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# Dominating set in circulant graphs

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## Abstract

A Cayley graph on a group  $\Gamma$  with the generating set  $S \subset \Gamma$  is a graph costructed out of  $\Gamma$ . A circulant graph is a Cayley graph on a cyclic group denoted by  $Circ(n, S)$ . In this paper, we address the problem *domination* in circulant graphs. We first give upper bounds for the domination number of certain circulant. The domination number of certain circulant graphs is obtained together with a corresponding dominating set. We also obtain the independent domination number and the total domination number for special cases of these circulant graphs.

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## 1. Introduction

Cayley graphs are usefull in various domains of theoretical computer sciences such as computer architecture, routing problem, networking and coding theory. In the coding theory, the most interesting graphs are those having perfect e-codes, particularly regular graphs, vertex transitive graphs and distance  
5 perfect 1-codes were introduced by Biggs[1] in the context of error correcting codes. A perfect 1-code is a single error correcting code, that is, it permits to correct one error. In the context of graph theory, a perfect 1-code is called an efficient dominating set. This is an independent dominating  
10 set in which every vertex not in the set is dominated by exactly one vertex of the set. Independent sets are also useful in the perspective of coding theory. In [2] for example, Chellali *et al.* introduce the independent  $[1,k]$ -sets and state that these sets are usefull in the context of error correcting codes as well. In

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fact, an independent  $[1,2]$ -set will permit to correct most errors and those not  
15 corrected come from at most 2 sources. Cayley graphs are such regular graphs  
usable in coding theory. As example, hypercubes are used to represent Ham-  
ming codes. They are defined out of groups. In fact, for a group (even infinite)  $\Gamma$   
and a set  $S \subset \Gamma$  such that  $s \in S \equiv s^{-1} \in S$  and  $e(\Gamma) \not\subset S$ , the Cayley graph  
 $G$  on  $\Gamma$  with the generating set  $S \subset \Gamma$  is a graph with vertex set elements of  $\Gamma$   
20  $(V)$  and edge set  $E(G) = \{(g, gs) | g \in \Gamma, s \in S\}$ . Cayley graphs on finite cyclic  
groups are called circulant graphs and denoted by  $Circ(n, S)$ .

*Definitions.* In a graph  $G = (V, E)$ , a dominating is a set  $D$  of vertices such  
that every vertex of  $G$  is either in  $D$  or has a neighbor in  $D$ . The DOMINATING  
SET problem asks to find in a graph  $G = (V, E)$  a dominating set with minimum  
25 cardinality denoted  $\gamma(G)$ . A dominating set is efficient if it is independent (i.e.  
no two vertices of  $D$  are adjacent) and each vertex  $v \notin D$  has exactly one  
neighbor in  $D$ . More generally, an independent exact  $k$ -dominating set  $D$  of  
a graph  $G = (V, E)$  is a set of vertices such that every vertex not in  $D$  has  
exactly  $k$  neighbors in  $D$ .  $k$ -dominating sets are useful in the context of fault-  
30 tolerance application. In this paper, we are investigating independent exact  
2-dominating sets of circulant graphs.

*State of art.* Minimum Dominating Set problem asks to find a dominating set of  
minimum cardinality. This problem is NP-complete for general graphs even  
when restricted to the class of planar graphs of maximum degree 3 [3]. When  
35 restricted to certain classes of graphs such as trees, we can get polynomial time  
algorithms even linear. Many authors address the problem in the class of circu-  
lant graphs [4, 5, 6, 7, 8]. In [5], Dejter and Serra give a method to construct  
countable family of circulant graphs each admitting an efficient dominating set.  
Obradovic *et al.* [6] give conditions based on the relationship between cordes  
40 length for Cayley graphs of degree at most 4, to admit efficient dominating set  
together with the structure of these sets. In [7] is given a necessary condition  
for a subgroup to be an efficient dominating set. In [9], Kumar and MacGillivray  
investigate efficient dominating sets in circulant graphs of large degree. They  
prove that if such graphs admit efficient dominating sets, then either its elements  
45 are equally spaced, or the graph is wreath product of a smaller circulant graph  
with an efficient dominating set and a complete graph. Yung-Ping [4] generalise  
the condition given by [6] together with techniques in [9] to completely describe  
efficient dominating set in circulant graphs with domination number prime.  
The problem of determining efficient dominating sets in circulant graphs is so  
50 largely study, but no general upper bounds are known for the domination num-  
ber of circulant graphs. In this paper, we are investigating the problem for  
graphs with generating sets of the form  $S = \{a, a + 1, \dots, a + k - 1, n - a, \dots, n -$   
 $(a + k - 1)\}$  where  $a, k, n \in \mathbb{N}$ . We also get independent number in certain cases  
of these graphs.

## 55 2. Preliminaries

In the article by Deng [4], the following theorem has been stated:

**Theorem 2.1** (Theorem 3.1, [4]). *Let  $n = |S_0|d$ , where  $d$  is an integer. Then  $\text{Circ}(n, S)$  admits an efficient dominating set  $D$  which is a coset of some subgroup of  $\mathbb{Z}_n$  if and only if  $s_i - s_j \not\equiv 0 \pmod{|S_0|}$ ,  $\forall s_i, s_j \in S_0 = \{0\} \cup S$ .*

60 **Remark 1.** *For any circulant graph  $G = \text{circ}(n, S)$  such that the connection set  $S$  is a symmetric generating set of  $\mathbb{Z}_n$  and  $D$  is a subgroup generated by  $d$  (usually noted  $D = \langle d \rangle$ ), we have:*

1.  $\forall s_i \in S, s_i \equiv 0 \pmod d$  is equivalent to the fact that vertex  $0 \in D$  is adjacent to vertex  $s_i$  which is also an element of  $D$ , hence, the set  $D$  is not independent.
- 65 2.  $\forall s_i, s_j \in S$ , if  $s_i - s_j \equiv 0 \pmod d$  then the vertices  $s_i$  and  $s_j$  have a common neighbor in  $D$ , hence the set  $D$  cannot be an exact 1-domination set (every vertex not in  $D$  has exactly one neighbor in  $D$ ).
- 70 3.  $\forall s_i, s_j \in S$ , if  $s_i - s_j \not\equiv 0 \pmod d$  then the vertices  $s_i$  and  $s_j$  are neither in  $D$  nor has common neighbors in  $D$ . Hence they are dominated by different vertices in  $D$  if we want  $D$  to be a dominating set.

**Remark 2.** *In the paper by Tamizh and Mutharasu[8], the theorem 2.4 stipulate that if a circulant graph  $G = \text{circ}(n, S)$  with  $S = \{a_1, a_2, \dots, a_k, n - a_1, \dots, n - a_k\}$  such that  $d_0 = a$ , if  $d_i$  are all same and  $2a_k + d$  divides  $n$  then  $G$  admits an efficient dominating set. It is easy to see that circulant graphs describe here are not connected when  $d \neq 1$ . In fact,  $a_k = kd$  since  $a_1 = d$  and so  $2a_k + d = (2k + 1)d$  then  $2a_k + d$  divides  $n$  implies that  $\gcd(n, S) \geq d \neq 1$ . We can reduce such cases of circulant graphs to general cases by just treating the connected components since these connected components are circulant graphs of the same degree. The corresponding subgraphs are all isomorphic to  $\text{circ}(n/d, S')$  with  $S' = a_i/d, a_i \in S$ . More generally, such reductions can always be applied when  $\gcd(n, S) \geq 1$ .*

### 3. Dominating sets of circulant graphs

Using the theorem 2.1, we obtain the following corollary.

85 **Corollary 1.** *Let  $n, k \in \mathbb{N}$  such that  $n = (2k + 1)d$ . For each  $a$  such that  $a \equiv 1 \pmod{2k + 1}$  or  $a \equiv k + 1 \pmod{2k + 1}$ ,  $G = \text{circ}(n, S)$ ,  $\gamma(G) = \frac{n}{2k + 1}$*

*Proof.* Since  $n \equiv 0 \pmod{2k + 1}$ , that is  $n = (2k + 1)d, d \in \mathbb{N}$ . In fact, we have for  $a \equiv 1 \pmod{2k + 1}$  or  $a \equiv k + 1 \pmod{2k + 1}$ , and  $S_0 = \{0\} \cup \{a, a + 1, \dots, a + k - 1, n - a, \dots, n - (k + a - 1)\}$  the evidence that  $\forall s_i, s_j \in S_0, s_i - s_j \not\equiv 0 \pmod{2k + 1}$ . By applying theorem 3.1 of [4], the conclusion follows.

□

It is a bit surprising that the domination of such circulant graph is not bounded by a constant integer ( $\gamma(G) \leq \frac{n}{2k + 1} + c$ ) when  $(2k + 1) \nmid n$  and generating set still those described in the previous corollary. The following theorem give an upper bounds for the domination number of circulant graphs where connection set contains consecutive numbers as describe early.

**Theorem 3.1.** Let  $a, k, n \in \mathbb{N}$  such that  $2k + 1 \leq n$ ,  $S = \{a, a + 1, \dots, a + k - 1, n - a, \dots, n - (a + k - 1)\}$ . Let  $d = \min\{2a - 1, n - 2a - 2k + 1\}$  and  $\gamma_0 = \lceil \frac{d}{k} \rceil + 1$ . The domination number of the circulant graph  $\text{circ}(n, S)$  is upper bounded by  $\gamma \leq \gamma_0 \lceil \frac{n}{(2a-1)+(\gamma_0+1)k} \rceil$ . 100

To prove this theorem we will construct a dominating set of these graphs.

*Proof.* First of all, suppose that  $2a - 1$  is the minimum of the two quantities that is  $d = \min\{2a - 1, n - 2a - 2k + 1\} = 2a - 1$ . Without loss of generality, suppose that the vertex  $0 \in D$ , the dominating set been in construction. By adding 0 as a dominator, vertices  $[a, a + k - 1]$  and  $[n - a, n - (a + k - 1)]$  are dominated together by 0. To dominate vertices lying into the set  $I_0 = [1, a - 1] \cup [n - (a - 1), n - 1]$  we use the following strategy: take vertices  $v_i = v_{i-1} + k$  with  $v_0 = 0$  until all vertices of  $I_0$  are dominated. Let  $\gamma_0 = 1 + \lceil \frac{2a-1}{k} \rceil$ .  $\gamma_0$  is the number of vertices to use to dominate all vertices in  $I_0$ . Moreover, vertices  $v_i$  also dominate  $(\gamma_0 + 1)k$  vertices other than the one in  $I_0$ . Finally, the number of vertices dominated by the set  $V = \{v_i | 0 \leq i \leq \gamma_0 - 1\}$  is  $nd = (2a - 1) + (\gamma_0 + 1)k$ . To dominate the rest of the graph, let consider the sets  $I_1$  to  $I_d$  with  $d = \lceil \frac{n}{nd} \rceil = \lceil \frac{n}{(2a-1)+(\gamma_0+1)k} \rceil$ . Vertices each  $I_i, i = 1..d - 2$  can be dominated with the same process the one used for  $I_0$  with the vertex 0 replaced by vertex .... of each  $I_i$ . Doing so, all  $I_i$   $0 \leq i \leq d - 2$  have the same number of dominators namely  $\gamma_0$ . For the sets  $I_{d-1}$ , the number of dominators is less or equal  $\gamma_0$ . Since each set  $I_i$  described bellow has at most  $\gamma_0$  dominators, the whole graphs has at most  $\gamma_0 d$  dominators hence,  $\gamma(G) \leq d\gamma_0 = \gamma_0 \lceil \frac{n}{(2a-1)+(\gamma_0+1)k} \rceil$  □

As special cases of graphs describe in the preceding theorem, the next theorems gived most tight bounds for circulant graphs with connection set containing small consecutive numbers relatively to  $(a \leq k + 1 \leq n)$ . In theorem 3.2 the dominating sets found have equally spaced vertices but are not independent. 120

**Theorem 3.2.** Let  $n, k, a \in \mathbb{N}$  s.t  $\lceil \frac{k}{2} \rceil + 1 \leq a \leq k + 1$ ,  $a + k \leq \frac{n+1}{2}$  and  $S = \{a, a + 1, \dots, a + k - 1, n - a, n - (a + 1), \dots, n - (a + k - 1)\}$ . The domination number of the circulant graph  $G = \text{circ}(n, S)$  is upper-bounded by  $\gamma(G) \leq \lceil \frac{n}{k+a} \rceil$ . 125

*Proof.* Let  $d = \lceil \frac{n}{k+a} \rceil$  and  $D = \{0, k + a, 2(k + a), \dots, (d - 1)(k + a)\}$ . Vertices of the graph  $G = \text{Circ}(n, S)$  can be partioned into sets  $I_0 = [1, a + k - 1]$ ,  $I_1 = [a + k + 1, 2(a + k) - 1]$ , ...,  $I_{d-2} = [(d - 2)(a + k) + 1, (d - 1)(a + k) - 1]$ ,  $I_{d-1} = [(d - 1)(a + k) + 1, n - 1]$ . For each  $0 \leq i \leq d - 2$ , element of  $I_i$  are dominated by  $i(k + a)$  and  $(i + 1)(k + a)$ . More precisely, the set  $I_i, 0 \leq i \leq d - 2$ , contains two subparts. the upper side  $[i(a + k) + a, (i + 1)(a + k) - 1]$  dominated by the vertex  $i(a + k)$  and the lower part  $[i(a + k), i(a + k) + k]$  dominated by the vertex  $(i + 1)(a + k)$ . for the set  $I_{d-1}$  □

**Corollary 2.** Let  $n, k, a \in \mathbb{N}$  s.t  $\lceil \frac{k}{2} \rceil + 1 \leq a \leq k + 1$ ,  $a + k \leq \frac{n+1}{2}$  and  $S = \{a, a + 1, \dots, a + k - 1, n - a, n - (a + 1), \dots, n - (a + k - 1)\}$ . If  $n$  divides  $k + a$  then  $\gamma(G) = \frac{n}{k+a}$  and the subgroups  $\langle k + a \rangle$  is an independent domination set for  $\text{Circ}(n, S)$ . 135

*Proof.* Let  $d = \frac{n}{k+a}$  and  $D = \langle k+a \rangle = \{0, k+a, 2(k+a), \dots, (d-1)(k+a)\}$ .  
 140 As we did in the proof of theorem 3.2, we partitioned the vertices of the graph  
 into following sets  $I_0 = [1, a+k-1]$ ,  $I_1 = [a+k+1, 2(a+k)-1]$ , ...,  $I_{d-2} =$   
 $[(d-2)(a+k)+1, (d-1)(a+k)-1]$ ,  $I_{d-1} = [(d-1)(a+k)+1, n-1]$  with  
 $n-1 = d(a+k)-1$ . For each  $0 \leq i \leq d-2$ , element of  $I_i$  are dominated by  
 $i(k+a)$  and  $(i+1)(k+a)$ . Thus  $D$  is a dominating set. To see the independence  
 145 of the set  $D$ , one can use the remark 1. In fact it is easy to see that between  $a$   
 and  $a+k-1$  there is no divisor of  $k+a$  since these numbers are all less than  
 $k+a$  in the settings of the theorem.  $\square$

We assume the minimality of domination sets found in the two preceding  
 theorems even in those coming down. However, we think it is provable using  
 150 remark 1 and techniques inspired by Yung[4].

**Theorem 3.3.** *Let  $n, k, a \in \mathbb{N}$  s.t  $2 \leq a \leq \lceil \frac{k}{2} \rceil$ ,  $n = 2(3k+2a-1)d$ ,  $d \in \mathbb{N}$  and  
 $S = \{a, a+1, \dots, a+k-1, n-a, \dots, n-(a+k-1)\}$ . The domination number  
 of  $Circ(n, S)$  is  $\gamma(G) = 2d$  and  $\langle 3k+2a-1 \rangle \cup \langle 3k+2a-1 \rangle + k$  is a  
 dominating set.*

*Proof.* Let  $d = \frac{n}{3k+2a-1}$  and  $v = \lceil \frac{d-1}{2} \rceil$ . Vertices of  $G$  can be partioned into  
 155 sets  $I_i$  as follows.  $I_0 = [0, 3k+2a-1]$ ,  $I_1 = [3k+2a, 2(3k+2a-1)-1]$ ,  $I_{v-2} =$   
 $[(v-2)(3k+2a-1)+1, (v-1)(3k+2a-1)]$  and  $I_{v-1} = [(v-1)(3k+2a-1)+1, n-1]$   
 where  $n-1 = v(3k+2a-1)-1$ . Each interval  $I_i = [i(3k+2a-1), (i+1)(3k+2a-1)-1]$ ,  $0 \leq i \leq v-1$  is dominated completely by two vertices  
 160 namely  $x_{i1} = i(3k+2a-1) + k+a-1$  and  $x_{i2} = i(3k+2a-1) + 2k+a-1$ .  
 Since Cayley graphs are vertex-transitive, we remove  $(4k+3a-2)$  from each  
 dominator we will get the set describes in the theorem.  $\square$

The following theorem generalise the situations of theorem 3.3 to situations

165 .

**Theorem 3.4.**

#### 4. Subgroups as independent domination sets

Dominating sets of graphs described in the theorem 3.3 are not independent  
 dominating sets. To obtain independent dominating sets in certain cases, it is  
 170 possible to use the following theorem.

**Theorem 4.1.** *Let  $n, k, a \in \mathbb{N}$  such that  $k+a$  divides  $n$ ,  $a \leq k+1$  and  $n \geq 2k+1$ .  
 Let  $S = \{a, a+1, \dots, a+k-1, n-a, \dots, n-(a+k-1)\}$  and  $G = Circ(n, S)$ .  
 The independent domination number of  $G$  is  $i(G) = \frac{n}{k+a}$ .*

*Proof.* Using techniques of theorem 3.2, it is easy to show that the subgroup  
 175 generated by  $k+a$  is a dominating sets. to see the indepenence, recall that these  
 sets have equally spaced vertices hence, the distance between two consecutive  
 elements is allways the same. This distance has the value  $a+k$  which is greater

than the maximum distance between two adjacent vertices  $(a+k-1)$ . Then the subgroup  $\langle a+k \rangle$  is also an independent sets for these circulant graphs.  $\square$

180 The theorem bellow gives a constructive tool to get circulant graphs that admit subgroups as independent domination sets. We thing that it is possible to give a characterization as those gives in [4] for circulant graphs admitting subgroups as independent dominating set. The next theorem will give a carac-  
185 terisation of those graphs with independent dominating sets that are subgroups and their cosets. Using notations of theorem 2.1, we can make the following remarks. Base on these remarks, we can formulate the following theorem.

**Theorem 4.2.** *Let  $n \in \mathbb{Z}$ ,  $S$  with  $|S| = 2k$  a symmetric generating set of  $\mathbb{Z}_n$ , and  $D$  be a subgroup of  $\mathbb{Z}_n$  generated by some integer  $d = \max_{a=2}^{k+1} \{k+a\}$ .  $D$  is an independent dominating set if the following conditions are meet.*

- 190 1.  $\forall s_i \in S, s_i \neq 0 \pmod d$ ;  
2. the number of tuples  $s_i, s_j$  such that  $s_i - s_j \neq 0 \pmod d$  is at least  $2d$

The prof of this theorem is based on the reamark 1. The first condition ensures the independence of the subgroup and the second ensures that the set  $D$  is a dominating set. It worth noticed that in certain cases, well choosed  
195  $D$  will also be a minimum dominating set. Theorem 3.1 describes such cases for graph with generating sets that are consecutive integers. The following corollary complete theorem 3.1 and theorem 4.2 by giving a criterion like the one in corollary 1 for circulant graphs with generating that are consecutive integers to admit subgroups as minimum independent set.

200 **Corollary 3.** *Let  $n, a, k \in \mathbb{N}$  such that  $a+k \leq \frac{n+1}{2}$ ,  $a \leq k+1$ , and  $(k+a) \mid n$ . Let  $a' \in \mathbb{N}$  such that  $a' \equiv a \pmod{(a+k)}$  or  $a' \equiv 1 \pmod{(a+k)}$ , and  $a'+k \leq \frac{n+1}{2}$ . The circulant graph  $\text{Circ}(n, S)$  where  $S = \{a', a'+1, \dots, a'+k-1, n-a', \dots, n-(a'+k-1)\}$  admits the subgroup generated by  $a+k$  and its cosets as minimum independent dominating set.*

205 It is easy to see that this theorem is clearly an extension of corollary 1, in fact, by setting  $a = k+1$  we clearly get the corollary. In this corollary, if the order  $n$  of the graph is not a multiple of

## 5. Conclusion and future work

In this work we give some bounds for the domination number of circulant  
210 graphs where connection sets are consecutive integers. We got some subgroups that are minimum dominating sets and independent dominating sets for these circulant graphs. We are planning to use these results and those in [4] in the perspective of coding theory. The problem of finding domination sets of circulant graphs remains not completely solve. Is this problem solvable or MDS is also  
215 NP-Complete for circulant graphs?

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