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Random dynamical systems with systematic drift competing with heavy-tailed randomness

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Abstract

Motivated by the study of the time evolution of random dynamical systems arising in a vast variety of domains — ranging from physics to ecology —, we establish conditions for the occurrence of a non-trivial asymptotic behaviour for these systems in the absence of an ellipticity condition. More precisely, we classify these systems according to their type and — in the recurrent case — provide with sharp conditions quantifying the nature of recurrence by establishing which moments of passage times exist and which do not exist. The problem is tackled by mapping the random dynamical systems into Markov chains on \mathbb{R} with heavy-tailed innovation and then using powerful methods stemming from Lyapunov functions to map the resulting Markov chains into positive semi-martingales.

Keywords: Markov chains, recurrence, heavy tails, moments of passage times, random dynamical systems.

AMS 2010 subject classifications: Primary: 60J05; (Secondary: 37H10, 34F05).

1 Introduction

1.1 Motivation and description of the model

Dynamical systems arise in several applied domains (economy, ecology, etc.) as models of evolution. We study in this paper the combined action of randomness and systematic deterministic bias that leads to a subtle competition of two antagonistic effects.

Suppose that there exists a universal mapping $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ — verifying certain conditions that will be precised later — and consider the following random dynamical system $X_{n+1} = A_{n+1}X_n f(A_{n+1}X_n)$, where $(A_n)_{n \geq 1}$ are a sequence of independent and identically distributed \mathbb{R}^+ -valued random variables with law ν and $X_n \in \mathbb{R}^+$ for all $n \geq 0$. Not to complicate unnecessarily the model, we assume that ν has always a density, with respect to either the Lebesgue measure on the non-negative axis or the counting measure of some infinitely denumerable unbounded subset of \mathbb{R}^+ . We address the question about the asymptotic behaviour of X_n , as $n \rightarrow \infty$. The situation $\lim_{n \rightarrow \infty} X_n = 0$ has a special significance since can be interpreted as the extinction of certain natural resources, or the bankruptcy of certain financial assets, etc. The dual situation of $\lim_{n \rightarrow \infty} X_n = \infty$ can also be interpreted as the proliferation of certain species, or the creation of instabilities due to the formation of speculative bubbles, etc. (see [2] for instance).

Since the previous Markov chain is multiplicative, it is natural to work at logarithmic scale and consider the additive version of the dynamical system $\xi_{n+1} = \xi_n + \alpha_{n+1} + \psi(\xi_n + \alpha_{n+1})$, with $\alpha_n = \ln A_n$ and $\psi(z) = \ln f(e^z)$, for $z \in \mathbb{R}$. Therefore, the Markov chain becomes now an \mathbb{R} -valued one reading $\xi_{n+1} = \xi_n + \alpha_{n+1} + \psi(\xi_n + \alpha_{n+1})$. Obviously, $\xi_n \rightarrow +\infty$ a.s. $\Leftrightarrow X_n \rightarrow +\infty$ a.s. and $\xi_n \rightarrow -\infty$ a.s. $\Leftrightarrow X_n \rightarrow 0$ a.s.

An important class of non-uniformly elliptic random dynamical systems are those (X_n) that — when considered at logarithmic scale as above — have $\psi(t) = \pm |t|^\gamma$, for $0 < \gamma < 1$ and $t \in \mathbb{R}^+$. Now using the elementary inequalities (see [5, §19, p. 28], for instance) $a^\gamma - |b|^\gamma \leq (a+b)^\gamma \leq a^\gamma + |b|^\gamma$, it turns out that the dynamical system reads $\xi_{n+1} = \xi_n + \alpha_{n+1} \pm |\xi_n + \alpha_{n+1}|^\gamma = \xi_n + \alpha_{n+1} \pm |\xi_n|^\gamma + \mathcal{O}(\alpha_{n+1}^\gamma)$. Now, for $\gamma \in]0, 1[$, the term $\mathcal{O}(\alpha_{n+1}^\gamma)$ in the above expression turns out to be subdominant.

For the aforementioned reasons, we study in this paper the Markov

chains on $\mathbb{X} = \mathbb{R}^+$ defined by one of the following recursions

$$\begin{aligned}\zeta_{n+1} &= (\zeta_n + \alpha_{n+1} - \zeta_n^\gamma)^+, \text{ or} \\ \zeta_{n+1} &= (\zeta_n + \alpha_{n+1} + \zeta_n^\gamma)^+, \end{aligned}$$

with $\gamma \in]0, 1[$ and $\zeta_0 = x$ a.s.; here $z^+ = \max(0, z)$ and $x \in \mathbb{X}$. The sequence $(\alpha_n)_{n \geq 1}$ are a family of independent \mathbb{R} -valued random variables having common distribution. This distribution can be supposed discrete or continuous but will always be assumed having one- or two-sided heavy tails. The heaviness of the tails is quantified by the order of the fractional moments failing to exist.

1.2 Main results

In all statements below, we make the

Global assumption 1.1. The sequence $(\alpha_n)_{n \in \mathbb{N}}$ are independent and identically distributed real random variables. The common law is denoted by μ and is supposed to be $\mu \ll \lambda$ where λ is a reference measure on \mathbb{R} ; we denote by $m = \frac{d\mu}{d\lambda}$ the corresponding density. Additionally, μ is supposed to be heavy-tailed (preventing thus integrability of the random variables α_n).

Let $(\zeta_n)_{n \in \mathbb{N}}$ be a Markov chain on a measurable space $(\mathbb{X}, \mathcal{X})$; denote, as usual, by \mathbb{P}_x the probability on the trajectory space conditioned to $\zeta_0 = x$ and, for $A \in \mathcal{X}$, define $\tau_A = \inf\{n \geq 1 : \zeta_n \in A\}$. Our paper is devoted in establishing conditions under which the time τ_A is finite (a.s.) or infinite (with strictly positive probability) and in case it is a.s. finite which of its moments exist. These results constitute the first step toward establishing more general results on the Markov chain like recurrence or transience, positive recurrence and existence of invariant probability, etc. However, the latter need more detailed conditions on the communication structure of states of the chain like ϕ -accessibility, ϕ -recurrence, maximal irreducibility measures and so on (see [9, 8, 7] for instance). All those questions are important but introduce some technicalities that blur the picture that we wish to reveal here, namely that questions on τ_A can be answered with extreme parsimony on the hypotheses imposed on the Markov chain, by using Lyapunov functions. As a matter of fact, the only communication

property imposed on the Markov chain is mere accessibility whose definition is recalled here for the sake of completeness.

Definition 1.2. Let (Z_n) be a Markov chain on $(\mathbb{X}, \mathcal{X})$ with stochastic kernel P and $A \in \mathcal{X}$. Denote by \mathbb{P} the probability on its trajectory space induced by P and by \mathbb{P}_x the law of trajectories conditioned on $\{Z_0 = x\}$. We say that A is **accessible** from $x \notin A$, if $\mathbb{P}_x(\tau_A < \infty) > 0$.

Theorem 1.3. Let (ζ_{n+1}) be the Markov chain defined by the recursion

$$\zeta_{n+1} = \zeta_n - \zeta_n^\gamma + \alpha_{n+1}, \quad n \geq 0,$$

where $0 < \gamma < 1$ and the random variables (α_n) have a common law μ supported by \mathbb{R}_+ , satisfying the condition $\mu([0, 1]) > 0$ and whose density with respect to the Lebesgue measure, for large $y > 0$, reads $m(y) = \mathbb{1}_{\mathbb{R}_+}(y)c_y y^{-1-\theta}$, with $\theta \in]0, 1[$. Let $a > 1$ and denote by $A := A_a = [0, a]$. Then A is accessible from any point $x > a$. Additionally, the following hold.

1. Assume that there exist constants $0 < b_1 < b_2 < \infty$ such that $b_1 \leq c_y \leq b_2$ for all $y \in \mathbb{X}$.
 - (a) If $\theta > 1 - \gamma$ then $\mathbb{P}_x(\tau < \infty) = 1$. Additionally,
 - if $q < \frac{\theta}{1-\gamma}$ then $\mathbb{E}_x(\tau_A^q) < \infty$, and
 - if $q \geq \frac{\theta}{1-\gamma}$ then $\mathbb{E}_x(\tau_A^q) = \infty$.
 - (b) If $\theta < 1 - \gamma$ then $\mathbb{P}_x(\tau_A < \infty) < 1$, and this implies transience.
2. Assume further that $\lim_{y \rightarrow \infty} c_y = c > 0$ and $\theta = 1 - \gamma$.
 - (a) If $c\pi \csc(\pi\theta) < \theta$ then $\mathbb{P}_x(\tau_A < \infty) = 1$. Additionally, there exists a unique $\delta_0 \in]0, \theta[$ such that
 - if $q < \frac{\delta_0}{1-\gamma}$ then $\mathbb{E}_x(\tau_A^q) < \infty$, and
 - if $q > \frac{\delta_0}{1-\gamma}$ then $\mathbb{E}_x(\tau_A^q) = \infty$.
 - (b) If $c\pi \csc(\pi\theta) > \theta$ then $\mathbb{P}_x(\tau_A < \infty) < 1$, and this implies transience.

Theorem 1.4. Let (ζ_{n+1}) be the Markov chain defined by the recursion

$$\zeta_{n+1} = (\zeta_n + \zeta_n^\gamma - \alpha_{n+1})^+$$

for $n \geq 0$, where $0 < \gamma < 1$ and the common law of the random variables (α_n) is supported by \mathbb{R}_+ and has density m with respect to the Lebesgue measure verifying $m(y) = \mathbb{1}_{\mathbb{R}_+}(y)c_y y^{-1-\theta}$ for large $y > 0$, with $\theta \in]0, 1[$. Assume further that $\lim_{y \rightarrow \infty} c_y = c > 0$. Then the state 0 is accessible and

1. If $\theta < 1 - \gamma$ then $\mathbb{E}_x(\tau_0^q) < \infty$, for all $q > 0$.
2. If $\theta = 1 - \gamma$ then $\mathbb{P}_x(\tau_0 < \infty) = 1$. Additionally, there exists a unique $\delta_0 \in]0, \infty[$ such that
 - If $q < \frac{\delta_0}{\theta}$ then $\mathbb{E}_x(\tau_0^q) < \infty$, and
 - if $q > \frac{\delta_0}{\theta}$ then $\mathbb{E}_x(\tau_0^q) = \infty$.
3. If $\theta > 1 - \gamma$ then $\mathbb{P}_x(\tau_0 < \infty) < 1$, and this implies transience.

Remark 1.5. If $b_1 \leq c_y \leq b_2$ but $c_y \not\rightarrow c$ then the conclusions established in the cases of strict inequalities $\theta < 1 - \gamma$ or $\theta > 1 - \gamma$ remain valid. Nevertheless, we are unable to treat the critical case $\theta = 1 - \gamma$.

Remark 1.6. In both the above theorems, the boundedness or existence of limit conditions on (c_y) imply that the tails have power decay, i.e. there exists C such that the tail estimate $\mathbb{P}(\alpha > y) \geq \frac{C}{y^\theta}$ holds. Nevertheless, the control we impose is much sharper because we wish to treat the critical case. If we are not interested in the critical case, the control on (c_y) can be considerably weakened by assuming only the tail estimate. Results established with such weakened control on the tails are given in theorems 1.8 and 1.9 below.

Remark 1.7. Although, in both the above theorems, the existence and uniqueness of δ_0 , occurring in the critical case $\theta = 1 - \gamma$, can be established by general convexity arguments, the sharp determination of its value requires the complete knowledge of the distribution function of (α_n) . Nevertheless, if we define $K_{\delta, \theta} = \frac{\Gamma(1-\theta)\Gamma(\theta-\delta)}{\theta\Gamma(1-\delta)}$, the value of δ_0 in 1.3 can be approximated by the solution of the transcendental equation $cK_{\delta_0, \theta} = 1$. Similarly, if we define¹ by $L_{\delta, \theta} = \frac{\Gamma(1+\delta)\Gamma(-\theta)}{\Gamma(1-\theta+\delta)}$, the value of δ_0 in 1.4 can be approximated by the solution of the transcendental equation $cL_{\delta_0, \theta} + \delta_0 = 0$.

¹It is recalled that the transcendental function Γ , defined by $\Gamma(z) := \int_0^\infty \exp(-t)t^{z-1}dt$ for $\Re z > 0$, can be analytically continued on $\mathbb{C} \setminus \{0, -1, -2, -3, \dots\}$; its analytic continuation can be expressed by $\Gamma(z) = \int_0^\infty [\exp(-t) - \sum_{m=0}^n \frac{(-t)^m}{m!}]t^{z-1}dt$ for $-(n+1) < \Re z < -n$ and $n \in \mathbb{N}$ (see [3, §1.1 (9), p. 2] for instance).

The faster the convergence $c_y \rightarrow c$, the better are these approximations; the determination becomes exact when c_y is constant (depending on θ). Not to complicate the study in the sequel, we sketch the proofs (for the critical cases) of the theorems 1.3 and 1.4 for c_y is constant.

Theorem 1.8. *Let (ζ_n) be the Markov chain defined by the recursive relation*

$$\zeta_{n+1} = (\zeta_n - \zeta_n^\gamma + \alpha_{n+1})^+, \quad n \geq 0, \quad (1)$$

where $0 < \gamma < 1$ and the random variables (α_n) have common law with support extending to both negative and positive parts of the real axis. Let $a > 1$ and denote by $A := A_a = [0, a]$. Then A is accessible and the following statements hold.

1. *Suppose that there exist a positive constant C and a parameter $\theta \in]0, 1[$ such that $\mathbb{P}(\alpha_1 > y) \leq Cy^{-\theta}$. If $\theta > 1 - \gamma$, then $\forall q < \frac{\theta}{1-\gamma}$, $\mathbb{E}_x(\tau_A^q) < \infty$.*
2. *Suppose that there exist a positive constants C, C' and parameters θ, θ' with $0 < \theta < \theta' < 1$ such that $\mathbb{P}(\alpha_1 > y) \geq C'y^{-\theta}$ and $\mathbb{P}(\alpha_1 < -y) \leq Cy^{-\theta'}$ (the right tails are heavier than the left ones). If $\theta < 1 - \gamma$, then $\mathbb{P}_x(\tau_A < \infty) < 1$, and this implies transience.*

Theorem 1.9. *Assume that the Markov chain (ζ_n) is defined by the recursive relation*

$$\zeta_{n+1} = (\zeta_n + \zeta_n^\gamma + \alpha_{n+1})^+, \quad n \geq 0,$$

where $0 < \gamma < 1$ and the random variables (α_n) have common law with support extending to both negative and positive parts of the real axis. Let $a > 1$ and suppose that the set $A := A_a = [0, a]$ is accessible.

1. *Suppose there exist a positive constant C and a parameter θ with $0 < \theta < 1$, such that $\mathbb{P}(\alpha_1 < -y) \leq Cy^{-\theta}$. If $\theta > 1 - \gamma$, then $\mathbb{P}_x(\tau_A < \infty) < 1$, and this implies transience.*
2. *Suppose there exist positive constant C, C' and parameters θ and θ' , with $0 < \theta < \theta' < 1$, such that $\mathbb{P}(\alpha_1 > y) \leq C'y^{-\theta'}$ and $\mathbb{P}(\alpha_1 < -y) \geq Cy^{-\theta}$. If $\theta < 1 - \gamma$ then the state 0 is recurrent and $\forall q < 1$, $\mathbb{E}_x(\tau_A^q) < \infty$.*

Remark 1.10. Some comments are due:

1. The systematic drift term ζ_n^γ , although subdominant with respect to ζ_n , is far from being trivial. As a matter of fact, this term is responsible for the failure of uniform ellipticity when we consider the system at exponential scale. In particular if the random innovation α_{n+1} is integrable, then the asymptotic behaviour of the random dynamical system is determined solely from the systematic drift term. Only when the innovation has heavy tails, a competition between systematic drift and random perturbation can built-up leading to interesting critical phenomena.
2. It is worth noting that the previous results establish not only recurrence or transience properties but a fine stratification of the parameter space according to which moments of the passage time τ_A exist. For the cases where moments $\mathbb{E}(\tau^q) < \infty$, for $q \geq 1$, exist (see for instance theorem 1.4) the above results lead immediately to the existence of an invariant probability.

2 Proofs

2.1 Results from the constructive theory of Markov chains

The Markov chains we consider evolve on the set $\mathbb{X} = \mathbb{R}_+$. Our proofs rely on the possibility of constructing measurable functions $g : \mathbb{X} \rightarrow \mathbb{R}^+$ (with some special properties regarding their asymptotic behaviour) that are superharmonic with respect to the discrete Laplacian operator $D = P - I$; consequently, the image of the Markov chain under g becomes a supermartingale outside some specific sets. For the convenience of the reader, we state here the principal theorems from the constructive theory, developed in [4] and in [1], rephrased and adapted to the needs and notation of the present paper. We shall use repeatedly these theorems in the sequel.

In the sequel (Z_n) denotes a Markov chain on \mathbb{X} , having stochastic kernel P . We denote by

$$\text{Dom}_+(P) : \{f : \mathbb{X} \rightarrow \mathbb{R}^+ : f \text{ measurable s.t. } \forall x \in \mathbb{X}, \int_{\mathbb{X}} P(x, dy) f(y) < \infty\}.$$

We denote by $D = P - I$ the Markov operator whose action $\text{Dom}_+(P) \ni$

$g \mapsto Dg$ reads

$$Dg(x) = \int_{\mathbb{X}} P(x, dy)g(y) - g(x) = \mathbb{E}(g(Z_{n+1}) - g(Z_n)|Z_n = x).$$

Notice that when g is P -superharmonic, then $(g(Z_n))$ is a positive supermartingale.

Let $f : \mathbb{X} \rightarrow \mathbb{R}_+$ and $a > 0$. We denote by $S_a(f) = \{x \in \mathbb{X} : f(x) \leq a\}$, the **sublevel set** of f . We say that the function **tends to infinity**, $f \rightarrow \infty$, if $\forall n \in \mathbb{N}, \text{card}S_n(f) < \infty$.

Theorem 2.1 (Fayolle, Malyshev, Menshikov [4, Theorems 2.2.1 and 2.2.2]). *Let (Z_n) be a Markov chain on \mathbb{X} with kernel P and for $a \geq 0$, denote by $A := A_a = [0, a]$.*

1. *If there exist a pair (f, x_0) , where $x_0 > 0$ and $f \in \text{Dom}_+(P)$ such that $f \rightarrow \infty$, $Df(x) \leq 0$ for all $x \geq x_0$, and $A := A_{x_0}$ is accessible, then $\mathbb{P}_{x_0}(\tau_A < \infty) = 1$.*
2. *If there exist a pair (f, A) , where A is a subset of \mathbb{X} and $f \in \text{Dom}_+(P)$ such that*

- (a) *$Df(x) \leq 0$ for $x \notin A$, and*
- (b) *there exists $y \in A^c : f(y) < \inf_{x \in A} f(x)$,*

then $\mathbb{P}_{x_0}(\tau_A < \infty) < 1$.

Theorem 2.2 (Aspandiiarov, Iasnogorodski, Menshikov [1, Theorems 1 and 2]). *Let (Z_n) be a Markov chain on \mathbb{X} with kernel P and $f \in \text{Dom}_+(P)$ such that $f \rightarrow \infty$.*

1. *If there exist strictly positive constants a, p, c such that the set $A := S_a(f)$ is accessible, $f^p \in \text{Dom}_+(P)$, and $Df^p(x) \leq -cf^{p-2}(x)$ on A^c , then $\mathbb{E}_x(\tau_A^q) < \infty$ for all $q < p/2$.*
2. *It there exist $g \in \text{Dom}_+(P)$ and*
 - (a) *a constant $b > 0$ such that $f \leq bg$,*
 - (b) *constants $a, c_1 > 0$ such that $Dg(x) \geq -c_1$ on $\{g > a\}$,*
 - (c) *constants $c_2 > 0$ and $r > 1$ such that $g^r \in \text{Dom}_+(P)$ and $Dg^r(x) \leq c_2g^{r-1}(x)$ on $\{g > a\}$,*

(d) a constant $p > 0$ such that $f^p \in \text{Dom}_+(P)$ and $Df^p(x) \geq 0$ on $\{f > ab\}$,

then $\mathbb{E}_x(\tau_{S_{ab}(f)}^q) = \infty$ for all $q > p$.

Notation 2.3. For $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+, \rho \in \mathbb{R}$, we write $h(x) \asymp x^\rho$, if $\lim_{x \rightarrow \infty} h(x)x^{-\rho} = 1$ and $h(x) \prec x^\rho$, if there exist a function h_1 such that $h(x) \leq h_1(x)$ and $h_1(x) \asymp x^\rho$.

2.2 Proof of the theorems 1.3 and 1.4

The main theorems are stated under the condition that the reference measure λ is the Lebesgue measure on \mathbb{R} (or on \mathbb{R}_+). To simplify notation, we write $\lambda(dy) = dy$ for Lebesgue measure. The case of μ having a density with respect to the counting measure on \mathbb{Z} requires a small technical additional step as will be explained in the remark 2.11 below.

In the sequel, we shall use a Lyapunov function, g , depending on a parameter $\delta \neq 0$, reading

$$g(x) = \begin{cases} x^\delta, & x \geq 1 \\ 1, & x < 1 \end{cases} \quad (\text{if } \delta < 0)$$

and

$$g(x) = x^\delta \quad (\text{if } \delta > 0).$$

in general the choice $\delta > 0$ is made to prove recurrence and $\delta < 0$ to prove transience. The range of values of δ will be determined from the specific context as explained below.

Lemma 2.4. Let (ζ_n) be the Markov chain of the theorem 1.3 and suppose that x is very large. For arbitrary $y_0 \geq 1$ and $\delta < \theta$,

$$Dg(x) \prec (x - x^\gamma)^\delta \left[\int_{[y_0, \infty[} \left(\left(1 + \frac{y}{x - x^\gamma} \right)^\delta - 1 \right) m(y) dy - \delta \frac{x^\gamma}{x - x^\gamma} \right].$$

Proof. Assume everywhere in the sequel that x is very large. The parameter δ is allowed to be positive or negative.

$$\begin{aligned} Dg(x) &= \int_{\mathbb{R}^+} [(x - x^\gamma + y)^\delta - x^\delta] m(y) dy \\ &= (x - x^\gamma)^\delta \int_{\mathbb{R}^+} \left[\left(1 + \frac{y}{x - x^\gamma}\right)^\delta - \left(1 + \frac{x^\gamma}{x - x^\gamma}\right)^\delta \right] m(y) dy \\ &\asymp (x - x^\gamma)^\delta \left[\int_{\mathbb{R}^+} \left(1 + \frac{y}{x - x^\gamma}\right)^\delta m(y) dy - 1 - \delta \frac{x^\gamma}{x - x^\gamma} \right]. \end{aligned}$$

For arbitrary $y_0 \in \mathbb{R}^+$, the integral $\int_{\mathbb{R}^+}$ in the previous formula can be split into $\int_{]0, y_0[} + \int_{[y_0, \infty[}$. In the sequel we shall consider only the case $x \gg y_0$. If $\delta < 0$ then the function $y \mapsto \left(1 + \frac{y}{x - x^\gamma}\right)^\delta$ is decreasing, hence $\sup_{y \in]0, y_0[} \left(1 + \frac{y}{x - x^\gamma}\right)^\delta \leq 1$. On the contrary, when $\delta > 0$, the corresponding function is increasing and we have $\sup_{y \in]0, y_0[} \left(1 + \frac{y}{x - x^\gamma}\right)^\delta \leq \left(1 + \frac{y_0}{x - x^\gamma}\right)^\delta \asymp 1 + \delta \frac{y_0}{x - x^\gamma}$. In any situation,

$$\int_{]0, y_0[} \left(1 + \frac{y}{x - x^\gamma}\right)^\delta m(y) dy \prec \mu(]0, y_0[) + |\delta| \frac{y_0}{x - x^\gamma}.$$

The remaining integral can be written as

$$\int_{[y_0, \infty[} \left(1 + \frac{y}{x - x^\gamma}\right)^\delta m(y) dy = \int_{[y_0, \infty[} \left[\left(1 + \frac{y}{x - x^\gamma}\right)^\delta - 1 \right] m(y) dy + \mu([y_0, \infty[).$$

Replacing these expressions into the formula for $Dg(x)$ yields

$$Dg(x) \prec (x - x^\gamma)^\delta \left[\int_{[y_0, \infty[} \left(\left(1 + \frac{y}{x - x^\gamma}\right)^\delta - 1 \right) m(y) dy - \delta \frac{x^\gamma}{x - x^\gamma} \right],$$

because, for x sufficiently large, $\frac{y_0}{x - x^\gamma}$ is negligible compared to $\frac{x^\gamma}{x - x^\gamma}$. \square

Remark 2.5. Note that since $0 < \gamma < 1$, the asymptotic majorisation $Dg(x) \prec x^\delta \left[\int_{[y_0, \infty[} \left(\left(1 + \frac{y}{x - x^\gamma}\right)^\delta - 1 \right) m(y) dy - \delta \frac{x^\gamma}{x - x^\gamma} \right]$ is equivalent to the one established in lemma 2.4.

Lemma 2.6. Let $\delta < \theta < 1$. Suppose further that there exist constants $0 < b_1 \leq b_2 < \infty$ such that for all $y \geq y_0$, for some $y_0 > 0$, we have $b_1 \leq c_y \leq b_2$. Then, the integral

$$I(x) := \int_{[y_0, \infty[} \left(\left(1 + \frac{y}{x - x^\gamma} \right)^\delta - 1 \right) m(y) dy,$$

asymptotically for large x , satisfies

$$\delta B_1 K_{\delta, \theta} x^{-\theta} < I(x) < \delta B_2 K_{\delta, \theta} x^{-\theta},$$

where $(B_1, B_2) = (b_1, b_2)$ if $\delta > 0$, and $(B_1, B_2) = (b_2, b_1)$ when $\delta < 0$.

Proof. Write

$$I(x) := \int_{[y_0, \infty[} \left(\left(1 + \frac{y}{x - x^\gamma} \right)^\delta - 1 \right) m(y) dy = \int_{[y_0, \infty[} c_y \frac{\left(1 + \frac{y}{x - x^\gamma} \right)^\delta - 1}{y^{1+\theta}} dy.$$

Consider first $\delta > 0$; in this case the integrand is positive, hence

$$b_1 I_1(x) \leq I(x) \leq b_2 I_1(x),$$

where $I_1(x) := \int_{[y_0, \infty[} \frac{\left(1 + \frac{y}{x - x^\gamma} \right)^\delta - 1}{y^{1+\theta}} dy$. We estimate then, for fixed y_0 and large x (so, y_0 is small compared to x) and performing the change of variable $u = \frac{y}{x - x^\gamma}$,

$$\begin{aligned} I_1(x) &:= \int_{[y_0, \infty[} \frac{\left(1 + \frac{y}{x - x^\gamma} \right)^\delta - 1}{y^{1+\theta}} dy \\ &= (x - x^\gamma)^{-\theta} \int_{\frac{y_0}{x - x^\gamma}}^{\infty} \frac{(1+u)^\delta - 1}{u^{1+\theta}} du < x^{-\theta} \int_0^{\infty} \frac{(1+u)^\delta - 1}{u^{1+\theta}} du \end{aligned}$$

Now for $\delta < \theta < 1$ (recall that $\theta > 0$)

$$\begin{aligned} \int_0^{\infty} \frac{(1+u)^\delta - 1}{u^{1+\theta}} du &= -\frac{1}{\theta} \int_{u=0}^{u=\infty} [(1+u)^\delta - 1] d(u^{-\theta}) \\ &= \left[\frac{1}{\theta} \frac{(1+u)^\delta - 1}{u^\theta} \right]_{u=0}^{u=\infty} + \frac{\delta}{\theta} \int_0^{\infty} \frac{(1+u)^{\delta-1}}{u^\theta} du \\ &= 0 + \frac{\delta}{\theta} \frac{\Gamma(1-\theta)\Gamma(\theta-\delta)}{\Gamma(1-\delta)} = \delta K_{\delta, \theta}. \end{aligned}$$

The claimed majorisation $I_1(x) \prec x^{-\theta} \delta K_{\delta, \theta}$ is obtained immediately. The minoration is obtained similarly. If $\delta < 0$, the integrand is negative, hence the role of b_1 and b_2 must be interchanged. \square

Lemma 2.7. *Let $\delta < \theta < 1$. Suppose further that $c_y \rightarrow c$. Then for all $\epsilon > 0$, there exists a y_0 such that for $x \gg y_0$ we have*

$$I(x) := \int_{[y_0, \infty[} \left(\left(1 + \frac{y}{x - x^\gamma} \right)^\delta - 1 \right) m(y) dy = c \delta K_{\delta, \theta} (x - x^\gamma)^{-\theta} (1 + \epsilon \mathcal{O}(1)).$$

Proof. Observe that $I(x) = c I_1(x) + \int_{[y_0, \infty[} (c_y - c) \frac{(1 + \frac{y}{x - x^\gamma})^\delta - 1}{y^{1+\theta}} dy$. Now, since $c_y \rightarrow c$, it follows that for all $\epsilon > 0$ one can choose y_0 such that for $y \geq y_0$, we have $|c_y - c| \leq \epsilon$. We then immediately conclude that the absolute value of the above integral is majorised by $\epsilon I_1(x)$. \square

Lemma 2.8. 1. *Let $\theta \in]0, 1[$ and $c > 0$. If $c \pi \csc(\pi \theta) < \theta$ then there exists a unique $\delta_0 := \delta_0(c, \theta) \in]0, \theta[$ such that $c K_{\delta_0, \theta} - 1 = 0$.*

2. *Let $\theta \in]0, 1[$, $c > 0$. Then for every fixed $\theta \in]0, 1[$, $L_{\delta, \theta} + \frac{1}{\theta} < 0$ for all $\delta > 0$ and there exists a unique $\delta_0 := \delta_0(c, \theta) \in]0, \infty[$ such that $c L_{\delta_0, \theta} + \delta_0 = 0$.*

Proof. For all $\delta \in]-\infty, \theta[$, the quantities $K_{\delta, \theta}$ and $L_{\delta, \theta}$ are well defined.

1. By standard results² on Γ functions, $K(0, \theta) = \frac{1}{\theta} \csc(\pi \theta)$. For fixed θ , the function $K_{\delta, \theta}$ is strictly increasing and continuous in δ — as follows from its integral representation — and $\lim_{\delta \uparrow \theta} K_{\delta, \theta} = \infty$, from which follows the existence of $\delta_0 \in]0, \theta[$ verifying the claimed equality.
2. From [3, §1.2, formulæ (4) and (1)] follow immediately that $L_{0, \theta} = -\frac{1}{\theta} < 0$ and $\lim_{\delta \rightarrow \infty} \frac{L_{\delta, \theta}}{\Gamma(-\theta) \delta^\theta} = 1$. The strict monotonicity and continuity (in δ) of the function $L_{\delta, \theta}$ follows from its integral representation: $L_{\delta, \theta} + \frac{1}{\theta} = \int_0^1 \frac{((1-u)^\delta - 1)}{u^{1+\theta}} du$. Hence $L_{\delta, \theta} \asymp \Gamma(-\theta) \delta^\theta$ for large δ . Since the asymptotic behaviour of $L_{\delta, \theta}$ is negative and sublinear in δ , it follows that there exists a sufficiently large δ_0 for which the claimed equality holds. Additionally, the strict monotonicity of $L_{\delta, \theta}$ combined with the fact that $L_{0, \theta} = -\frac{1}{\theta} < 0$ guarantees that $L_{\delta, \theta} + \frac{1}{\theta} < 0$ for all $\delta > 0$.

²We used the identity $\Gamma(z) \Gamma(1-z) = \pi \csc(\pi z)$, (see [3, formula §1.2 (6)] for instance).

□

Lemma 2.9. *Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a given function; define the dynamical system $(X_t)_{t \in \mathbb{N}}$ by $X_0 = x_0$ and recursively $X_{t+1} = f(X_t)$, for $t \in \mathbb{N}$. For $a > 0$ define $T_{[0,a]}(x_0) = \inf\{t \geq 1 : X_t \leq a\}$.*

1. *If $f(x) = x - x^\gamma$ for some $\gamma \in]0, 1[$ and $x_0 \gg 1$, then $T_{[0,a]}(x_0) \asymp \frac{x_0^{1-\gamma}}{1-\gamma}$.*
2. *If $f(x) = x - x^\gamma + 1$ for some $\gamma \in]0, 1[$, $a > 1$, and $x_0 \gg a$, then $T_{[0,a]}(x_0) \asymp \frac{x_0^{1-\gamma}}{1-\gamma}$.*

Proof. 1. The derivative of the function f satisfies $0 < f'(x) < 1$ for all $x > 1$. Therefore, successive iterates $f^{on}(x_0)$ eventually reach the interval $[0, 1]$ for all $x_0 > 1$ in a finite number of steps $T_{[0,1]}(x_0)$. To estimate this number, start by approximating, for $X_t = x$ and $1 < x < x_0$, the difference $X_{t+1} - X_t = \Delta X_t = -X_t^\gamma$ by the differential $dX_t = -X_t^\gamma dt$. Then

$$T_{[0,a]}(x_0) = \int_0^{T_{[0,a]}(x_0)} dt = - \int_{x_0}^a X_t^{-\gamma} dX_t = \frac{1}{1-\gamma} (x_0^{1-\gamma} - a^{1-\gamma}) \asymp \frac{x_0^{1-\gamma}}{1-\gamma}.$$

2. Using the same arguments, and denoting by F the hypergeometric function, we estimate (see [3] for instance);

$$T_{[0,a]}(x_0) = \int_{x_0}^a \frac{dX_t}{1 - X_t^\gamma} = x_0 F\left(1, \frac{1}{\gamma}, 1 + \frac{1}{\gamma}, x_0^\gamma\right) - a F\left(1, \frac{1}{\gamma}, 1 + \frac{1}{\gamma}, a^\gamma\right) \asymp \frac{1}{1-\gamma} x_0^{1-\gamma}.$$

□

Proof of the theorem 1.3: First we need to prove accessibility of $A = [0, a]$, with $a > 1$ from any point $x > a$. Denote by $r := \mu([0, 1]) > 0$. Since the dynamical system evolving according to the iteration of the function $f(x) = x - x^\gamma + 1$ reaches A in finite time $T_A(x)$, as proven in lemma 2.9, the Markov chain can reach A in time τ_A verifying $\mathbb{P}_x(\tau_A \leq T_A + 1) \geq Cr^{T_A(x)} > 0$, for all $x \gg a$.

We substitute the estimates obtained in lemmata 2.6 and 2.7 into the expression for Dg obtained in lemma 2.4.

1. Assume that $b_1 \leq c_y \leq b_2$.

(a) Choose $0 < \delta < \theta$. Then

$$\begin{aligned} Dg(x) &= (x - x^\gamma)^\delta \left[-\delta \frac{x^\gamma}{x - x^\gamma} + b_2 \delta K_{\delta, \theta} (x - x^\gamma)^{-\theta} + \mathcal{O}(x^{-1}) \right] \\ &= -\delta x^{\delta + \gamma - 1} + \delta b_2 K_{\delta, \theta} x^{\delta - \theta} + \mathcal{O}(x^{\delta - \theta - 1}). \end{aligned}$$

If $\theta > 1 - \gamma$, the dominant term reads $-\delta x^{\delta + \gamma - 1}$ which is negative. Hence, $(g(\zeta_n))$ is a supermartingale tending to infinity if $\zeta_n \rightarrow \infty$. We conclude then by theorem 2.1.

- To prove finiteness of moments up to $\theta/(1 - \gamma)$, consider p such that $0 < p\delta < \theta$. Then

$$Dg^p(x) < -\delta p x^{\delta p + \gamma - 1} = -\delta p g(x)^{p - \frac{1 - \gamma}{\delta}} \leq -C g(x)^{p - 2},$$

provided that $\frac{1}{\delta} < \frac{2}{1 - \gamma}$. The latter, combined with the inequality $p\delta < \theta$, establishes the majorisation by $-C g(x)^{p - 2}$. This allows to conclude by theorem 2.2.

- To prove the non existence of moments for $q \geq \theta/(1 - \gamma)$, denote by $f(x) = x - x^\gamma$. Define $Z_0 = x$ and recursively $Z_{n+1} = f(Z_n)$ as in lemma 2.9; similarly the Markov chain can be rewritten $\zeta_0 = x$ and recursively $\zeta_{n+1} = f(\zeta_n + \alpha_{n+1})$ as long as $\zeta_n > 1$. Now remark that $Z_1 = f(x) < f(x + \alpha_1) = \zeta_1$; a simple recursion shows that $Z_{n+1} = f^{\circ n}(x + \alpha_1) < \zeta_{n+1}$. Obviously $T_{[0,1]}(x + \alpha_1, 0) < \tau_0$. Hence $\tau_0 > C(x + \alpha_1)^{1 - \gamma} > C(\alpha_1)^{1 - \gamma}$ by lemma 2.9 and subsequently $\mathbb{E}_x(\tau_0^q) > C \mathbb{E}(\alpha_1)^{q(1 - \gamma)} = \infty$ whenever $q(1 - \gamma) \geq \theta$.

(b) Choose now $\delta < 0$. Using the same arguments as above, we see that the dominant term is $\delta b_1 K_{\delta, \theta} x^{\delta - \theta}$ which is again negative. Hence $(g(\zeta_n))$ is a bounded supermartingale. We conclude by using theorem 2.1.

2. Assume now that $\theta = 1 - \gamma$ and $c_y \rightarrow c > 0$. In this situation, for every $\epsilon > 0$ we can choose y_0 such that for $y \geq y_0$, we have asymptotically, for $x \gg y_0$ and every $\delta \neq 0$,

$$Dg(x) = \delta x^{\delta + \gamma - 1} \left(c K_{\delta, \theta} - 1 + \mathcal{O}(x^{-1}) + \epsilon \mathcal{O}(1) \right).$$

Therefore, the dominant term is $\delta(cK_{\delta,\theta} - 1)x^{\delta+\gamma-1}$. The sign of δ will thus be multiplied by the sign of the difference $cK_{\delta,\theta} - 1$.

(a) If $c\pi \csc(\pi\theta) < \theta$, by lemma 2.8, we can chose $\delta \in]0, \delta_0[$, so that that $Dg(x) \leq 0$ while g tends to infinity. We conclude by theorem 2.1.

- To prove finiteness of moments of the time τ_A , for the δ chosen to establish recurrence, we can further choose $p > 1$ so that $p\delta < \delta_0$. Then

$$Dg^p(x) \prec -p\delta x^{p\delta+\gamma-1} = -p\delta g(x)^{p-\frac{1-\gamma}{\delta}} \leq -Cg(x)^{p-2}$$

whenever $\frac{1-\gamma}{\delta} > 2$ or $\frac{1}{\delta} \leq \frac{2}{1-\gamma}$. Combining with the condition $p\delta < \delta_0$ we get $p < \frac{2\delta_0}{1-\gamma}$ and we conclude by theorem 2.2 that all moments up to $\frac{\delta_0}{1-\gamma}$ are finite.

- To prove non-existence of moments for $q > \frac{\delta_0}{1-\gamma}$, for any $\delta \in]0, \delta_0[$, we check immediately $Dg(x) \geq -\epsilon$. Now, choose $r > 1$ such that $r\delta > \delta_0$ and determine under which circumstances $Dg^r(x) \leq Cg(x)^{r-1}$. Computing explicitly, we get

$$Dg^r(x) \asymp r\delta(cK_{r\delta,\theta} - 1)x^{r\delta+\gamma-1} \leq Cg(x)^{r-\frac{1-\gamma}{\delta}} \leq Cg(x)^{r-1}$$

whenever $\frac{1-\gamma}{\delta} > 1$ or equivalently $\frac{1}{\delta} > \frac{1}{1-\gamma}$. But the latter inequalities are always verified for $0 < \delta < \delta_0$. Similarly, for any p such that $p\delta > \delta_0$, i.e. for $p > \frac{\delta_0}{\delta} > \frac{\delta_0}{1-\gamma}$, we get $Dg^p(x) \geq 0$. We conclude, by theorem 2.2, that all moments $q > \frac{\delta_0}{1-\gamma}$ of τ_A fail to exist.

(b) If $\delta < 0$ and $c\pi \csc(\pi\theta) > \theta$, then $(g(\zeta_n))$ is a bounded supermartingale. We conclude by theorem 2.1.

□

Lemma 2.10. *Let (ζ_n) be the Markov chain of the theorem 1.4 and assume that x is very large. For the Lyapunov function g with $\delta < \theta < 1$, we have*

$$\begin{aligned} Dg(x) &= (x + x^\gamma)^\delta \int_0^{x+x^\gamma} \left[\left(1 - \frac{y}{x + x^\gamma}\right)^\delta - 1 \right] \mu(dy) \\ &\quad + \left((x + x^\gamma)^\delta - x^\delta \right) \mu([0, x + x^\gamma]) - x^\delta \mu([x + x^\gamma, \infty[). \end{aligned}$$

Proof. Write simply

$$\begin{aligned}
Dg(x) &= \int_{\mathbb{R}^+} ((x + x^\gamma - y)^+)^{\delta} \mu(dy) - x^{\delta} \\
&= (x + x^\gamma)^{\delta} \int_0^{x+x^\gamma} \left(1 - \frac{y}{x + x^\gamma}\right)^{\delta} \mu(dy) - x^{\delta} \\
&= (x + x^\gamma)^{\delta} \int_0^{x+x^\gamma} \left[\left(1 - \frac{y}{x + x^\gamma}\right)^{\delta} - 1 \right] \mu(dy) \\
&\quad + (x + x^\gamma)^{\delta} \mu([0, x + x^\gamma]) - x^{\delta}.
\end{aligned}$$

□

Proof of the theorem 1.4: First we need to establish accessibility of the state 0. But this is obvious since from any $x > 0$ the $\mathbb{P}(\alpha_1 > x + x^\gamma) > 0$.

We only sketch the proof since it uses the same arguments as the proof of the theorem 1.3. It is enough to consider the case $c_y = c$ since the case $c_y \rightarrow c$ will give rise to an additional corrective term that will be negligible. With this proviso, the integral appearing in the right hand side of the expression for $Dg(x)$ in the previous lemma 2.10 reads

$$\begin{aligned}
\int_0^{x+x^\gamma} \left[\left(1 - \frac{y}{x + x^\gamma}\right)^{\delta} - 1 \right] \mu(dy) &= c(x + x^\gamma)^{-\theta} \int_0^1 \frac{((1-u)^{\delta} - 1)}{u^{1+\theta}} du \\
&= c(x + x^\gamma)^{-\theta} (L_{\delta,\theta} + \frac{1}{\theta}),
\end{aligned}$$

where $L_{\delta,\theta}$ is defined in lemma 2.8.. It is further worth noting that $L_{\delta,\theta} \leq 0$, for all $\delta \in \mathbb{R}_+$. Therefore,

1. If $\theta < 1 - \gamma$, then the dominant terms in the expression of Dg are those with $x^{\delta-\theta}$, hence, choosing $\delta > 0$, we get $Dg(x) \leq cx^{\delta-\theta}L_{\delta,\theta}$. Since the value of $Dg(x)$ is always negative i.e. the process $(g(\zeta_n))$ is a supermartingale tending to infinity. We conclude by theorem 2.1.

To establish the existence of all moments, it is enough to check that

$$Dg^p(x) < cx^{p\delta-\theta}L_{p\delta,\theta} < -Cg(x)^{p-\frac{\theta}{\delta}} \leq -Cg(x)^{p-2}$$

whenever $\delta > \theta/2$. But since $L_{\delta,\theta}$ is defined and negative for all positive δ , we conclude that all positive moments of τ_0 exist by theorem 2.2.

2. When $\theta = 1 - \gamma$, then all terms are of the same order and $Dg(x) \asymp x^{\delta-\theta}(cL_{\delta,\theta} + \delta)$. From lemma 2.8, for fixed θ and $c > 0$, there exists $\delta_0 > 0$ such that $cL_{\delta_0,\theta} + \delta_0 = 0$. We conclude then that asymptotically, for large x ,

$$Dg(x) \asymp x^{\delta-\theta}(cL_{\delta,\theta} - 1),$$

the sign of the discrete Laplacian is negative (positive) depending on the value of δ being smaller (larger) than δ_0 .

Choose $\delta > 0$ and p such that $p\delta < \delta_0$. Then $Dg^p(x) \asymp -Cg(x)^{p-\frac{\theta}{\delta}} \leq -Cg(x)^{p-2}$ whenever $\frac{1}{\delta} < \frac{2}{\theta}$ and, consequently, $p < \frac{2\delta_0}{\theta}$. Then we conclude by theorem 2.2 that $\mathbb{E}_x(\tau_0^q) < \infty$ for all $q < \frac{\delta_0}{\theta}$ as claimed.

To show that moments higher than $\frac{\delta_0}{\theta}$ fail to exist, choose $\delta < \delta_0$. It is then evident that $Dg(x) \asymp -Cx^{\delta-\theta} \geq -\epsilon$, for some $\epsilon > 0$. There exists then $r > 1$ such that $r\delta > \delta_0$; estimating then $Dg^r(x) \asymp Cg(x)^{r-\frac{\theta}{\delta}}$ we conclude immediately that $0 \leq Dg^r(x) \leq Cg(x)^{r-1}$ whenever $\frac{\theta}{\delta} > 1$. We conclude then by theorem 2.2 that for all $q > \frac{\delta_0}{\theta}$, we have $\mathbb{E}_x(\tau_0^q) = \infty$.

3. If $\theta > 1 - \gamma$, the dominant term is $\delta x^{\delta-\gamma+1}\mu([0, x + x^\gamma])$ that can be made negative by choosing $\delta < 0$ and x sufficiently large. We conclude by theorem 2.1.

□

Remark 2.11. In this subsection, we assumed that the law μ of the random variables (α_n) is absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^+ . If instead the law is absolutely continuous with respect to the counting measure on the positive integers, the integrals in the expression of Dg become sums. Now, the sums over the positive integers can be replaced by integrals. It turns out that the error committed in such a replacement is *always* a subleading term in the expression of Dg , leaving the conclusion unaffected.

Remark 2.12. The two previous theorems have been established by assuming that the random variables (α_n) are always positive and act in the opposite direction of the systematic drift x^γ . By examining the proofs of the theorems however, it is evident that nothing will change if the random variables are both sided, even with both sided heavy tails, provided that

the heaviest tail is the one acting in the opposite direction of the systematic drift x^γ .

2.3 Proof of the theorems 1.8 and 1.9

Here the control is only through the tail decay and consequently, the estimates are considerably more involved. The subsection relies on methods developed in [6] to deal with heavy tails when only tail control is available.

Lemma 2.13. *Let Z be a positive random variable, $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ an increasing function, and $0 \leq a < b \leq \infty$. Then*

$$\mathbb{E}(\phi(Z)\mathbb{1}_{[a,b[}(Z)) = \int_{[\phi(a),\phi(b)[} \mathbb{P}(Z > \phi^{-1}(t))dt - \phi(b)\mathbb{P}(Z \geq b) + \phi(a)\mathbb{P}(Z \geq a).$$

Proof. Denote by ν the law of Z . Then

$$\begin{aligned} \int_{[a,b[} \mathbb{P}(Z > t)dt &= \int_{[a,b[} \mathbb{E}(\mathbb{1}_{]t,\infty[}(Z))dt \\ &= \int_{\mathbb{R}^+ \times \mathbb{R}^+} \mathbb{1}_{]t,\infty[}(z)\mathbb{1}_{[a,b[}(t) dt \nu(dz) \\ &= \int_{\mathbb{R}^+ \times \mathbb{R}^+} \mathbb{1}_{[a,z \wedge (b-)]}(t) dt \nu(dz) \\ &= \int_{[a,\infty[} [z \wedge (b-) - a]\nu(dz) \\ &= \mathbb{E}\left(Z\mathbb{1}_{[a,b[}(Z)\right) + b\mathbb{P}(Z \geq b) - a\mathbb{P}(Z \geq a). \end{aligned}$$

On denoting $Y = \phi(Z)$, we conclude by remarking that

$$\mathbb{E}\left(\phi(Z)\mathbb{1}_{[a,b[}(Z)\right) = \mathbb{E}\left(Y\mathbb{1}_{[\phi(a),\phi(b)[}(Y)\right).$$

□

Remark 2.14. When $b = \infty$ in the above formula and the random variable Z is almost surely finite, then the term $b\mathbb{P}(Z \geq b)$ reads $\infty \mathbb{P}(Z = \infty) = 0$; otherwise the value is ∞ and the random variable Z cannot be then almost surely finite.

In the sequel, we shall partition the real axis into $\mathbb{R} = \sqcup_{i=1}^4 A_i$ with

$$A_1 =]-\infty, -x^\beta[, \quad A_2 = [-x^\beta, 0[, \quad A_3 = [0, x^\beta[, \quad A_4 = [x^\beta, \infty[,$$

with some parameter β (verifying $0 < \gamma < \beta < 1$) that will be specified later. On denoting, for every choice of the Lyapunov function g , by $d_i = \mathbb{E}(g(\zeta_{n+1})\mathbb{1}_{A_i}(\alpha_{n+1})|\zeta_n = x)$, the above partition induces a decomposition of the conditional increment as

$$Dg(x) = \sum_{i=1}^4 (d_i - g(x)\mu(A_i)).$$

Proof of the theorem 1.9.

1. Let $\beta \in]\gamma, 1[$ and $\delta > 0$ and define

$$g(x) = \begin{cases} x^{-\delta}, & x \geq 1, \\ 1, & x < 1. \end{cases}$$

The parameter δ (together with β) will be chosen later; we get then

$$d_i = \int_{A_i} g((x + x^\gamma + y)^+) \mu(dy).$$

For x sufficiently large we have

$$\begin{aligned} d_1 &\leq \mu(A_1), \\ d_2 &= x^{-\delta} \int_{A_2} \left(1 + \frac{x^\gamma + y}{x}\right)^{-\delta} \mu(dy) \\ &\asymp x^{-\delta} (1 - \delta x^{\gamma-1}) \mu(A_2) - \delta x^{-\delta-1} \int_{A_2} y \mu(dy) \\ &\asymp x^{-\delta} \mu(A_2) - \delta x^{-\delta+\gamma-1} \mu(A_2) + \delta x^{-\delta-1} \int_{A_2} |y| \mu(dy), \\ d_3 &\asymp x^{-\delta} \mu(A_3) - \delta x^{-\delta+\gamma-1} \mu(A_3) - \delta x^{-\delta-1} \int_{A_3} y \mu(dy), \\ d_4 &< (x + x^\gamma + x^\beta)^{-\delta} \mu(A_4) \\ &\asymp x^{-\delta} \mu(A_4) - \delta x^{-\delta+\gamma-1} \mu(A_4) - \delta x^{-\delta+\beta-1} \mu(A_4). \end{aligned}$$

Replacing into the expression for Dg , we get

$$Dg(x) \prec \mu(A_1) + \delta x^{-\delta-1} \int_{A_2} |y| \mu(dy) \\ - \delta x^{-\delta+\gamma-1} [\mu(A_2) + \mu(A_3) + \mu(A_4)] - \delta x^{-\delta-1} \int_{A_3} y \mu(dy).$$

Note that in the previous inequality, the terms on the first line are positive, while the terms appearing in the second line are negative. In order that Dg be negative, we need to show that the positive terms are subdominant in the expression of $Dg(x)$ for sufficiently large x . Now, $\mu(A_1) = \mathbb{P}(\alpha_1 < -x^\beta) \leq Cx^{-\beta\theta}$, while, by lemma 2.13,

$$\int_{A_2} |y| \mu(dy) = \mathbb{E}(\alpha_1^- \mathbb{1}_{]0, x^\beta]}(\alpha_1^-) \leq \int_0^{x^\beta} \mathbb{P}(\alpha_1^- > t) dt - x^\beta \mathbb{P}(\alpha_1^- > x^\beta) \leq x^\beta - Cx^{\beta(1-\theta)} \prec x^\beta.$$

Combining, we see that we get a supermartingale if we satisfy simultaneously the inequalities

$$-\theta\beta < \gamma - \delta - 1 \text{ and } -\delta - 1 + \beta < \gamma - \delta - 1$$

that — for $\theta > 1 - \gamma$ — have a solution for $\beta \in]\frac{1-\gamma}{\theta}, 1[$ and $\delta \in]0, \beta\theta - (1 - \gamma)[$. We conclude by theorem 2.2.

2. Let $\beta > 0$, $\delta \in]0, \theta[$, and $g(x) = x^\delta$. The possible values of the parameters β and δ will be further delimited later. We proceed now with the partition $\mathbb{R} = \sqcup_{i=1}^4 A_i$, where $A_1 =]-\infty, -x^\beta[$, $A_2 = [-x^\beta, 0[$, $A_3 = [0, x^\beta[$, and $A_4 = [x^\beta, \infty[$; we introduce also the sets $A_0 = [-x - x^\gamma, -x^\beta[\subset A_1$ and $B = A_1 \setminus A_0 =]-\infty, -x - x^\gamma[$. Using similar arguments as in the first part of the theorem we estimate

$$d_1 = \int_{A_1} ((x + x^\gamma + y)^+)^{\delta} \mu(dy) = \int_{A_0} (x + x^\gamma + y)^{\delta} \mu(dy) \\ \leq (x + x^\gamma - x^\beta)^{\delta} \mu(A_0) \asymp x^{\delta} \mu(A_0) + \delta x^{\delta+\gamma-1} \mu(A_0) - \delta x^{\delta+\beta-1} \mu(A_0),$$

leading further to the estimate

$$d_1 - x^{\delta} \mu(A_1) \prec -x^{\delta} \mu(B) + \delta x^{\delta+\gamma-1} \mu(A_0) - \delta x^{\delta+\beta-1} \mu(A_0).$$

The estimates of the other terms are obtained using the similar arguments:

$$\begin{aligned} d_2 - x^\delta \mu(A_2) &\asymp -\delta x^{\delta-1} \mathbb{E}(|\alpha_1| \mathbb{1}_{A_2}(\alpha_1)) + \delta x^{\delta-1+\gamma} \mu(A_2) \leq \delta x^{\delta-1+\gamma} \mu(A_2), \\ d_3 - x^\delta \mu(A_3) &\asymp \delta x^{\delta-1} \mathbb{E}(\alpha_1 \mathbb{1}_{A_3}(\alpha_1)) + \delta x^{\delta-1+\gamma} \mu(A_3), \\ d_4 - x^\delta \mu(A_4) &\asymp \mathbb{E}(\alpha_1^\delta \mathbb{1}_{A_4}(\alpha)) + x^{\gamma\delta} \mu(A_4) \leq \mathbb{E}(\alpha_1^\delta \mathbb{1}_{A_4}(\alpha)) + C' x^{\gamma\delta-\beta\theta'}, \end{aligned}$$

where, we have used [5, §2.10, p. 28] to establish the inequality $(a + b + c)^\delta \leq a^\delta + b^\delta + c^\delta$ that has been used to obtain the estimate for d_4 . Using lemma 2.13, we get

$$\begin{aligned} \mathbb{E}(\alpha_1 \mathbb{1}_{A_3}(\alpha_1)) &= \mathbb{E}(\alpha_1^+ \mathbb{1}_{[0, x^\beta]}(\alpha_1^+)) = \left(\int_0^{x^\beta} \mathbb{P}(\alpha_1^+ > t) dt - x^\beta \mathbb{P}(\alpha_1^+ > x^\beta) \right) \\ &\leq C' x^\beta (1 - \mu(A_4)), \\ \mathbb{E}(\alpha_1^\delta \mathbb{1}_{A_4}(\alpha_1)) &= \mathbb{E}((\alpha_1^+)^\delta \mathbb{1}_{[x^\beta, \infty]}(\alpha_1^+)) = \int_{x^\beta}^\infty \mathbb{P}(\alpha^+ > t^{1/\delta}) dt + x^{\beta\delta} \mathbb{P}(\alpha^+ > x^\beta) \\ &\leq C' \int_{x^\beta}^\infty t^{-\theta'/\delta} dt + C' x^{\beta(\delta-\theta')} \asymp K x^{\beta(\delta-\theta')}, \end{aligned}$$

where $K = C' \frac{\theta'}{\theta' - \delta}$. (Mind that $\delta < \theta < \theta'$). Using the fact that $\mu(B) \prec Cx^{-\theta}$ and $\mu(A_4) \leq C' x^{-\beta\theta'}$ and grouping the terms together, we get

$$Dg(x) \prec -Cx^{\delta-\theta} + \delta x^{\delta-1+\gamma} + \delta x^{\delta-1+\beta} + C' x^{\gamma\delta-\beta\theta'} + Kx^{\beta(\delta-\theta')}.$$

This conditional increment will be negative for sufficiently large x , provided that the following inequalities

$$\begin{aligned} \delta - \theta > \delta - 1 + \gamma &\Leftrightarrow \theta < 1 - \gamma \\ \delta - \theta > \delta - 1 + \beta &\Leftrightarrow \beta < 1 - \theta \\ \delta - \theta > \gamma\delta - \beta\theta' &\Leftrightarrow \frac{\theta - \beta\theta'}{1 - \gamma} < \delta \\ \delta - \theta > \beta(\delta - \theta') &\Leftrightarrow \frac{\theta - \beta\theta'}{1 - \beta} < \delta \end{aligned}$$

have a non-empty set of solutions. Now, the first inequality is automatically verified by the hypothesis of the theorem. Recalling that

$\delta < \theta$, the inequalities $\frac{\theta - \beta\theta'}{1 - \gamma} < \delta < \theta$ have a non-empty set of solutions for δ provided that $\beta \in I :=]\gamma\frac{\theta}{\theta'}, 1 - \theta[$; but $I \neq \emptyset$, hence such δ 's exist. Finally, the inequalities $\frac{\theta - \beta\theta'}{1 - \beta} < \delta < \theta$ have automatically a non-empty set of solutions since $\theta' > \theta$. Therefore, $\forall \beta \in J :=]\gamma\frac{\theta}{\theta'}, 1 - \theta[$, we can choose $\delta \in]b, \theta[$ — where $b := \max(\frac{\theta - \beta\theta'}{1 - \gamma}, \frac{\theta - \beta\theta'}{1 - \beta})$ so that $Dg(x) \prec -Cx^{\delta - \theta}$.

To establish the existence of moments, choose $p > 0$ such that $g^p \in \text{Dom}_+(P)$, i.e. $\delta p < \theta$. From the previous statements, we can choose $\beta \in J$ for b to be arbitrarily close to 0. Now $Dg^p(x) \prec -Cx^{\delta p - \theta} \leq -Cg(x)^{p-2}$ provided that $p - \frac{\theta}{\delta} > p - 2$ or equivalently $\frac{1}{\delta} < \frac{2}{\theta}$. From the condition $p\delta < \theta$ we get $p < 2$ hence, by theorem 2.2, $\mathbb{E}_x(\tau_A^q) < \infty$ for all $q < 1$.

□

Proof of the theorem 1.8. Accessibility of A follows using the same arguments as those used in the proof of theorem 1.4. We use again the partition $\mathbb{R} = \sqcup_{i=1}^4 A_i$, with $A_1 =] - \infty, -x^\beta[$, $A_2 = [-x^\beta, 0[$, $A_3 = [0, x^\beta[$, and $A_4 =]x^\beta, \infty[$, with provisional choice of the parameter $\beta \in]0, 1[$; its domain of variation will be further delimited later. For appropriately chosen g , we decompose the conditional drift $Dg(x) = \sum_{i=1}^4 (d_i - g(x))\mu(A_i)$, where $d_i = \int_{A_i} g((x - x^\gamma + y)^+) \mu(dy)$.

1. Let $g(x) = x^\delta$, with $\delta \in]0, \theta[$ (the domain of δ will be further delimited later). We get

$$\begin{aligned} d_1 &\leq (x - x^\gamma - x^\beta)^\delta \mu(A_1) \asymp x^\delta \mu(A_1) - \delta x^{\delta-1+\gamma} \mu(A_1) - \delta x^{\delta-1+\beta} \mu(A_1), \\ d_i &\asymp x^\delta \mu(A_i) - \delta x^{\delta-1+\gamma} \mu(A_i) + \delta x^{\delta-1} \int_{A_i} y \mu(dy), \text{ for } i = 2, 3, \\ d_4 &\leq (x - x^\gamma)^\delta + \int_{A_4} y^\delta \mu(dy) \\ &\asymp x^\delta \mu(A_4) - \delta x^{\delta-1+\gamma} \mu(A_4) + \left[\int_{x^\beta}^\infty \mathbb{P}(\alpha_1^+ > t^{1/\delta}) dt + x^{\beta\delta} \mathbb{P}(\alpha_1^+ > x^\beta) \right] \\ &\leq x^\delta \mu(A_4) - \delta x^{\delta-1+\gamma} \mu(A_4) + Kx^{\beta\delta - \beta\theta}, \text{ where } K = C(1 + \frac{\delta}{\theta - \delta}). \end{aligned}$$

Now

$$\int_{A_2} y\mu(dy) \leq 0$$

$$\int_{A_3} y\mu(dy) = \int_0^{x^\beta} \mathbb{P}(\alpha_1^+ > t)dt - x^\beta\mu(A_4) \leq x^\beta(1 - \mu(A_4)),$$

so that, grouping all terms together, we get

$$Dg(x) \prec -\delta x^{\delta-1+\gamma} + C\delta x^{\delta-1+\beta} + Kx^{\beta\delta-\beta\theta}.$$

This conditional increment will lead to a supermartingale tending to infinity whenever the system of inequalities

$$\delta - 1 + \gamma > \delta - 1 + \beta \Leftrightarrow \beta < \gamma, \text{ and}$$

$$\delta - 1 + \gamma > \beta\delta - \beta\theta \Leftrightarrow \delta > \frac{1 - \gamma - \beta\theta}{1 - \theta}$$

have a non-empty set of solutions. Recalling that $\delta < \theta$, the second inequality defines a non-empty domain for δ provided that $\frac{1-\gamma-\beta\theta}{1-\theta} < \theta \Leftrightarrow \theta > 1 - \gamma$ which is satisfied by hypothesis. Hence, picking any $\beta \in]0, \gamma[$ and $\delta \in J :=]\frac{1-\gamma-\beta\theta}{1-\theta}, \theta[$ guarantees that $Dg(x) \prec -\delta x^{\gamma-1-\delta}$ and shows that $(g(\zeta_n))_n$ is a positive supermartingale, while $g \rightarrow \infty$. We conclude from theorem 2.1 that the chain is recurrent.

To establish the existence of moments, pick again any $\beta \in]0, \gamma[$ and $\delta, p > 0$ such that $\delta p \in J$. Then, by the previous results,

$$Dg^{\delta p}(x) \prec -\delta p x^{\gamma-1+\delta p} = -\delta p g(x)^{p-\frac{1-\gamma}{\delta}} \leq -\delta p g(x)^{p-2}$$

whenever $\frac{1}{\delta} < \frac{2}{1-\gamma}$. We conclude by the theorem 2.2 that the moments $\mathbb{E}_x(\tau_A^q) < \infty, \forall q < \frac{\theta}{1-\gamma}$. Since $\theta > 1 - \gamma$, this result establishes in particular that the passage time is integrable.

2. Let now $g(x) = x^{-\delta}\mathbb{1}_{[1,\infty[}(x) + \mathbb{1}_{[0,1[}(x)$, with $\delta > 0$ and choose $\beta \in]0, 1[$ (the domains of δ and θ will be further delimited later).

Estimate then

$$\begin{aligned}
d_1 &\leq \mu(A_1), \\
d_2 &= \int_{A_2} (x - x^\gamma + y)^{-\delta} \mu(dy) \asymp x^{-\delta} \mu(A_2) + \delta x^{-\delta-1+\gamma} \mu(A_2) + \delta x^{\delta-1} \int_{A_2} |y| \mu(dy), \\
d_3 &= \int_{A_3} (x - x^\gamma + y)^{-\delta} \mu(dy) \asymp x^{-\delta} \mu(A_3) + \delta x^{-\delta-1+\gamma} \mu(A_3) - \delta x^{-\delta-1} \int_{A_3} y \mu(dy), \\
d_4 &\leq (x - x^\gamma + x^\beta)^{-\delta} \mu(A_4) \asymp x^{-\delta} \mu(A_4) + \delta x^{-\delta-1+\gamma} \mu(A_4) - \delta x^{-\delta-1+\beta} \mu(A_4).
\end{aligned}$$

The integrals appearing in the above majorisations can be further estimated — using lemma 2.13 — as

$$\begin{aligned}
\int_{A_2} |y| \mu(dy) &= \int_0^{x^\beta} \mathbb{P}(\alpha_1^- > t) dt + x^\beta \mathbb{P}(\alpha_1^- > x^\beta) \leq x^\beta (1 + \mu(A_1)) \\
\int_{A_3} y \mu(dy) &= \int_0^{x^\beta} \mathbb{P}(\alpha_1^+ > t) dt + x^\beta \mathbb{P}(\alpha_1^+ > x^\beta) \geq x^\beta (1 + \mu(A_4)).
\end{aligned}$$

Grouping all terms together, we obtain

$$Dg(x) \leq C' x^{-\beta\theta'} + \delta x^{-\delta-1+\gamma} + C' \delta x^{-\delta-1+\beta(1-\theta')} - 2C\delta x^{-\delta-1+\beta(1-\theta)}.$$

Only the last term in the above expression is negative. For the image of (ζ_n) through g to be a supermartingale, we must choose the parameters β and δ so that $Dg(x) \leq 0$ for x large enough. The set of solutions to the following inequalities

$$\begin{aligned}
-\delta - 1 + \beta(1 - \theta) &> -\delta - 1 + \beta(1 - \theta') \Leftrightarrow \theta < \theta', \\
-\delta - 1 + \beta(1 - \theta) &> -\delta - 1 + \gamma \Leftrightarrow \beta > \frac{\gamma}{1 - \theta}, \\
-\delta - 1 + \beta(1 - \theta) &> -\beta\theta' \Leftrightarrow \delta < \beta(1 - (\theta' - \theta)) - 1
\end{aligned}$$

have a non-empty set of solutions. In fact, the first inequality is satisfied by hypothesis; the second imposes reducing the initial domain of β to $\beta > \frac{\gamma}{1-\theta}$. Since δ must be strictly positive, the last inequality defines a non-empty domain for δ provided that $\beta > \frac{1}{1-(\theta'-\theta)}$.

Hence, picking any $\beta \in]b, 1[$ with $b = \max\{\gamma/(1 - \theta), 1/[1 - (\theta' - \theta)]\}$ and any $\delta \in]0, \beta(1 + \theta - \theta') - 1[$ guarantees that $(g(\zeta_n)_n)$ is a bounded positive supermartingale. We conclude from theorem 2.1.

□

3 Conclusion and open problems

We have examined the asymptotic behaviour of the chains (ζ_n) evolving on \mathbb{R}^+ . The cases we reported in this paper demonstrate an interesting phenomenon of antagonism between the heaviness of the tail (quantified by θ) of the innovation part of the Markov chain and the strength of the systematic drift (quantified by γ). It is precisely this antagonism that makes the model non trivial; if instead of heavy-tailed random variables, integrable ones are used, then the systematic drift totally determines the asymptotic behaviour of (ζ_n) .

Note also that the study of the chain (ζ_n) is sufficient for determining whether the limiting behaviour of the original random dynamical system (X_n) is towards 0 or ∞ . Nevertheless, the Markov chain obtained by looking at the (X_n) on logarithmic scales is not (ζ_n) (evolving on \mathbb{R}^+) but $(\tilde{\zeta}_n)$ (evolving on \mathbb{R}). Interesting problems concern random dynamical systems in higher dimension driven by non-integrable random matrices.

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