Dominating set in circulant graphs
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Abstract

A Cayley graph on a group $\Gamma$ with the generating set $S \subset \Gamma$ is a graph constructed out of $\Gamma$. A circulant graph is a Cayley graph on a cyclic group denoted by $\text{Circ}(n, S)$. In this paper, we address the problem domination in circulant graphs. We first give upper bounds for the domination number of certain circulant graphs. The domination number of certain circulant graphs is obtained together with a corresponding dominating set. We also obtain the independent domination number and the total domination number for special cases of these circulant graphs.

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1. Introduction

Cayley graphs are useful in various domains of theoretical computer sciences such as computer architecture, routing problem, networking and coding theory. In the coding theory, the most interesting graphs are those having perfect e-codes, particularly regular graphs, vertex transitive graphs and distance transitive graphs. Perfect 1-codes were introduced by Biggs[1] in the context of error correcting codes. A perfect 1-code is a single error correcting code, that is, it permits to correct one error. In the context of graph theory, a perfect 1-code is called an efficient dominating set. This is an independent dominating set in which every vertex not in the set is dominated by exactly one vertex of the set. Independent sets are also useful in the perspective of coding theory. In [2] for example, Chellali et al. introduced the independent $[1,k]$-sets and state that these sets are useful in the context of error correcting codes as well.
fact, an independent \([1,2]\)-set will permit to correct most errors and those not
corrected come from at most 2 sources. Cayley graphs are such regular graphs
usable in coding theory. As example, hypercubes are used to represent Ham-
ing codes. They are define out of groups. In fact, for a group (even infinite) \(\Gamma\)
and a set \(S \subseteq \Gamma\) such that \(s \in S \equiv s^{-1} \in S\) and \(e(\Gamma) \notin S\), the Cayley graph
\(G\) on \(\Gamma\) with the generating set \(S \subseteq \Gamma\) is a graph with vertex set elements of \(\Gamma\)
\((V)\) and edge set \(E(G) = \{(g, gs) \mid g \in \Gamma, s \in S\}\). Cayley graphs on finite cyclic
groups are called circulant graphs and denoted by \(Circ(n, S)\).

Definitions. In a graph \(G = (V, E)\), a dominating is a set \(D\) of vertices such
that every vertex of \(G\) is either in \(D\) or has a neighbor in \(D\). The DOMINATING
SET problem ask to find in a graph \(G = (V, E)\) a dominating set with minimum
cardinality denoted \(\gamma(G)\). A dominating set is efficient if it is independent (i.e.
no two vertices of \(D\) are adjacent) and each vertex \(v \notin D\) has exactly one
neighbor in \(D\). More generally, an independent exact \(k\)-dominating set \(D\) of
a graph \(G = (V, E)\) is a set of vertices such that every vertex not in \(D\) has
exactly \(k\) neighbors in \(D\). \(k\)-dominating set are useful in the context of fault-
tolerance application. In this paper, we are investigating independent exacts
2-dominating sets of circulant graphs.

State of art. Minimum Dominating SET problem ask to find a dominating set of
minimum cardinality. This problem is an NP-complete for general graphs even
when restricted to the class of planar graphs of maximum degree 3\(^3\). When
restricted to certain classes of graphs such as trees, we can get polynomial time
algorithms even linear. Many authors address the problem in the class of circu-
lant graphs \([1, 5, 6, 7, 8]\). In \([5]\), Dejter and Serra give a method to construct
countable family of circulant graphs each admitting an efficient dominating set.
Obradovic et al.\([6]\) give conditions based on the relationship between cords
length for cayley graphs of degree at most 4, to admit efficient dominating set
together with the structure of these sets. In \([7]\) is given a necessarily condition
for a subgroup to be an efficient dominating set. In \([9]\), Kumar and MacGillivray
investigate efficient dominating sets in circulant graphs of large degree. They
prove that if such graphs admit efficient dominating sets, then either its elements
are equally spaced, or the graph is wreath product of a smaller circulant graph
with an efficient dominating set and a complete graph. Yung-Ping\([4]\) generalise
the condition given by \([6]\) together with techniques in \([9]\) to completely describe
efficient dominating set in circulant graphs with domination number prime.
The problem of determining efficient dominating sets in circulant graphs is so
largely study, but no general upper bounds are known for the domination num-
ber of circulant graphs. In this paper, we are investigating the problem for
graphs with generating sets of the form \(S = \{a, a + 1, \ldots, a + k - 1, n - a, \ldots, n -
(a + k - 1)\}\) where \(a, k, n \in \mathbb{N}\). We also get independent number in certain cases of
these graphs.

2. Preliminaries

In the article by Deng \([4]\), the following theorem has been stated:
Theorem 2.1 (Theorem 3.1, [3]). Let \( n = |S_0|d \), where \( d \) is an integer. Then \( \text{Circ}(n, S) \) admits an efficient dominating set \( D \) which is a coset of some subgroup of \( \mathbb{Z}_n \) if and only if \( s_i - s_j \neq 0 \mod (|S_0|) \), \( \forall s_i, s_j \in S_0 = \{0\} \cup S \).

Remark 1. For any circulant graph \( G = \text{circ}(n, S) \) such that the connection set \( S \) is a symmetric generating set of \( \mathbb{Z}_n \) and \( D \) is a subgroup generated by \( d \) (usually noted \( D = \langle d \rangle \)), we have:

1. \( \forall s_i \in S, s_i \equiv 0 \mod d \) is equivalent to the fact that vertex 0 \( \in D \) is adjacent to vertex \( s_i \), which is also an element of \( D \), hence, the set \( D \) is not independent.
2. \( \forall s_i, s_j \in S, \) if \( s_i - s_j \equiv 0 \mod d \) then the vertices \( s_i \) and \( s_j \) have a common neighbor in \( D \), hence the set \( D \) cannot be an exact 1-domination set (every vertex not in \( D \) has exactly one neighbor in \( D \)).
3. \( \forall s_i, s_j \in S, \) if \( s_i - s_j \neq 0 \mod d \) then the vertices \( s_i \) and \( s_j \) are neither in \( D \) nor have common neighbors in \( D \). Hence they are dominated by different vertices in \( D \) if we want \( D \) to be a dominating set.

Remark 2. In the paper by Tamizh and Matharasi [3], the theorem 2.4 stipulate that if a circulant graph \( G = \text{circ}(n, S) \) with \( S = \{a_1, a_2, \ldots, a_k, n - a_1, \ldots, n - a_k\} \) such that \( d_0 = a_1 \), if \( d_i \) are all same and \( 2a_k + d \) divides \( n \) then \( G \) admits an efficient dominating set. It is easy to see that circulant graphs describe here are not connected when \( d \neq 1 \). In fact, \( a_k = kd \) since \( a_1 = d \) and so \( 2a_k + d = (2k + 1)d \) then \( 2a_k + d \) divides \( n \) implies that \( \gcd(n, S) \geq d \neq 1 \). We can reduce such cases of circulant graphs to general cases by just treating the connected components since these connected components are circulant graphs of the same degree. The corresponding subgraphs are all isomorphic to \( \text{circ}(n/d, S') \) with \( S' = a_i/d, a_i \in S \). More generally, such reductions can always be applied when \( \gcd(n, S) \geq 1 \).

3. Dominating sets of circulant graphs

Using the theorem 2.1, we obtain the following corollary.

Corollary 1. Let \( n, k \in \mathbb{N} \) such that \( n = (2k + 1)d \). For each \( a \) such that \( a \equiv 1 \mod (2k + 1) \) or \( a \equiv k + 1 \mod (2k + 1) \), \( G = \text{circ}(n, S), \gamma(G) = \frac{n}{2k + 1} \).

Proof. Since \( n \equiv 0 \mod (2k + 1) \), that is \( n = (2k + 1)d, d \in \mathbb{N} \). In fact, we have for \( a \equiv 1 \mod (2k + 1) \) or \( a \equiv k + 1 \mod (2k + 1) \), and \( S_0 = \{0\} \cup \{a, a + 1, \ldots, a + k - 1, n - a, \ldots, n - (k + a - 1)\} \) the evidence that \( \forall s_i, s_j \in S_0, s_i - s_j \neq 0 \mod (2k + 1) \). By applying theorem 3.1 of [3], the conclusion follows.

It is a bit surprising that the domination of such circulant graph is not bounded by a constant integer \( \gamma(G) \leq \frac{n}{2k + 1} + c \) when \( (2k + 1) \nmid n \) and generating set still those described in the preview corollary. The following theorem give an upper bounds for the domination number of circulant graphs where connection set contains consecutive numbers as describe early.
Theorem 3.1. Let \( a, k, n \in \mathbb{N} \) such that \( 2k + 1 \leq n \), \( S = \{a, a+1, ..., a+k-1, n-a, ..., n-(a+k-1)\} \). Let \( d = \min\{2a - 1, n - 2a - 2k + 1\} \) and \( \gamma_0 = \lceil \frac{n}{2} \rceil + 1 \). The domination number of the circulant graph \( \text{circ}(n, S) \) is upper bounded by \( \gamma \leq \gamma_0 \left[ \frac{n}{(2a - 1) + (\gamma_0 - 1)k} \right] \).

To prove this theorem we will construct a dominating set of these graphs.

Proof. First of all, suppose that \( 2a - 1 \) is the minimum of the two quantities that is \( d = \min\{2a - 1, n - 2a - 2k + 1\} = 2a - 1 \). Without loss of generality, suppose that the vertex 0 \( \in D \), the dominating set been in construction. By adding 0 as a dominator, vertices \( [a, a+k-1] \) and \([n-a, n-(a+k-1)]\) are dominated together by 0. To dominate vertices lying into the set \( I_0 = [1, a-1] \cup [n-(a-1), n-1] \) we use the following strategy: take vertices \( v_i = v_{i-1} + k \) with \( v_0 = 0 \) until all vertices of \( I_0 \) are dominated. Let \( \gamma_0 = 1 + \lceil \frac{2a-1}{k} \rceil \). \( \gamma_0 \) is the number of vertices to use to dominate all vertices in \( I_0 \). Moreover, vertices \( v_i \) also dominate \( (\gamma_0 + 1)k \) vertices other than the one in \( I_0 \). Finally, the number of vertices dominated by the set \( V = \{v_i | 0 \leq i \leq \gamma_0 - 1\} \) is \( nd = (2a - 1) + (\gamma_0 + 1)k \)

To dominate the rest of the graph, let consider the sets \( I_1 \) to \( I_d \) with \( d = \lceil \frac{n}{\gamma_0} \rceil = \left\lfloor \frac{n}{(2a - 1) + (\gamma_0 - 1)k} \right\rfloor \). Vertices each \( I_i \), \( i = 1, d - 2 \) can be dominated with the same process the one used for \( I_0 \) with the vertex 0 replaced by vertex ... of each \( I_i \). Doing so, all \( I_i \) \( 0 \leq i \leq d - 2 \) have the same number of dominators namely \( \gamma_0 \). For the sets \( I_d - 1 \), the number of dominators is less or equal \( \gamma_0 \). Since each set \( I_i \) described bellow has at most \( \gamma_0 \) dominators, the whole graphs has at most \( \gamma_0 d \) dominators hence, \( \gamma(G) \leq d\gamma_0 = \gamma_0 \left[ \frac{n}{(2a-1) + (\gamma_0 - 1)k} \right] \)

As special cases of graphs describe in the preceding theorem, the next theorems give most tight bounds for circulant graphs with connection set containing small consecutive numbers relatively to \( a \leq k + 1 \leq n \). In theorem 3.2 the dominating sets found have equally spaced vertices but are not independent.

Theorem 3.2. Let \( n, k, a \in \mathbb{N} \) s.t \( \lceil \frac{n}{a+1} \rceil + 1 \leq a \leq k + 1 \), \( a + k \leq \frac{n+1}{2} \) and \( S = \{a, a+1, ..., a+k-1, n-a, n-(a+1), ..., n-(a+k-1)\} \). The domination number of the circulant graph \( G = \text{circ}(n, S) \) is upper-bounded by \( \gamma(G) \leq \lceil \frac{n}{k+a} \rceil \).

Proof. Let \( d = \lceil \frac{n}{k+a} \rceil \) and \( D = \{0, k+a, 2(k+a), ..., (d-1)(k+a)\} \). Vertices of the graph \( G = \text{circ}(n, S) \) can be partitioned into sets \( I_0 = [1, a+k-1] \), \( I_1 = [a+k+1, 2(a+k)-1], ..., I_{d-2} = [(d-2)(a+k)+1, (d-1)(a+k)-1], I_{d-1} = [(d-1)(a+k) + 1, n-1] \). For each \( 0 \leq i \leq d - 2 \), element of \( I_i \) are dominated by \( i(k+a) \) and \( (i+1)(k+a) \). More precisely, the set \( I_i, 0 \leq i \leq d - 2 \), contains two subparts: the upper side \( i(a+k) + a, (i+1)(a+k) - 1 \) dominated by the vertice \( i(a+k) \) and the lower part \( i(a+k), i(a+k) + k \) dominated by the vertice \( (i+1)(a+k) \) for the set \( I_{d-1} \)

Corollary 2. Let \( n, k, a \in \mathbb{N} \) s.t \( \lceil \frac{n}{k+a} \rceil + 1 \leq a \leq k + 1 \), \( a + k \leq \frac{n+1}{2} \) and \( S = \{a, a+1, ..., a+k-1, n-a, n-(a+1), ..., n-(a+k-1)\} \). If \( n \) divides \( k + a \) then \( \gamma(G) = \frac{n}{k+a} \) and the subgroups \( < k + a > \) is an independent domination set for \( \text{Circ}(n, S) \).
Let $d = \frac{n}{k+a}$ and $D = < k+a >= \{0, k+a, 2(k+a), ..., (d-1)(k+a)\}$. As we did in the proof of theorem 3.2, we partitioned the vertices of the graph into following sets $I_0 = [1, a+k-1]$, $I_1 = [a+k+1, 2(a+k) - 1]$, ..., $I_{d-2} = [(d-2)(a+k) + 1, (d-1)(a+k)-1]$, $I_{d-1} = [(d-1)(a+k) + 1, n-1]$ with $n-1 = d(a+k) - 1$. For each $0 \leq i \leq d-2$, element of $I_i$ are dominated by $i(k+a)$ and $(i+1)(k+a)$. Thus $D$ is a dominating set. To see the independence of the set $D$, one can use the remark 1. In fact it is easy to see that between $a$ and $a+k-1$ there is no divisor of $k+a$ since these numbers are all less than $k+a$ in the settings of the theorem.

We assume the minimality of domination sets found in the two preceding theorems even in those coming down. However, we think it is provable using remark 1 and techniques inspired by Yung.  

Theorem 3.3. Let $n, k, a \in \mathbb{N}$ s.t $2 \leq a \leq \left\lfloor \frac{n}{k} \right\rfloor$, $n = 2(3k+2a-1)d, d \in \mathbb{N}$ and $S = \{a, a+1, ..., a+k-1, n-a, ..., n-(a+k-1)\}$. The domination number of $Circ(n, S)$ is $\gamma(G) = 2d$ and $< 3k+2a-1 > \cup < 3k+2a-1 > + k$ is a dominating set.

Proof. Let $d = \frac{n}{3k+2a-1}$ and $v = \left\lceil \frac{n-1}{2} \right\rceil$. Vertices of $G$ can be partitioned into sets $I_i$ as follows. $I_0 = [0, 3k+2a-1]$, $I_1 = [3k+2a, 2(3k+2a-1)-1]$, $I_{d-2} = [(v-2)(3k+2a-1) + 1, (v-1)(3k+2a-1)]$ and $I_{d-1} = [(v-1)(3k+2a-1)+1, n-1]$ where $n = v(3k+2a-1) - 1$. Each interval $I_i = [i(3k+2a-1), (i+1)(3k+2a-1)-1], 0 \leq i \leq v-1$ is dominated completely by two vertices namely $x_{i1} = i(3k+2a-1)+k+a-1$ and $x_{i2} = i(3k+2a-1)+2k+a-1$. Since Cayley graphs are vertex-transitive, we remove $(4k+3a-2)$ from each dominator we will get the set describes in the theorem.

The following theorem generalise the situations of theorem 3.3 to situations 3.3 .

Theorem 3.4.

4. Subgroups as independent domination sets

Dominating sets of graphs described in the theorem 3.3 are not independent dominating sets. To obtain independent dominating sets in certain cases, it is possible to use the following theorem.

Theorem 4.1. Let $n, k, a \in \mathbb{N}$ such that $k+a$ divides $n$, $a \leq k+1$ and $n \geq 2k+1$. Let $S = \{a, a+1, ..., a+k-1, n-a, ..., n-(a+k-1)\}$ and $G = Circ(n, S)$. The independent domination number of $G$ is $\gamma(G) = \frac{n}{k+a}$.

Proof. Using techniques of theorem 3.2 it is easy to show that the subgroup generated by $k+a$ is a dominating sets. to see the independency, recall that these sets have equally spaced vertices hence, the distance between two consecutive elements is always the same. This distance has the value $a+k$ which is greater
than the maximum distance between two adjacent vertices \((a + k - 1)\). Then the subgroup \(<a + k>\) is also an independent sets for these circulant graphs.

The theorem bellow gives a constructive tool to get circulant graphs that admit subgroups as independent domination sets. We thing that it is possible to give a caracterization as those gives in \([4]\) for circulant graphs admitting subgroups as independent dominating set. The next theorem will give a caracterisation of those graphs with independent dominating sets that are subgroups and their cosets. Using notations of theorem \([2,1]\), we can make the following remarks. Base on these remarks, we can formulate the following theorem.

**Theorem 4.2.** Let \(n \in \mathbb{Z}, S\) with \(|S| = 2k\) a symmetric generating set of \(\mathbb{Z}_n\), and \(D\) be a subgroup of \(\mathbb{Z}_n\) generated by some integer \(d = \max_{a \leq 2} (k + a)\). \(D\) is an independent dominating set if the following conditions are meet.

1. \(\forall s_i \in S, s_i \neq 0 \mod d\);
2. the number of tuples \(s_i, s_j\) such that \(s_i - s_j \neq 0 \mod d\) is at least \(2d\)

The prof of this theorem is based on the reamark \([1]\). The first condition ensures the independence of the subgroup and the second ensures that the set \(D\) is a dominating set. It worth noticed that in certain cases, well choosed \(D\) will also be a minimum dominating set. Theorem \([3,1]\) describes such cases for graph with generating sets that are consecutive integers. The following corollary complete theorem \([3,1]\) and theorem \([4,2]\) by giving a criterion like the one in corollary \([1]\) for circulant graphs with generating that are consecutive integers to admit subgroups as minimum independent set.

**Corollary 3.** Let \(n, a, k \in \mathbb{N}\) such that \(a + k \leq \frac{n+1}{2}, a \leq k + 1,\) and \((k + a) | n\). Let \(a' \in \mathbb{N}\) such that \(a' \equiv a \mod (a + k)\) or \(a' \equiv 1 \mod (a + k)\), and \(a' + k \leq \frac{n+1}{2}\). The circulant graph \(\text{Circ}(n, S)\) where \(S = \{a', a' + 1, ..., a' + k - 1, n - a', ..., n - (a' + k - 1)\}\) admits the subgroup generated by \(a + k\) and its cosets as minimum independent dominating set.

It is easy to see that this theorem is clearly an extension of corollary \([1]\). In fact, by setting \(a = k + 1\) we clearly get the corollary. In this corollary, if the order \(n\) of the graph is not a multiple of

5. **Conclusion and future work**

In this work we give some bounds for the domination number of circulant graphs where connection sets are consecutive integers. We got some subgroups that are minimum dominating sets and independent dominating sets for these circulant graphs. We are planning to use these results and those in \([4]\) in the perspective of coding theory. The problem of finding domination sets of circulant graphs remains not completely solve. Is this problem solvable or MDS is also NP-Complete for circulant graphs?
References


