EASY ISING OR CURIE-WEISS MODEL EXPLAINED TO ERGODICISTS

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Abstract. We state a dictionary for thermodynamic formalism within the Curie-Weiss model between the Probability-Statistical Mechanics and the Ergodic viewpoints. Limits for probabilistic Gibbs measures are identified as combinations of, say $N(\beta)$, dynamical conformal measures. Phase transitions are then related to a change in $N(\beta)$ as β moves.

More surprising we point out that, to be closer to what is done in Statistical Mechanics, ergodicists should study the supremum of the sum of the entropy and the square of the integral of the potential instead of the sum of the entropy and the integral of the potential.

1. Introduction

1.1. **Background, main motivations and results.** The notion of Gibbs measure comes from Statistical Mechanics. It has been studied a lot from the probabilistic viewpoint (see [13, 7, 9, 10]). This notion was introduced in Ergodic Theory in the 70's by Sinai, Ruelle and Bowen (see [24, 25, 23, 22, 3]). Since that moment, the thermodynamic formalism became in Dynamical Systems a pure mathematical question and somehow disconnected from the original physical questions.

Since the 00's, ergodicists started to reconnect the questions in Dynamical Systems with the original physical ones. Notions as ground states and phase transitions were re-defined and investigated from the ergodic viewpoint. From that moment, the sharing of the vocabulary started to be a source of confusion, and the main motivation for this paper is to make clear some similitudes and differences between Gibbs measures within the probabilistic or the ergodic viewpoints.

Beyond stating a kind of dictionary between thermodynamic formalisms in Statistical Mechanics and Probability on the one hand, and Ergodic Theory on the other hand, the main byproduct of our work is a very strange discovery within the ergodic viewpoint.

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Actually, and using the Ergodic viewpoint and vocabulary, Theorem 2 shows that, in Statistical Mechanics and Probability, one maximizes the entropy plus the square of an integral, whereas in Ergodic Theory one maximizes the entropy plus an integral.

This has a lot of nice consequences and further possible research questions. Some of them are discussed later (see Subsubsection 1.2.3).

Our dictionary works as follows. In Statistical Mechanics and in Probability theory, the system is a finite set of sites. There is an interaction between sites given by an *Hamiltonian*. A *Gibbs measure* is a probability on the finite set of configurations which is optimal with respect to some quantity. All the objects depend on a parameter β equal to the inverse of the temperature. Then, the issue is to define the possible accumulation points for the Gibbs measures as the number of sites goes to $+\infty$. A *phase transition* occurs if, as moving the parameter β, this set of accumulation points admits a discontinuity for some β_c.

In Ergodic Theory (on symbolic dynamics) one immediately considers the set of infinite configurations Σ. Instead of having an Hamiltonian one considers a *potential* $\psi : \Sigma \to \mathbb{R}$. We consider here the case where ψ is Hölder continuous. The Gibbs measure is an invariant measure, also called equilibrium state, which maximizes the *pressure* (see below). The pressure function is the pressure for $\beta.\psi$ considered as a function of β. Then, a phase transition occurs if the pressure function is not analytic at some β_c. It is noteworthy that in this setting, the number of equilibrium states is almost independent of the regularity of the pressure function (see [27, 18]).

Roughly speaking, our dictionary identifies accumulation points of Gibbs measures in Probability with conformal measures in Ergodic Theory. Conformal measures are not Gibbs measures (within the Ergodic viewpoint) but are strongly related to them. More precisely, we show that for the Curie-Weiss model (see Theorem 1) and the Curie-Weiss-Potts model (see Theorem 3) the Gibbs measures converge to a convex combination of conformal measures. A (probabilistic) phase transition occurs when the number of conformal measures in the limit changes. This notion of phase transition is thus different from the one used in Ergodic Theory.

Although some of the convergences we prove here were already known (see [19]), we point out that the identification of the limit with ergodic quantities was not stated. We claim that the novelty here does not concern the convergence but the dictionary and identification/similitude/differences between Gibbs measures and phase transitions in Probability Theory or Ergodic Theory. We point out that this dictionary works well for the Curie-Weiss and Curie-Weiss-Potts models probably because for these models, the Hamiltonian turns out to be easily writable as a function of a Birkhoff sum. This is the core of Theorem 2.

At last, we mention works of Cioletti and Lopes (see [5, 4, 6]). There, for different settings (non Hölder continuous potentials and not for the Curie-Weiss models), the

connection between DLR-Gibbs measures (within the Statistical Mechanics viewpoint) and the conformal measures (within the Ergodic viewpoint) is also done.

1.2. **Precise settings and results.**

1.2.1. *Ergodic and Dynamical settings.* We consider a finite set Λ with cardinality bigger or equal to 2. It is called the alphabet. Then we consider the one-sided full shift $\Sigma = \Lambda^{\mathbb{N}}$ over Λ. A point x in Σ is a sequence $x_0, x_1, \dots$ (also called an infinite word) where the x_i are in Λ. Most of the times we shall use the notation $x = x_0x_1x_2\dots$. A $x_i \in \Lambda$ can either be called a letter, or a digit or a symbol.

The shift map σ is defined by

$$\sigma(x_0x_1x_2\dots) = x_1x_2\dots.$$

The distance between two points $x = x_0x_1\dots$ and $y = y_0y_1\dots$ is given by

$$d(x,y) = \frac{1}{2^{\min\{n,\ x_n \neq y_n\}}}.$$

A finite string of symbols $x_0\dots x_{n-1}$ is also called a *word*, of length n. For a word w, its length is $|w|$. A *cylinder* (of length n) is denoted by $[x_0\dots x_{n-1}]$. It is the set of points y such that $y_i = x_i$ for $i = 0, \dots n-1$. We shall also talk about n-cylinder instead of cylinder of length n.

If w is the word of finite length $w_0\dots w_{n-1}$ and x is a word, the concatenation wx is the new word $w_0w_1\dots w_{n-1}x_0x_1\dots$.

For $\psi : \Sigma \to \mathbb{R}$ continuous and $\beta > 0$, the *pressure function* is defined by

$$\mathcal{P}(\beta\psi) := \sup_{\mu} \left\{ h_\mu + \beta \int_\Sigma \psi \, d\mu \right\}, \tag{1}$$

where the supremum is taken among the set $\mathcal{M}_\sigma(\Sigma)$ of σ-invariant probabilities on Σ and h_μ is the Kolmogorov-Sinaï entropy of μ. The real parameter β is assumed to be positive because it represents the inverse of the temperature in statistical mechanics. It is known that the supremum is actually a maximum and any measure for which the maximum is attained in (1) is called an *equilibrium state for* $\beta\psi$. We refer the reader to [3, 23] for basic notions on thermodynamic formalism in ergodic theory.

If ψ is Lipschitz continuous then the Ruelle-Griffith theorem (see [14]) states that for every β, there is a unique equilibrium state for $\beta\psi$, which is denoted by $\widetilde{\mu}_{\beta\psi}$. It is *ergodic* and it shall be called the *dynamical Gibbs measure* (DGM for short[1]). It

[1]We prefer the adjective "dynamical" instead of "ergodic" to avoid the discussion if an ergodic Gibbs measure is ergodic or not.

is the unique σ-invariant probability measure which satisfies the property that for every $x = x_0x_1\dots$ and for every n,

$$e^{-C_\beta} \leq \frac{\widetilde{\mu}_{\beta\psi}([x_0\dots x_{n-1}])}{e^{\beta.S_n(\psi)(x)-n\mathcal{P}(\beta\psi)}} \leq e^{C_\beta}, \tag{2}$$

where C_β is a positive real number depending only on β and ψ (but not on x or n), and $S_n(\psi)$ stands for $\psi + \psi\circ\sigma + \ldots + \psi\circ\sigma^{n-1}$.

With these settings, the *$\beta\psi$-conformal measure* is the unique probability measure such that for every x and for every n,

$$\nu_{\beta\psi}([x_0\dots x_{n-1}]) = \int e^{\beta S_n(\psi)(x_0\dots x_{n-1}y)-n\mathcal{P}(\beta\psi)}\, d\nu_{\beta\psi}(y). \tag{3}$$

A precise (and more technical) definition of conformal measure is given in page 19.

Again, we refer the reader to [3, 23] to see the connection between conformal measures and DGM. We emphasize that in our settings, conformal measures and DGM are equivalent measures and one can obtain one from the other.

If the choice of ψ is clear we shall drop the ψ and write $\widetilde{\mu}_\beta$, ν_β and $\mathcal{P}(\beta)$.

1.2.2. *The Curie-Weiss model. Probabilistic settings 1 and results.* We consider the case $\Lambda = \{-1, +1\}$; Σ will be denoted by Σ_2.

If $\omega_0\dots\omega_{n-1}$ is a finite word, we set

$$H_n(\omega) := -\frac{1}{2n}\sum_{i,j=0}^{n-1}\omega_j\omega_i. \tag{4}$$

It is called the *Curie-Weiss Hamiltonian*. The *empirical magnetization* for ω is $m_n(\omega) := \dfrac{1}{n}\displaystyle\sum_{j=0}^{n-1}\omega_j$. Then we have

$$H_n(\omega) = -\frac{n}{2}(m_n(\omega))^2. \tag{5}$$

We denote by $\mathbb{P} := \rho^{\otimes\mathbb{N}}$ the product measure on Σ_2, where ρ is the uniform measure on $\{-1, 1\}$, i.e. $\rho(\{1\}) = \rho(\{-1\}) = \frac{1}{2}$, and we define the *probabilistic Gibbs measure* (PGM for short) $\mu_{n,\beta}$ on Σ_2 by

$$\mu_{n,\beta}(d\omega) := \frac{e^{-\beta H_n(\omega)}}{Z_{n,\beta}}\mathbb{P}(d\omega), \tag{6}$$

where $Z_{n,\beta}$ is the normalization factor

$$Z_{n,\beta} = \frac{1}{2^n}\sum_{\omega',\ |\omega'|=n} e^{-\beta H_n(\omega')}.$$

Note that $\mu_{n,\beta}$ can also be viewed as a probability defined on Λ^n.

The measure $\mathbb{P}$ is a Bernoulli measure and is σ-invariant. In Ergodic Theory it is usually called the Parry-measure (see [20]) and turns out to be the unique measure with maximal entropy. With our previous notations it corresponds to the DGM $\widetilde{\mu}_0$.

If P_n, P are probability measures on the Borel sets of a metric space S, we say that P_n converges weakly to P if $\int_S f\,dP_n \to \int_S f\,dP$ for each f in the class $C_b(S)$ of bounded, continuous real functions f on S. In this case we write $P_n \xrightarrow[n\to+\infty]{w} P$.

Our first result concerns the weak convergence of the measures $\mu_{n,\beta}$.

Theorem 1. Weak convergence for the CW model

Let ξ_β be the unique point in $[0,1]$ which realizes the maximum for $\varphi_I(x) := \log(\cosh(\beta x)) - \frac{\beta}{2}x^2$. Let $\widetilde{\mu}_b^+$ and $\widetilde{\mu}_b^-$ be the dynamical Gibbs measures for $b\mathbb{1}_{[+1]}$ and $b\mathbb{1}_{[-1]}$ respectively. Then

$$\mu_{n,\beta} \xrightarrow[n\to+\infty]{w} \begin{cases} \widetilde{\mu}_0 & \text{if } \beta \le 1, \\ \frac{1}{2}\left[\widetilde{\mu}^+_{2\beta\xi_\beta} + \widetilde{\mu}^-_{2\beta\xi_\beta}\right] & \text{if } \beta > 1. \end{cases} \tag{7}$$

Remark 1. *Actually $\mu_{n,\beta}$ converges towards $\frac{1}{2}\left[\widetilde{\mu}^+_{2\beta\xi_\beta} + \widetilde{\mu}^-_{2\beta\xi_\beta}\right]$ for every $\beta > 0$ since we shall see that for $\beta \le 1$ we have $\xi_\beta = 0$, and it is clear that $\widetilde{\mu}_0^+ = \widetilde{\mu}_0^- = \widetilde{\mu}_0 = \rho^{\otimes\mathbb{N}}$.* ■

We refer to [8], sections IV.4 and V.9, for a discussion of the Curie-Weiss model and historical references (see also [21], section 3.4). In Theorem IV.4.1 it is proved, using large deviations, that the sequence of image probabilities $\mathbb{Q}_n$ defined by

$$\mathbb{Q}_n(A) := \mu_{n,\beta}(m_n \in A) \tag{8}$$

converges to an atomic measure as n goes to $+\infty$. By using this theorem, Orey ([19], Corollary 1.2) proved by a nice simple probabilistic argument the weak convergence of $\mu_{n,\beta}$ towards an explicit atomic measure. We were not aware of the work of Orey when we proved our theorem 1. We think it is still interesting because it makes the link with dynamical Gibbs measures, and furthermore our proof is direct and does not use large deviations.

We emphasize the equality

$$m_n(\omega) := \frac{1}{n} S_n(\mathbb{1}_{[+1]} - \mathbb{1}_{[-1]})(\omega) \tag{9}$$

which shows that m_n can be written as a Birkhoff mean of a continuous function.

A consequence of (9) is that (5) can be rewritten under the form

$$H_n(\omega) = -\frac{n}{2}\left(\frac{1}{n} S_n(\psi)(\omega)\right)^2,$$

where $\psi := \mathbb{1}_{[+1]} - \mathbb{1}_{[-1]}$. We are thus led to investigate about results similar to those of Theorem 1 but for a more general Hamiltonian. From that point we discovered a very strange result: to export the Curie-Weiss Model in Ergodic Theory, one should focus on the invariant measures μ which maximize

$$h_\mu + \frac{\beta}{2}\left(\int \psi \, d\mu\right)^2$$

instead of measures which maximize $h_\mu + \beta \int \psi \, d\mu$.

Theorem 2. Convergence for a more general Hamiltonian

Let ψ be a locally constant function[2] *on Σ_2. Let H_n be the Hamiltonian defined by*

$$H_n(\omega) = -\frac{n}{2}\left(\frac{1}{n} S_n(\psi)(\omega)\right)^2.$$

Let $\mu_{n,\beta}$ be the PGM defined by (6) *with this new value for H_n. Then, for each $\beta > 0$,*

1. *there are finitely many real numbers $t_1, \dots t_J$, and J depends on β such that the DGM $\widetilde{\mu}_{\beta t_j}$'s associated to $\beta t_j \psi$ are the unique measures which maximize $h_\mu + \frac{\beta}{2}\left(\int \psi \, d\mu\right)^2$.*
2. *As n goes to $+\infty$, $\mu_{n,\beta}$ converges weakly to a convex combination of the conformal measures $\nu_{\beta t_j}$'s associated to $\beta t_j \psi$.*

From this theorem it makes sense to define a new concept of pressure as

$$P_2(\beta\psi) := \max\left\{h_\mu + \frac{\beta}{2}\left(\int \psi \, d\mu\right)^2\right\},$$

and to study measures which realize this maximum. This quantity shall be referred to as the *quadratic pressure*. Note that upper semi-continuity for entropy shows that the maximum is well defined. The function $\beta \mapsto P_2(\beta\psi)$ is obviously convex (thus continuous). Theorem 1 shows that it can be piecewise analytic and that the number of measures which realize the maximum may change with respect to β.

Remark 2. *We emphasize that Theorem 1 is a particular case of Theorem 2 with $\psi = \mathbb{1}_{[+]} - \mathbb{1}_{[-]}$. Note that for this particular case, the DGM is also the conformal measure.* ■

[2]That is ψ is of the form $\psi = \sum_{j=0}^{k} a_j \mathbb{1}_{C_j}$ where the C_j's are disjoint cylinders

1.2.3. *Some consequences of Theorem 2.* Several questions arise from Theorem 2. Actually, the main goal for Thermodynamic Formalism is to furnish a way to select one (or finitely many) invariant ergodic measure via the variational principle (see (1)). In terms of mathematics, it does not make neither more nor less sense to select measures which maximize $h_\mu + \beta \int \psi \, d\mu$ instead of $h_\mu + \frac{\beta}{2} \left(\int \psi \, d\mu \right)^2$.

Therefore, we are naturally led to redo the Thermodynamic Formalism for all known Dynamical Systems but inquiring for measures μ which maximize $h_\mu + \frac{\beta}{2} \left(\int \psi \, d\mu \right)^2$. In particular, a natural question is to inquire whether Theorem 2 holds for any sufficiently regular potential ψ and not only the locally constant ones. We point out that our proof cannot be easily adapted to that more general case. Moreover, the usual tool in Thermodynamic Formalism —that is the *Transfer operator*— does not seem to be well adapted to this question.

For more geometric dynamical systems, one usually considers or studies the special class of *physical or/and SRB-measures*. These measures are usually considered as the most natural ones with the measures of maximal entropy. It is clear that measures of maximal entropy also maximize $h_\mu + \left(\int \psi \, d\mu \right)^2$ for $\psi \equiv 0$. A natural question is thus to know if SRB-measures can also be characterized as measures maximizing $h_\mu + \left(\int \psi \, d\mu \right)^2$.

1.2.4. *The Curie-Weiss-Potts model. Probabilistic settings 2 and result.* The *Curie-Weiss-Potts model* will be for $\Lambda = \{\theta^1, \dots, \theta^q\}$ with $q > 2$. In that case we shall write Σ_q instead of Σ.

The Curie-Weiss-Potts Hamiltonian is defined for a finite word $\omega = \omega_0 \cdots \omega_{n-1}$ by

$$H_n(\omega) := -\frac{1}{2n} \sum_{i,j=0}^{n-1} \mathbb{1}_{\omega_j = \omega_i}. \tag{10}$$

We define the vector $L_n(\omega) = (L_{n,1}(\omega), \cdots, L_{n,q}(\omega))$ where

$$L_{n,k}(\omega) = \sum_{i=0}^{n-1} \mathbb{1}_{\omega_i = \theta^k}$$

is the number of digits of ω which take the value θ^k, so that we can write

$$\sum_{i,j=0}^{n-1} \mathbb{1}_{\omega_j = \omega_i} = \sum_{k=1}^{q} \left(\sum_{i=0}^{n-1} \mathbf{1}_{\omega_i = \theta^k} \right)^2 = \|L_n(\omega)\|^2,$$

where $\| \cdot \|$ stands for the euclidean norm on $\mathbb{R}^q$.

We denote by $\mathbb{P} := \rho^{\otimes\mathbb{N}}$ the product measure on Σ_q, where ρ is the uniform measure on Λ, i.e. $\rho = \frac{1}{q}\sum_{k=1}^{q}\delta_{\theta^k}$, and we define the probabilistic Gibbs measure $\mu_{n,\beta}$ on Σ_q by

$$\mu_{n,\beta}(d\omega) := \frac{e^{-\beta H_n(\omega)}}{Z_{n,\beta}}\mathbb{P}(d\omega) = \frac{e^{\frac{\beta}{2n}\|L_n(\omega)\|^2}}{Z_{n,\beta}}\mathbb{P}(d\omega), \tag{11}$$

where $Z_{n,\beta}$ is the normalization factor

$$Z_{n,\beta} = \frac{1}{q^n}\sum_{\omega',\ |\omega'|=n} e^{\frac{\beta}{2n}\|L_n(\omega')\|^2}.$$

Now we can state the analog of Theorem 1.

Theorem 3. Weak convergence for the CWP model

For $1 \leq k \leq q$, $b \in \mathbb{R}$, let $\widetilde{\mu}_b^k$ be the dynamical Gibbs measure for $b\mathbb{1}_{[\theta^k]}$. Let $\beta_c = \frac{2(q-1)\log(q-1)}{q-2}$. For $0 < \beta < \beta_c$ set $s_\beta = 0$ and for $\beta \geq \beta_c$ let s_β be the largest solution of the equation

$$s = \frac{e^{\beta s} - 1}{e^{\beta s} + q - 1}. \tag{12}$$

Then,

$$\mu_{n,\beta} \xrightarrow[n\to+\infty]{w} \begin{cases} \rho^{\otimes\mathbb{N}} & \text{if } 0 < \beta < \beta_c, \\ \dfrac{1}{q}\displaystyle\sum_{k=1}^{q}\widetilde{\mu}^k_{\beta s_\beta} & \text{if } \beta > \beta_c, \\ \dfrac{A\,\widetilde{\mu}_0^1 + B\,\sum_{k=1}^{q}\widetilde{\mu}^k_{\beta_c s_{\beta_c}}}{A+qB} & \text{if } \beta = \beta_c, \end{cases} \tag{13}$$

with $A = \left(1 - \frac{\beta_c}{q(q-1)}\right)^{\frac{q-2}{2}}$ and $B = \left(1 - \frac{\beta_c}{q}\right)^{\frac{q-2}{2}}$.

Remark 3. *Actually $\mu_{n,\beta}$ converges towards $\frac{1}{q}\sum_{k=1}^{q}\widetilde{\mu}^k_{\beta s_\beta}$ for every $\beta \neq \beta_c$ since $s_\beta = 0$ for $\beta < \beta_c$, and it is clear that $\widetilde{\mu}_0^k = \rho^{\otimes\mathbb{N}}$ for each $1 \leq k \leq q$.* ■

We refer to [11] for a discussion of the Curie-Weiss-Potts model and historical references. Orey ([19], Theorem 4.4) mentions the weak convergence of $\mu_{n,\beta}$ towards an explicit atomic measure, but he makes a mistake concerning the case $\beta = \beta_c$, as pointed out in [11].

It is highly probable that a similar result to Theorem 2 holds for the Curie-Weiss-Potts model. Note that as for Theorem 1, in the settings of Theorem 3 conformal and Gibbs measures are equal.

1.3. **Plan of the paper.** The paper is composed as follows.

In Section 2 we prove Theorem 1. In Section 3 we prove Theorem 3. Both proofs are very similar and are based on the convergence of $\mu_{n,\beta}(C)$ where C is a cylinder in Σ.

Theorem 2 is proved in Section 4. The proof is similar to the ones of Theorem 1 and 3. The main difference is that the auxiliary function is more delicate to control. For that, we need to use the Transfer Operator.

2. Proof of Theorem 1

To prove the convergence of $\mu_{n,\beta}$ towards μ, it is enough to show that for every cylinder C,

$$\lim_{n\to\infty} \mu_{n,\beta}(C) = \mu(C). \tag{14}$$

First we justify that φ_I admits a unique maximum in $[0,1]$ and use this point to get convergence for $\mu_{n,\beta}(C)$, where C is any cylinder. In the second subsection we show that this limit is equal to the right convex combinations of DGM's.

2.1. **The auxiliary function φ_I and limit for $\mu_{n,\beta}$.** We recall that we set $\varphi_I(x) := \log(\cosh(\beta x)) - \frac{\beta}{2}x^2$.

Lemma 2.1. Maxima for φ_I

The function φ_I attains its maximum on $\mathbb{R}^+$ at a unique point ξ_β which is the unique non-negative solution of the equation $\tanh(\beta x) = x$. If $\beta \le 1$, then $\xi_\beta = 0$.

Proof. Note that $\varphi_I'(x) = \beta\,(\tanh(\beta x) - x)$ and $\varphi_I''(x) = \beta(\beta - 1 - \beta\tanh^2(\beta x))$. If $\beta \le 1$, φ_I'' is non-positive, thus φ_I' decreases and $\varphi_I'(0) = 0$ yields that φ_I is a decreasing function. The maximum is then attained for $\xi_\beta = 0$.

If $\beta > 1$, then φ_I'' is positive and then negative, which yields that φ_I' is first an increasing and then a decreasing function. Note that $\varphi_I'(0) = 0$ and $\varphi_I'(1) < 0$, which shows that φ_I' is positive on some interval $]0, \xi_\beta[$ with $0 < \xi_\beta < 1$ and negative on $]\xi_\beta, +\infty[$. Consequently, φ_I reaches its maximal value on $\mathbb{R}$ at the points $\pm\xi_\beta$ defined by

$$\tanh(\beta\xi_\beta) = \xi_\beta. \tag{15}$$ □

Now we are ready to compute the limit of a fixed cylinder. Let $\omega = \omega_0 \dots \omega_{p-1}$ be a finite word of length p. We denote by $S_p(\omega) = \sum_{i=0}^{p-1} \omega_i$ the sum of the p digits of ω.

Lemma 2.2. Limit of the measure of a fixed cylinder

$$\lim_{n\to\infty} \mu_{n,\beta}([\omega_0 \dots \omega_{p-1}]) = \begin{cases} \frac{1}{2^p} \text{ if } \beta \leq 1, \\ \frac{1}{2}\left(f(\xi_\beta) + f(-\xi_\beta)\right) \text{ if } \beta > 1, \end{cases} \tag{16}$$

where

$$f(y) = \frac{e^{\beta y S_p(\omega)}}{(e^{\beta y} + e^{-\beta y})^p}.$$

Proof. For big n,

$$[\omega] = \bigsqcup [\omega_0 \dots \omega_{p-1}\alpha_p \dots \alpha_{n-1}] = \bigsqcup_{\alpha,\ |\alpha|=n-p} [\omega\alpha],$$

and

$$\mu_{n,\beta}([\omega]) = \sum_{\alpha,\ |\alpha|=n-p} \mu_{n,\beta}([\omega\alpha]) = \frac{\sum\limits_{\alpha,\ |\alpha|=n-p} e^{-\beta H_n(\omega\alpha)}}{\sum\limits_{\alpha,\ |\alpha|=n} e^{-\beta H_n(\alpha)}}.$$

For any word $x = x_0x_1x_2\dots$ and any $n \in \mathbb{N}^*$, we denote by $S_n(x) = \sum_{i=0}^{n-1} x_i$ the sum of the n first digits of x. Then

$$e^{-\beta H_n(\omega\alpha)} = e^{\left(\sqrt{\frac{\beta}{2n}} S_n(\omega\alpha)\right)^2} \text{ and } S_n(\omega\alpha) = S_p(\omega) + S_{n-p}(\alpha).$$

We use the equality

$$e^{a^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2} + \sqrt{2}ax}\, dx,$$

sometimes called the Hubbard-Stratonovich transformation ([16],[26]), to compute the following.

$$\begin{aligned} \sum_{\alpha,\ |\alpha|=n-p} e^{-\beta H_n(\omega\alpha)} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} \sum_{\alpha} e^{\sqrt{\frac{\beta}{n}} x S_n(\omega\alpha)}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2} + \sqrt{\frac{\beta}{n}} x S_p(\omega)} \sum_{\alpha} e^{\sqrt{\frac{\beta}{n}} x S_{n-p}(\alpha)}\, dx \\ &= \frac{\sqrt{n\beta}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-n\frac{\beta y^2}{2} + \beta y S_p(\omega)} \sum_{\alpha} e^{\beta y S_{n-p}(\alpha)}\, dy \end{aligned}$$

where we made the change of variable $\beta y := \sqrt{\frac{\beta}{n}} x$. Now we notice that the sum $\sum_{\alpha,\ |\alpha|=n-p} e^{\beta y S_{n-p}(\alpha)}$ is equal to $2^{n-p}\mathbb{E}(e^{\beta y \sum_{i=0}^{n-p-1} X_i})$ where the X_i's are independent identically distributed with law $\mathbb{P}(X_i = -1) = \mathbb{P}(X_i = 1) = \frac{1}{2}$, and we deduce that

$$\sum_{\alpha,\ |\alpha|=n-p} e^{\beta y S_{n-p}(\alpha)} = (e^{\beta y} + e^{-\beta y})^{n-p} = 2^n \frac{\cosh(\beta y)^n}{(e^{\beta y} + e^{-\beta y})^p}.$$

Eventually we obtain

$$\sum_{\alpha,\ |\alpha|=n-p} e^{-\beta H_n(\omega\alpha)} = \frac{2^n\sqrt{n\beta}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{n\varphi_I(y)} f(y)\, dy. \tag{17}$$

Similarly, $p = 0$ yields

$$\sum_{\alpha,\ |\alpha|=n} e^{-\beta H_n(\alpha)} = \frac{2^n\sqrt{n\beta}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{n\varphi_I(y)}\, dy,$$

therefore we obtain that

$$\mu_{n,\beta}([\omega]) = \frac{\int_{-\infty}^{+\infty} e^{n\varphi_I(y)} f(y)\, dy}{\int_{-\infty}^{+\infty} e^{n\varphi_I(y)}\, dy}. \tag{18}$$

We recall that the Laplace method shows that if φ_I' vanishes on a single point ξ in the interior of the interval I and if $\varphi_I''(\xi) < 0$ and $f(\xi) \neq 0$, then

$$\int_I e^{n\varphi_I(y)} f(y) dy \sim_{n\to\infty} \frac{\sqrt{2\pi}}{\sqrt{|\varphi_I''(\xi)|}} e^{n\varphi_I(\xi)} f(\xi) n^{-1/2}. \tag{19}$$

We remind that $u_n \sim_{n\to+\infty} v_n$ means $u_n = v_n(1 + \varepsilon(n))$ with $\lim_{n\to+\infty} \varepsilon(n) = 0$.

If $\beta < 1$: we may consider $I = \mathbb{R}$ and $\xi = \xi_\beta = 0$. We thus get $\varphi_I''(0)) = \beta(\beta - 1)$, $f(0) = \frac{1}{2^p}$, and

$$\mu_{n,\beta}([\omega]) \sim_{n\to\infty} \frac{f(0)}{1} = \frac{1}{2^p}.$$

If $\beta = 1$: in this case $\varphi_I''(0)) = 0$ but the Laplace method still works if we consider the least integer k such that $\varphi_I^{(k)}(0) \neq 0$. We do not need to calculate it because we have as in the preceding case that

$$\int_{\mathbb{R}} e^{n\varphi_I(y)} f(y) dy \sim_{n\to\infty} \frac{1}{2^p} \int_{\mathbb{R}} e^{n\varphi_I(y)} dy,$$

therefore we still have $\mu_{n,\beta}([\omega]) \sim_{n\to+\infty} \frac{1}{2^p}$.

If $\beta > 1$: we may consider two intervals $\mathbb{R}_-$ and $\mathbb{R}_+$ and $\xi = \pm\xi_\beta$. Then we get

$$\int_{-\infty}^{+\infty} e^{n\varphi_I(y)} f(y) dy \sim_{n\to\infty} \frac{\sqrt{2\pi}}{\sqrt{|\varphi_I''(\xi_\beta)|}}\, e^{n\varphi_I(\xi_\beta)} \left(f(\xi_\beta) + f(-\xi_\beta)\right) n^{-1/2},$$

which yields

$$\mu_{n,\beta}([\omega]) \sim_{n\to\infty} \frac{f(\xi_\beta) + f(-\xi_\beta)}{2}. \qquad \square$$

2.2. **Identification of the limit as a convex combination of DGM's.** First, we point out that Lemma 2.2 yields $\lim_{n\to+\infty} \mu_{n,\beta} = \widetilde{\mu}_0$ if $\beta \leq 1$. We thus have to deal with the case $\beta > 1$.

Let us first compute the dynamical Gibbs measures $\widetilde{\mu}_b^+$ and $\widetilde{\mu}_b^-$. We denote by $L_{p,+}(\omega) := \sum_{k=0}^{p-1} \mathbf{1}_{\omega_k=1}$ the number of digits of ω which take the value 1, and similarly $L_{p,-}(\omega) := \sum_{k=0}^{p-1} \mathbf{1}_{\omega_k=-1}$ is the number of digits which take the value -1.

Lemma 2.3. Computation for $\widetilde{\mu}_b^{\pm}$

$$\widetilde{\mu}_b^+([\omega]) = \frac{e^{bL_{p,+}(\omega)}}{(e^b+1)^p} \quad \text{and} \quad \widetilde{\mu}_b^-([\omega]) = \frac{e^{bL_{p,-}(\omega)}}{(e^b+1)^p}. \tag{20}$$

Proof. The function $b\mathbb{1}_{[+1]}$ depends only on the zero coordinate. It is shown for instance in Example 4.2.2 of [17] that in this case the supremum in (1) is attained for the product measure $\mathbb{P}_p := \rho_p^{\otimes\mathbb{N}}$, where $p \in [0,1]$, $\rho_p = p\delta_1 + (1-p)\delta_{-1}$, and p satisfies

$$-p\log p - (1-p)\log(1-p) + bp = \sup_{q\in[0,1]} \left(-q\log q - (1-q)\log(1-q) + bq\right).$$

It is easy to show that $p = \frac{e^b}{1+e^b}$, and then

$$\widetilde{\mu}_b^+([\omega]) = \frac{e^{bL_{p,+}(\omega)}}{(e^b+1)^p}.$$

Exchanging the roles of $+1$ and -1 gives the equality

$$\widetilde{\mu}_b^-([\omega]) = \frac{e^{bL_{p,-}(\omega)}}{(e^b+1)^p}. \qquad \square$$

We recall that $S_p(\omega) = \sum_{k=0}^{p-1} \omega_k = L_{p,+}(\omega) - L_{p,-}(\omega) = 2L_{p,+}(\omega) - p = p - 2L_{p,-}(\omega)$. Then

$$\begin{aligned} f(\xi_\beta) &= \frac{e^{\beta\xi_\beta S_p(\omega)}}{(e^{\beta\xi_\beta}+e^{-\beta\xi_\beta})^p} \\ &= \frac{e^{\beta\xi_\beta(2L_{p,+}(\omega)-p)}}{(e^{\beta\xi_\beta}+e^{-\beta\xi_\beta})^p} \\ &= \frac{e^{2\beta\xi_\beta L_{p,+}(\omega)}}{(e^{2\beta\xi_\beta}+1)^p}. \end{aligned}$$

Similarly we have $f(-\xi_\beta) = \dfrac{e^{2\beta\xi_\beta L_{p,-}(\omega)}}{(e^{2\beta\xi_\beta}+1)^p}$ and replacing these values in (16) we get

$$\lim_{n\to+\infty} \mu_{n,\beta}([\omega]) = \frac{1}{2}(\widetilde{\mu}^+_{2\beta\xi_\beta}([\omega]) + \widetilde{\mu}^-_{2\beta\xi_\beta}([\omega])),$$

and the theorem is proved.

3. Proof of Theorem 3

The proof follows the same lines as the proof of Theorem 1. In the first step we use an auxiliary function φ_P. Note that the function was already studied by Ellis and Wang in [11]. Then we deduce that $\mu_{n,\beta}(C)$ converge for any cylinder C. In the second step we identify the limit as the relevant convex combination of dynamical measures.

3.1. Auxiliary function φ_P and convergence for $\mu_{n,\beta}$.

We shall need the function φ_P defined on $\mathbb{R}^q$ by

$$\varphi_P(z) = -\frac{\beta}{2}\|z\|^2 + \log \sum_{k=1}^{q} e^{\beta z_k}. \tag{21}$$

This function attains its maximum on $\mathbb{R}^q$ since $\varphi_P(z) \leq -c\|z\|^2$ as $\|z\|$ tends to ∞. We recall Theorem 2.1 of [11], which describes precisely the global maximum points of φ_P.

Theorem 4. (Ellis Wang [11])

Let $\beta_c = \frac{2(q-1)\log(q-1)}{q-2}$. For $0 < \beta < \beta_c$ set $s_\beta = 0$ and for $\beta \geq \beta_c$ let s_β be the largest solution of the equation

$$s = \frac{e^{\beta s} - 1}{e^{\beta s} + q - 1}. \tag{22}$$

The function $\beta \mapsto s_\beta$ is strictly increasing on the interval $[\beta_c, +\infty[$, $s(\beta_c) = \frac{q-2}{q-1}$, and $\lim_{\beta\to\infty} s_\beta = 1$.

Denote by ϕ the function from $[0,1]$ into $\mathbb{R}^q$ defined by

$$\phi(s) = \left(\frac{1+(q-1)s}{q}, \frac{1-s}{q}, \cdots, \frac{1-s}{q}\right),$$

the last $(q-1)$ components all equal $\frac{1-s}{q}$. Let K_β denote the set of global maximum points of the symmetric function φ_P. Define $\nu^0 = \phi(0) = \left(\frac{1}{q}, \cdots, \frac{1}{q}\right)$. For $\beta \geq \beta_c$, define $\nu^1(\beta) = \phi(s_\beta)$ and let $\nu^i(\beta)$, $i = 2, \cdots, q$ denote the points in $\mathbb{R}^q$ obtained by interchanging the first and ith coordinates of $\nu^1(\beta)$. Then

$$K_\beta = \begin{cases} \{\nu^0\} & \text{for } 0 < \beta < \beta_c, \\ \{\nu^1(\beta), \nu^2(\beta), \cdots, \nu^q(\beta)\} & \text{for } \beta > \beta_c, \\ \{\nu^0, \nu^1(\beta_c), \nu^2(\beta_c), \cdots, \nu^q(\beta_c)\} & \text{for } \beta = \beta_c. \end{cases}$$

For $\beta \geq \beta_c$ the points in K_β are all distinct.

We fix a finite word $\omega = \omega_0 \cdots \omega_{p-1}$ of length p and we compute the limit of $\mu_{n,\beta}([\omega])$.

Lemma 3.1.

$$\lim_{n\to\infty}\mu_{n,\beta}([\omega])=\begin{cases}\dfrac{1}{q^p}\ \textit{if } \beta<\beta_c,\\ \dfrac{1}{q}\dfrac{1}{(e^{\beta s_\beta}+q-1)^p}\displaystyle\sum_{k=1}^{q}e^{\beta s_\beta L_{p,k}(\omega)}\ \textit{if } \beta>\beta_c,\\ \dfrac{\frac{A}{q^p}+\frac{B}{(e^{\beta s_\beta}+q-1)^p}\sum_{k=1}^{q}e^{\beta_c s_{\beta_c}L_{p,k}(\omega)}}{A+qB}\ \textit{if } \beta=\beta_c.\end{cases}$$

Proof. The proof is very much the same as the proof of Lemma 2.2, but the Hamiltonian has changed and we are in dimension q instead of dimension one. We want to evaluate the limit of

$$\mu_{n,\beta}([\omega])=\sum_{\alpha,\ |\alpha|=n-p}\mu_{n,\beta}([\omega\alpha])=\frac{\displaystyle\sum_{\alpha,\ |\alpha|=n-p}e^{\frac{\beta}{2n}\|L_n(\omega\alpha)\|^2}}{\displaystyle\sum_{\alpha,\ |\alpha|=n}e^{\frac{\beta}{2n}\|L_n(\alpha)\|^2}}.$$

With the help of the identity

$$e^{\|u\|^2}=\frac{1}{(2\pi)^{q/2}}\int_{\mathbb{R}^q}\exp\left(-\frac{1}{2}\|y\|^2+\sqrt{2}\langle y,u\rangle\right)dy, \tag{23}$$

and noticing that $L_n(\omega\alpha)=L_p(\omega)+L_{n-p}(\alpha)$, we write

$$\begin{aligned}\sum_{\alpha,\ |\alpha|=n-p}e^{\frac{\beta}{2n}\|L_n(\omega\alpha)\|^2} &= \frac{1}{(2\pi)^{q/2}}\int_{\mathbb{R}^q}e^{-\frac{1}{2}\|y\|^2}\sum_{\alpha}e^{\sqrt{\frac{\beta}{n}}\langle y,L_n(\omega\alpha)\rangle}\,dy\\ &= \frac{1}{(2\pi)^{q/2}}\int_{\mathbb{R}^q}e^{-\frac{1}{2}\|y\|^2+\sqrt{\frac{\beta}{n}}\langle y,L_p(\omega)\rangle}\sum_{\alpha}e^{\sqrt{\frac{\beta}{n}}\langle y,L_{n-p}(\alpha)\rangle}\,dy.\end{aligned}$$

It is easily seen that

$$\sum_{\alpha,|\alpha|=n-p}e^{\sqrt{\frac{\beta}{n}}\langle y,L_{n-p}(\alpha)\rangle}=\left(\sum_{k=1}^{q}e^{\sqrt{\frac{\beta}{n}}y_k}\right)^{n-p},$$

therefore we get

$$\sum_{\alpha,\ |\alpha|=n-p}e^{\frac{\beta}{2n}\|L_n(\omega\alpha)\|^2}=$$
$$\frac{1}{(2\pi)^{q/2}}\int_{\mathbb{R}^q}\exp\left(-\frac{1}{2}\|y\|^2+\sqrt{\frac{\beta}{n}}\langle y,L_p(\omega)\rangle+(n-p)\log\left(\sum_{k=1}^{q}e^{\sqrt{\frac{\beta}{n}}y_k}\right)\right)dy.$$

Now we make the change of variable $\beta z=\sqrt{\frac{\beta}{n}}y$, and we obtain

$$\sum_{\alpha,\ |\alpha|=n-p}e^{\frac{\beta}{2n}\|L_n(\omega\alpha)\|^2}=\left(\frac{n\beta}{2\pi}\right)^{q/2}\int_{\mathbb{R}^q}e^{n\varphi_P(z)}f(z)\,dz, \tag{24}$$

where φ_P was defined in (21) and f is defined on $\mathbb{R}^q$ by

$$f(z) = \exp\left(\beta\langle z, L_p(\omega)\rangle - p\log\left(\sum_{k=1}^{q} e^{\beta z_k}\right)\right). \tag{25}$$

Similarly, $p = 0$ yields

$$\sum_{\alpha,\ |\alpha|=n} e^{\frac{\beta}{2n}\|L_n(\alpha)\|^2} = \left(\frac{n\beta}{2\pi}\right)^{q/2} \int_{\mathbb{R}^q} e^{n\varphi_P(z)}\, dz,$$

hence

$$\mu_{n,\beta}([\omega]) = \frac{\sum_{\alpha,\ |\alpha|=n-p} e^{\frac{\beta}{2n}\|L_n(\omega\alpha)\|^2}}{\sum_{\alpha,\ |\alpha|=n} e^{\frac{\beta}{2n}\|L_n(\alpha)\|^2}} = \frac{\int_{\mathbb{R}^q} e^{n\varphi_P(z)} f(z)\, dz}{\int_{\mathbb{R}^q} e^{n\varphi_P(z)}\, dz}.$$

We denote by $D\varphi_P(z)$, respectively $\mathcal{H}(z)$, the gradient, respectively the Hessian matrix, of φ_P at z. It is proved in Proposition 2.2 of [11] that the Hessian matrix of φ_P is negative definite at each global maximum point of φ_P. Now if $D\varphi_P$ vanishes at a single point z_0 in an open set O of $\mathbb{R}^q$, if $\mathcal{H}(z_0)$ is negative definite and if $f(z_0) \neq 0$, then we know by Laplace's method that

$$\int_0 e^{n\varphi_P(z)} f(z)\, dz \sim_{n\to\infty} \frac{(2\pi)^{q/2} f(z_0) e^{n\varphi_P(z_0)}}{n^{q/2}\sqrt{|\det \mathcal{H}(z_0)|}}.$$

If $0 < \beta < \beta_c$: according to Theorem 4, φ_P attains its maximum at the unique point ν^0 so applying Laplace's method yields

$$\mu_{n,\beta}([\omega]) \sim_{n\to\infty} \frac{f(\nu^0)}{1} = \frac{1}{q^p}.$$

If $\beta > \beta_c$: Theorem 4 states that φ_P attains its maximum at exactly q points $\nu^i(\beta)$, $i = 1, \cdots, q$, where $\nu^i(\beta)$, $i = 2, \cdots, q$ is obtained by interchanging the first and ith coordinates of $\nu^1(\beta)$. Due to the symmetry of the function φ_P it is clear that $\det \mathcal{H}(\nu^i) = \det \mathcal{H}(\nu^1)$, $i = 2, \cdots, q$. Considering a family of disjoint open sets $(O_i)_{1\leq i\leq q}$ such that O_i contains ν^i and $\mathbb{R}^q = \cup_{i=1}^q O_i \cup N$, where N is a set of measure zero, Laplace's method yields

$$\mu_{n,\beta}([\omega]) \sim_{n\to\infty} \frac{1}{q}\sum_{i=1}^{q} f(\nu^i).$$

Recall that

$$f(\nu^i) = \frac{e^{\beta\langle \nu^i, L_p(\omega)\rangle}}{\left(\sum_{k=1}^q e^{\beta\nu_k^i}\right)^p}$$

with

$$\nu_k^i = \begin{cases} \dfrac{1 - s_\beta}{q} & \text{if } k \neq i, \\ \dfrac{1 + (q-1)s_\beta}{q} & \text{if } k = i. \end{cases}$$

As $\sum_{k=1}^{q} L_{p,k}(\omega) = p$ it is easily seen that

$$e^{\beta\langle \nu^i, L_p(\omega)\rangle} = \exp\left(\frac{\beta p(1-s_\beta)}{q} + \beta s_\beta L_{p,i}(\omega)\right). \tag{26}$$

As ν^i is a critical point of φ_P and $\frac{\partial \varphi_P}{\partial z_i}(z) = \frac{\beta e^{\beta z_i}}{\sum_{k=1}^{q} e^{\beta z_k}} - \beta z_i$, we know that

$$\sum_{k=1}^{q} e^{\beta \nu_k^i} = \frac{e^{\beta \nu_j^i}}{\nu_j^i} = \frac{q}{1-s_\beta}\, e^{\frac{\beta(1-s_\beta)}{q}}. \tag{27}$$

Putting together (26) and (27) we obtain

$$f(\nu^i) = \left(\frac{1-s_\beta}{q}\right)^p e^{\beta s_\beta L_{p,i}(\omega)},$$

which can also be written

$$f(\nu^i) = \frac{1}{(e^{\beta s_\beta} + q - 1)^p}\, e^{\beta s_\beta L_{p,i}(\omega)} \tag{28}$$

since s_β is solution of the equation (12). Therefore

$$\mu_{n,\beta}([\omega]) \sim_{n\to\infty} \frac{1}{q}\frac{1}{(e^{\beta s_\beta} + q - 1)^p} \sum_{i=1}^{q} e^{\beta s_\beta L_{p,i}(\omega)}.$$

If $\beta = \beta_c$: the function φ_P admits exactly $q+1$ maximun points $\nu^i(\beta)$, $i = 0, \cdots, q$ but $\det \mathcal{H}(\nu^0) \neq \det \mathcal{H}(\nu^1)$, therefore Laplace's method yields

$$\mu_{n,\beta}([\omega]) \sim_{n\to\infty} \frac{|\det \mathcal{H}(\nu^0)|^{-1/2} f(\nu^0) + |\det \mathcal{H}(\nu^1)|^{-1/2} \sum_{i=1}^{q} f(\nu^i)}{|\det \mathcal{H}(\nu^0)|^{-1/2} + q\, |\det \mathcal{H}(\nu^1)|^{-1/2}}. \tag{29}$$

In the proof of Proposition 2.2 of [11] it is proved that $\mathcal{H}(\nu^0)$ has a simple eigenvalue at β and an eigenvalue of multiplicity $(q-1)$ at $\beta q^{-1}(q-\beta)$ whereas $\mathcal{H}(\nu^1)$ has simple eigenvalues at β and $\beta - \beta^2 qab$ and an eigenvalue of multiplicity $(q-2)$ at $\beta - \beta^2 b$, where $a = q^{-1}(1 + (q-1)s_\beta)$ and $b = q^{-1}(1 - s_\beta)$. Recalling that $s(\beta_c) = \frac{q-2}{q-1}$ we deduce that

$$|\det \mathcal{H}(\nu^0)| = \beta_c^q (1 - q^{-1}\beta_c)^{q-1},$$

$$|\det \mathcal{H}(\nu^1)| = \beta_c^q (1 - q^{-1}\beta_c)\left(1 - \frac{\beta_c}{q(q-1)}\right)^{q-2}.$$

Reporting in (29) and recalling (28) we get the result. □

3.2. **Identification of the limit.** We can already deduce from Lemma 3.1 that $\mu_{n,\beta} \xrightarrow[n\to+\infty]{w} \widetilde{\mu}_0$ if $\beta < \beta_c$.

Lemma 3.2. Computation for $\widetilde{\mu}_b^k$

For $k = 1, \dots, q$,

$$\widetilde{\mu}_b^k([\omega]) = \frac{e^{bL_{p,k}(\omega)}}{(e^b + q - 1)^p}. \tag{30}$$

Proof. The function $b\mathbb{1}_{[\theta^k]}$ depends only on the zero coordinate, therefore the supremum in (1) is attained for the product measure $(m^k)^{\otimes\mathbb{N}}$, where the probability vector $(m_j^k)_{1\le j\le q}$ on Λ maximizes the quantity

$$-\sum_{j=1}^{q} p_j \log p_j + bp_k$$

over all the probability vectors $(p_j)_{1\le j\le q}$ on Λ, and is given by $m_k^k = \frac{e^b}{e^b+q-1}$, $m_j^k = \frac{1}{e^b+q-1}$ if $j \neq k$ (see for instance Example 4.2.2 of [17]). The result is then clear. □

The limit in (13) is now a direct consequence of the lemmas 3.1 and 3.2.

4. Proof of Theorem 2

We consider a function ψ of the form

$$\psi := \sum_i a_i \mathbb{1}_{C_i},$$

where the sum is finite, the C_i's are disjoint cylinders, and the a_i's are real numbers which are not all equal to zero. We can always assume that all the cylinders have the same length, say q.

In the following we assume that we have chosen an order on q-cylinders and we consider the vector $\mathbf{a} := (a_1, \dots, a_{2^q})$.

4.1. **Convergence of $\mu_{n,\beta}$.** Let $\omega = \omega_0 \dots \omega_{p-1}$ be a finite word of length p. To fix the ideas, we assume that p is bigger than q. Note that if the convergence holds for any cylinder with length larger than q then it holds for any cylinder.

For n very big and $x \in \Sigma_2$, $S_n(\psi)(x)$ involves digits $x_0, \dots, x_{n+q-2}$. Then, we get

$$\mu_{n,\beta}([\omega]) = \sum_{\alpha,\ |\alpha|=n-p+q-1} \mu_{n,\beta}([\omega_0 \dots \omega_{p-1}\alpha_0 \dots \alpha_{n-p+q-2}]).$$

For simplicity we write $S_n(\omega\alpha)$ for $S_n(\psi)(\omega\alpha)$ (which is well defined if $|\alpha| \geq n-p+q-1$). Again, using the Hubbard-Stratonovich transformation we get

$$\begin{aligned} \sum_{\alpha,\ |\alpha|=n-p+q-1} e^{\frac{\beta}{2n}S_n^2(\omega\alpha)} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} \sum_{\alpha} e^{\sqrt{\frac{\beta}{n}}xS_n(\omega\alpha)}\,dx, \\ &= \sqrt{\frac{\beta n}{2\pi}}\int_{-\infty}^{+\infty} e^{-n\frac{\beta}{2}z^2} \sum_{\alpha} e^{\beta z S_n(\omega\alpha)}\,dz, \end{aligned} \tag{31}$$

where we have made the change of variable $\beta z = \sqrt{\frac{\beta}{n}}x$.

We point out the equality

$$S_n(\omega\alpha) = S_{p-q+1}(\omega) + S_{q-1}(\sigma^{p+1-q}(\omega\alpha)) + S_{n-p}(\alpha). \tag{32}$$

This equality means that the first $p+1-q$ terms in the Birkhoff sum $S_n(\omega\alpha)$ only involve the ω_i's, then come $q-1$ terms which involve the tail of ω and the head of α, and then comes the tail of the sum which only depends on α.

Let us define the operator $\mathcal{L}_\xi$, depending on a real or complex parameter ξ, by

$$\mathcal{L}_\xi(T)(x) := e^{\xi\psi(-1x)}T(-1x) + e^{\xi\psi(1x)}T(1x).$$

Here x is a point in Σ_2, T is a continuous function on Σ_2, and we remind that ix, with $i = \pm 1$, is the infinite word $ix_0x_1x_2\ldots$. $\mathcal{L}_\xi$ is the transfer operator[3] associated to the potential $\xi.\psi$. We notice that for any $m \in \mathbb{N}^*$,

$$\mathcal{L}_\xi^m(T)(x) := \sum_{\alpha,\ |\alpha|=m} e^{\xi S_m(\alpha x)}T(\alpha x).$$

Now, we write $\alpha = \alpha'\alpha''$, with $|\alpha'| = n-p$ and $|\alpha''| = q-1$. Using the equality (32) it is easy to check that

$$\sum_{\alpha} e^{\beta z S_n(\omega\alpha)} = e^{\beta z S_{p-q+1}(\omega)} \sum_{\alpha''\ |\alpha''|=q-1} \mathcal{L}_{\beta.z}^{n-p}(e^{\beta z S_{q-1}(\sigma^{p+1-q}(\omega.))})(\alpha''x). \tag{33}$$

We point out that the x on the right hand side of this equality is just there to make sense but the expression actually does not depend on x because $|\alpha''| = q-1$.

It is known (see [3]) that $\mathcal{L}_\xi$ acts on continuous and Hölder continuous functions. Moreover on the space of Hölder continuous functions it is quasi-compact: its spectral radius, say $\lambda(\xi)$, is a simple dominating eigenvalue associated to a positive eigenfunction, say H_ξ. The adjoint operator acts on measures and it admits a unique eigenmeasure associated to the spectral radius, say ν_ξ. Moreover $\int H_\xi d\nu_\xi = 1$ and for every continuous function T,

$$\lim_{n\to+\infty} \max_{x\in\Sigma_2} \left| \lambda(\xi)^{-n}\mathcal{L}_\xi^n(T)(x) - \left(\int T d\nu_\xi\right) H_\xi(x) \right| = 0.$$

[3]also called the Ruelle Perron Frobenius operator.

Remark 4. *The measure ν_ξ is exactly the conformal measure associated to $\xi\psi$.* ■

We refer the reader to [15] for properties of quasi-compact operators ,in particular regularity of the maximal eigenvalue and the eigenvectors with respect to ξ.

Furthermore, $\mathcal{L}_\xi$ acts on functions constant on $(q-1)$-cylinders because ψ is constant on q-cylinders. In that case we can give a simple characterization for $\lambda(\xi)$. First, we order the $(q-1)$-cylinders, say $K_1, \dots, K_{2^{q-1}}$, and the q-cylinders, say $C_1, \dots, C_{2^q}$. Let $\mathbf{z} = (z_1, \dots, z_{2^q})$ be a vector in $\mathbb{R}^{2^q}$. We consider the $2^{q-1} \times 2^{q-1}$ matrix $M(\mathbf{z})$ whose entries are defined as follows: From the $(q-1)$-cylinder $K_i = [\omega_0^i, \dots \omega_{q-2}^i]$ we can construct two new $(q-1)$-cylinders

$$[\omega_1^i, \dots, \omega_{q-2}^i, +1] \text{ and } [\omega_1^i, \dots, \omega_{q-2}^i, -1],$$

corresponding respectively to say $K_{j_1(i)}$ and $K_{j_2(i)}$. We can also define two q-cylinders $C_{k_1(i)} = [\omega_0^i, \dots, \omega_{q-2}^i, +1]$ and $C_{k_2(i)} = [\omega_0^i, \dots, \omega_{q-2}^i, -1]$. Then we set

$$M(\mathbf{z})_{ij} = \begin{cases} 0 \text{ if } j \notin \{j_1(i), j_2(i)\}, \\ e^{z_{k_1(i)}} \text{ if } j = j_1(i), \\ e^{z_{k_1(i)}} \text{ if } j = j_2(i). \end{cases}$$

A simple computation shows that

$$\mathcal{L}_\xi(\mathbb{1}_{K_i}) = e^{\xi a_{k_1(i)}} \mathbb{1}_{K_{j_1(i)}} + e^{\xi a_{k_2(i)}} \mathbb{1}_{K_{j_2(i)}}.$$

Let us call E_{q-1} the vector space of functions constant on $(q-1)$-cylinders. The set $\{\mathbb{1}_{K_i}; 1 \le i \le 2^{q-1}\}$ is a basis of E_{q-1}, which can be identified with the canonical basis $(e_i)_{1 \le i \le 2^{q-1}}$ of $\mathbb{R}^{2^{q-1}}$. With this identification we get that the restriction of $\mathcal{L}_\xi$ to E_{q-1} is a linear operator whose matrix is exactly the transpose ${}^tM(\xi.\mathbf{a})$ of $M(\xi.\mathbf{a})$. Therefore, $\lambda(\xi)$ is the dominating eigenvalue for $M(\xi.\mathbf{a})$. We emphasize that this matrix is aperiodic and then the Perron Frobenius theorem on matrices holds. Furthermore, H_ξ is a left eigenvector for $M(\xi.\mathbf{a})$ associated to $\lambda(\xi)$.

Let us denote by T_ξ the function $x \mapsto e^{\xi S_{q-1}(\sigma^{p+1-q}(\omega x))}$. This function T_ξ belongs to E_{q-1}, and

$$\|T_\xi\|_\infty \le e^{(q-1)|\xi|A},$$

where $A := \max\{|a_i|; 1 \le i \le 2^q\}$. The spectral properties of $\mathcal{L}_\xi$ then yield that

$$\left\| \frac{\mathcal{L}_\xi^{n-p}(T_\xi)}{\lambda(\xi)^{n-p}} - \int T_\xi \, d\nu_\xi \; . H_\xi \right\|_\infty \le \frac{C(\xi)}{(1+\varepsilon(\xi))^{n-p}} e^{(q-1)|\xi|A}, \tag{34}$$

where $\varepsilon(\xi)$ is continuous and positive (in ξ) and $C(\xi)$ does only depend on ξ.

Lemma 4.1. *Let φ_{OS} be defined be $\varphi_{OS}(t) = -\frac{\beta}{2}t^2 + \log\lambda(\beta t)$. Then, for each β, φ_{OS} attains its maximum on $\mathbb{R}$ at finitely many points.*

Proof. Recall that $A = \max|a_i|$. We do the case $t \ge 0$, the case $t \le 0$ being the same with $-A$ instead of A.

Inequalities between eigenvalues of matrices and coefficients yield

$$\lambda(\beta t) \leq 2e^{A\beta t}.$$

This shows that for $t \geq 0$, $\varphi_{OS}(t) \leq -\frac{\beta}{2}t^2 + A\beta t + \log 2$, and this upper bound goes to $-\infty$ if t goes to $+\infty$. On the other hand, $\varphi_{OS}(0) = \log 2$. Therefore, φ_{OS} reaches its maximum on $\mathbb{R}^+$ only on some compact set $[0, \kappa(\beta)]$.

Furthermore, any t which realizes the maximum must be a critical point of the derivative. Note that $\xi \mapsto \lambda(\xi)$ admits an analytic continuation in a complex neighborhood of $\mathbb{R}$ which prohibits to get infinitely many zeros of φ'_{OS} on the segment $[0, \kappa(\beta)]$. □

Proposition 4.2. *Let β be fixed. Let $(t_i)_{1\leq i\leq J}$ be the finitely many real numbers where φ_{OS} reaches its maximum*[4]*. Then, there exist finitely many positive b_i, $1 \leq i \leq J$ such that $\sum_{i=1}^{J} b_i = 1$ and such that $\mu_{n,\beta}([\omega])$ converges to*

$$\sum_{i=1}^{J} b_i \nu_{\beta t_i}([\omega]).$$

Proof. From the definition of $\mu_{n,\beta}$ we know that

$$\mu_{n,\beta}([\omega]) = \frac{\sum_{\alpha,\ |\alpha|=n-p+q-1} e^{\frac{\beta}{2n}S_n^2(\omega\alpha)}}{\sum_{\gamma,\ |\gamma|=n+q-1} e^{\frac{\beta}{2n}S_n^2(\gamma)}}.$$

Using (31) and (33) we get

$$\mu_{n,\beta}([\omega]) = \frac{\sum_{|\alpha|=q-1} \int_{-\infty}^{+\infty} e^{-n\frac{\beta}{2}z^2} e^{\beta z S_{p-q+1}(\omega)} \mathcal{L}_{\beta z}^{n-p}(T_{\beta z})(\alpha)\, dz}{\sum_{|\alpha|=q-1} \int_{-\infty}^{+\infty} e^{-n\frac{\beta}{2}z^2} \mathcal{L}_{\beta z}^{n}(\mathbb{1})(\alpha)\, dz}. \tag{35}$$

We can rewrite the numerator as a sum of q terms of the form

$$\int_{-\infty}^{+\infty} e^{n\varphi_{OS}(\beta z)} e^{\beta z S_{p-q+1}(\omega) - p\log\lambda(\beta z)} \frac{\mathcal{L}_{\beta z}^{n-p}}{\lambda(\beta z)^{n-p}}(T_{\beta z})(\alpha)\, dz,$$

and the denominator as a sum of q terms of the form

$$\int_{-\infty}^{+\infty} e^{n\varphi_{OS}(\beta z)} \frac{\mathcal{L}_{\beta z}^{n}}{\lambda(\beta z)^{n}}(\mathbb{1})(\alpha)\, dz.$$

We want to use the Laplace method for these terms. Lemma 4.1 shows that the exponential term (in n) has only finitely many maximum. But we are not yet done and the problem comes from the last term in the integral which depends on n. Inequality (34) shows that this term converge as n goes to $+\infty$ but the speed of convergence depends on z and z may goes to $\pm\infty$.

[4]We emphasize that the t_i's depend on β but we drop the index β to get lighter notations.

To deal with this fact, we recall that A is $\max|a_i|$ and we notice that for every continuous T,

$$||\mathcal{L}_{\beta z}(T)||_\infty \leq 2e^{A\beta|z|}||T||_\infty,$$

therefore

$$||\mathcal{L}_{\beta z}(T_{\beta z})||_\infty \leq 2e^{q\beta|z|A},$$

and

$$|e^{-n\frac{\beta}{2}z^2}e^{\beta z S_{p-q+1}(\omega)}\mathcal{L}_{\beta z}^{n-p}(T_{\beta z})(\alpha)| \leq e^{n(-\frac{\beta}{2}z^2+\beta|z|A(p+1)+\log 2)}.$$

As $-\frac{\beta}{2}z^2+|z|(\beta A(p+1)+1)+\log 2$ goes to $-\infty$ as $|z|$ goes to $+\infty$, we can deduce that there exists $\kappa(\beta)>0$ such that for every $n\geq 2$,

$$\int_{|z|\geq\kappa(\beta)} e^{-n\frac{\beta}{2}z^2}e^{\beta z S_{p-q+1}(\omega)}\mathcal{L}_{\beta z}^{n-p}(T_{\beta z})(\alpha)\,dz \leq e^{-n\kappa(\beta)}. \tag{36}$$

We choose $\kappa(\beta)$ big enough so that all the points t_j, $1\leq j\leq J$, where φ_{OS} reaches its maximum, are in the interior of the interval $[-\kappa(\beta),\kappa(\beta)]$. Then we can write the integral over the segment $[-\kappa(\beta),\kappa(\beta)]$ as a finite sum of integrals over segments $[a_j,b_j]$ where each segment $[a_j,b_j]$ contains exactly one of the points t_j, $1\leq j\leq J$. We state the following lemma, which is an immediate adaptation of the Laplace method.

Lemma 4.3. *Let $\varphi:[a,b]\to\mathbb{R}$ a function of class C^2, with φ' vanishing on a single point c in $]a,b[$ and $\varphi''(c)<0$. Let $(f_n)_{n\geq 1}$, f some continuous functions from $[a,b]$ to $\mathbb{R}$ such that f_n converges to f uniformly on $[a,b]$, and $f(c)\neq 0$. Then as $n\to\infty$*

$$\int_a^b e^{n\varphi(x)}f_n(x)\,dx \sim \sqrt{\frac{\pi}{2|\varphi''(c)|}}e^{n\varphi(c)}f(c)n^{-1/2}.$$

We apply this lemma on every $[a_j,b_j]$ to the functions f_n defined by

$$f_n(z) = e^{\beta z S_{p-q+1}(\omega)-p\log\lambda(\beta z)}\frac{\mathcal{L}_{\beta z}^{n-p}}{\lambda(\beta z)^{n-p}}(T_{\beta z})(\alpha).$$

Thanks to (34) the functions f_n converge uniformly on $[a_j,b_j]$ to f defined by

$$\begin{aligned} f(z) &= \frac{e^{\beta z S_{p+1-q}(\omega)}}{\lambda(\beta z)^p}\int e^{\beta z S_{q-1}(\sigma^{p+1-q}(\omega x))}\,d\nu_{\beta z}(x)\;.H_{\beta z}(\alpha) \\ &= \frac{1}{\lambda(\beta z)^p}\int e^{\beta z S_p(\omega x)}\,d\nu_{\beta z}(x)\;.H_{\beta z}(\alpha) \\ &= \frac{1}{\lambda(\beta z)^p}\int \mathcal{L}_{\beta z}^p(\mathbb{1}_{[\omega]})(x)\,d\nu_{\beta z}(x)\;.H_{\beta z}(\alpha) \\ &= H_{\beta z}(\alpha)\nu_{\beta z}([\omega]). \end{aligned}$$

Putting together (36) and the result of Lemma 4.3 applied to every $[a_j, b_j]$ we obtain that the numerator of $\mu_{n\beta}([\omega])$ in (35) is equivalent to

$$\sum_{|\alpha|=q-1} \sum_{j=1}^{J} \sqrt{\frac{\pi}{2|\varphi''_{OS}(t_j)|}} e^{n\varphi_{OS}(t_j)} H_{\beta t_j}(\alpha) \nu_{\beta t_j}([\omega]) n^{-1/2}.$$

Similarly, the denominator of $\mu_{n\beta}([\omega])$ in (35) is equivalent to

$$\sum_{|\alpha|=q-1} \sum_{j=1}^{J} \sqrt{\frac{\pi}{2|\varphi''_{OS}(t_j)|}} e^{n\varphi_{OS}(t_j)} H_{\beta t_j}(\alpha) n^{-1/2}.$$

We then get that $\mu_{n\beta}([\omega])$ is equivalent to

$$\frac{\sum_{j=1}^{J} \sum_{|\alpha|=q-1} \frac{H_{\beta t_j}(\alpha)}{\sqrt{|\varphi''_{OS}(t_j)|}} \nu_{\beta t_j}([\omega])}{\sum_{j=1}^{J} \sum_{|\alpha|=q-1} \frac{H_{\beta t_j}(\alpha)}{\sqrt{|\varphi''_{OS}(t_j)|}}}.$$

Let us set

$$b_i := \frac{\sum_{\alpha,|\alpha|=q-1} \frac{H_{\beta t_i}(\alpha)}{\sqrt{|\varphi''_{OS}(t_i)|}}}{\sum_{j,\alpha,|\alpha|=q-1} \frac{H_{\beta t_j}(\alpha)}{\sqrt{|\varphi''_{OS}(t_j)|}}}.$$

Then the b_i's are positive, they satisfy $\sum_i b_i = 1$, and $\mu_{n,\beta}([\omega])$ converges to

$$\sum_i b_i \nu_{\beta t_i}([\omega]).$$

□

4.2. **End of the Proof of Theorem 2.**

4.2.1. *Some properties for Pressure function.* We recall here some properties for the pressure function. If g is Hölder continuous on Σ_2 then $t \mapsto \mathcal{P}(tg)$ is convex thus continuous. The Ruelle theorem also yields

$$\frac{\partial}{\partial t} \mathcal{P}(tg)_{t=t_0} = \int g \, d\widetilde{\mu}_{t_0}, \tag{37}$$

where $\widetilde{\mu}_{t_0}$ is the DGM for $t_0 g$.

We recall also that for $g = \psi = \sum_{i=1}^{2^q} a_i \mathbb{1}_{C_i}$, $\mathcal{P}(\beta t \psi) = \log \lambda(\beta t)$, where $\lambda(\beta t)$ is the spectral radius for the matrix $M(\beta t \mathbf{a})$.

4.2.2. *Maximizing the quadratic pressure.* There are 2^q q-cylinders in Σ_2. If μ and ν are invariant measures, then

$$\mu \approx \nu \iff \mu([\omega_0 \dots \omega_{q-1}]) = \nu([\omega_0 \dots \omega_{q-1}]) \text{ for every } q\text{-cylinder.}$$

defines an equivalence relation over the set of invariant probabilities. If $\mu \approx \nu$ then $\int \psi \, d\mu = \int \psi \, d\nu$ holds because ψ is constant on q-cylinders.

Note that $h_\mu + \dfrac{\beta}{2}\left(\int \psi \, d\mu\right)^2$ is maximal among elements of the same class for $\approx$ if and only if h_μ is maximal.

Because we are considering probability measures, it is sufficient to know $\mu(C_i)$ for all $i < 2^q$ to know all of the measure μ. If y_i, $i = 1, \dots 2^q - 1$ are non-negative real numbers such that $\sum y_i \leq 1$, then we denote by $\approx_{\mathbf{y}}$, the class of measures such that $\mu(C_i) = y_i$ for all $i < 2^q$.

Now, we state a lemma that is an adapted version of results from [2] (th. 7, 8) and [12] (th 4.8). These papers focus on multifractal analysis and they have different ways to state the result we need here. It is not clear to us if their statements can be directly applied in our case.

Lemma 4.4. *Let* $\mathbf{y}$ *be such that* $\approx_{\mathbf{y}}$ *is non-empty. Then*

$$\max\{h_\mu, \mu \in \approx_{\mathbf{y}}\} \leq \inf_{\mathbf{z} \in \mathbb{R}^{2^q-1},\ \mathbf{z} \neq 0} \left\{ \mathcal{P}\left(\sum z_i \mathbb{1}_{C_i}\right) - \sum z_i y_i \right\}$$

Proof. Note the immediate equality:

$$\inf_{t \in \mathbb{R}_+,\ \mathbf{z} \in \mathbb{S}^{2^q-2}} \left\{ \mathcal{P}(t \sum z_i \mathbb{1}_{C_i}) - t \sum z_i y_i \right\} = \inf_{\mathbf{z} \in \mathbb{R}^{2^q-1},\ \mathbf{z} \neq 0} \left\{ \mathcal{P}\left(\sum z_i \mathbb{1}_{C_i}\right) - \sum z_i y_i \right\}$$

For our proof, we prefer to use the left hand side characterization. Let us pick $\mathbf{z} \in \mathbb{S}^{2^q-2}$. First we assume that $\sum z_i y_i = \max \int \sum z_i \mathbb{1}_{C_i} d\mu$, where the maximum is taken among all invariant measures. Then, it is well known (see [1]) that $t \mapsto \mathcal{P}(t \sum z_i \mathbb{1}_{C_i}) - t \sum z_i y_i$ is a convex decreasing function which converges to the residual entropy:

$$\lim_{t \to +\infty} \mathcal{P}(t \sum z_i \mathbb{1}_{C_i}) - t \sum z_i y_i = \max \left\{ h_\mu,\ \int \sum z_i \mathbb{1}_{C_i} \text{ is maximal} \right\}.$$

Clearly the residual entropy is bigger than $\max\{h_\mu, \mu \in \approx_{\mathbf{y}}\}$.

Let us now assume that the class $\approx_{\mathbf{y}}$ is not included into the set of maximizing measures for $\sum z_i \mathbb{1}_{C_i}$. Then $\lim_{t \to +\infty} \mathcal{P}(t \sum z_i \mathbb{1}_{C_i}) - t \sum z_i y_i = +\infty$ and the convex function admits a unique minimum on R_+.

If this minimum is reached at $t = 0$, then its value is $h_\top = \log 2$ which is bigger than any metric entropy.

The last case holds if the minimum is reached at a critical point, say t_0. Note that if ν_{t_0} is the equilibrium state for $t_0 \sum z_i \mathbb{1}_{C_i}$, then

$$\frac{\partial}{\partial t}\mathcal{P}(t\sum z_i \mathbb{1}_{C_i})_{|t=t_0} = \int \sum z_i \mathbb{1}_{C_i} d\nu_{t_0}.$$

The fact that t_0 is a critical point yields

$$\int \sum z_i \mathbb{1}_{C_i} d\nu_{t_0} = \sum z_i y_i,$$

and then for any μ in $\approx_{\mathbf{y}}$, we get

$$h_\mu + t_0 \sum z_i y_i = h_\mu + t_0 \int \sum z_i \mathbb{1}_{C_i} d\mu \leq h_{\nu_{t_0}} + t_0 \int \sum z_i \mathbb{1}_{C_i} d\nu_{t_0} = \mathcal{P}(t_0 \sum z_i \mathbb{1}_{C_i}),$$

which concludes the proof. □

Let $\mathbf{y}$ be such that $\approx_{\mathbf{y}}$ is non-empty. Note that for every $\mu \in \approx_{\mathbf{y}}$, $\int \psi \, d\mu$ is the constant $a_{2^q} + \sum (a_i - a_{2^q}) y_i$. For simplicity we denote this quantity by $\langle \psi | \mathbf{y} \rangle$. Upper semi-continuity for the entropy yields the existence of measures $\mu \in \approx_{\mathbf{y}}$ such that

$$h_\mu = \max\{h_\nu, \nu \in \approx_{\mathbf{y}}\}.$$

Then we set $h_{\mathbf{y}} := h_\mu$.

Lemma 4.5. *For every $\mathbf{y}$ such that $\approx_{\mathbf{y}}$ is non-empty, set $t = \langle \psi | \mathbf{y} \rangle$. Then,*

$$h_{\mathbf{y}} + \frac{\beta}{2} t^2 \leq \varphi_{OS}(t).$$

Proof. Let $\widetilde{\mu}_t$ be the DGM associated to the potential $\beta t \sum (a_i - a_{2^q}) \mathbb{1}_{C_i}$. Note the following equality

$$\beta t \sum (a_i - a_{2^q}) \mathbb{1}_{C_i} = \beta t \psi - \beta t a_{2^q}.$$

Set

$$\mathbf{z} := (z_1, \ldots, z_{2^q - 1}), \text{ with } z_i := \beta t (a_i - a_{2^q}).$$

The properties of the Pressure function (see Subsubsec. 4.2.1) yield that the pressure for $\beta t \sum (a_i - a_{2^q}) \mathbb{1}_{C_i}$ is equal to

$$\mathcal{P}(\beta t \psi) - \beta t a_{2^q} = \log \lambda(\beta t) - \beta t a_{2^q}.$$

On the other hand Lemma 4.4 yields

$$h_{\mathbf{y}} \leq \mathcal{P}(\sum z_i \mathbb{1}_{C_i}) - \sum z_i y_i. \tag{38}$$

We immediately get the following sequence of inequalities:

$$\begin{aligned} h_{\mathbf{y}} + \frac{\beta}{2}t^2 &\leq \log\lambda(\beta t) - \beta t a_{2^q} - \beta t \sum (a_i - a_{2^q}) y_i + \frac{\beta}{2}t^2 \\ &\leq \log\lambda(\beta t) + \frac{\beta}{2}t^2 - \beta t\langle\psi|\mathbf{y}\rangle \\ &\leq \log\lambda(\beta t) + \frac{\beta}{2}t^2 - \beta t^2 = \varphi_{OS}(t), \end{aligned}$$

where the last inequality uses $t = \langle\psi|\mathbf{y}\rangle$. □

Lemma 4.6. *If t is a critical point for φ_{OS}, we consider the DGM $\widetilde{\mu}_{t\beta\psi}$ associated to $t\beta\psi$. We also set $y_i = \widetilde{\mu}_{t\beta\psi}(C_i)$ for $1 \leq i \leq 2^q$ and*

$$\mathbf{y} := (y_1, \dots y_{2^q-1}).$$

Then, $t = \displaystyle\int \psi \, d\widetilde{\mu}_{t\beta\psi}$ and $h_{\mathbf{y}} + \frac{\beta}{2}t^2 = \varphi_{OS}(t)$.

Proof. The DGM $\widetilde{\mu}_{t\beta\psi}$ has pressure $\log\lambda(\beta t)$. Moreover Equality (37) yields

$$\frac{\lambda'(\beta t)}{\lambda(\beta t)} = \int \psi \, d\widetilde{\mu}_{\beta t\psi}.$$

As t is a critical point for φ_{OS} we have

$$\beta t = \beta . \frac{\lambda'(\beta t)}{\lambda(\beta t)},$$

and then $t = \displaystyle\int \psi \, d\widetilde{\mu}_{t\beta\psi}$ holds. We deduce that

$$\mathcal{P}(\beta t\psi) - \beta t\langle\psi|\mathbf{y}\rangle = \log\lambda(\beta t) - \beta t^2 = \varphi_{OS}(t) - \frac{\beta}{2}t^2.$$

Now by definition of the DGM we know that

$$\mathcal{P}(\beta t\psi) - \beta t\langle\psi|\mathbf{y}\rangle = h_{\widetilde{\mu}_{t\beta\psi}}$$

which is smaller than $h_{\mathbf{y}}$ by definition of this last term, therefore

$$\varphi_{OS}(t) \leq h_{\mathbf{y}} + \frac{\beta}{2}t^2.$$

The reverse inequality holds true thanks to Lemma 4.5, which finishes the proof. □

We can now finish the proof of Theorem 2. We have seen that the PGM $\mu_{\beta,n}$ converges to $\sum_j b_j \nu_{\beta t_j}$, where the t_j's are the maxima for φ_{OS}.

For these t_j's, Lemmas 4.5 and 4.6 show that

$$h_\mu + \frac{\beta}{2}\left(\int \psi \, d\mu\right)^2 \leq \varphi_{OS}(t_j),$$

with equality if μ is one of the DGM's, $\widetilde{\mu}_{\beta t_k\psi}$.

It remains to prove that the DGM's are the only maximizers of the quadratic pressure. We define a new functional

$$\phi_r(t) := \max\left\{ h_\mu + \frac{\beta}{2}t^2,\ \int \psi\, d\mu = t \right\}.$$

This function is defined on some interval $\mathcal{M}(\psi)$. If μ is a measure, Lemma 4.5 shows for $\int \psi\, d\mu =: t$

$$h_\mu + \frac{\beta}{2}t^2 \leq \varphi_{OS}(t).$$

This yields that for all $t \in \mathcal{M}(\psi)$,

$$\phi_r(t) \leq \varphi_{OS}(t). \tag{39}$$

Therefore, Lemma 4.6 shows that the maxima for ϕ_r are the maxima for φ_{OS}. If a measure μ maximizes the quadratic pressure, then we have

$$\mathcal{P}_2(\beta\psi) = h_\mu + \frac{\beta}{2}\left(\int \psi\, d\mu\right)^2$$

and setting $t := \int \psi\, d\mu$ we must have $\phi_r(t) = \max \varphi_{OS}(t)$ which yields that t is one of the t_j's. Now we have

$$\int \psi\, d\mu = t_j = \int \psi\, d\widetilde{\mu}_{\beta t_j \psi},$$

and then

$$\mathcal{P}_2(\beta\psi) = h_\mu + \frac{\beta}{2}\left(\int \psi\, d\mu\right)^2 = \varphi_{OS}(t_j) = h_{\widetilde{\mu}_{\beta t_j\psi}} + \frac{\beta}{2}\left(\int \psi\, d\widetilde{\mu}_{\beta t_j\psi}\right)^2$$

hence $h_\mu = h_{\widetilde{\mu}_{\beta t_j\psi}}$. Therefore

$$h_\mu + \frac{\beta}{2}\int \psi\, d\mu = h_{\widetilde{\mu}_{\beta t_j\psi}} + \frac{\beta}{2}\int \psi\, d\widetilde{\mu}_{\beta t_j\psi}$$

and μ must be the DGM $\widetilde{\mu}_{\beta t_j\psi}$. This finishes the proof of Theorem 2.

References

[1] Alexandre Baraviera, Renaud Leplaideur, and Artur Lopes. *Ergodic optimization, zero temperature limits and the max-plus algebra*. Publicações Matemáticas do IMPA. [IMPA Mathematical Publications]. Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2013. 29o Colóquio Brasileiro de Matemática.

[2] Luis Barreira, Benoît Saussol, and Jörg Schmeling. Higher-dimensional multifractal analysis. *J. Math. Pures Appl. (9)*, 81(1):67–91, 2002.

[3] R. Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*. Lecture Notes in Mathematics, Vol. 470. Springer-Verlag, Berlin, 1975. 2nd ed. - 2008 by JR Chazottes.

[4] L. Cioletti and A. O. Lopes. Interactions, Specifications, DLR probabilities and the Ruelle Operator in the One-Dimensional Lattice. *ArXiv e-prints*, April 2014.

[5] L. Cioletti and A. O. Lopes. Phase transitions in one-dimensional translation invariant systems: a Ruelle operator approach. *J. Stat. Phys.*, 159(6):1424–1455, 2015.
[6] L. Cioletti and A. O. Lopes. Ruelle Operator for Continuous Potentials and DLR-Gibbs Measures. *ArXiv e-prints*, August 2016.
[7] Marius Costeniuc, Richard S. Ellis, and Hugo Touchette. Complete analysis of phase transitions and ensemble equivalence for the Curie-Weiss-Potts model. *J. Math. Phys.*, 46(6):063301, 25, 2005.
[8] Richard S. Ellis. *Entropy, large deviations, and statistical mechanics*. Classics in Mathematics. Springer-Verlag, Berlin, 2006. Reprint of the 1985 original.
[9] Richard S. Ellis and Charles M. Newman. Limit theorems for sums of dependent random variables occurring in statistical mechanics. *Z. Wahrsch. Verw. Gebiete*, 44(2):117–139, 1978.
[10] Richard S. Ellis and Charles M. Newman. The statistics of Curie-Weiss models. *J. Statist. Phys.*, 19(2):149–161, 1978.
[11] Richard S. Ellis and Kongming Wang. Limit theorems for the empirical vector of the Curie-Weiss-Potts model. *Stochastic Processes Appl.*, 35(1):59–79, 1990.
[12] De-Jun Feng and Wen Huang. Lyapunov spectrum of asymptotically sub-additive potentials. *Comm. Math. Phys.*, 297(1):1–43, 2010.
[13] Hans-Otto Georgii. *Gibbs measures and phase transitions*, volume 9 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, second edition, 2011.
[14] R. B. Griffiths and D. Ruelle. Strict convexity (continuity) of the pressure in lattice systems. *Commun. Math. Phys.*, 23:169–175, 1971.
[15] Hubert Hennion and Loïc Hervé. *Limit theorems for Markov chains and stochastic properties of dynamical systems by quasi-compactness*, volume 1766 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.
[16] J. Hubbard. Calculation of partition functions. *Phys. Rev. Lett.*, 3:77–78, Jul 1959.
[17] Gerhard Keller. *Equilibrium states in ergodic theory.* Cambridge: Cambridge University Press, 1998.
[18] Renaud Leplaideur. Chaos: butterflies also generate phase transitions. *J. Stat. Phys.*, 161(1):151–170, 2015.
[19] Steven Orey. Large deviations for the empirical field of Curie-Weiss models. *Stochastics*, 25(1):3–14, 1988.
[20] W. Parry and M. Pollicott. *Zeta functions and the periodic orbit structure of hyperbolic dynamics*, volume 187-188 of *Astérisque*. SMF, 1990.
[21] Firas Rassoul-Agha and Timo Seppäläinen. *A course on large deviations with an introduction to Gibbs measures*, volume 162 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2015.
[22] D. Ruelle. *Statistical mechanics*. World Scientific Publishing Co. Inc., River Edge, NJ, 1999. Rigorous results, Reprint of the 1989 edition.
[23] D. Ruelle. *Thermodynamic formalism*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2004. The mathematical structures of equilibrium statistical mechanics.
[24] Y. Sinai. Gibbs measures in ergodic theory,. *Uspehi Mat. Nauk*, 27(4(166)):21–64, 1972.
[25] Y. Sinaĭ. *Theory of phase transitions: rigorous results*, volume 108 of *International Series in Natural Philosophy*. Pergamon Press, Oxford, 1982. Translated from the Russian by J. Fritz, A. Krámli, P. Major and D. Szász.
[26] R. L. Stratonovič. A method for the computation of quantum distribution functions. *Dokl. Akad. Nauk SSSR (N.S.)*, 115:1097–1100, 1957.
[27] Maximilian Thaler. Estimates of the invariant densities of endomorphisms with indifferent fixed points. *Israel J. Math.*, 37(4):303–314, 1980.

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