A Pseudo-Markov Property for Controlled Diffusion Processes

Julien Claisse, Denis Talay, Xiaolu Tan

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Abstract

In this note, we propose two different approaches to rigorously justify a pseudo-Markov property for controlled diffusion processes which is often (explicitly or implicitly) used to prove the dynamic programming principle in the stochastic control literature. The first approach develops a sketch of proof proposed by Fleming and Souganidis [9]. The second approach is based on an enlargement of the original state space and a controlled martingale problem. We clarify some measurability and topological issues raised by these two approaches.

Key words. Stochastic control, martingale problem, dynamic programming principle, pseudo-Markov property.

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1 Introduction

The dynamic programming principle (DPP) is a key step in the mathematical analysis of optimal stochastic control problems. For various formulations and approaches, among hundreds of references, see, e.g., Fleming and Rishel [7], Krylov [13], El Karoui [5], Borkar [1], Fleming and Soner [8], Yong and Zhou [22], and the more recent monographs by Pham [15] or Touzi [21].

The DPP has a very intuitive meaning but its rigorous proof for controlled diffusion processes is a difficult issue in all cases where the set of admissible controls is the set of all progressively measurable processes taking values in a given domain. In particular this occurs in the stochastic analysis of Hamilton-Jacobi-Bellman equations. The highly technical approach to the DPP developed by Krylov in [13] consists in establishing a continuity property of expected cost functions w.r.t. controls (see [13, Cor.3.2.8]) in order to be in a position to only prove the DPP in the simpler case where the controls are piecewise constant. When the controls belong to this

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restricted set, the DPP is obtained by proving that the corresponding controlled diffusions enjoy a pseudo-Markov property which easily results from the classical Markov property enjoyed by uncontrolled diffusions. See [13, Lem.3.2.14].

In their context of stochastic differential games problems, Fleming and Souganidis [9, Lem.1.11] propose to avoid the control approximation procedure by establishing the pseudo-Markov property without restriction on the controls. Although natural, this way to establish the DPP leads to various difficulties. In particular, one needs to restrict the state space to the standard canonical space. Unfortunately the proof of the crucial lemma 1.11 is only sketched.

Many other authors had implicitly or explicitly followed the same way and partially justified the pseudo-Markov property in the case of general admissible controls: see, e.g., Tang and Yong [20, Lem.2.2], Yong and Zhou [22, Lem.4.3.2], Bouchard and Touzi [2] and Nutz [14]. However, to the best of our knowledge, there is no full and precise proof available in the literature except in restricted situations (see, e.g., Kabanov and Klüppelberg [10]).

When completing the elements of proof provided by Fleming and Souganidis and the other above mentioned authors, we found several subtle technical difficulties to overcome. For example, the stochastic integrals and solutions to stochastic differential equations are usually defined on a complete probability space equipped with a filtration satisfying the usual conditions; on the other hand, the completion of the canonical σ–field is undesirable when one needs to use a family of regular conditional probabilities. See more discussion in Section 2.3.

We therefore find it useful to provide a precise formulation and a detailed full justification of the pseudo-Markov property for general controlled diffusion processes, and to clarify measurability and topological issues, particularly the importance of setting stochastic control problems in the canonical space rather to an abstract non Polish probability space if the pseudo-Markov property is used to prove the DPP.

We provide two proofs. The first one develops the arguments sketched by Fleming and Souganidis [9]. The second one is simpler in some aspects but requires some more heavy notions; it is based on an enlargement of the original state space and a suitable controlled martingale problem.

The rest of the paper is organized as follows. In Section 2, we introduce the classical formulation of controlled SDEs. We then precisely state the pseudo-Markov property for controlled diffusions, which is our main result. We discuss its use to establish the DPP. To prepare its two proofs, technical lemmas are proven in Section 3. Finally, in Sections 4 and 5 we provide our two proofs.

Notations. We denote by $\Omega := C(\mathbb{R}^+, \mathbb{R}^d)$ the canonical space of continuous functions from $\mathbb{R}^+$ to $\mathbb{R}^d$, which is a Polish space under the locally uniform convergence metric. B denotes the canonical process and $\mathbb{F} = (\mathcal{F}_s,\ s \geq 0)$ denotes the canonical filtration. The Borel σ-field of Ω is denoted by $\mathcal{F}$ and coincides with $\bigvee_{s\geq 0} \mathcal{F}_s$. We denote by $\mathbb{P}$ the Wiener measure on $(\Omega, \mathcal{F})$ under which the canonical process B is a $\mathbb{F}$-Brownian motion, and by $\mathcal{N}^{\mathbb{P}}$ the collection of all $\mathbb{P}$–negligible sets of $\mathcal{F}$, i.e. all sets A included in some $N \in \mathcal{F}$ satisfying $\mathbb{P}(N) = 0$. We denote by $\mathbb{F}^{\mathbb{P}} = (\mathcal{F}^{\mathbb{P}}_s,\ s \geq 0)$ the $\mathbb{P}$–augmented filtration, that is, $\mathcal{F}^{\mathbb{P}}_s := \mathcal{F}_s \vee \mathcal{N}^{\mathbb{P}}$. We also set $\mathcal{F}^{\mathbb{P}} := \mathcal{F} \vee \mathcal{N}^{\mathbb{P}} = \bigvee_{s\geq 0} \mathcal{F}^{\mathbb{P}}_s$.

Finally, for all $(t, \omega) \in \mathbb{R}^+ \times \Omega$, the stopped path of ω at time t is denoted by $[\omega]_t := (\omega(t \wedge s), s \geq 0)$.

2 A pseudo-Markov property for controlled diffusion processes

2.1 Controlled stochastic differential equations

Let U be a Polish space and $S_{d,d}$ be the collection of all square matrices of order d.

Consider two Borel measurable functions $b : \mathbb{R}^+ \times \Omega \times U \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^+ \times \Omega \times U \to S_{d,d}$ satisfying the following condition: for all $u \in U$, the processes $b(\cdot, \cdot, u)$ and $\sigma(\cdot, \cdot, u)$ are $\mathbb{F}$–progressively measurable or, equivalently, $(t, \mathbf{x}) \mapsto b(t, \mathbf{x}, u)$ and $(t, \mathbf{x}) \mapsto \sigma(t, \mathbf{x}, u)$ are $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}$–measurable and

$$b(t, \mathbf{x}, u) = b(t, [\mathbf{x}]_t, u), \quad \sigma(t, \mathbf{x}, u) = \sigma(t, [\mathbf{x}]_t, u), \quad \forall (t, \mathbf{x}) \in \mathbb{R}^+ \times \Omega$$

(see Proposition 3.4 below). In addition, we suppose that there exists $C > 0$ such that, for all $(t, \mathbf{x}, \mathbf{y}, u) \in \mathbb{R}^+ \times \Omega^2 \times U$,

$$\begin{cases} |b(t, \mathbf{x}, u) - b(t, \mathbf{y}, u)| + \|\sigma(t, \mathbf{x}, u) - \sigma(t, \mathbf{y}, u)\| \le C \sup_{0 \le s \le t} |\mathbf{x}(s) - \mathbf{y}(s)|, \\ \sup_{u \in U} \left(|b(t, \mathbf{x}, u)| + \|\sigma(t, \mathbf{x}, u)\|\right) \le C \left(1 + \sup_{0 \le s \le t} |\mathbf{x}(s)|\right). \end{cases} \tag{1}$$

Denote by $\mathcal{U}$ the collection of all U–valued $\mathbb{F}$–predictable (or progressively measurable, see Proposition 3.4 below) processes. Then, given a control $\nu \in \mathcal{U}$, consider the controlled stochastic differential equation (SDE)

$$dX_s \;=\; b(s, X, \nu_s)ds \;+\; \sigma(s, X, \nu_s)dB_s. \tag{2}$$

As B is still a Brownian motion on $(\Omega, \mathcal{F}^{\mathbb{P}}, \mathbb{P}, \mathbb{F}^{\mathbb{P}})$ (see, e.g., [12, Thm.2.7.9]) we may and do consider (2) on this filtered probability space.

Under Condition (1), for all control $\nu \in \mathcal{U}$ and initial condition $(t, \mathbf{x}) \in \mathbb{R}^+ \times \Omega$, there exists a unique (up to indistinguishability) continuous and $\mathbb{F}^{\mathbb{P}}$–progressively measurable process $X^{t,\mathbf{x},\nu}$ in $(\Omega, \mathcal{F}^{\mathbb{P}}, \mathbb{P})$, such that, for all θ in $\mathbb{R}^+$,

$$X^{t,\mathbf{x},\nu}_\theta \;=\; \mathbf{x}(t \wedge \theta) + \int_t^{t \vee \theta} b(s, X^{t,\mathbf{x},\nu}, \nu_s)ds + \int_t^{t \vee \theta} \sigma(s, X^{t,\mathbf{x},\nu}, \nu_s)dB_s, \qquad \mathbb{P} - \text{a.s.} \tag{3}$$

Our two proofs of a pseudo-Markov property for $X^{t,\mathbf{x},\nu}$ use an identity in law which makes it necessary to reformulate the controlled SDE (2) as a standard SDE.

Given $(t, \nu) \in \mathbb{R}^+ \times \mathcal{U}$, we define $\bar{b}^{t,\nu} : \mathbb{R}^+ \times \Omega^2 \to \mathbb{R}^{2d}$ and $\bar{\sigma}^{t,\nu} : \mathbb{R}^+ \times \Omega^2 \to S_{2d,d}$ as follows : for all $s \in \mathbb{R}^+$, $\bar{\omega} = (\omega, \omega') \in \Omega^2$,

$$\bar{b}^{t,\nu}(s, \bar{\omega}) \;:=\; \begin{pmatrix} 0 \\ b(s, \omega', \nu_s(\omega))\mathbb{I}_{s \ge t} \end{pmatrix}, \quad \bar{\sigma}^{t,\nu}(s, \bar{\omega}) \;:=\; \begin{pmatrix} \mathrm{Id}_d \\ \sigma(s, \omega', \nu_s(\omega))\mathbb{I}_{s \ge t} \end{pmatrix}.$$

Then, given $\mathbf{x}$ in Ω, $Y^{t,\mathbf{x},\nu} := (B, X^{t,\mathbf{x},\nu})$ is the unique (up to indistinguishability) continuous and $\mathbb{F}^{\mathbb{P}}$–progressively measurable process in $(\Omega, \mathcal{F}^{\mathbb{P}}, \mathbb{P})$ such that, for all θ in $\mathbb{R}^+$,

$$Y^{t,\mathbf{x},\nu}_\theta \;=\; \begin{pmatrix} 0 \\ \mathbf{x}(t \wedge \theta) \end{pmatrix} + \int_0^\theta \bar{b}^{t,\nu}(s, Y^{t,\mathbf{x},\nu})ds + \int_0^\theta \bar{\sigma}^{t,\nu}(s, Y^{t,\mathbf{x},\nu})dB_s, \quad \mathbb{P}\text{–a.s.} \tag{4}$$

Define pathwise uniqueness and uniqueness in law for standard SDEs as, respectively, in Rogers and Williams [18, Def.V.9.4] and [18, Def.V.16.3]). In view of the celebrated Yamada and Wanabe's Theorem (see, e.g., [18, Thm.V.17.1]), the former implies the latter and the SDE (4) satisfies uniqueness in law.

Remark 2.1. (i) *The stochastic integrals and the solution to the SDE* (2) *are defined as continuous and adapted to the augmented filtration* $\mathbb{F}^{\mathbb{P}}$.
(ii) *In an abstract probability space equipped with a standard Brownian motion* W, *our formulation is equivalent to choosing controls as predictable processes w.r.t. the filtration generated by* W. *Indeed such a process can always be represented as* $\nu(W.)$ *for some* ν *in* $\mathcal{U}$ *(see Proposition 3.5 below). This fact plays a crucial role in our justification of the pseudo-Markov property for controlled diffusions. In their analyses of stochastic control problems, Krylov [13] or Fleming and Soner [8] do not need to use this property and deal with controls adapted to an arbitrary filtration.*
(iii) *We could have defined controls as* $\mathbb{F}^{\mathbb{P}}$*–predictable processes since any* $\mathbb{F}^{\mathbb{P}}$*–predictable process is indistinguishable from an* $\mathbb{F}$*–predictable process (see Dellacherie and Meyer [4, Thm.IV.78 and Rk.IV.74]). Notice also that the notions of predictable, optional and progressively measurable process w.r.t.* $\mathbb{F}$ *coincide (see Proposition 3.4 below).*

2.2 A pseudo-Markov property for controlled diffusion processes

Before formulating our main result, we introduce the class of the shifted control processes constructed by concatenation of paths: for all ν in $\mathcal{U}$ and (t, w) in $\mathbb{R}^+ \times \Omega$ we set

$$\nu_s^{t,\mathsf{w}}(\omega) := \nu_s(\mathsf{w} \otimes_t \omega), \quad \forall (s,\omega) \in \mathbb{R}^+ \times \Omega,$$

where $\mathsf{w} \otimes_t \omega$ is the function in Ω defined by

$$(\mathsf{w} \otimes_t \omega)(s) \; := \; \begin{cases} \mathsf{w}(s), & \text{if } 0 \le s \le t, \\ \mathsf{w}(t) + \omega(s) - \omega(t), & \text{if } s \ge t. \end{cases}$$

Our main result is the following pseudo-Markov property for controlled diffusion processes. As already mentioned, it is most often considered as classical or obvious; however, to the best of our knowledge, its precise statement and full proof are original.

Theorem 2.2. *Let* $\Phi : \Omega \to \mathbb{R}^+$ *be a bounded Borel measurable function and let* J *be defined as*

$$J\big(t, \mathbf{x}, \nu\big) \; := \; \mathbb{E}^{\mathbb{P}}\big[\Phi\big(X^{t,\mathbf{x},\nu}\big)\big], \quad \forall (t, \mathbf{x}, \nu) \in \mathbb{R}^+ \times \Omega \times \mathcal{U}.$$

Under Condition (1), for all $(t, \mathbf{x}, \nu) \in \mathbb{R}^+ \times \Omega \times \mathcal{U}$ *and* $\mathbb{F}^{\mathbb{P}}$*–stopping time* τ *taking values in* $[t, +\infty)$ *we have*

$$\mathbb{E}^{\mathbb{P}}\left[\Phi\left(X^{t,\mathbf{x},\nu}\right) \Big| \mathcal{F}_\tau^{\mathbb{P}}\right](\omega) \; = \; J\left(\tau(\omega), \left[X^{t,\mathbf{x},\nu}\right]_\tau(\omega), \nu^{\tau(\omega),\omega}\right), \quad \mathbb{P}(d\omega) - \text{a.s.} \quad (5)$$

Remark 2.3. (i) *Our formulation slightly differs from Fleming and Souganidis [9] who consider deterministic times* $\tau = s$ *and conditional expectations given the non-augmented* σ*–field* $\mathcal{F}_s$.
(ii) *We work with the augmented* σ*–fields* $\mathbb{F}^{\mathbb{P}}$ *in order to make the solutions* $X^{t,\mathbf{x},\nu}$ *adapted.*
(iii) *One motivation to consider stopping times* τ *rather than deterministic times is that, to study viscosity solutions to Hamilton-Jacobi-Bellman equations, one often uses first exit times of* $X^{t,\mathbf{x},\nu}$ *from Borel subsets of* $\mathbb{R}^d$.
(iv) *It is not clear to us whether the function* J *is measurable w.r.t. all its arguments. However Theorem 2.2 implies that the r.h.s. of* (5) *is a* $\mathcal{F}_\tau^{\mathbb{P}}$*–measurable random variable.*

In Section 2.3 we discuss technical subtleties hidden in Equality (5) and point out some of the difficulties which motivate us to propose a detailed proof. Before further discussions, we show how the pseudo-Markov property is used to prove parts of the DPP.

Consider the value function

$$V(t,\mathbf{x}) \ := \ \sup_{\nu\in\mathcal{U}} \mathbb{E}^{\mathbb{P}}\left[\Phi(X^{t,\mathbf{x},\nu})\right], \quad \forall (t,\mathbf{x})\in\mathbb{R}^+\times\Omega. \tag{6}$$

Proposition 2.4. (i) *For all $(t,\mathbf{x})$ in $\mathbb{R}^+\times\Omega$, it holds that*

$$V(t,\mathbf{x}) \ = \ \sup_{\mu\in\mathcal{U}^t} \mathbb{E}^{\mathbb{P}}\left[\Phi(X^{t,\mathbf{x},\mu})\right], \tag{7}$$

where $\mathcal{U}^t$ denotes the set of all $\nu\in\mathcal{U}$ which are independent of $\mathcal{F}_t$ under $\mathbb{P}$.
(ii) *Suppose in addition that the value function V is measurable. Then, for all $(t,\mathbf{x})\in\mathbb{R}^+\times\Omega$, $\varepsilon>0$, there exists $\nu\in\mathcal{U}$ such that for all $\mathbb{F}^{\mathbb{P}}$–stopping times τ taking values in $[t,\infty)$, one has*

$$V(t,\mathbf{x})-\varepsilon \ \le \ \mathbb{E}^{\mathbb{P}}\left[V\left(\tau,\left[X^{t,\mathbf{x},\nu}\right]_\tau\right)\right]. \tag{8}$$

Proof. (i) For (7), it is enough to prove that

$$V(t,\mathbf{x}) \ \le \ \sup_{\mu\in\mathcal{U}^t} \mathbb{E}^{\mathbb{P}}\left[\Phi(X^{t,\mathbf{x},\mu})\right],$$

since the other inequality results from $\mathcal{U}^t\subset\mathcal{U}$. Now, let ν be an arbitrary control in $\mathcal{U}$. Apply Theorem 2.2 with $\tau\equiv t$. It comes:

$$J\big(t,\mathbf{x},\nu\big) \ = \ \int_\Omega J\left(t,[\mathbf{x}]_t,\nu^{t,\omega}\right)\mathbb{P}(d\omega) \ = \ \int_\Omega J\left(t,\mathbf{x},\nu^{t,\omega}\right)\mathbb{P}(d\omega).$$

Then (7) follows from the fact that, for all fixed $\omega\in\Omega$, the control $\nu^{t,\omega}$ belongs to $\mathcal{U}^t$.
(ii) Notice that, for all $\varepsilon>0$, one can choose an ε–optimal control ν, that is,

$$V(t,\mathbf{x})-\varepsilon \ \le \ J(t,\mathbf{x},\nu).$$

Apply Theorem 2.2 with this control ν. It comes:

$$V(t,\mathbf{x})-\varepsilon \ \le \ \int_\Omega J\left(\tau(\omega),\left[X^{t,\mathbf{x},\nu}\right]_\tau(\omega),\nu^{\tau(\omega),\omega}\right)\mathbb{P}(d\omega) \ \le \ \mathbb{E}^{\mathbb{P}}\Big[V\left(\tau,\left[X^{t,\mathbf{x},\nu}\right]_\tau\right)\Big].$$

□

Remark 2.5. *Inequality (8) is the 'easy' part of the DPP. Equality (7), combined with the continuity of the value function, is a key step in classical proofs of the 'difficult' part of the DPP, that is: for all control ν in $\mathcal{U}$ and all $\mathbb{F}^{\mathbb{P}}$–stopping time τ taking values in $[t,\infty)$, one has*

$$V(t,\mathbf{x}) \ \ge \ \mathbb{E}^{\mathbb{P}}\Big[V\left(\tau,\left[X^{t,\mathbf{x},\nu}\right]_\tau\right)\Big].$$

2.3 Discussion on Theorem 2.2

On the hypothesis on b and σ. In Theorem 2.2 the coefficients b and σ are supposed to satisfy Condition (1) and thus the solutions to the SDE (4) are pathwise unique. However, we only need uniqueness in law in our second proof of Theorem 2.2 which is based on controlled martingale problems related to the SDE (4). Hence Condition (1) can be relaxed to weaker conditions which imply weak uniqueness. However we have not followed this way here to avoid too heavy notations and to remain within the classical family of controlled SDEs with pathwise unique solutions.

Intuitive meaning and measurability issues. The intuitive meaning of Theorem 2.2 is as follows. Notice that $X^{t,\mathbf{x},\nu}$ satisfies: for all $\theta \in \mathbb{R}^+$,

$$X_\theta^{t,\mathbf{x},\nu} \;=\; X_{\tau\wedge\theta}^{t,\mathbf{x},\nu} + \int_\tau^{\tau\vee\theta} b(s, X^{t,\mathbf{x},\nu}, \nu_s)ds + \int_\tau^{\tau\vee\theta} \sigma(s, X^{t,\mathbf{x},\nu}, \nu_s)dB_s, \quad \mathbb{P}-\text{a.s.} \tag{9}$$

If a regular conditional probability (r.c.p.) $(\mathbb{P}_{\mathsf{w}}, \mathsf{w} \in \Omega)$ of $\mathbb{P}$ given $\mathcal{F}_\tau^{\mathbb{P}}$ were to exist (see, e.g., Karatzas and Shreve [12, Sec.5.3.C]), then Equation (9) would be satisfied $\mathbb{P}_{\mathsf{w}}$–almost surely and, in addition, we would have

$$\mathbb{P}_{\mathsf{w}}\Big(\tau = \tau(\mathsf{w}), \left[X^{t,\mathbf{x},\nu}\right]_\tau = \left[X^{t,\mathbf{x},\nu}\right]_\tau(\mathsf{w}), \nu = \nu^{\tau(\mathsf{w}),\mathsf{w}}\Big) = 1. \tag{10}$$

Hence, Theorem 2.2 would follow from the uniqueness in law of solutions to Equation (4). Unfortunately the situation is not so simple for the following reasons. First, since $\mathcal{F}_\tau^{\mathbb{P}}$ is complete, a r.c.p. of $\mathbb{P}$ given $\mathcal{F}_\tau^{\mathbb{P}}$ does not exist (see [6, Thm.10]). Second, even if τ were a $\mathbb{F}$–stopping time and $(\mathbb{P}_{\mathsf{w}}, \mathsf{w} \in \Omega)$ were a r.c.p. of $\mathbb{P}$ given $\mathcal{F}_\tau$, then $\mathbb{P}_{\mathsf{w}}$ would be defined as a measure on $\mathcal{F}$ whereas $X^{t,\mathbf{x},\nu}$ is adapted to the $\mathbb{P}$–augmented filtration $\mathbb{F}^{\mathbb{P}}$: hence, Equality (10) needs to be rigorously justified. Third, one needs to check that the stochastic integral in (9), which is constructed under $\mathbb{P}$, agrees with the stochastic integral constructed under $\mathbb{P}_{\mathsf{w}}$ for $\mathbb{P}$-a.a. w.

3 Technical Lemmas

In order to solve the measurability issues mentioned above, we establish three technical lemmas which, to the best of our knowledge, are not available in the literature.

The first lemma improves the classical Dynkin theorem which states that, given an arbitrary probability space and arbitrary filtration $(\mathcal{H}_t)$, a stopping time w.r.t. to the augmented filtration of $(\mathcal{H}_t)$ is a.s. equal to a stopping time w.r.t. $(\mathcal{H}_{t+})$: see, e.g., [17, Thm.II.75.3]. The improvement here is allowed by the fact that we are dealing with the augmented Brownian filtration (more generally, we could deal with the natural augmented filtration generated by a Hunt process with continuous paths: see Chung and Williams [3, Sec.2.3,p.30-32]).

Lemma 3.1. *Let τ be a $\mathbb{F}^{\mathbb{P}}$–stopping time. There exists a $\mathbb{F}$–stopping time η such that*

$$\mathbb{P}\big(\tau = \eta\big) = 1 \quad \textit{and} \quad \mathcal{F}_\tau^{\mathbb{P}} \;=\; \mathcal{F}_\eta \;\vee\; \mathcal{N}^{\mathbb{P}}. \tag{11}$$

Proof. First step. As $\mathbb{F}^{\mathbb{P}}$ is the augmented Brownian filtration, all $\mathbb{F}^{\mathbb{P}}$–optional process is predictable and hence all $\mathbb{F}^{\mathbb{P}}$–stopping time is predictable (see, e.g., Revuz and Yor [16, Cor.V.3.3]). It follows from Dellacherie and Meyer [4, Thm.IV.78] that there exists a $\mathbb{F}$–stopping time (actually, a predictable time) η such that

$$\mathbb{P}(\tau = \eta) \ = \ 1.$$

Second step. We now prove $\mathcal{F}_\eta \vee \mathcal{N}^{\mathbb{P}} \subseteq \mathcal{F}^{\mathbb{P}}_\tau$. First, since $\mathcal{F}^{\mathbb{P}}_0 \subset \mathcal{F}^{\mathbb{P}}_\tau$, we have $\mathcal{N}^{\mathbb{P}} \subset \mathcal{F}^{\mathbb{P}}_\tau$. Second, η is a $\mathbb{F}^{\mathbb{P}}$–stopping time. Since $\tau = \eta$, $\mathbb{P}-$ a.s. and $\mathcal{F}^{\mathbb{P}}_\tau$, $\mathcal{F}^{\mathbb{P}}_\eta$ are both $\mathbb{P}$–complete, one has $\mathcal{F}^{\mathbb{P}}_\tau = \mathcal{F}^{\mathbb{P}}_\eta$, from which $\mathcal{F}_\eta \subset \mathcal{F}^{\mathbb{P}}_\eta = \mathcal{F}^{\mathbb{P}}_\tau$.

Last step. It therefore only remains to prove $\mathcal{F}^{\mathbb{P}}_\tau \subset \mathcal{F}_\eta \vee \mathcal{N}^{\mathbb{P}}$. Let $A \in \mathcal{F}^{\mathbb{P}}_\tau$. In view of [4, Thm.IV.64], there exists a $\mathbb{F}^{\mathbb{P}}$–optional (or equivalently $\mathbb{F}^{\mathbb{P}}$–predictable) process X such that $\mathbb{I}_A = X_\tau$. By using Theorem IV.78 and Remark IV.74 in [4], there exists a $\mathbb{F}$–predictable process Y which is indistinguishable from X. It follows that $\mathbb{I}_A = X_\tau = Y_\tau = Y_\eta$, $\mathbb{P}-$ a.s., which implies that $A \in \mathcal{F}_\eta \vee \mathcal{N}^{\mathbb{P}}$ since Y_η is $\mathcal{F}_\eta$–measurable. This ends the proof. □

The two next lemmas are used in Section 4 only. They concern the r.c.p. $(\mathbb{P}_{\mathsf{w}})_{\mathsf{w}\in\Omega}$ of $\mathbb{P}$ given a σ–algebra $\mathcal{G} \subset \mathcal{F}$. They allow us to circumvent the difficulties raised by the fact that $\mathbb{P}_{\mathsf{w}}$ is not a measure on $\mathcal{F}^{\mathbb{P}}$. The key idea is to notice that, if N in $\mathcal{F}$ satisfies $\mathbb{P}(N) = 0$, then there is a $\mathbb{P}$–null set $M \in \mathcal{F}$ (depending on N) such that $\mathbb{P}_{\mathsf{w}}(N) = 0$ for all $\mathsf{w} \in \Omega \setminus M$. Hence any subset of N belongs to $\mathcal{N}^{\mathbb{P}}$ and, for all $\mathsf{w} \in \Omega \setminus M$, to the set $\mathcal{N}^{\mathbb{P}_{\mathsf{w}}}$ of all $\mathbb{P}_{\mathsf{w}}$–negligible sets of $\mathcal{F}$.

Lemma 3.2. *Define $\mathcal{G}^{\mathbb{P}} := \mathcal{G} \vee \mathcal{N}^{\mathbb{P}}$ and $\mathcal{G}^{\mathbb{P}_{\mathsf{w}}} := \mathcal{G} \vee \mathcal{N}^{\mathbb{P}_{\mathsf{w}}}$. Let E be a Polish space and $\xi : \Omega \to E$ be a $\mathcal{G}^{\mathbb{P}}$–random variable. Then*
(i) *There exists a $\mathcal{G}$–random variable $\xi^0 : \Omega \to E$ such that*

$$\mathbb{P}\left(\xi = \xi^0\right) = 1.$$

(ii) *For $\mathbb{P}$–a.a. $\mathsf{w} \in \Omega$, ξ is a $\mathcal{G}^{\mathbb{P}_{\mathsf{w}}}$–random variable and*

$$\mathbb{P}_{\mathsf{w}}\left(\xi = \xi(\mathsf{w})\right) = 1.$$

(iii) *Let $\mathcal{F}^{\mathbb{P}_{\mathsf{w}}} := \mathcal{F} \vee \mathcal{N}^{\mathbb{P}_{\mathsf{w}}}$ and ζ be a $\mathbb{P}$–integrable $\mathcal{F}^{\mathbb{P}}$–random variable. Then for $\mathbb{P}$–a.a. $\mathsf{w} \in \Omega$, ζ is $\mathcal{F}^{\mathbb{P}_{\mathsf{w}}}$–measurable and*

$$\mathbb{E}[\zeta \mid \mathcal{G}^{\mathbb{P}}](\mathsf{w}) = \int_\Omega \zeta(\mathsf{w}')\mathbb{P}_{\mathsf{w}}(d\mathsf{w}').$$

Proof. (i) As E is Polish, there exists a Borel isomorphism between E and a subset of $\mathbb{R}$. This allows us to only consider real-valued random variables. A monotone class argument allows us to deal with random variables of the type $\mathbb{I}_G$ with G in $\mathcal{G}^{\mathbb{P}}$, from which the result (i) follows since

$$\mathcal{G}^{\mathbb{P}} \ = \ \left\{G \in \mathcal{F}^{\mathbb{P}};\ \exists\, G^0 \in \mathcal{G},\ G \triangle G^0 \in \mathcal{N}^{\mathbb{P}}\right\}, \tag{12}$$

where $\triangle$ denotes the symmetric difference (see, e.g., Rogers and Williams [17, Ex.V.79.67a]).
(ii) Let ξ^0 be a $\mathcal{G}$–random variable such that $\mathbb{P}(\xi = \xi^0) = 1$. In other words, there is a $\mathbb{P}$–null set $N \in \mathcal{F}$ such that $\xi^0(\omega) = \xi(\omega)$ for all $\omega \in \Omega \setminus N$. Hence, for $\mathbb{P}$–a.a. w, we have $\mathbb{P}_{\mathsf{w}}(N) = 0$ and $\{\xi \in A\} \triangle \{\xi^0 \in A\} \subset N$ belongs to $\mathcal{N}^{\mathbb{P}_{\mathsf{w}}}$ for all Borel set A. It results from (12) with $\mathbb{P}_{\mathsf{w}}$ in place of $\mathbb{P}$ that ξ is $\mathcal{G}^{\mathbb{P}_{\mathsf{w}}}$–measurable. Moreover, since ξ^0 is an $\mathcal{G}$–random variable taking values in a Polish space, we have

$$\mathbb{P}_{\mathsf{w}}\left(\xi^0 = \xi^0(\mathsf{w})\right) = 1,$$

for $\mathbb{P}$–a.a. $\mathsf{w} \in \Omega$. Then, for all $\mathsf{w} \in \Omega \setminus N$ such that

$$\mathbb{P}_{\mathsf{w}}(N) = 0 \ \ and \ \ \mathbb{P}_{\mathsf{w}}\left(\xi^0 = \xi^0(\mathsf{w})\right) = 1,$$

we have

$$\mathbb{P}_{\mathsf{w}}\left(\xi = \xi(\mathsf{w})\right) = 1.$$

(iii) Using the same arguments as in the proof of (ii), we get that ζ is $\mathcal{F}^{\mathbb{P}_{\mathsf{w}}}$–measurable. Now, let χ be a bounded $\mathcal{G}^{\mathbb{P}}$–measurable random variable. In view of (i), there exists a $\mathcal{G}$–measurable (resp. $\mathcal{F}$–measurable) χ_0 (resp. ζ_0) random variable such that $\chi = \chi_0$ (resp. $\zeta = \zeta_0$) $\mathbb{P}$–a.s. Therefore,

$$\mathbb{E}^{\mathbb{P}}\left[\zeta\chi\right] = \mathbb{E}^{\mathbb{P}}\left[\zeta_0\chi_0\right] = \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}[\zeta_0 \mid \mathcal{G}]\chi\right].$$

Hence it holds

$$\mathbb{E}^{\mathbb{P}}[\zeta \mid \mathcal{G}^{\mathbb{P}}](\mathsf{w}) = \int_{\Omega} \zeta_0(\mathsf{w}')\mathbb{P}_{\mathsf{w}}(d\mathsf{w}'), \qquad \mathbb{P}(d\mathsf{w}) - a.s.$$

We then conclude by using that ζ is $\mathcal{F}^{\mathbb{P}_{\mathsf{w}}}$–measurable and $\mathbb{P}_{\mathsf{w}}(\zeta = \zeta_0) = 1$ for $\mathbb{P}$–a.a $\mathsf{w} \in \Omega$.

□

Lemma 3.3. *Let $\mathbb{F}^{\mathbb{P}_{\mathsf{w}}}$ be the $\mathbb{P}_{\mathsf{w}}$–augmented filtration of $\mathbb{F}$. Let E be a Polish space and $Y : \mathbb{R}^+ \times \Omega \to E$ be a $\mathbb{F}^{\mathbb{P}}$–predictable process. Then, for $\mathbb{P}$–a.a. $\mathsf{w} \in \Omega$, Y is $\mathbb{F}^{\mathbb{P}_{\mathsf{w}}}$–predictable.*

Proof. As already noticed in the proof of Lemma 3.1, there exists a $\mathbb{F}$–predictable process indistinguishable from Y under $\mathbb{P}$. The result thus follows from our first arguments in the proof of Lemma 3.2 (ii). □

We end this section by two propositions which will not be used in the proof of the main results but were implicitly used in the formulation of stochastic control problems in Section 2.1.

The proposition below ensures that classical notions of measurability coincide for processes defined on the canonical space Ω. This is no longer true when Ω is the space of càdlàg functions. For a precise statement, see Dellacherie and Meyer [4, Thm.IV.97].

Proposition 3.4. *Let E be a Polish space and $Y : \mathbb{R}^+ \times \Omega \to E$ be a stochastic process. Then the following statements are equivalent :*
(i) *Y is $\mathbb{F}$–predictable.*
(ii) *Y is $\mathbb{F}$–optional.*
(iii) *Y is $\mathbb{F}$–progressively measurable.*
(iv) *Y is $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}$–measurable and $\mathbb{F}$–adapted.*
(v) *Y is $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}$–measurable and satisfies*

$$Y_s(\omega) \ = \ Y_s([\omega]_s), \quad \forall (s,\omega) \in \mathbb{R}^+ \times \Omega.$$

Proof. It is well known that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$. Now we show that $(iv) \Rightarrow (v)$. Remember that $\mathcal{F}_s = \sigma([\cdot]_s : \omega \mapsto [\omega]_s)$ for all $s \in \mathbb{R}^+$. Since Y_s is $\mathcal{F}_s$–measurable and E is Polish, Doob's functional representation Theorem (see, e.g., Kallenberg [11, Lem.1.13]) implies that there exists a random variable Z_s such that $Y_s(\omega) = Z_s([\omega]_s)$ for all $\omega \in \Omega$. By noticing that $[\cdot]_s \circ [\cdot]_s = [\cdot]_s$, we deduce that

$$Y_s([\omega]_s) = Z_s([\omega]_s) = Y_s(\omega), \quad \forall (s,\omega) \in \mathbb{R}^+ \times \Omega.$$

To conclude it remains to prove that $(v) \Rightarrow (i)$. Clearly it is enough to show that $[\cdot] : (s,\omega) \mapsto [\omega]_s$ is a predictable process. In other words, we have to show that for any $C \in \mathcal{F}$, $[\cdot]^{-1}(C)$ is a predictable event. When C is of the type $B_t^{-1}(A)$ with $t \in \mathbb{R}^+$ and $A \in \mathcal{B}(\mathbb{R}^d)$, the result holds true since $B_t \circ [\cdot] : (s,\omega) \mapsto \omega(t \wedge s)$ is predictable as a continuous and adapted $\mathbb{R}^d$–valued process. We conclude by observing that the former events generate the σ–algebra $\mathcal{F}$. □

Now, let $(\Omega^*, \mathcal{F}^*)$ be an abstract measurable space equipped with a measurable, continuous process $X^* = (X_t^*)_{t \geq 0}$. Denote by $\mathbb{F}^* = (\mathcal{F}_t^*)_{t\geq 0}$ the filtration generated by X^*, i.e. $\mathcal{F}_t^* = \sigma\big\{X_s^*,\ s \leq t\big\}$.

Proposition 3.5. *Let E be a Polish space. Then a process $Y : \mathbb{R}^+ \times \Omega^* \to E$ is $\mathbb{F}^*$-predictable if and only if there exists a process $\phi : \mathbb{R}^+ \times \Omega \to E$ defined on the canonical space Ω and $\mathbb{F}$–predictable such that*

$$Y_t(\omega^*) \;=\; \phi\big(t, X^*(\omega^*)\big) \;=\; \phi\big(t, [X^*]_t\,(\omega^*)\big), \qquad \forall (t,\omega^*) \in \mathbb{R}^+ \times \Omega^*. \tag{13}$$

Proof. First step. Let $\phi : \mathbb{R}^+ \times \Omega \to E$ be a $\mathbb{F}$–predictable process on the canonical space. Notice that $(t,\omega^*) \mapsto t$ and $(t,\omega^*) \mapsto [X^*]_t(\omega^*)$ are both $\mathbb{F}^*$–predictable. It follows that $(t,\omega^*) \mapsto Y_t(\omega^*) := \phi(t, [X^*]_t(\omega^*))$ is also $\mathbb{F}^*$–predictable.

Second step. We now prove the converse relation. Suppose that Y is a $\mathbb{F}^*$–predictable process of the type $Y_s(\omega^*) = \mathbb{I}_{(t_1,t_2]\times A}(s,\omega^*)$, where $0 < t_1 < t_2$ and $A \in \mathcal{F}_{t_1}^*$. There exists C in $\mathcal{F}$ such that $A = ([X^*]_{t_1})^{-1}(C)$, so that (13) holds true with

$$\phi(s, \mathsf{w}) := \mathbb{I}_{(t_1,t_2]\times C}(s, [\mathsf{w}]_{t_1}).$$

In view of Proposition 3.4, it is clear that the above ϕ is $\mathbb{F}$–predictable. The same arguments show that (13) holds also true if Y is of the form $Y_s(\omega^*) = \mathbb{I}_{\{0\}\times A}$ with $A \in \mathcal{F}_0^*$. Notice that the predictable σ-field on $\mathbb{R}^+ \times \Omega^*$ w.r.t. $\mathbb{F}^*$ is generated by the collection of all sets of the form $\{0\} \times A$ with $A \in \mathcal{F}_0^*$ and of the form $(t_1,t_2] \times A$ with $0 < t_1 < t_2$ and $A \in \mathcal{F}_{t_1}^*$ (see, e.g., [4, Thm.IV.64]). It then remains to use the monotone class theorem to conclude. □

4 A first proof of Theorem 2.2

In this section we develop the arguments sketched by Fleming and Souganidis [9] in the proof of their Lemma 1.11.

Let τ be a $\mathbb{F}^{\mathbb{P}}$–stopping time taking values in $[t, +\infty)$. We have

$$X_\theta^{t,\mathbf{x},\nu} = X_{\tau\wedge\theta}^{t,\mathbf{x},\nu} + \int_\tau^{\tau\vee\theta} b(s, X^{t,\mathbf{x},\nu}, \nu_s)ds + \int_\tau^{\tau\vee\theta} \sigma(s, X^{t,\mathbf{x},\nu}, \nu_s)dB_s,\ \forall\theta \geq 0,\ \mathbb{P}\text{–a.s.}$$

Now, it follows from Lemma 3.1 that there is a $\mathbb{F}$–stopping time η such that (11) holds true. Let $(\mathbb{P}_{\mathsf{w}})_{\mathsf{w}\in\Omega}$ be a family of r.c.p. of $\mathbb{P}$ given $\mathcal{F}_\eta$. In view of Lemma 3.2, for $\mathbb{P}$–a.a. $\mathsf{w} \in \Omega$,

$$\mathbb{E}^{\mathbb{P}}\left[\Phi\left(X^{t,\mathbf{x},\nu}\right) \,\middle|\, \mathcal{F}_\tau^{\mathbb{P}}\right](\mathsf{w}) \;=\; \mathbb{E}^{\mathbb{P}_{\mathsf{w}}}\left[\Phi\left(X^{t,\mathbf{x},\nu}\right)\right], \tag{14}$$

and

$$\mathbb{P}_{\mathsf{w}}\big(\tau=\tau(\mathsf{w}),\ [X^{t,\mathbf{x},\nu}]_\tau=[X^{t,\mathbf{x},\nu}]_\tau(\mathsf{w}),\ \nu=\nu^{\tau(\mathsf{w}),\mathsf{w}}\big)\ =\ 1. \tag{15}$$

In view of Lemma 4.1 below we have, for $\mathbb{P}$–a.a. $\mathsf{w}\in\Omega$,

$$\int_{[\tau,\theta]}^{\mathbb{P}}\sigma\left(s,X^{t,\mathbf{x},\nu},\nu_s\right)dB_s\ =\ \int_{[\tau(\mathsf{w}),\theta]}^{\mathbb{P}_{\mathsf{w}}}\sigma\left(s,X^{t,\mathbf{x},\nu},\nu_s\right)dB_s,\quad \forall\,\theta\geq\tau(\mathsf{w}),\ \mathbb{P}_{\mathsf{w}}-\text{a.s.},$$

where the l.h.s. (resp. r.h.s.) term denotes the stochastic integral constructed under $\mathbb{P}$ (resp. $\mathbb{P}_{\mathsf{w}}$). It follows from (15) that, for $\mathbb{P}$–a.a. $\mathsf{w}\in\Omega$,

$$\begin{aligned} X^{t,\mathbf{x},\nu}_\theta\ &=\ X^{t,\mathbf{x},\nu}_{\tau\wedge\theta}(\mathsf{w})\ +\ \int_{\tau(\mathsf{w})}^{\tau(\mathsf{w})\vee\theta}b(s,X^{t,\mathbf{x},\nu},\nu_s^{\tau(\mathsf{w}),\mathsf{w}})ds\\ &\quad+\int_{\tau(\mathsf{w})}^{\tau(\mathsf{w})\vee\theta}\sigma(s,X^{t,\mathbf{x},\nu},\nu_s^{\tau(\mathsf{w}),\mathsf{w}})dB_s,\qquad \forall\theta\geq 0,\ \mathbb{P}_{\mathsf{w}}-\text{a.s.}\end{aligned}$$

Notice that, by Lemma 3.3, $X^{t,\mathbf{x},\nu}$ is $\mathbb{F}^{\mathbb{P}_{\mathsf{w}}}$–adapted, for $\mathbb{P}$–a.a. $\mathsf{w}\in\Omega$. Hence, $X^{t,\mathbf{x},\nu}$ is the solution of SDE (2) with initial condition $(\tau(\mathsf{w}),[X^{t,\mathbf{x},\nu}]_\tau(\mathsf{w}))$ and control $\nu^{\tau(\mathsf{w}),\mathsf{w}}$ in $(\Omega,\mathbb{F}^{\mathbb{P}_{\mathsf{w}}},\mathbb{P}_{\mathsf{w}})$ for $\mathbb{P}$–a.a. $\mathsf{w}\in\Omega$. As the SDE (4) satisfies uniqueness in law, the law of $X^{t,\mathbf{x},\nu}$ under $\mathbb{P}_{\mathsf{w}}$ coincides with the law of $X^{\tau(\mathsf{w}),[X^{t,\mathbf{x},\nu}]_\tau(\mathsf{w}),\nu^{\tau(\mathsf{w}),\mathsf{w}}}$ under $\mathbb{P}$. We then conclude the proof by using (14). $\square$

Lemma 4.1. *Let H be a $\mathbb{F}^{\mathbb{P}}$–predictable process such that*

$$\int_0^\theta (H_s)^2ds<+\infty,\quad \forall\theta\geq 0,\ \mathbb{P}-a.s.$$

Then, using the same notation as above for the stochastic integrals, we have: For $\mathbb{P}$–a.a. $\mathsf{w}\in\Omega$, $\int_{[\tau,\theta]}^{\mathbb{P}}H_sdB_s$ is $\mathcal{F}^{\mathbb{P}_{\mathsf{w}}}$–measurable and

$$\int_{[\tau,\theta]}^{\mathbb{P}}H_sdB_s\ =\ \int_{[\tau(\mathsf{w}),\theta]}^{\mathbb{P}_{\mathsf{w}}}H_sdB_s,\quad \forall\,\theta\geq\tau(\mathsf{w}),\ \mathbb{P}_{\mathsf{w}}-a.s. \tag{16}$$

Proof. In this proof we implicitly use Lemma 3.3 to consider $\mathbb{F}^{\mathbb{P}}$–predictable processes, such as the stochastic integrals defined under $\mathbb{P}$, as $\mathbb{F}^{\mathbb{P}_{\mathsf{w}}}$–predictable processes for $\mathbb{P}$–a.a. $\mathsf{w}\in\Omega$. In particular, the l.h.s. of (16) is $\mathcal{F}^{\mathbb{P}_{\mathsf{w}}}$–measurable for $\mathbb{P}$–a.a. $\mathsf{w}\in\Omega$.

A standard localizing procedure allows us to assume that

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^{+\infty}(H_s)^2ds\right]<+\infty.$$

Now let $(H^{(n)})_{n\in\mathbb{N}}$ be a sequence of simple processes such that

$$\lim_{n\to\infty}\mathbb{E}^{\mathbb{P}}\left[\int_0^{+\infty}\left(H_s^{(n)}-H_s\right)^2ds\right]=0. \tag{17}$$

We then write

$$\begin{aligned}\int^{\mathbb{P}}H_sdB_s-\int^{\mathbb{P}_{\mathsf{w}}}H_sdB_s&=\int^{\mathbb{P}}H_s^{(n)}dB_s-\int^{\mathbb{P}_{\mathsf{w}}}H_s^{(n)}dB_s\\ &\quad+\int^{\mathbb{P}}(H_s-H_s^{(n)})dB_s-\int^{\mathbb{P}_{\mathsf{w}}}(H_s-H_s^{(n)})dB_s\end{aligned}$$

and notice that the first difference is null since the stochastic integral is defined pathwise when the integrand is a simple process.

By taking conditional expectations in (17), we get that there exists a subsequence such that

$$\lim_{n\to\infty} \mathbb{E}^{\mathbb{P}_{\mathsf{w}}}\left[\int_0^{+\infty}\left(H_s^{(n)}-H_s\right)^2 ds\right]=0,$$

for $\mathbb{P}$–a.a. $\mathsf{w}\in\Omega$. Hence, by Doob's inequality and Itô's isometry,

$$\lim_{n\to\infty} \mathbb{E}^{\mathbb{P}_{\mathsf{w}}}\left[\sup_{\theta\geq\tau(\mathsf{w})}\left\{\left(\int_{[\tau(\mathsf{w}),\theta]}^{\mathbb{P}_{\mathsf{w}}} H_s^{(n)}dB_s-\int_{[\tau(\mathsf{w}),\theta]}^{\mathbb{P}_{\mathsf{w}}} H_s dB_s\right)^2\right\}\right]=0.$$

To conclude, it thus suffices to prove that

$$\lim_{n\to\infty} \mathbb{E}^{\mathbb{P}_{\mathsf{w}}}\left[\sup_{\theta\geq\tau(\mathsf{w})}\left\{\left(\int_{[\tau(\mathsf{w}),\theta]}^{\mathbb{P}} H_s^{(n)}dB_s-\int_{[\tau,\theta]}^{\mathbb{P}} H_s dB_s\right)^2\right\}\right]=0.$$

for $\mathbb{P}$–a.a. $\mathsf{w}\in\Omega$. We cannot use Doob's inequality and Itô's isometry without care because the stochastic integrals are built under $\mathbb{P}$ and the expectation is computed under $\mathbb{P}_{\mathsf{w}}$. However we have

$$\lim_{n\to\infty} \mathbb{E}^{\mathbb{P}}\left[\sup_{\theta\geq 0}\left\{\left(\int_{[0,\theta]}^{\mathbb{P}} H_s^{(n)}\mathbb{I}_{s\geq\tau}dB_s-\int_{[0,\theta]}^{\mathbb{P}} H_s\mathbb{I}_{s\geq\tau} dB_s\right)^2\right\}\right]=0.$$

Thus we can proceed as above by taking conditional expectation and extracting a new subsequence. That ends the proof. □

Remark 4.2. *To avoid the technicalities of this first proof of Theorem 2.2, a natural attempt consists in solving the equations* (2) *in the space* Ω *equipped with the non augmented filtration. This may be achieved by following Stroock and Varadhan's approach [19] to stochastic integration. This way leads to right-continuous and only* $\mathbb{P}$*–a.s continuous stochastic integrals and solutions. As they are not valued in a Polish space, new delicate technical issues arise: for example,* (14) *and* (15) *need to be revisited.*

5 A second proof of Theorem 2.2

In this section we provide a second proof of Theorem 2.2. Compared to the above first proof, the technical details are lighter. However we emphasize that it uses that weak uniqueness for Brownian SDEs is equivalent to uniqueness of solutions to the corresponding martingale problems. Therefore it may be more difficult to extend this second methodology than the first one to stochastic control problems where the noise involves a Poisson random measure.

Our second proof of Theorem 2.2 is based on the notion of controlled martingale problems on the enlarged canonical space $\overline{\Omega}:=\Omega^2$. Denote by $\overline{\mathbb{F}}=(\overline{\mathcal{F}}_t)_{t\geq 0}$ the canonical filtration, and by (B,X) the canonical process.

Fix $(t,\nu)\in\mathbb{R}^+\times\mathcal{U}$. Let $\bar{b}^{t,\nu}$ and $\bar{\sigma}^{t,\nu}$ be defined as in Section 2.1. For all functions φ in $\mathcal{C}_c^2(\mathbb{R}^{2d})$, let the process $(\overline{M}_\theta^{t,\nu,\varphi},\theta\geq t)$ be defined on the enlarged space $\overline{\Omega}$ by

$$\overline{M}_\theta^{t,\nu,\varphi}(\bar{\omega}) \;:=\; \varphi(\bar{\omega}_\theta) \;-\; \int_t^\theta \mathcal{L}_s^{t,\nu}\varphi(\bar{\omega})ds,\quad \theta\geq t, \tag{18}$$

where $\mathcal{L}^{t,\nu}$ is the differential operator

$$\mathcal{L}_s^{t,\nu}\varphi(\bar{\omega}) \;\; := \;\; \bar{b}^{t,\nu}(s,\bar{\omega})\cdot D\varphi(\bar{\omega}_s) \; + \; \frac{1}{2}\,\bar{a}^{t,\nu}(s,\bar{\omega}) : D^2\varphi(\bar{\omega}_s);$$

here, we set $\bar{a}^{t,\nu}(s,\bar{\omega}) := \bar{\sigma}^{t,\nu}(s,\bar{\omega})\bar{\sigma}^{t,\nu}(s,\bar{\omega})^*$ and the operator ":" is defined by $p : q := \mathrm{Tr}(pq^*)$ for all p and q in $S_{2d,d}$. It is clear that the process $\overline{M}^{t,\nu,\varphi}$ is $\overline{\mathbb{F}}$–progressively measurable.

Let $(t,\mathbf{x},\nu) \in \mathbb{R}^+ \times \Omega \times \mathcal{U}$. Denote by $\overline{\mathbb{P}}^{t,\mathbf{x},\nu}$ the probability measure on $\overline{\Omega}$ induced by $(B, X^{t,\mathbf{x},\nu})$ under the Wiener measure $\mathbb{P}$. For all φ in $\mathcal{C}_c^2(\mathbb{R}^{2d})$, the process $\overline{M}^{t,\nu,\varphi}$ is a martingale under $\overline{\mathbb{P}}^{t,\mathbf{x},\nu}$ and

$$\overline{\mathbb{P}}^{t,\mathbf{x},\nu}\big(X_s = \mathbf{x}(s),\; 0 \le s \le t\big) \;=\; 1.$$

Let τ be a $\mathbb{F}^{\mathbb{P}}$–stopping time taking values in $[t,+\infty)$ and let η be the $\mathbb{F}$–stopping time defined in Lemma 3.1. Set $\bar{\nu}(\bar{\mathsf{w}}) := \nu(\mathsf{w})$ and $\bar{\eta}(\bar{\mathsf{w}}) := \eta(\mathsf{w})$ for all $\bar{\mathsf{w}} = (\mathsf{w},\mathsf{w}') \in \overline{\Omega}$. It is clear that $\bar{\eta}$ is a $\overline{\mathbb{F}}$–stopping time. Then there is a family of r.c.p. of $\overline{\mathbb{P}}^{t,\mathbf{x},\nu}$ given $\overline{\mathcal{F}}_{\bar{\eta}}$ denoted by $(\overline{\mathbb{P}}^{t,\mathbf{x},\nu}_{\bar{\mathsf{w}}})_{\bar{\mathsf{w}}\in\overline{\Omega}}$, and a $\overline{\mathbb{P}}^{t,\mathbf{x},\nu}$–null set $\overline{N} \subset \overline{\Omega}$ such that

$$\overline{\mathbb{P}}^{t,\mathbf{x},\nu}_{\bar{\mathsf{w}}}\big(\bar{\eta} = \eta(\mathsf{w}), B_s = \mathsf{w}(s), X_s = \mathsf{w}'(s), 0 \le s \le \eta(\mathsf{w})\big) \;=\; 1 \tag{19}$$

for all $\bar{\mathsf{w}} = (\mathsf{w},\mathsf{w}') \in \overline{\Omega} \setminus \overline{N}$. In particular, one has

$$\overline{\mathbb{P}}^{t,\mathbf{x},\nu}_{\bar{\mathsf{w}}}\big(\bar{\nu}_s = \nu_s^{\eta(\mathsf{w}),\mathsf{w}},\; \forall\, s \ge 0\big) \;=\; 1$$

for all $\bar{\mathsf{w}} = (\mathsf{w},\mathsf{w}') \in \overline{\Omega}\setminus\overline{N}$. Moreover, Lemma 6.1.3 in [19] combined with a standard localization argument implies that for $\overline{\mathbb{P}}^{t,\mathbf{x},\nu}$–a.a. $\bar{\mathsf{w}} \in \overline{\Omega}$, for all $\varphi \in C_c^2(\mathbb{R}^{2d})$, the process

$$\varphi(\bar{\omega}_\theta) \;-\; \int_{\eta(\mathsf{w})}^{\theta} \mathcal{L}_s^{t,\nu}\varphi(\bar{\omega})ds, \quad \theta \ge \eta(\mathsf{w}),$$

is a martingale under $\overline{\mathbb{P}}^{t,\mathbf{x},\nu}_{\bar{\mathsf{w}}}$. It follows by (19) that for $\overline{\mathbb{P}}^{t,\mathbf{x},\nu}$–a.a. $\bar{\mathsf{w}} \in \overline{\Omega}$, for all $\varphi \in C_c^2(\mathbb{R}^{2d})$, $\overline{M}^{\eta(\mathsf{w}),\nu^{\eta(\mathsf{w}),\mathsf{w}},\varphi}$ is a martingale under $\overline{\mathbb{P}}^{t,\mathbf{x},\nu}_{\bar{\mathsf{w}}}$.

As weak uniqueness is equivalent to uniqueness of solutions to martingale problem[1], for $\overline{\mathbb{P}}^{t,\mathbf{x},\nu}$–a.a. $\bar{\mathsf{w}} \in \overline{\Omega}$, $\overline{\mathbb{P}}^{t,\mathbf{x},\nu}_{\bar{\mathsf{w}}}$ coincides with the probability measure on $\overline{\Omega}$ induced by $\big(\mathsf{w} \otimes_{\eta(\mathsf{w})} B, X^{\eta(\mathsf{w}),\mathsf{w}',\nu^{\eta(\mathsf{w}),\mathsf{w}}}\big)$ under the Wiener measure $\mathbb{P}$. Therefore, for all bounded random variable Y in $\mathcal{F}_\eta$ we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\Phi\left(X^{t,\mathbf{x},\nu}\right)Y\right] &= \int_{\overline{\Omega}} \big(\Phi(\mathsf{w}')Y(\mathsf{w})\big)\,\overline{\mathbb{P}}^{t,\mathbf{x},\nu}(d\bar{\mathsf{w}}) \\
&= \int_{\overline{\Omega}} \Big(\mathbb{E}^{\overline{\mathbb{P}}^{t,\mathbf{x},\nu}_{\bar{\mathsf{w}}}}\left[\Phi(X)\right]\,Y(\mathsf{w})\Big)\,\overline{\mathbb{P}}^{t,\mathbf{x},\nu}(d\bar{\mathsf{w}}) \\
&= \int_{\overline{\Omega}} \Big(\mathbb{E}^{\mathbb{P}}\Big[\Phi\Big(X^{\eta(\mathsf{w}),\mathsf{w}',\nu^{\eta(\mathsf{w}),\mathsf{w}}}\Big)\Big]\,Y(\mathsf{w})\Big)\,\overline{\mathbb{P}}^{t,\mathbf{x},\nu}(d\bar{\mathsf{w}}) \\
&= \int_{\Omega} \Big(J\Big(\eta(\omega), X^{t,\mathbf{x},\nu}(\omega), \nu^{\eta(\omega),\omega}\Big)\,Y(\omega)\Big)\,\mathbb{P}(d\omega).
\end{aligned}$$

Since $\eta = \tau$ $\mathbb{P}$ – a.s., we have completed the proof. ∎

[1] Here the SDE to consider is : for all $\theta \in \mathbb{R}^+$,

$$Z_\theta \;=\; \bar{\mathsf{w}}(\eta(\mathsf{w}) \wedge \theta) + \int_{\eta(\mathsf{w})}^{\eta(\mathsf{w})\vee\theta} \bar{b}^{\eta(\mathsf{w}),\nu^{\eta(\mathsf{w}),\mathsf{w}}}(s,Z)ds + \int_{\eta(\mathsf{w})}^{\eta(\mathsf{w})\vee\theta} \bar{\sigma}^{\eta(\mathsf{w}),\nu^{\eta(\mathsf{w}),\mathsf{w}}}(s,Z)dB_s.$$

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