdot S)_N + hg \geq 0, \mathcal{P}\text{-q.s.} \implies (H \cdot S)_N + hg = 0, \mathcal{P}\text{-q.s.}$$

Let  $\mathcal{M}$  be the collection of all probability measures  $\mathbb{Q}$  on  $\Omega$  under which  $S$  is a  $\mathbb{F}$ -martingale, and  $\mathbb{Q} \ll \mathbb{P}$  for some  $\mathbb{P} \in \mathcal{P}$ , we then denote

$$\mathcal{M}_e := \{ \mathbb{Q} \in \mathcal{M} : \mathbb{E}^{\mathbb{Q}}[g^i(X)] = 0, i = 1, \dots, e \}.$$

It is proved in [10] that the condition  $\text{NA}(\mathcal{P})$  is equivalent to the existence of a martingale measure  $\mathbb{Q} \in \mathcal{M}_e$ .

**Superhedging of American options** While the duality result provided in [10] concerns the superhedging cost of a European type option using semi-static strategies, we will study the same problem for an American type option.

Without loss of generality, we assume that the studied American option is allowed to be exercised at time  $k \in \mathbb{T} := \{1, \dots, N\}$ , but not at time 0. Let the payoff function of the American option is given by  $\Phi = (\Phi_k)_{1 \leq k \leq N}$ , where  $\Phi_k : \Omega \rightarrow \mathbb{R}$  is upper semianalytic.

**Remark 2.2.** Here  $\Phi_k$  represents the payoff of the American option, at the maturity  $N$ , if the holder exercises it at time  $k$ . For a standard American option,  $\Phi_k$  should be assumed to be  $\mathcal{F}_k$ -measurable. However, we will allow that  $\Phi_k$  be  $\mathcal{F}_N$ -measurable for generality. In particular, it allows to study a portfolio including an American option and some European options.

Since the American option's holder can exercise it at any time, a superhedging portfolio should dominate the payoff of the option, regardless of the exercise time  $k = 1, \dots, N$ . In the contrast, the hedger should be allowed to adjust his dynamic strategy, at each time  $k = 1, \dots, N - 1$ , using the information  $\mathcal{F}_k$  as well as the exercise time if it happens before (or at) time  $k$ . Therefore, we introduce  $\overline{\mathcal{H}}$  as the collection of all process  $\overline{H} = (\overline{H}^1, \dots, \overline{H}^N) \in (\mathcal{H})^N$  such that for any  $1 \leq i \leq j \leq k \leq N$  one has  $\overline{H}_i^j = \overline{H}_i^k$ . Then for any  $\overline{H} \in \overline{\mathcal{H}}$  and  $k \in \{1, \dots, N\}$ , we define

$$(\overline{H} \cdot S)_N^k := (\overline{H}^k \cdot S)_N = \sum_{i=1}^N \overline{H}_i^k \Delta S_i. \quad (2.3)$$

Notice that for  $i \leq k$ ,  $\overline{H}_i^k$  only depends on the information in  $\mathcal{F}_{i-1}$ . Then our minimum superhedging cost of the American option  $\Phi$  using semi-static strategy is given by

$$\pi_e(\Phi) := \inf \{x : \exists (\overline{H}, h) \in \overline{\mathcal{H}} \times \mathbb{R}^e \text{ s.t. } x + (\overline{H} \cdot S)_N^k + hg \geq \Phi_k, \mathcal{P}\text{-q.s.}, \forall k \in \mathbb{T}\}. \quad (2.4)$$

**A reformulation of the superhedging problem** We next introduce an enlarged probability space, and reformulate the superhedging problem (2.4) for American options as a problem for European options.

Let us define an enlarged space  $\overline{\Omega} := \Omega \times \mathbb{T}$  with  $\mathbb{T} := \{1, \dots, N\}$ , a canonical process  $X = (X_k)_{0 \leq k \leq N}$  and a (canonical) random variable  $T : \overline{\Omega} \rightarrow \mathbb{T}$  by  $X_k(\bar{\omega}) := \omega_k$  and  $T(\bar{\omega}) := \theta$ , for all  $\bar{\omega} := (\omega, \theta) \in \overline{\Omega}$ . Moreover, we extend naturally the definition of  $S$  from  $\Omega$  to  $\overline{\Omega}$ , i.e.  $S(\bar{\omega}) := S(\omega)$  for  $\bar{\omega} := (\omega, \theta) \in \overline{\Omega}$ . We next introduce an enlarged filtration  $\overline{\mathbb{F}}^0 = (\overline{\mathcal{F}}_k^0)_{0 \leq k \leq N}$  by

$$\overline{\mathcal{F}}_0^0 := \{\emptyset, \overline{\Omega}\} \quad \text{and} \quad \overline{\mathcal{F}}_k^0 := \sigma\{X_i, \{T \leq i\}, i = 1, \dots, k\},$$

and the universally completed filtration  $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_k)_{0 \leq k \leq N}$  by defining  $\overline{\mathcal{F}}_k$  as the universally completion of  $\overline{\mathcal{F}}_k^0$ . It follows that the random variable  $T : \overline{\Omega} \rightarrow \mathbb{T}$  is automatically an  $\overline{\mathbb{F}}^0$ -stopping time. Moreover, we can define the class  $\overline{\mathcal{H}}$  of dynamic strategy in the following equivalent way

$$\overline{\mathcal{H}} := \{\overline{H} : \overline{\Omega} \times \mathbb{T} \rightarrow \mathbb{R}^d : \overline{H} \text{ is } \overline{\mathbb{F}}\text{-predictable}\}.$$

Then by abusing of notations, we denote also

$$(\overline{H} \cdot S)_N(\bar{\omega}) := \sum_{i=1}^N \overline{H}_i(\bar{\omega}) \Delta S_i(\bar{\omega}), \quad \text{for all } \bar{\omega} = (\omega, \theta).$$

Let us denote by  $\overline{\mathcal{P}}$  the collection of all probability measures  $\overline{\mathbb{P}}$  on  $(\overline{\Omega}, \overline{\mathcal{F}}_N^0)$  such that, for every  $k = 0, 1, \dots, N - 1$ ,  $\overline{\mathbb{P}}_{\bar{\omega}} \circ X_{k+1}^{-1} \in \mathcal{P}_k(\omega)$  for  $\overline{\mathbb{P}}$ -a.e.  $\bar{\omega} = (\omega, \theta) \in \overline{\Omega}$ , where  $(\overline{\mathbb{P}}_{\bar{\omega}})_{\bar{\omega} \in \overline{\Omega}}$  is a family of conditional probability measures of  $\overline{\mathbb{P}}$  w.r.t.  $\overline{\mathcal{F}}_k^0$ . Let

$$\Phi(\bar{\omega}) := \Phi_\theta(\omega), \quad \text{for every } \bar{\omega} = (\omega, \theta) \in \overline{\Omega},$$

we then obtain the following equivalent reformulation:

**Proposition 2.3.** *The superhedging problem  $\pi_e(\Phi)$  in (2.4) is equivalent to*

$$\pi'_e(\Phi) := \inf \{x : \exists(\bar{H}, h) \in \bar{\mathcal{H}} \times \mathbb{R}^e \text{ s.t. } x + (\bar{H} \cdot S)_N + hg \geq \Phi, \bar{\mathcal{P}}\text{-q.s.}\}. \quad (2.5)$$

*Proof.* (i) First, we notice that  $\mathcal{P}$  in (2.2) can be defined equivalently as collection of all probability measures  $\mathbb{P}$  on  $(\Omega, \mathcal{F}_N^0)$  such that, for every  $k = 0, 1, \dots, N-1$ ,  $\mathbb{P}_\omega \circ X_{k+1}^{-1} \in \mathcal{P}_k(\omega)$  for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ , where  $(\mathbb{P}_\omega)_{\omega \in \Omega}$  is a family of conditional probability measures of  $\mathbb{P}$  w.r.t.  $\mathcal{F}_k^0$ .

Let  $x \in \mathbb{R} \cup \{\infty\}$  and  $(\bar{H}, h) \in \bar{\mathcal{H}} \times \mathbb{R}^e$  be such that  $x + (\bar{H} \cdot S)_N + hg \geq \Phi$ ,  $\bar{\mathcal{P}}$ -q.s. Then for every  $\mathbb{P} \in \mathcal{P}$  and  $k = 1, \dots, N$ , one has  $\bar{\mathbb{P}}(d\omega, d\theta) := \mathbb{P}(d\omega) \otimes \delta_k(d\theta) \in \bar{\mathcal{P}}$ , and hence  $x + (\bar{H} \cdot S)_N + hg \geq \Phi$ ,  $\bar{\mathbb{P}}$ -a.s. It follows that  $x + (\bar{H} \cdot S)_N^k + hg \geq \Phi_k$ ,  $\mathbb{P}$ -a.s. for all  $k \in \mathbb{T}$  and  $\mathbb{P} \in \mathcal{P}$ . Therefore, one has  $\pi_e(\Phi) \leq \pi'_e(\Phi)$ .

(ii) Next, let  $\bar{\mathbb{P}} \in \bar{\mathcal{P}}$ , and  $\mathbb{P} := \bar{\mathbb{P}}|_\Omega$  be the marginal distribution of  $\bar{\mathbb{P}}$  on  $\Omega$ . Remember that  $\mathcal{P}_k(\omega)$  is assumed to be convex as in [10], it follows that  $\mathbb{P} \in \mathcal{P}$ .

Therefore, let  $x \in \mathbb{R} \cup \{\infty\}$  and  $(\bar{H}, h) \in \bar{\mathcal{H}} \times \mathbb{R}^e$  satisfy  $x + (\bar{H} \cdot S)_N^k + hg \geq \Phi_k$ ,  $\mathcal{P}$ -q.s., for each  $k \in \mathbb{T}$ , one has immediately that  $x + (\bar{H} \cdot S)_N + hg \geq \Phi$ ,  $\bar{\mathbb{P}}$ -a.s., for every  $\bar{\mathbb{P}} \in \bar{\mathcal{P}}$ , since  $\bar{\mathbb{P}}(d\omega, d\theta) = \mathbb{P}(d\omega) \otimes \bar{\mathbb{P}}_\omega(d\theta)$  where  $\mathbb{P} := \bar{\mathbb{P}}|_\Omega$  and  $(\bar{\mathbb{P}}_\omega)_{\omega \in \Omega}$  is a family of conditional probability measures of  $\bar{\mathbb{P}}$  w.r.t.  $\sigma(X_k, k \in \mathbb{T})$ . It follows then  $\pi'_e(\Phi) \leq \pi_e(\Phi)$ .  $\square$

**Remark 2.4.** (i) *The above reformulation (2.5) can be considered as a minimum superhedging cost of a European option on a filtered space  $(\bar{\Omega}, \bar{\mathcal{F}}_N, \bar{\mathbb{F}})$  with payoff  $\Phi : \bar{\Omega} \rightarrow \mathbb{R}$ , but under the model uncertainty  $\bar{\mathcal{P}}$ .*

(ii) *The above reformulation technique (from an American option superreplication problem to a European option problem) can be easily extended to a more general context (such as the continuous time case).*

(iii) *In this discrete time context, if we choose  $\mathcal{P} = \mathfrak{B}(\Omega)$  to be the collection of all Borel probability measures on  $\Omega$ , then  $\bar{\mathcal{P}}$  turns to be the collection of all Borel probability measures on  $\bar{\Omega}$ . In this case, the superreplication in (2.4) (and (2.5)) is in fact in a point-wise sense.*

**The dual formulation** Recall that for a European option defined on  $\Omega$ , its minimum superhedging cost equals to the supremum over a suitable family of martingale measures on  $\Omega$  as shown in [10]. In view of Remark 2.4, the dual formulation for the superhedging problem of American option on  $\Omega$  (or equivalently, that of European option on  $\bar{\Omega}$ ) should be the supremum of the expected value of  $\Phi$  under a suitable family of martingale measures on the enlarged space  $\bar{\Omega}$ .

Let

$$\begin{aligned} \bar{\mathcal{M}}_e &:= \{ \bar{\mathbb{Q}} \in \mathfrak{B}(\bar{\Omega}) : \bar{\mathbb{Q}} \lll \mathcal{P}, \mathbb{E}^{\bar{\mathbb{Q}}}[g^i(X)] = 0, i = 1, \dots, e \\ &\quad \text{and } S \text{ is a } (\bar{\mathbb{F}}^0, \bar{\mathbb{Q}})\text{-martingale} \}, \end{aligned}$$

where  $\bar{\mathbb{Q}} \lll \mathcal{P}$  means that for every  $k = 0, 1, \dots, N-1$ , and any family of conditional probability measures  $(\bar{\mathbb{Q}}_{\bar{\omega}})_{\bar{\omega} \in \bar{\Omega}}$  of  $\bar{\mathbb{Q}}$  w.r.t.  $\bar{\mathcal{F}}_k^0$ , one has  $\bar{\mathbb{Q}}_{\bar{\omega}} \circ X_{k+1}^{-1} \in \mathcal{M}_k(\omega)$  (i.e.  $\bar{\mathbb{Q}}_{\bar{\omega}} \circ X_{k+1}^{-1} \lll \mathbb{P} \circ X_{k+1}^{-1}$  for some  $\mathbb{P} \in \mathcal{P}_k(\omega)$ ) for  $\bar{\mathbb{Q}}$ -a.e.  $\bar{\omega} = (\omega, \theta) \in \bar{\Omega}$ . We then introduce the dual formulation of problem (2.4) by

$$\sup_{\bar{\mathbb{Q}} \in \bar{\mathcal{M}}_e} \mathbb{E}^{\bar{\mathbb{Q}}}[\Phi_T(X)]. \quad (2.6)$$



**Remark 2.5.** We should see the probability measure  $\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_e$  as a market model, consistent with the market price of options  $(g^i)_{i=1, \dots, e}$ , together with a stopping time  $T$ . However, since  $T$  is a priori not a stopping time w.r.t. the filtration generated by  $X$  (neither by  $S$ ), it is in fact a weak formulation of the model-free pricing problem in (2.6) (see more discussions in Section 2.3).

## 2.2 Main results

Our main result is the following duality under the no-arbitrage condition in Definition 2.1.

**Theorem 2.6.** Let  $NA(\mathcal{P})$  hold true, and  $\Phi_k : \Omega \rightarrow \mathbb{R}$  is upper semianalytic, for each  $k \in \mathbb{T}$ . Then the martingale measures set  $\overline{\mathcal{M}}_e$  is nonempty, and

$$\pi_e(\Phi) = \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_e} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi_T(X)]. \quad (2.7)$$

Moreover, one has  $(\overline{H}, h) \in \overline{\mathcal{H}} \times \mathbb{R}^e$  such that

$$\pi_e(\Phi) + (\overline{H} \cdot S)_N^k + hg \geq \Phi_k, \quad \mathcal{P}\text{-q.s. for every } k \in \mathbb{T}.$$

We emphasize that we do not assume that  $\Phi_k$  is  $\mathcal{F}_k$ -measurable throughout the whole paper.

**A martingale optimal transport (MOT) problem** We next restrict to the case  $\Omega_1 := \mathbb{R}^d$ ,

$$\Omega := (\mathbb{R}^d)^N, \quad S = X, \quad \text{and } \mathcal{P}_k(\omega) := \mathfrak{B}(\mathbb{R}^d), \text{ for all } \omega \in \Omega. \quad (2.8)$$

Moreover, we assume that the liquid options on the market are all vanilla options, and are numerous enough so that one can recover the marginal distribution of the underlying process  $S$  at some maturity times  $\mathbb{T}_0 = \{t_1, \dots, t_{N_0}\} \subseteq \mathbb{T}$ , where  $t_{N_0} = N$ . More precisely, we are given a family  $\mu = (\mu_{t_1}, \dots, \mu_{t_{N_0}})$  of marginal distributions, such that  $\mu_k(|x|) < \infty$  for all  $k \in \mathbb{T}_0$ , and

$$\mu_i(\phi) \leq \mu_j(\phi) \text{ for all } 1 \leq i \leq j \leq n \text{ and convex function } \phi : \mathbb{R}^d \rightarrow \mathbb{R}.$$

Then the maximization problem (2.6) turns to be a MOT problem:

$$P(\mu) := \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}(\mu)} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi_T(X)], \quad (2.9)$$

where

$$\overline{\mathcal{M}}(\mu) := \{\overline{\mathbb{Q}} \in \mathfrak{B}(\overline{\Omega}) : S_k \sim^{\overline{\mathbb{Q}}} \mu_k, k \in \mathbb{T}_0, \text{ and } S \text{ is a } (\overline{\mathbb{F}}^0, \overline{\mathbb{Q}})\text{-martingale}\}.$$

Let  $\Lambda_0$  be the class of all Lipschitz function  $\lambda : \mathbb{R}^d \rightarrow \mathbb{R}$ , and we denote  $\Lambda := \Lambda_0^{N_0}$ . We then introduce the replication problem as follows:

$$D(\mu) := \inf \{ \mu(\lambda) : \exists (\overline{H}, \lambda) \in \overline{\mathcal{H}} \times \Lambda \text{ s.t. } \lambda(\omega) + (\overline{H} \cdot S)_N^k(\omega) \geq \Phi_k(\omega), \\ \text{for all } k \in \mathbb{T}, \omega \in \Omega \}, \quad (2.10)$$

where  $\lambda(\omega) := \sum_{i=1}^{N_0} \lambda_i(\omega_{t_i})$ .



**Theorem 2.7.** *Suppose that  $\Phi_k : \Omega \rightarrow \mathbb{R}$  is bounded from above and upper semicontinuous for each  $k \in \mathbb{T}$ , Then in the above context of martingale optimal transport problem, one has the existence of the optimal martingale  $\overline{\mathbb{Q}}^* \in \overline{\mathcal{M}}(\mu)$  in (2.9), and the duality holds true:*

$$P(\mu) = D(\mu).$$

## 2.3 Further discussions

**The weak stopping time and the equivalence** We should see a probability measure  $\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_e$  on  $\overline{\Omega}$  as a weak (or randomized) stopping time. We say a weak stopping term  $\alpha$  is a term

$$\alpha = (\Omega^\alpha, \mathcal{F}^\alpha, \mathbb{P}^\alpha, \mathbb{F}^\alpha = (\mathcal{F}_k^\alpha)_{0 \leq k \leq N}, (X_k^\alpha)_{0 \leq k \leq N}, (S_k^\alpha)_{0 \leq k \leq N}, \tau^\alpha)$$

such that  $(\Omega^\alpha, \mathcal{F}^\alpha, \mathbb{P}^\alpha, \mathbb{F}^\alpha)$  is a filtered probability space, equipped with a  $\mathbb{T}$ -valued stopping time  $\tau^\alpha$ , an adapted  $\Omega_1$ -valued process  $X^\alpha$  and an  $\mathbb{R}^d$ -valued  $\mathbb{F}^\alpha$ -martingale  $S^\alpha$ , and moreover, the transition kernel from  $X_k^\alpha$  to  $X_{k+1}^\alpha$  lies in  $\mathcal{P}_k(\omega)$  conditioning on  $(X_1^\alpha, \dots, X_k^\alpha) = \omega$ . Denote by  $\mathcal{A}_e$  the collection of all weak stopping terms  $\alpha$  such that  $\mathbb{E}^{\mathbb{P}^\alpha}[g_i(X_T^\alpha)] = 0$  for each  $i = 1, \dots, e$ , and by  $\mathcal{A}(\mu)$  the collection of all weak stopping terms  $\alpha$  such that  $S_k^\alpha \sim \mu_k$  for all  $k \in \mathbb{T}_0$ , in the context of (2.8).

**Proposition 2.8.** *Let  $\Phi : \overline{\Omega} \rightarrow \mathbb{R}$  be universally measurable, then*

$$\sup_{\alpha \in \mathcal{A}_e} \mathbb{E}^{\mathbb{P}^\alpha}[\Phi_{\tau^\alpha}(X^\alpha)] = \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_e} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi_T(X)] \text{ and } \sup_{\alpha \in \mathcal{A}(\mu)} \mathbb{E}^{\mathbb{P}^\alpha}[\Phi_{\tau^\alpha}(X^\alpha)] = \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}(\mu)} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi_T(X)]$$

*Proof.* We will only prove the first equality since the second follows by the same arguments. First, given an arbitrary stopping term  $\alpha \in \mathcal{A}_e$ , it is clear that  $(S^\alpha, X^\alpha, \tau^\alpha, \mathbb{P}^\alpha)$  induces a probability measures  $\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_e$  such that  $\mathbb{E}^{\mathbb{P}^\alpha}[\Phi_{\tau^\alpha}(X^\alpha)] = \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi_T(X)]$ . Then one obtains the inequality “ $\leq$ ”. For the inverse inequality, it is enough to notice that a probability measure  $\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_e$  together with the canonical space  $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{F}})$  and the canonical process  $(X, S)$  provides a weak stopping term in  $\mathcal{A}_e$ .  $\square$

If the optimization problem in (2.6) can be seen as a weak formulation of the problem, a natural question is whether it is equivalent to the strong formulation, i.e. by considering only the stopping time  $\tau \in \mathcal{T}$ , where  $\mathcal{T}$  denotes the class of all  $\mathbb{T}$ -valued  $\mathbb{F}$ -stopping time in  $\Omega$ . Let us consider the following “strong” formulation:

$$\sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_e} \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi_\tau(X)] \text{ and } \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}(\mu)} \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi_\tau(X)].$$

The following example shows that under marginal constraint, the weak formulation and strong formulation are not equivalent.

**Example 2.9.** *Let us consider the case  $N = 2$ ,  $\mathbb{T}_0 = \mathbb{T} = \{1, 2\}$ ,  $\mu_1 = \delta_{\{0\}}$  and  $\mu_2 = \frac{1}{4}(\delta_{\{-2\}} + \delta_{\{-1\}} + \delta_{\{1\}} + \delta_{\{2\}})$ . Let  $\Phi_1(0) = 1$ ,  $\Phi_2(\pm 1) = 0$  and  $\Phi_2(\pm 2) = 2$ . Then  $\mathcal{M}(\mu)$  contains only one probability measure, and by direct computation, one has*

$$\mathbb{E}^{\mathbb{P}}[\Phi_\tau(X)] = 1, \text{ for all } \tau \in \mathcal{T}.$$

Let us now construct a martingale measure  $\overline{\mathbb{Q}}_0$  by

$$\overline{\mathbb{Q}}_0(d\omega, d\theta) := \frac{1}{4}\delta_{\{1\}}(d\theta) \otimes (\delta_{(0,1)} + \delta_{(0,-1)})(d\omega) + \frac{1}{4}\delta_{\{2\}}(d\theta) \otimes (\delta_{(0,2)} + \delta_{(0,-2)})(d\omega).$$

Then one can check that  $\overline{\mathbb{Q}}_0 \in \overline{\mathcal{M}}(\mu)$  and it follows that

$$\sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}(\mu)} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi_T(X)] \geq \mathbb{E}^{\overline{\mathbb{Q}}_0}[\Phi_T(X)] = \frac{3}{2} > 1 = \sup_{\mathbb{Q} \in \mathcal{M}(\mu)} \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}}[\Phi_\tau(X)].$$

The next natural question is whether one can obtain the equivalence under certain conditions. We provide below two cases where the answer is affirmative. The first is the case without marginal constraints (i.e.  $e = 0$ ), and the second case is to add, in the weak formulation, an immersion condition as introduced in Blanchet-Scalliet, Jeanblanc & Romero [8, Section 3.1.2]. Let us define

$$\overline{\mathcal{M}}_e^0 := \{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_e : \overline{\mathbb{Q}}[T > k | \overline{\mathcal{F}}_n^X] = \overline{\mathbb{Q}}[T > k | \overline{\mathcal{F}}_k^X], \text{ for any } 0 \leq k \leq n \leq N\}.$$

Notice that (see Section 3.1.2 of [8]) the condition

$$\overline{\mathbb{Q}}[T > k | \overline{\mathcal{F}}_n^X] = \overline{\mathbb{Q}}[T > k | \overline{\mathcal{F}}_k^X], \text{ for all } 0 \leq k \leq n \leq N \quad (2.11)$$

is equivalent to say that any  $(\overline{\mathbb{F}}^X, \overline{\mathbb{Q}})$ -martingale is a  $(\overline{\mathbb{F}}, \overline{\mathbb{Q}})$ -martingale, where  $\overline{\mathbb{F}}^X$  denote the natural filtration generated by  $X$  in  $\overline{\Omega}$ .

**Proposition 2.10.** *Let  $\Phi : \overline{\Omega} \rightarrow \mathbb{R}$  be universally measurable, then*

$$\sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_e^0} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi_T(X)] = \sup_{\mathbb{Q} \in \mathcal{M}_e} \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}}[\Phi_\tau(X)]. \quad (2.12)$$

*Suppose that  $\Phi_k$  is upper semianalytic for each  $k \in \mathbb{T}$  and  $e = 0$ , then*

$$\sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_0} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi_T(X)] = \sup_{\mathbb{P} \in \mathcal{M}_0} \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{P}}[\Phi_\tau(X)]. \quad (2.13)$$

The proof will be completed in Section 3.2.

**Comparison with Neuberger [27], and Hobson & Neuberger [24]**  
 Neuberger [27] and Hobson & Neuberger [24] studied the same superhedging problem in a Markovian setting, where the underlying process  $S$  takes value in a discrete lattice  $\mathcal{X}$ . By considering the weak formulation (which is equivalent to our formulation, as shown by Proposition 2.8 above), they obtain similar duality results. Moreover, in this Markovian discrete space context, the optimization problem (2.6) and the dual problem turn to be the linear programming problems under linear constraints, which can be solved numerically. Their arguments have also been extended to a more general context, where  $S$  takes value in  $\mathbb{R}^+$ .

Comparing to [27, 24], our idea to consider the “weak” stopping terms turns to be essentially the same. But our arguments to prove the duality are completely different. Our setting is more general, as a heritage from the context of Bouchard and Nutz [10]. Nevertheless, we do not discuss on the numerical computation of the value  $\pi(\Phi)$  nor that of the superhedging strategies.

**Comparison with Bayraktar, Huang & Zhou [2] and Bayraktar & Zhou [3]** In [2], the authors considered the same superhedging problem (2.4), and established the duality

$$\pi(\Phi) = \inf_{h \in \mathbb{R}^e} \sup_{\tau \in \mathcal{T}} \sup_{\mathbb{Q} \in \mathcal{M}_0} \mathbb{E}^{\mathbb{Q}}[\Phi_{\tau} - hg], \quad (2.14)$$

under some regularity conditions (see Proposition 3.1 in [2]). Our duality in Theorem 2.6 is more general and more complete, and moreover, together with Proposition 2.10, it induces the above duality (2.14). Moreover, we do not assume that  $\Phi_k$  is  $\mathcal{F}_k$ -measurable, which permits to study the superhedging problem for a portfolio containing an American option and some European options. Another subhedging problem  $\sup_{\tau \in \mathcal{T}} \inf_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\Phi_{\tau}]$  has also been studied in [2]. The corresponding weak formulation is not clear for our techniques and we will not address to this problem.

More recently, Bayraktar and Zhou [3] consider the “randomized” stopping times, and obtain a more complete duality for problem (2.4). The dual formulations in [3] and in our results are more or less in the same spirit (as in [27, 24]). Nevertheless, the duality in [3] is established under some integrability conditions and an abstract condition which is checked under regularity conditions (see their Assumption 2.1 and Remark 2.1). Technically, they use the duality in [10] together with a minimax theorem to prove their results. Our main idea is to introduce an enlarged canonical space (together with an enlarged canonical filtration), to reformulate it as a superhedging problem for European options. Then by adapting the arguments in [10], we establish our duality under general conditions as in [10].

### 3 Proofs

In preparation of the proofs of Theorem 2.6, let us introduce

$$\begin{aligned} \overline{\mathcal{M}}_e^{loc} := \{ \overline{\mathbb{Q}} : \overline{\mathbb{Q}} \lll \mathcal{P}, \mathbb{E}^{\overline{\mathbb{Q}}}[g^i(X)] = 0, i = 1, \dots, e \\ \text{and } S \text{ is a } (\overline{\mathbb{F}}^0, \overline{\mathbb{Q}})\text{-local martingale} \}. \end{aligned}$$

Then by Proposition 2.8 and Lemma A.1 in Appendix, one can easily obtain the weak duality:

$$\sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_e} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi_T(X)] \leq \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_e^{loc}} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi_T(X)] \leq \pi_e(\Phi) \quad \text{and} \quad P(\mu) \leq D(\mu). \quad (3.15)$$

#### 3.1 Proof of Theorem 2.6 (the case $e = 0$ )

To prepare the proof, let us introduce another filtration  $\overline{\mathbb{F}}^- = (\overline{\mathcal{F}}_k^-)_{0 \leq k \leq N}$  on  $\overline{\Omega}$  by

$$\overline{\mathcal{F}}_0^- := \{\emptyset, \overline{\Omega}\} \quad \text{and} \quad \overline{\mathcal{F}}_k^- := \sigma\{T \wedge k, X_i, i \leq k\}.$$

Notice that  $\overline{\mathcal{F}}_k^-$  contains all sets  $\{T \leq i\}$  for  $i = 1, \dots, k$ , but  $\overline{\mathcal{F}}_k^-$  contains only these sets for  $i = 1, \dots, k - 1$ . Therefore one has, in general,  $\overline{\mathcal{F}}_k^- \subseteq \overline{\mathcal{F}}_k^0$  and  $T$  is not a  $\overline{\mathbb{F}}^-$ -stopping time. We also define a restricted enlarged space, for every  $k = 1, \dots, N$ ,

$$\overline{\Omega}_k := \Omega_k \times \{1, \dots, k\} = (\Omega_1)^k \times \{1, \dots, k\}.$$

For each  $1 \leq i \leq j \leq N$ , we introduce an application from  $\Omega_j$  to  $\Omega_i$  (resp.  $\bar{\Omega}_j$  to  $\bar{\Omega}_i$ ) by

$$[\omega]_i := (\omega_1, \dots, \omega_i), \quad [\bar{\omega}]_i := ([\omega]_i, \theta \wedge i), \quad \text{for all } \bar{\omega} = (\omega_1, \dots, \omega_j, \theta) \in \bar{\Omega}_j.$$

Then an  $\bar{\mathcal{F}}_k^-$  is the smallest  $\sigma$ -field on  $\bar{\Omega}$  generated by  $[\cdot]_k : \bar{\Omega} \rightarrow \bar{\Omega}_k$ ; or equivalently, a  $\bar{\mathcal{F}}_k^-$ -measurable random variable  $f$  defined on  $\bar{\Omega}$  can be identified as a Borel measurable function on  $\bar{\Omega}_k$ . The canonical processes  $X$  and  $S$  are naturally defined on the restricted spaces  $\Omega_k$  and  $\bar{\Omega}_k$ .

Recall that for  $\omega \in \Omega_k$ ,  $\mathcal{M}_k(\omega)$  denotes the collection of all martingale measures  $\mathbb{Q}$  on  $\Omega_1$  such that  $\mathbb{Q} \ll \mathbb{P}$  for some  $\mathbb{P} \in \mathcal{P}_k(\omega)$ . Then for an upper semianalytic function  $f : \bar{\Omega}_{k+1} \rightarrow \bar{\mathbb{R}}$ , we define for each  $\bar{\omega} = (\omega, \theta) \in \bar{\Omega}_k$ ,  $\bar{\mathcal{E}}_k(f) : \bar{\Omega}_k \rightarrow \bar{\mathbb{R}}$ ,

$$\bar{\mathcal{E}}_k(f)(\bar{\omega}) := \sup_{\mathbb{Q} \in \mathcal{M}_k(\omega)} \left( \mathbb{E}^{\mathbb{Q}}[f(\omega, \cdot, \theta)] \mathbf{1}_{\{\theta < k\}} + \mathbb{E}^{\mathbb{Q}}[f(\omega, \cdot, k)] \vee \mathbb{E}^{\mathbb{Q}}[f(\omega, \cdot, k+1)] \mathbf{1}_{\{\theta = k\}} \right).$$

**Lemma 3.1.** *Let  $f : \bar{\Omega}_{k+1} \rightarrow \bar{\mathbb{R}}$  be upper semianalytic, then  $\bar{\mathcal{E}}_k(f) : \bar{\Omega}_k \rightarrow \bar{\mathbb{R}}$  is still upper semianalytic. Moreover, there exist two universally measurable functions  $(y_1, y_2) : \bar{\Omega}_k \rightarrow \mathbb{R}^d \times \mathbb{R}^d$  such that*

$$\begin{aligned} \bar{\mathcal{E}}_k(f)(\bar{\omega}) + y_1(\bar{\omega}) \Delta S_{k+1}(\omega, \cdot, \theta) &\geq f(\omega, \cdot, \theta), \\ \text{and } \bar{\mathcal{E}}_k(f)(\bar{\omega}) + y_2(\bar{\omega}) \Delta S_{k+1}(\omega, \cdot, \theta) &\geq f(\omega, \cdot, k+1), \quad \mathcal{P}_k(\omega)\text{-q.s.} \end{aligned}$$

for all  $\bar{\omega} = (\omega, \theta) \in \bar{\Omega}_k$  such that  $NA(\mathcal{P}_k(\omega))$  holds and  $f(\bar{\omega}, \cdot) > -\infty$ ,  $\mathcal{P}_k(\omega)$ -q.s.

*Proof.* Notice that  $f_1 \vee f_2$  is upper semianalytic whenever  $f_1$  and  $f_2$  are both upper semianalytic. Then the above lemma is an immediate consequence of Lemma 4.10 of [10] by the definition of  $\bar{\mathcal{E}}_k$ .  $\square$

Recall that  $\bar{\mathcal{M}}_0$  (resp.  $\bar{\mathcal{M}}_0^{loc}$ ) means  $\bar{\mathcal{M}}_e$  (resp.  $\bar{\mathcal{M}}_e^{loc}$ ) for the case  $e = 0$ . We next provide a dynamic programming principle result.

**Lemma 3.2.** *Let  $\Phi : \bar{\Omega} \rightarrow \mathbb{R}$  be upper semianalytic and bounded from above, then*

$$\sup_{\mathbb{Q} \in \bar{\mathcal{M}}_0} \mathbb{E}^{\mathbb{Q}}[\Phi_T(X)] = \bar{\mathcal{E}}[\Phi] := \bar{\mathcal{E}}_0 \circ \dots \circ \bar{\mathcal{E}}_{N-1}[\Phi],$$

*Proof.* (i) First, let  $\bar{\mathbb{Q}}$  be an arbitrary martingale measure in  $\bar{\mathcal{M}}_0$ , and  $(\bar{\mathbb{Q}}_{\bar{\omega}}^{N-1})_{\bar{\omega} \in \bar{\Omega}}$  be a family of conditional probability measures w.r.t.  $\bar{\mathcal{F}}_{N-1}^0$ , then for  $\bar{\mathbb{Q}}$ -a.e.  $\bar{\omega} \in \bar{\Omega}$ , one has  $\bar{\mathbb{Q}}_{\bar{\omega}}^{N-1} \circ S_N^{-1} \in \mathcal{M}([\omega]_{N-1})$ . Notice also  $\bar{\mathcal{F}}_{N-1}^- \subset \bar{\mathcal{F}}_{N-1}^0$ , it follows immediately that

$$\mathbb{E}^{\bar{\mathbb{Q}}}[\Phi_T(X) | \bar{\mathcal{F}}_{N-1}^-] \leq \bar{\mathcal{E}}_{N-1}[\Phi], \quad \bar{\mathbb{Q}}\text{-a.s.}$$

Next, considering conditional probability measures w.r.t.  $\bar{\mathcal{F}}_{N-2}^0$  and then  $\bar{\mathcal{F}}_{N-2}^-$ , one obtains

$$\mathbb{E}^{\bar{\mathbb{Q}}}[\Phi_T(X) | \bar{\mathcal{F}}_{N-2}^-] \leq \bar{\mathcal{E}}_{N-2} \circ \bar{\mathcal{E}}_{N-1}[\Phi], \quad \bar{\mathbb{Q}}\text{-a.s.}$$

Repeating the procedure, we then obtain that, for any  $\bar{\mathbb{Q}} \in \bar{\mathcal{M}}_0$ ,  $\mathbb{E}^{\bar{\mathbb{Q}}}[\Phi_T(X)] \leq \bar{\mathcal{E}}[\Phi]$ .

(ii) To prove the inverse inequality “ $\geq$ ”, we first consider, for a fixed  $\bar{\omega} = (\omega, \theta) \in \bar{\Omega}_k$ , two optimization problems:

$$V_1(\bar{\omega}) := \sup_{\mathbb{Q} \in \mathcal{M}(\omega)} \mathbb{E}^{\mathbb{Q}}[\bar{\mathcal{E}}^{k+1}(\Phi)(\omega, \cdot, \theta)] \quad \text{and} \quad V_2(\bar{\omega}) := \sup_{\mathbb{Q} \in \mathcal{M}(\omega)} \mathbb{E}^{\mathbb{Q}}[\bar{\mathcal{E}}^{k+1}(\Phi)(\omega, \cdot, k+1)].$$

Notice that  $\{(\omega, \mathbb{Q}) : \mathbb{Q} \in \mathcal{M}_k(\omega), \omega \in \Omega_k\}$  is analytic, then following exactly the same arguments as in Lemma 4.13 of [10], one can choose a universally measurable selector  $(\overline{\mathbb{Q}}_k^\varepsilon = (\mathbb{Q}_k^{1,\varepsilon}(\bar{\omega}), \mathbb{Q}_k^{2,\varepsilon}(\bar{\omega})))_{\bar{\omega} \in \overline{\Omega}_k}$  such that

$$\mathbb{E}^{\mathbb{Q}_k^{1,\varepsilon}(\bar{\omega})}[\overline{\mathcal{E}}^{k+1}(\Phi)(\omega, \cdot, \theta)] \geq V_1(\bar{\omega}) - \varepsilon \quad \text{and} \quad \mathbb{E}^{\mathbb{Q}_k^{2,\varepsilon}(\bar{\omega})}[\overline{\mathcal{E}}^{k+1}(\Phi)(\omega, \cdot, k+1)] \geq V_2(\bar{\omega}) - \varepsilon.$$

Given a probability  $\overline{\mathbb{P}}$  on  $(\overline{\Omega}_k, \mathcal{B}(\overline{\Omega}_k))$ , we can define a concatenated probability measures  $\overline{\mathbb{P}} \otimes \overline{\mathbb{Q}}_k^\varepsilon$  on  $(\overline{\Omega}_{k+1}, \mathcal{B}(\overline{\Omega}_{k+1}))$  by

$$\begin{aligned} \mathbb{E}^{\overline{\mathbb{P}} \otimes \overline{\mathbb{Q}}_k^\varepsilon}[f] := & \int_{\overline{\Omega}} \left[ \mathbb{E}^{\mathbb{Q}_k^{1,\varepsilon}(\bar{\omega})}[f(\omega, \cdot, \theta)] \mathbf{1}_{\{\theta < k\}} + \mathbb{E}^{\mathbb{Q}_k^{1,\varepsilon}(\bar{\omega})}[f(\omega, \cdot, \theta)] \mathbf{1}_{\{\theta = k, V_1(\bar{\omega}) \geq V_2(\bar{\omega})\}} \right. \\ & \left. + \mathbb{E}^{\mathbb{Q}_k^{2,\varepsilon}(\bar{\omega})}[f(\omega, \cdot, k+1)] \mathbf{1}_{\{\theta = k, V_1(\bar{\omega}) < V_2(\bar{\omega})\}} \right] d\overline{\mathbb{P}}(\bar{\omega}). \end{aligned}$$

Let  $\overline{\mathbb{Q}}^\varepsilon := \overline{\mathbb{Q}}_0^\varepsilon \otimes \cdots \otimes \overline{\mathbb{Q}}_{N-1}^\varepsilon$ , under which  $S$  is a generalized martingale (and equivalently a local martingale) w.r.t.  $\overline{\mathbb{F}}$ . In particular, one has

$$\overline{\mathcal{E}}(\Phi) \leq \mathbb{E}^{\overline{\mathbb{Q}}^\varepsilon}[\Phi] + N\varepsilon \leq \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_0^{loc}} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi] + N\varepsilon.$$

Finally, notice that  $\Phi$  is bounded from above, then by Lemma A.2 in Appendix as well as the arbitrariness of  $\varepsilon > 0$ , we conclude the proof.  $\square$

**Proof of Theorem 2.6 (the case  $e = 0$ ).** First, one has the weak duality as in (3.15)

$$\sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_e} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi_T(X)] \leq \pi_e(\Phi).$$

Next, for the inverse inequality, we can assume, with loss of generality, that  $\Phi$  is bounded from above. Indeed, by Lemma A.3, one has  $\lim_{n \rightarrow \infty} \pi_e(\Phi \wedge n) = \pi_e(\Phi)$  (see also the proof of Theorem 3.4 of [10]). Besides, the other approximation  $\lim_{n \rightarrow \infty} \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_e} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi_T(X) \wedge n] = \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_e} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi_T(X)]$  is an easy consequence of the monotone convergence theorem.

When  $\Phi$  is bounded from, by Lemma 3.2, it is enough to prove that there is some  $\overline{H} \in \overline{\mathcal{H}}$  such that

$$\overline{\mathcal{E}}[\Phi] + (\overline{H} \cdot S)_N^k \geq \Phi_k, \quad \text{for each } k = 1, \dots, N, \quad \mathcal{P}\text{-q.s.} \quad (3.16)$$

Define  $\overline{\mathcal{E}}^k(\Phi)(\bar{\omega}) := (\overline{\mathcal{E}}_k \circ \cdots \circ \overline{\mathcal{E}}_{N-1})(\Phi)(\bar{\omega})$ , then by Lemma 3.1, there exist two universally measurable functions  $(y_1^k, y_2^k) : \overline{\Omega}_k \rightarrow \mathbb{R}^d \times \mathbb{R}^d$  such that

$$\begin{aligned} y_1^k(\bar{\omega}) \Delta S_{k+1}(\omega, \cdot, \theta) & \geq \overline{\mathcal{E}}^{k+1}(\Phi)(\omega, \cdot, \theta) - \overline{\mathcal{E}}^k(\Phi)(\bar{\omega}), \\ \text{and } y_2^k(\bar{\omega}) \Delta S_{k+1}(\omega, \cdot, \theta) & \geq \overline{\mathcal{E}}^{k+1}(\Phi)(\omega, \cdot, k+1) - \overline{\mathcal{E}}^k(\Phi)(\bar{\omega}), \quad \mathcal{P}_k(\omega)\text{-q.s.} \end{aligned}$$

It follows that

$$\sum_{k=0}^{N-1} \overline{H}_{k+1} \Delta S_{k+1} \geq \sum_{k=0}^{N-1} (\overline{\mathcal{E}}^{k+1}(\Phi) - \overline{\mathcal{E}}^k(\Phi)) = \Phi - \overline{\mathcal{E}}(\Phi), \quad \mathcal{P}\text{-q.s.}$$

with  $\overline{H}_{k+1}(\bar{\omega}) := y_1^k([\bar{\omega}]_k) \mathbf{1}_{\{\theta \leq k\}} + y_2^k([\bar{\omega}]_k) \mathbf{1}_{\{\theta > k\}}$ .

Moreover, we inherit from [10] the existence of the optimal dual strategies.  $\square$

### 3.2 Proof of Proposition 2.10

(i) We follow the proof in Proposition 4.3 of El Karoui and Tan [17]. Let  $\bar{\mathbb{Q}} \in \bar{\mathcal{M}}_e^0$ , we denote by  $\mathbb{Q}$  its marginal distribution on  $\Omega$ . By the convexity of  $\mathcal{P}_k(\omega)$ , one has the convexity of  $\mathcal{M}_k(\omega)$ , and hence one has  $\mathbb{Q} \in \mathcal{M}_e$ . Let  $\bar{\mathcal{F}}_N^X := \sigma(X_1, \dots, X_N)$  be the  $\sigma$ -field generated by  $X_1, \dots, X_N$ , and  $(\bar{\mathbb{Q}}_\omega)_{\omega \in \Omega}$  be a family of regular conditional probability distribution (r.c.p.d.) of  $\bar{\mathbb{Q}}$  w.r.t.  $\bar{\mathcal{F}}_N^X$ . Denote then by  $F_\omega(t) := \bar{\mathbb{Q}}_\omega[T \leq t]$  the distribution function of  $T$  under  $\bar{\mathbb{Q}}_\omega$ , and by  $F_\omega^{-1}(u)$  the right-continuous generalized inverse function of  $t \mapsto F_\omega(t)$ . Then for fixed  $u \in [0, 1]$ ,  $\omega \mapsto F_\omega^{-1}(u)$  is an  $\mathbb{F}$ -stopping time under the immersion condition (2.11). Let  $U$  be an independent random variable of uniform distribution on  $[0, 1]$ , it follows that

$$\begin{aligned} \mathbb{E}^{\bar{\mathbb{Q}}}[\Phi_T(X)] &= \int_{\Omega} \mathbb{E}^{\bar{\mathbb{Q}}_\omega}[\Phi_T(\omega)] \mathbb{Q}(d\omega) = \int_{\Omega} \mathbb{E}[\Phi_{F_\omega^{-1}(U)}(\omega)] \mathbb{Q}(d\omega) \\ &= \int_0^1 \mathbb{E}^{\mathbb{Q}}[\Phi_{F_X^{-1}(u)}(X)] du \leq \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}}[\Phi_\tau(X)], \end{aligned}$$

where the last inequality follows from the fact that  $F_\omega^{-1}(u)$  is a  $\mathbb{F}$ -stopping time for every  $u \in [0, 1]$ . We hence obtain (2.12), since the inverse inequality is trivial.

(ii) To prove (2.13), it is enough to use the dynamic programming. First, by considering the law  $\bar{\mathbb{Q}}$  on  $\bar{\Omega}$  induced by  $(X, S, \tau)$  under some  $\mathbb{Q} \in \mathcal{M}_0$ , it is clear that one has inequality  $\sup_{\mathbb{Q} \in \mathcal{M}_0} \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}}[\Phi_\tau(X)] \leq \sup_{\bar{\mathbb{Q}} \in \bar{\mathcal{M}}_0} \mathbb{E}^{\bar{\mathbb{Q}}}[\Phi_T(X)]$ . Next, let us define a  $\mathbb{F}$ -stopping time by

$$\hat{\tau}(\omega) := \inf \left\{ k \geq 1 : \bar{\mathcal{E}}^k[\Phi] \geq \sup_{\mathbb{Q} \in \mathcal{M}_k(\omega)} \mathbb{E}^{\mathbb{Q}}[\bar{\mathcal{E}}^{k+1}[\Phi]] \right\}.$$

Then  $\hat{\tau}$  satisfies that  $\sup_{\mathbb{Q} \in \mathcal{M}_0} \mathbb{E}^{\mathbb{Q}}[\Phi_{\hat{\tau}}(X)] \leq \sup_{\bar{\mathbb{Q}} \in \bar{\mathcal{M}}_0} \mathbb{E}^{\bar{\mathbb{Q}}}[\Phi_T(X)]$ .  $\square$

### 3.3 Proof of Theorem 2.6 (the case $e \geq 1$ )

We will adapt the arguments in Section 5 of Bouchard & Nutz [10] to prove Theorem 2.6 in the context with finitely many options  $e \geq 1$ .

For technical reasons, we introduce

$$\varphi(T, X) := 1 + |g^1(X)| + \dots + |g^e(X)| + |\Phi_T(X)|,$$

and

$$\bar{\mathcal{M}}_e^\varphi := \{\bar{\mathbb{Q}} \in \bar{\mathcal{M}}_0 : \mathbb{E}^{\bar{\mathbb{Q}}}[\varphi] < \infty \text{ and } \mathbb{E}^{\bar{\mathbb{Q}}}[g^j] = 0 \text{ for } j = 1, \dots, e\}. \quad (3.17)$$

Moreover, in view of Lemma A.2, one has

$$\sup_{\bar{\mathbb{Q}} \in \bar{\mathcal{M}}_e} \mathbb{E}^{\bar{\mathbb{Q}}}[\Phi_T(X)] = \sup_{\bar{\mathbb{Q}} \in \bar{\mathcal{M}}_e^\varphi} \mathbb{E}^{\bar{\mathbb{Q}}}[\Phi_T(X)].$$

**Proof of Theorem 2.6 (the case  $e \geq 1$ ).** The existence of some  $\bar{\mathbb{Q}} \in \bar{\mathcal{M}}_e$  is an easy consequence of Theorem 5.1 of [10] under the NA( $\mathcal{P}$ ) assumption, we will then focus on the duality results.

First, the duality  $\pi_e(\Phi) = \sup_{\bar{\mathbb{Q}} \in \bar{\mathcal{M}}_e} \mathbb{E}^{\bar{\mathbb{Q}}}[\Phi_T(X)]$  in (2.7) has already been proved for the case  $e = 0$ , we will use the induction arguments: Suppose that the duality

(2.7) holds true for the case with  $e \geq 0$ , We aim to prove the duality with  $e + 1$  options:

$$\pi_{e+1}(\Phi) = \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_e^\varphi} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi_T(X)],$$

where the addition option has a Borel-measurable payoff function  $f \equiv g^{e+1}$  such that  $|f| \leq \varphi$ , and has an initial price  $f_0 = 0$ . By the weak duality, the “ $\geq$ ” side of the inequality holds true, we will focus on the “ $\leq$ ” side of the inequality:

$$\pi_{e+1}(\Phi) \leq \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_e^\varphi} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi_T(X)]. \quad (3.18)$$

If  $f(X)$  is replicable using strategies  $\overline{H} \in \overline{\mathcal{H}}$  and  $(g^1, \dots, g^e)$ , we can then reduce it to the case with  $e$  options and the result is trivial. Let us assume that  $f$  is not replicable, and we claim that there is a sequence  $(\overline{\mathbb{Q}}_n)_{n \geq 1} \subset \overline{\mathcal{M}}_e^\varphi$  such that

$$\mathbb{E}^{\overline{\mathbb{Q}}_n}[f(X)] \rightarrow f_0 \quad \text{and} \quad \mathbb{E}^{\overline{\mathbb{Q}}_n}[\Phi_T(X)] \rightarrow \pi_{e+1}(\Phi), \quad \text{as } n \rightarrow \infty. \quad (3.19)$$

Next, denote by  $\pi_e^s(f)$  the minimum superhedging cost of European option  $f$  in sense of [10], then since  $f$  is not replicable, it follows that  $0 = f_0 < \pi_e^s(f) = \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_e} \mathbb{E}^{\overline{\mathbb{Q}}}[f(X)]$ . Then there exists some  $\overline{\mathbb{Q}}_+ \in \overline{\mathcal{M}}_e$ , s.t.  $0 < \mathbb{E}^{\overline{\mathbb{Q}}_+}[f(X)] < \pi_e^s(f)$ . With the same argument on  $-f$ , we can find another  $\overline{\mathbb{Q}}_- \in \overline{\mathcal{M}}_e$  such that

$$-\pi^s(-f) < \mathbb{E}^{\overline{\mathbb{Q}}_-}[f(X)] < f_0 < \mathbb{E}^{\overline{\mathbb{Q}}_+}[f(X)] < \pi^s(f)$$

Then one can choose an appropriate sequence of weight  $\lambda_-^n, \lambda_0^n, \lambda_+^n$ , such that  $\lambda_-^n + \lambda_0^n + \lambda_+^n = 1$ ,  $\lambda_\pm^n \rightarrow 0$  and

$$\overline{\mathbb{Q}}'_n := \lambda_-^n \overline{\mathbb{Q}}_- + \lambda_0^n \overline{\mathbb{Q}}_n + \lambda_+^n \overline{\mathbb{Q}}_+ \in \overline{\mathcal{M}}_e, \quad \text{and} \quad \mathbb{E}^{\overline{\mathbb{Q}}'_n}[f] = f_0 = 0,$$

i.e.  $\overline{\mathbb{Q}}'_n \in \overline{\mathcal{M}}_{e+1}^\varphi$ . Moreover, since  $\lambda_\pm^n \rightarrow 0$ , it follows that  $\mathbb{E}^{\overline{\mathbb{Q}}'_n}[\Phi_T(X)] \rightarrow \pi_{e+1}(\Phi)$  and we hence have the inequality (3.18).

It is enough to prove the claim (3.19), for which we suppose without loss of generality that  $\pi_{e+1}(\Phi) = 0$ . Assume that (3.19) fails, then one has

$$0 \notin \overline{\{\mathbb{E}^{\overline{\mathbb{Q}}}[(f(X), \Phi_T(X))] : \overline{\mathbb{Q}} \in \overline{\mathcal{M}}_e^\varphi\}} \subseteq \mathbb{R}^2.$$

By the convexity of the above set and the separation argument, there is  $(y, z) \in \mathbb{R}^2$  with  $|(y, z)| = 1$ , such that

$$0 > \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_e^\varphi} \mathbb{E}^{\overline{\mathbb{Q}}}[yf(X) + z\Phi_T(X)] = \pi_e(yf + z\Phi) \geq \pi_{e+1}(z\Phi). \quad (3.20)$$

Now, if  $z > 0$ , we then have  $\pi_{e+1}(\Phi) < 0$ , which contradicts  $\pi_{e+1}(\Phi) = 0$ . If  $z < 0$ , then by (3.20), one has  $0 > \mathbb{E}^{\overline{\mathbb{Q}}}[yf(X) + z\Phi_T(X)] = \mathbb{E}^{\overline{\mathbb{Q}}}[z\Phi_T(X)]$  for some  $\overline{\mathbb{Q}}' \in \overline{\mathcal{M}}_{e+1} \subseteq \overline{\mathcal{M}}_e$  since  $\overline{\mathcal{M}}_{e+1}$  is nonempty under the NA( $\mathcal{P}$ ) assumption in the case of  $e + 1$  options. As  $z < 0$ , it follows that  $\mathbb{E}^{\overline{\mathbb{Q}}'}[\Phi_T(X)] > 0 = \pi_{e+1}(\Phi)$ , which contradicts the weak duality result (3.15), and we hence conclude the proof.  $\square$



### 3.4 Proof of Theorem 2.7

A first approach to prove Theorem 2.7 could be following the two steps arguments as in Guo, Tan and Touzi [21]. First, under the condition that  $\Phi_k$  is bounded from above and upper semicontinuous, we can prove that  $\mu \in \mathfrak{B}((\mathbb{R}^d)^{N_0}) \mapsto P(\mu) \in \mathbb{R}$  is concave and upper semicontinuous, where we equip  $\mathfrak{B}((\mathbb{R}^d)^{N_0})$  with the Wasserstein topology. Recall that  $S = X$ , then using Fenchel-Moreau theorem, it follows that

$$P(\mu) = D_0(\mu) := \inf_{\lambda \in \Lambda} \left\{ \mu(\lambda) + \sup_{\mathbb{Q} \in \overline{\mathcal{M}}_0} \mathbb{E}^{\mathbb{Q}}[\Phi_T(S) - \lambda(S)] \right\}. \quad (3.21)$$

Then by solving the maximization problem in (3.21) using Theorem 2.6, we can easily conclude the proof of Theorem 2.7.

In the following, we will provide another proof, which is based on an approximate argument. For simplicity, we suppose that  $\mathbb{T}_0 = \{N\}$ , where the same arguments work for more general  $\mathbb{T}_0$ . In preparation, let us provide a technical lemma. In the context of the martingale optimal transport problem (2.8), we introduce a sequence of basket options with payoff

$$g_k(\omega) := \left( \sum_{i=1}^d a_k^i S_N^i - K_k \right)_+ - c_k, \quad \text{with } c_k := \int_{\mathbb{R}^d} \left( \sum_{i=1}^d a_k^i x_i - K_k \right)_+ \mu(dx),$$

where  $(a_k)_{k \geq 1} \subset \mathbb{R}^d$  is a sequence of constant vectors and  $(K_k)_{k \geq 1}$  a sequence of constants. We assume that the sequence  $(a_k, K_k)_{k \geq 1}$  is dense in  $\mathbb{R}^d \times \mathbb{R}$ .

Next, let us introduce

$$\overline{\mathcal{M}}_m(\mu) := \left\{ \overline{\mathbb{Q}} \in \overline{\mathcal{M}} : \mathbb{E}^{\overline{\mathbb{Q}}}[g_k(S)] = 0, \text{ for } k = 1, \dots, m \right\},$$

and

$$P_m(\mu) := \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_m(\mu)} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi_T(S)].$$

Similarly, an approximate dual problem is given by

$$D_m(\mu) := \inf \left\{ x : \exists (\overline{H}, h) \in \overline{\mathcal{H}} \times \mathbb{R}^m \text{ s.t. for all } k \in \mathbb{T}, \omega \in \Omega, \right. \\ \left. x + \sum_{k=1}^m h_k g_k(\omega_N) + \overline{H} \cdot S_N^k(\omega) \geq \Phi_k(\omega) \right\}.$$

**Lemma 3.3.** *Let  $(\overline{\mathbb{Q}}_m)_{m \geq 1} \subset \mathcal{M}_0$  be a sequence of martingale measures such that  $\overline{\mathbb{Q}}_m \in \overline{\mathcal{M}}_m(\mu)$  for each  $m \geq 1$ . Then,*

- (i)  $(\overline{\mathbb{Q}}_m)_{m \geq 1}$  is relatively compact under the weak convergence topology.
- (ii) Moreover, the sequence  $(S_N, \overline{\mathbb{Q}}_m)_{m \geq 1}$  is uniformly integrable, and any accumulation point of  $(\overline{\mathbb{Q}}_m)_{m \geq 1}$  belongs to  $\mathcal{M}_0(\mu)$ .

*Proof.* (i) We notice that one has  $\sup_{m \geq 1} \mathbb{E}^{\overline{\mathbb{Q}}_m} \left[ \sum_{i=1}^d |S_N^i| \right] < \infty$ . Let us first prove the relative compactness of  $(\overline{\mathbb{Q}}_m)_{m \geq 1}$ . By Prokhorov theorem, it is enough to find, for every  $\varepsilon > 0$ , a compact set  $D_\varepsilon \subset \mathbb{R}^d$  such that  $\overline{\mathbb{Q}}_m[S_k \notin D_\varepsilon] \leq \varepsilon$  for all  $k = 1, \dots, N$ . It is then enough to find, for every  $\varepsilon > 0$ , a constant  $K_\varepsilon > 0$  such that  $\overline{\mathbb{Q}}_m[|S_k^i| \geq K_\varepsilon] \leq \varepsilon$  for all  $i = 1, \dots, d$  and  $k = 1, \dots, N$ . Next, by the martingale property, one has  $\mathbb{E}^{\overline{\mathbb{Q}}_m}[|S_k^i|] \leq \mathbb{E}^{\overline{\mathbb{Q}}_m}[|S_N^i|]$ . Then for every  $\varepsilon > 0$ ,

one can choose  $K_\varepsilon > 0$  such that  $\sup_{m \geq 1} \mathbb{E}^{\overline{\mathbb{Q}}_m} [\sum_{i=1}^d |S_N^i|] \leq K_\varepsilon \varepsilon$ . It follows that  $\overline{\mathbb{Q}}_m [|S_k^i| \geq K_\varepsilon] \leq \frac{\mathbb{E}^{\overline{\mathbb{Q}}_m} [|S_k^i|]}{K_\varepsilon} \leq \varepsilon$ , and hence  $(\overline{\mathbb{Q}}_m)_{m \geq 1}$  is relatively compact.

(ii) Let  $\overline{\mathbb{Q}}_0$  be an accumulation point of  $(\overline{\mathbb{Q}}_m)_{m \geq 1}$ . Since the sequence  $(a_k, K_k)_{k \geq 1}$  is supposed to be dense in  $\mathbb{R}^d \times \mathbb{R}$ , it is easy to obtain that  $\overline{\mathbb{Q}}_0 \circ S_N^{-1} = \mu$ .

(iii) To conclude the proof, it is enough to show that the martingale property is preserved for the limiting measure  $\overline{\mathbb{Q}}_0$ . By abstracting a subsequence, we assume that  $\overline{\mathbb{Q}}_m \rightarrow \overline{\mathbb{Q}}_0$  weakly, and we will prove that for all  $1 \leq k_1 < k_2 \leq N$ , for any bounded continuous function  $\varphi : (\mathbb{R}^d)^{k_1} \times \mathbb{T} \rightarrow \mathbb{R}$ , one has

$$\mathbb{E}^{\overline{\mathbb{Q}}_0} [\varphi(S_1, \dots, S_{k_1}, T \wedge (k_1 + 1))(S_{k_2} - S_{k_1})] = 0. \quad (3.22)$$

Let  $K > 0$ , and  $\chi_K : \mathbb{R}^d \rightarrow \mathbb{R}^d$  a continuous function uniformly bounded by  $K$  satisfying  $\chi_K(x) = x$  when  $\|x\| \leq K$ , and  $\chi_K(x) = 0$  when  $\|x\| \geq K + 1$ . Then for every  $m = 0$  or  $m \geq 1$ , one has

$$\begin{aligned} |\mathbb{E}^{\overline{\mathbb{Q}}_m} [\varphi(S, T)(S_{k_2} - S_{k_1})]| &\leq |\mathbb{E}^{\overline{\mathbb{Q}}_m} [\varphi(S, T)(\chi_K(S_{k_2}) - \chi_K(S_{k_1}))]| \\ &\quad + |\varphi|_\infty \mathbb{E}^{\overline{\mathbb{Q}}_m} [|S_{k_2}| \mathbf{1}_{|S_{k_2}| \geq K} + |S_{k_1}| \mathbf{1}_{|S_{k_1}| \geq K}], \end{aligned} \quad (3.23)$$

where we simplify  $\varphi(S_1, \dots, S_{k_1}, T \wedge (k_1 + 1))$  to  $\varphi(S, T)$ .

For every  $\varepsilon > 0$ , by uniform integrability of  $(S_N, \overline{\mathbb{Q}}_m)_{m \geq 1}$ , there is  $K_\varepsilon > 0$  such that

$$|\varphi|_\infty \mathbb{E}^{\overline{\mathbb{Q}}_m} [|S_{k_2}| \mathbf{1}_{|S_{k_2}| \geq K_\varepsilon} + |S_{k_1}| \mathbf{1}_{|S_{k_1}| \geq K_\varepsilon}] \leq \varepsilon, \quad \text{for all } m = 0, 1, \dots \quad (3.24)$$

Moreover, for  $m \geq 1$ ,  $\overline{\mathbb{Q}}_m$  is a martingale measure, then  $\mathbb{E}^{\overline{\mathbb{Q}}_m} [\varphi(S, T)(S_{k_2} - S_{k_1})] = 0$  and hence  $|\mathbb{E}^{\overline{\mathbb{Q}}_m} [\varphi(S, T)(\chi_K(S_{k_2}) - \chi_K(S_{k_1}))]| \leq \varepsilon$ . Then by taking the limit  $m \rightarrow \infty$ , it follows that

$$|\mathbb{E}^{\overline{\mathbb{Q}}_0} [\varphi(S, T)(\chi_K(S_{k_2}) - \chi_K(S_{k_1}))]| \leq \varepsilon.$$

Combining (3.23) and (3.24), and by the arbitrariness of  $\varepsilon > 0$ , it follows that (3.22) holds true and we hence conclude the proof.  $\square$

**Proof of Theorem 2.7.** We notice that by Theorem 2.6,  $P_m(\mu) \equiv D_m(\mu) \geq D(\mu)$ . Let  $(\overline{\mathbb{Q}}_m)_{m \geq 1}$  be a sequence of probability measures such that  $\overline{\mathbb{Q}}_m \in \overline{\mathcal{M}}_m(\mu)$  for each  $m \geq 1$  and

$$\limsup_{m \rightarrow \infty} \mathbb{E}^{\overline{\mathbb{Q}}_m} [\Phi_T(S)] = \limsup_{m \rightarrow \infty} P_m(\mu).$$

It follows by Lemma 3.3 that there is some  $\overline{\mathbb{Q}}_0 \in \overline{\mathcal{M}}(\mu)$  and a subsequence  $\overline{\mathbb{Q}}_{m_k} \rightarrow \overline{\mathbb{Q}}_0$  under the weak convergence topology. It leads the inequality

$$P(\mu) \geq \mathbb{E}^{\overline{\mathbb{Q}}_0} [\Phi_T(S)] \geq \limsup_{m \rightarrow \infty} P_m(\mu) \geq \limsup_{m \rightarrow \infty} D_m(\mu) \geq D(\mu),$$

and we hence conclude the proof by the weak duality (3.15)  $\square$

## A Appendix

**Lemma A.1.** *Let  $\Phi : \overline{\Omega} \rightarrow \mathbb{R}$  and  $g = (g^1, \dots, g^e) : \Omega \rightarrow \mathbb{R}^e$  be Borel measurable,  $\overline{\mathbb{Q}}$  be a probability measure on  $(\overline{\Omega}, \overline{\mathcal{F}}_N^0)$  under which  $S$  is a  $\overline{\mathbb{F}}$ -local martingale and such that  $\mathbb{E}^{\overline{\mathbb{Q}}} [g^i(X)] = 0$  for all  $i = 1, \dots, e$ . Then for any  $x \in \mathbb{R}$  and  $(\overline{H}, h) \in \overline{\mathcal{H}} \times \mathbb{R}^e$  such that  $x + (\overline{H} \cdot S)_N(\overline{\omega}) + hg(\omega) \geq \Phi(\overline{\omega})$ ,  $\overline{\mathbb{Q}}$ -a.s. one has  $\mathbb{E}^{\overline{\mathbb{Q}}} [\Phi] \leq x$ .*

*Proof.* The proof follows by exactly the same arguments as in Lemma A.2 of [10], using the discrete time local martingale characterization in Lemma A.1 of [10].  $\square$

**Lemma A.2.** *Let  $\bar{\mathbb{Q}}$  be a probability measure under which  $S$  is a local martingale, and  $\varphi : \Omega \rightarrow [1, \infty)$  be such that  $|\Phi_k(\omega)| \leq \varphi(\omega)$  for all  $k \in \mathbb{T}$ . Let us denote by  $\bar{\mathcal{M}}_\varphi^{\bar{\mathbb{Q}}}$  the collection of all probability measures  $\bar{\mathbb{Q}}'$  under which  $S$  is a martingale,  $\mathbb{E}^{\bar{\mathbb{Q}}'}[\varphi(X)] < \infty$ . Then  $\mathbb{E}^{\bar{\mathbb{Q}}}[\Phi_T(X)] \leq \sup_{\bar{\mathbb{Q}}' \in \bar{\mathcal{M}}_\varphi^{\bar{\mathbb{Q}}}} \mathbb{E}^{\bar{\mathbb{Q}}'}[\Phi_T(X)]$ .*

*Proof.* First, by Lemma 3.2, there exists a probability  $\bar{\mathbb{P}}_*$  equivalent to  $\bar{\mathbb{Q}}$  on  $(\bar{\Omega}, \bar{\mathcal{F}}_N)$  such that  $\mathbb{E}^{\bar{\mathbb{P}}_*}[\varphi(X)] < \infty$ . On the filtered probability space  $(\bar{\Omega}, \bar{\mathcal{F}}_N, \bar{\mathbb{F}}, \bar{\mathbb{P}}_*)$ , one defines  $\bar{\mathcal{M}}_*^{loc}$  as the collection of all probability measures  $\bar{\mathbb{Q}}'$  dominated by  $\bar{\mathbb{P}}_*$  and such that  $S$  is  $(\bar{\mathbb{F}}, \bar{\mathbb{Q}}')$ -local martingale. Then by the classical arguments for the dominated discrete time market (see e.g. the proof of Lemma A.3 of [10]), one can easily obtain the inequality

$$\mathbb{E}^{\bar{\mathbb{Q}}}[\Phi_T(X)] \leq \sup_{\bar{\mathbb{Q}}' \in \bar{\mathcal{M}}_*^{loc}} \mathbb{E}^{\bar{\mathbb{Q}}'}[\Phi_T(X)] \leq \sup_{\bar{\mathbb{Q}}' \in \bar{\mathcal{M}}_\varphi^{\bar{\mathbb{Q}}}} \mathbb{E}^{\bar{\mathbb{Q}}'}[\Phi_T(X)],$$

which concludes the proof.  $\square$

Using the same arguments as in Theorem 2.2 of [10], we can obtain a closeness result for the set of all payoffs which can be superreplicated from initial capital  $x = 0$ . Let us denote by  $\bar{\mathcal{L}}_+^0$  the set of all random variables defined on  $\bar{\Omega}$ , and define  $\bar{\mathcal{C}} := \{(\bar{H} \cdot S)_N : \bar{H} \in \bar{\mathcal{H}}\} - \bar{\mathcal{L}}_+^0 = \{(\bar{H} \cdot S)_N - \xi : \bar{H} \in \bar{\mathcal{H}}, \xi \in \bar{\mathcal{C}}_0^+\}$ .

**Lemma A.3.** *The set  $\bar{\mathcal{C}}$  is closed in the following sense: Let  $(W^n)_{n \geq 1} \subset \bar{\mathcal{C}}$  and  $W$  be a random variable such that  $W^n \rightarrow W$   $\bar{\mathbb{P}}$ -q.s., then  $W \in \bar{\mathcal{C}}$ .*

*Proof.* One only needs to adapt the proof of Theorem 2.2 of [10], together with the new induction technique in our context as in Section 3.1.  $\square$

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