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Borell's formula on a Riemannian manifold and applications

Joseph Lehec

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1 Introduction

Throughout the article $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space, equipped with a filtration $(\mathcal{F}_t)_{t \leq T}$ and carrying a standard n -dimensional Brownian motion $(B_t)_{t \leq T}$. The time horizon T shall vary along the article, most of the time it will be finite. By *standard* we mean that (B_t) starts from 0 and has quadratic covariation given by $[B]_t = tI_n$ for all $t \leq T$. Let \mathbb{H} be the Cameron-Martin space, namely the space of absolutely continuous paths $u: [0, T] \rightarrow \mathbb{R}^n$, starting from 0 and equipped with the norm

$$\|u\|_{\mathbb{H}} = \left(\int_0^T |\dot{u}_s|^2 ds \right)^{1/2},$$

where $|\dot{u}_s|$ denotes the Euclidean norm of the derivative of u at time s . In this context a *drift* is a process which is adapted to the filtration (\mathcal{F}_t) and which belongs to \mathbb{H} almost surely. Let γ_n be the standard Gaussian measure on \mathbb{R}^n . The starting point of the present article is the so called Borell formula: If $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is measurable and bounded from below then

$$\log \left(\int_{\mathbb{R}^n} e^f d\gamma_n \right) = \sup_U \left\{ \mathbb{E} \left[f(B_1 + U_1) - \frac{1}{2} \|U\|_{\mathbb{H}}^2 \right] \right\} \quad (1)$$

where the supremum is taken over all drifts U (here the time horizon is $T = 1$). Actually a more general formula holds true, where the function f is allowed to depend on the whole trajectory of (B_t) rather than just B_1 . More precisely, let $(\mathbb{W}, \mathcal{B}, \gamma)$ be the n -dimensional Wiener space: \mathbb{W} is the space of continuous paths from $[0, T]$ to \mathbb{R}^n , \mathcal{B} is the Borel σ -field associated to the topology given by the uniform convergence (uniform convergence on compact sets if $T = +\infty$) and γ is the law of the standard Brownian motion. If $F: \mathbb{W} \rightarrow \mathbb{R}$ is measurable and bounded from below then

$$\log \left(\int_{\mathbb{W}} e^F d\gamma \right) = \sup_U \left\{ \mathbb{E} \left[F(B + U) - \frac{1}{2} \|U\|_{\mathbb{H}}^2 \right] \right\} \quad (2)$$

Of course (1) is retrieved by applying the latter formula to a functional F of the form $F(w) = f(w_1)$. Formula (2) is due to Boué and Dupuis [6], we refer to our previous work [13] for more historical comments and an alternate proof of the formula. We are interested in the use of such formulas to prove functional inequalities. This was initiated by Borell in [5], in which he proved (1) and showed that it yields the Prékopa–Leindler inequality very easily. This was further developed in the author’s works [13, 14] where many other functional inequalities were derived from (1) and (2). The main purpose of the present article is to establish a version of Borell’s formula (1) for the Brownian motion on a Riemannian manifold and to give a couple of applications, including a new proof of the Brascamp–Lieb inequality on the sphere of Carlen, Lieb and Loss.

2 Borell’s formula for a diffusion

Let $\sigma: \mathbb{R}^n \rightarrow M_n(\mathbb{R})$, let $b: \mathbb{R}^n \rightarrow \mathbb{R}^n$ and assume that the stochastic differential equation

$$dX_t = \sigma(X_t) dB_t + b(X_t) dt \quad (3)$$

has a unique strong solution. We also assume for simplicity that the explosion time is $+\infty$. Then there exists a measurable functional

$$G: \mathbb{R}^n \times \mathbb{W} \rightarrow \mathbb{W}$$

(it is probably safer to complete the σ -field \mathcal{B} at this stage) such that for every $x \in \mathbb{R}^n$ the process $X = G(x, B)$ is the unique solution of (3) starting from x . This hypothesis is satisfied in particular if σ and b are locally Lipschitz and grow at most linearly, see for instance [11, Chapter IV]. The process (X_t) is then a diffusion with generator L given by

$$Lf = \frac{1}{2} \langle \sigma \sigma^T, \nabla^2 f \rangle + \langle b, \nabla f \rangle.$$

for every \mathcal{C}^2 -smooth function f . We denote the associated semigroup by (P_t) : for any test function f

$$P_t f(x) = \mathbb{E}_x [f(X_t)],$$

where as usual the subscript x denotes the starting point of (X_t) . Fix a finite time horizon T . Fix $x \in \mathbb{R}^n$, let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ and assume that f is bounded from below. Applying the representation formula (2) to the functional

$$F: w \in \mathbb{W} \mapsto f(G(x, w)_T)$$

we get

$$\log \int_{\mathbb{W}} e^{f(G(x, w)_T)} \gamma(dw) = \sup_U \left\{ \mathbb{E} \left[f(G(x, B + U)_T) - \frac{1}{2} \|U\|_{\mathbb{H}}^2 \right] \right\}$$

where the supremum is taken on all drifts U . By definition of the semigroup (P_t) we have

$$\log \int_{\mathbb{W}} e^{f(G(x, w)_T)} \gamma(dw) = \log P_T(e^f)(x).$$

Also, we have the following lemma.

Lemma 1. *Let $(U_t)_{t \leq T}$ be a drift. The process $X^U = G(x, B + U)$ is the unique process satisfying*

$$X_t^U = x + \int_0^t \sigma(X_s^U) (dB_s + dU_s) + \int_0^t b(X_s^U) ds, \quad t \leq T \quad (4)$$

almost surely.

Proof. Assume first that $\|U\|_{\mathbb{H}}$ is bounded. Then by Novikov's criterion the process (D_t) given by

$$D_t = \exp \left(- \int_0^t \langle \dot{U}_s, dB_s \rangle - \frac{1}{2} \int_0^t |\dot{U}_s|^2 ds \right)$$

is a uniformly integrable martingale. Moreover, according to Girsanov's formula, under the measure \mathbb{Q} given by $d\mathbb{Q} = D_T d\mathbb{P}$, the process $B + U$ is a standard Brownian motion on $[0, T]$, see for instance [12, section 3.5] for more details. Now since the stochastic differential equation (3) is assumed to have a unique strong solution and by definition of G , almost surely for \mathbb{Q} , the unique process satisfying (4) is $X^U = G(x, B + U)$. Since \mathbb{Q} and \mathbb{P} are equivalent this is the result. For general U , the result follows by applying the bounded case to $(U_t) = (U_{t \wedge T_n})$ where T_n is the stopping time

$$T_n = \inf \left\{ t \geq 0 : \int_0^t |\dot{U}_s|^2 ds \geq n \right\},$$

and letting n tend to $+\infty$. □

To sum up, we have established the following result.

Theorem 2. *For any function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ bounded from below we have*

$$\log P_T(e^f)(x) = \sup_U \left\{ \mathbb{E} \left[f(X_T^U) - \frac{1}{2} \|U\|_{\mathbb{H}}^2 \right] \right\},$$

where the supremum is taken over all drifts U and the process X^U is the unique solution of (4).

Remarks. This direct consequence of the representation formula (2) was already noted by Boué and Dupuis. They used it to recover Freidlin and Wentzell's large deviation principle as the diffusion coefficient is sent to 0. Let us also note that the non explosion hypothesis is not essential. One can consider $\mathbb{R}^n \cup \{\infty\}$, the one point compactification of \mathbb{R}^n , set $X_t = \infty$ after explosion time, and restrict to functions f that tend to 0 at infinity. In the same way we could also deal with a Dirichlet boundary condition.

3 Borell's formula on a Riemannian manifold

Let (M, g) be a complete Riemannian manifold of dimension n . In this section we wish to establish a Borell type formula for the Brownian motion on M . To do so we need to recall first the intrinsic construction of the Brownian motion on M .

Let us start with some definitions from Riemannian geometry. Recall that the orthonormal frame bundle $\mathcal{O}(M)$ is the set of $(n + 1)$ -tuples

$$\phi = (x, \phi^1, \dots, \phi^n)$$

where x is in M and (ϕ^1, \dots, ϕ^n) is an orthonormal basis of $T_x(M)$. Given an element $\phi = (x, \phi^1, \dots, \phi^n)$ of $\mathcal{O}(M)$ and a vector $v \in T_x(M)$, the *horizontal lift* of v at ϕ , denoted $\mathcal{H}(v)$, is an element of $T_\phi(\mathcal{O}(M))$ defined as follows: Choose a curve (x_t) starting from x with speed v and for $i \leq n$ let ϕ_t^i be the parallel translation of ϕ^i along (x_t) . Since parallel translation preserves the inner product, we thus obtain a smooth curve (ϕ_t) on $\mathcal{O}(M)$ and we can set

$$\mathcal{H}(v) = \dot{\phi}_0.$$

This is a lift of v in the sense that for any smooth f on M

$$\mathcal{H}(v)(f \circ \pi) = v(f),$$

where $\pi: \mathcal{O}(M) \rightarrow M$ is the canonical projection. Now for $i \leq n$ we define a vector field on $\mathcal{O}(M)$ by setting

$$\mathcal{H}^i(x, \phi^1, \dots, \phi^n) = \mathcal{H}(\phi^i).$$

The operator

$$\Delta_{\mathcal{H}} = \sum_{i=1}^n (\mathcal{H}^i)^2$$

is called the *horizontal Laplacian*. It is related to the Laplace–Beltrami operator on M , denoted Δ , through the following commutation property: for any smooth f on M we have

$$\Delta_{\mathcal{H}}(f \circ \pi) = \Delta(f) \circ \pi. \quad (5)$$

Note that the horizontal Laplacian is by definition a sum of squares of vector fields, and that this is typically not the case for the Laplace–Beltrami operator. We are now in a position to define the horizontal Brownian motion on $\mathcal{O}(M)$. Let

$$B_t = (B_t^1, \dots, B_t^n)$$

be a standard n -dimensional Brownian motion. We consider the following stochastic differential equation on $\mathcal{O}(M)$

$$d\Phi_t = \sum_{i=1}^d \mathcal{H}^i(\Phi_t) \circ dB_t^i. \quad (6)$$

Throughout the rest of the article, the notation $H \circ dM$ denotes the Stratonovitch integral. The equation (6) is a short way of saying that for any smooth function g on $\mathcal{O}(M)$ we have

$$g(\Phi_t) = g(\Phi_0) + \sum_{i=1}^n \int_0^t \mathcal{H}^i(g)(\Phi_t) \circ dB_t^i.$$

This always has a strong solution, see [11, Theorem V.1.1.]. Let us assume additionally that it does not explode in finite time. This is the case in particular if the Ricci curvature of M is bounded from below, see for instance [10, section 4.2], where a more precise criterion is given. Translating the equation above in terms of Itô increments we easily get

$$dg(\Phi_t) = \sum_{i=1}^n \mathcal{H}^i(g)(\Phi_t) dB_t^i + \frac{1}{2} \Delta_{\mathcal{H}} g(\Phi_t) dt.$$

Let (X_t) be the process given by $X_t = \pi(\Phi_t)$. Applying the previous formula and (5) we see that for any smooth f on M

$$df(X_t) = \sum_{i=1}^n \Phi_t^i(f)(X_t) dB_t^i + \frac{1}{2} \Delta f(X_t) dt. \quad (7)$$

In particular

$$f(X_t) - \frac{1}{2} \int_0^t \Delta f(X_s) ds, \quad t \geq 0$$

is a local martingale. This shows that (X_t) is a Brownian motion on M . The process (X_t) is called the *stochastic development* of (B_t) . In the sequel, it will be convenient to identify the orthogonal basis $\Phi_t^1, \dots, \Phi_t^n$ with the orthogonal map

$$x \in \mathbb{R}^n \rightarrow \sum_{i=1}^n x_i \Phi_t^i \in T_{X_t}(M).$$

Then the equation (7) can be rewritten

$$df(X_t) = \langle \nabla f(X_t), \Phi_t dB_t \rangle + \frac{1}{2} \Delta f(X_t) dt.$$

Similarly the equation (6) can be rewritten in a more concise form

$$d\Phi_t = \mathcal{H}(\Phi_t) \circ dB_t. \quad (8)$$

To sum up, the process (Φ_t) is an orthonormal basis above (X_t) which is used to map the Brownian increment dB_t from \mathbb{R}^n to the tangent space of M at X_t . Now we establish a Borell type formula for the process (X_t) . We know that there exists a measurable functional

$$G: \mathcal{O}(M) \times \mathbb{W} \rightarrow \mathcal{C}(\mathbb{R}, \mathcal{O}(M))$$

such that the process $\Phi = G(\phi, B)$ is the unique solution of (8) starting from ϕ . Let $\phi \in \mathcal{O}(M)$, let $T > 0$, let $f: M \rightarrow \mathbb{R}$ and assume that f is bounded from below. Applying the representation formula (2) to the functional

$$F: w \in \mathbb{W} \mapsto f \circ \pi(G(\phi, w)_T)$$

we get

$$\log \left(\int_{\mathbb{W}} e^{f \circ \pi(G(\phi, B)_T)} d\gamma \right) = \sup_U \left\{ \mathbf{E} \left[f \circ \pi(G(\phi, B + U)_T) - \frac{1}{2} \|U\|_{\mathbb{H}}^2 \right] \right\}.$$

Let $x = \pi(\phi)$. Since $\pi(G(\phi, B))$ is a Brownian motion on M starting from x we have

$$\log \left(\int_{\mathbb{W}} e^{f \circ \pi(G(\phi, B)_T)} d\gamma \right) = \log P_T(e^f)(x),$$

where (P_t) is the heat semigroup on M . Also, letting $\Phi^U = G(\phi, B + U)$ and reasoning along the same lines as in the proof of Lemma 1, we obtain that Φ^U is the only solution to

$$d\Phi_t^U = \mathcal{H}(\Phi_t^U) \circ (dB_t + dU_t)$$

starting from ϕ . We also let $X^U = \pi(\Phi^U)$ and call this process the stochastic development of $B + U$ starting from ϕ . To sum up, we have established the following result.

Theorem 3. *Let $f: M \rightarrow \mathbb{R}$, let $\phi \in \mathcal{O}(M)$, let $x = \pi(\phi)$ and let $T > 0$. If f is bounded from below then*

$$\log P_T(e^f)(x) = \sup_U \left\{ \mathbf{E} \left[f(X_T^U) - \frac{1}{2} \|U\|_{\mathbb{H}}^2 \right] \right\},$$

where the supremum is taken on all drifts U and where given a drift U , the process X^U is the stochastic development of $B + U$ starting from ϕ .

4 Brascamp–Lieb inequality on the sphere

In the article [14], we explained how to derive the Brascamp–Lieb inequality and its reversed version from Borell’s formula. In this section we extend this to the sphere, and give a proof based on Theorem 3 of the spherical version of the Brascamp–Lieb inequality, due to Carlen, Lieb and Loss in [7].

Theorem 4. *Let g_1, \dots, g_{n+1} be non-negative functions on the interval $[-1, 1]$. Let σ_n be the Haar measure on the sphere \mathbb{S}^n , normalized to be a probability measure. We have*

$$\int_{\mathbb{S}^n} \prod_{i=1}^{n+1} g_i(x_i) \sigma_n(dx) \leq \prod_{i=1}^{n+1} \left(\int_{\mathbb{S}^n} g_i(x_i)^2 \sigma_n(dx) \right)^{1/2}$$

Remark. The inequality does not hold if we replace the L^2 norm in the right-hand side by a smaller L^p norm, like the L^1 norm. Somehow this 2 accounts for the fact that the coordinates of a uniform random vector on \mathbb{S}^n are not independent. We refer to the introduction of [7] for a deeper insight on this inequality.

In addition to the Borell type formulas established in the previous two sections, our proof relies on a sole inequality, spelled out in the lemma below. Let $P_i: \mathbb{S}^n \rightarrow [-1; 1]$ be the application that maps x to its i -th coordinate x_i . The spherical gradient of P_i at x is the projection of the coordinate vector e_i onto x^\perp :

$$\nabla P_i(x) = e_i - x_i x.$$

Lemma 5. *Let $x \in \mathbb{S}^n$ and let $y \in x^\perp$. For $i \leq n+1$, if $\nabla P_i(x) \neq 0$ let*

$$\theta^i = \frac{\nabla P_i(x)}{|\nabla P_i(x)|}$$

and let θ_i be an arbitrary unit vector of x^\perp otherwise. Then for any $y \in x^\perp$ we have

$$\sum_{i=1}^{n+1} \langle \theta^i, y \rangle^2 \leq 2|y|^2.$$

Proof. Assume first that $\nabla P_i(x) \neq 0$ for every i . Since $\nabla P_i(x) = e_i - x_i x$ and y is orthogonal to x we then have

$$\langle \theta^i, y \rangle^2 = y_i^2 + x_i^2 \langle \theta^i, y \rangle^2 \leq y_i^2 + x_i^2 |y|^2.$$

Summing this over $i \leq n+1$ yields the result. On the other hand, if there exists i such that $\nabla P_i(x) = 0$ then $x = \pm e_i$ and it is almost immediate to check that the desired inequality holds true. \square

Proof of Theorem 4. Let us start by describing the behaviour of a given coordinate of a Brownian motion on \mathbb{S}^n . Let (B_t) be a standard Brownian motion on \mathbb{R}^n , let ϕ be a fixed element of $\mathcal{O}(\mathbb{S}^n)$ and let (Φ_t) be the horizontal Brownian motion given by

$$\Phi_0 = \phi \quad \text{and} \quad d\Phi_t = \mathcal{H}(\Phi_t) \circ dB_t.$$

We also let $X_t = \pi(\Phi_t)$ be the stochastic development of (B_t) and $X_t^i = P_i(X_t)$, for every $i \leq n+1$. We have

$$dX_t^i = \langle \nabla P_i(X_t), \Phi_t dB_t \rangle + \frac{1}{2} \Delta P_i(X_t) dt. \tag{9}$$

Let θ be an arbitrary unit vector of \mathbb{R}^n and define a process (θ_t^i) by

$$\theta_t^i = \begin{cases} \Phi_t^* \left(\frac{\nabla P_i(X_t)}{|\nabla P_i(X_t)|} \right), & \text{if } \nabla P_i(X_t) \neq 0, \\ \theta & \text{otherwise.} \end{cases}$$

Since Φ_t is an orthogonal map θ_t belongs to the unit sphere of \mathbb{R}^n . Consequently the process (W_t^i) defined by $dW_t^i = \langle \theta_t^i, dW_t \rangle$ is a one dimensional standard Brownian motion. Observe that $|\nabla P_i| = (1 - P_i^2)^{1/2}$ and recall that P_i is an eigenfunction for the spherical Laplacian: $\Delta P_i = -nP_i$. Equality (9) becomes

$$dX_t^i = (1 - (X_t^i)^2)^{1/2} dW_t^i - \frac{n}{2} X_t^i dt.$$

This stochastic differential equation is usually referred to as the Jacobi diffusion in the literature, see for instance [2, section 2.7.4]. What matters for us is that it does possess a unique strong solution. Indeed the drift term is linear and although the diffusion factor $(1 - x^2)^{1/2}$ is not locally Lipschitz, it is Hölder continuous with exponent 1/2, which is sufficient to insure strong uniqueness in dimension 1, see for instance [15, section V.40]. Let (Q_t) be the semigroup associated to the process (X_t^i) . The stationary measure ν_n is easily seen to be given by

$$\nu_n(dt) = c_n \mathbb{1}_{[-1,1]}(t)(1 - t^2)^{\frac{n}{2}-1} dt,$$

where c_n is the normalization constant. Obviously ν_n coincides with the push-forward of σ_n by P_i .

We now turn to the actual proof of the theorem. Let g_1, \dots, g_{n+1} be non negative functions on $[-1, 1]$ and assume (without loss of generality) that they are bounded away from 0. Let $f_i = \log(g_i)$ for all i and let

$$f: x \in \mathbb{S}^n \mapsto \sum_{i=1}^{n+1} f_i(x_i).$$

The functions f_i, f are bounded from below. Fix a time horizon T , let U be a drift and let (Φ_t^U) be the process given by

$$\begin{cases} \Phi_0^U = \phi \\ d\Phi_t^U = \mathcal{H}(\Phi_t^U) \circ (dB_t + dU_t), \quad t \leq T. \end{cases}$$

We also let $X_t^U = \pi(\Phi_t^U)$ be the stochastic development of $B + U$. These processes are well defined by the results of the previous section. We want to bound $f(X_T^U) - \frac{1}{2}\|U\|_{\mathbb{H}}^2$ from above. By definition

$$f(X_T^U) = \sum_{i=1}^{n+1} f_i(P_i X_T^U).$$

Let (θ_t^i) be the process given by

$$\theta_t^i = \Phi_t^* \left(\frac{\nabla P_i(X_t^U)}{|\nabla P_i(X_t^U)|} \right)$$

(again replace this by an arbitrary fixed unit vector if $\nabla P_i(X_t^U) = 0$). Then let (W_t^i) be the one dimensional Brownian motion given by $dW_t^i = \langle \theta_t^i, dW_t \rangle$

and let (U_t^i) be the one dimensional drift given by $dU_t^i = \langle \theta_t^i, dU_t \rangle$. The process $(P_i(X_t^U))$ then satisfies

$$dP_i(X_t^U) = (1 - P_i(X_t^U))^2 (dW_t^i + dU_t^i) - \frac{n}{2} P_i(X_t^U) dt. \quad (10)$$

Applying Lemma 5, we easily get

$$\sum_{i=1}^{n+1} \|U^i\|_{\mathbb{H}}^2 \leq 2\|U\|_{\mathbb{H}}^2,$$

almost surely (note that in the left hand side of the inequality \mathbb{H} is the Cameron–Martin space of \mathbb{R} rather than \mathbb{R}^n). Therefore

$$f(X_T^U) - \frac{1}{2}\|U\|_{\mathbb{H}}^2 \leq \sum_{i=1}^{n+1} \left(f_i(P_i(X_T^U)) - \frac{1}{4}\|U^i\|_{\mathbb{H}}^2 \right). \quad (11)$$

Recall (10) and apply Theorem 2 to the semigroup (Q_t) and to the function $2f_i$ rather than f_i . This gives

$$\mathbb{E} \left[f_i(P_i(X_t^U)) - \frac{1}{4}\|U^i\|_{\mathbb{H}}^2 \right] \leq \frac{1}{2} \log Q_T(e^{2f_i})(x_i),$$

for every $i \leq n + 1$. Taking expectation in (11) thus yields

$$\mathbb{E} \left[f(X_T^U) - \frac{1}{2}\|U\|_{\mathbb{H}}^2 \right] \leq \frac{1}{2} \sum_{i=1}^{n+1} \log Q_T(e^{2f_i})(x_i).$$

Taking the supremum over all drifts U and using Theorem 3 we finally obtain

$$P_T(e^f)(x) \leq \prod_{i=1}^{n+1} (Q_T(e^{2f_i})(x_i))^{1/2}.$$

The semigroup (P_t) is ergodic and converges to σ_n as t tends to $+\infty$. Similarly (Q_t) converges to ν_n . So letting T tend to $+\infty$ in the previous inequality gives

$$\int_{\mathbb{S}^n} e^f d\sigma_n \leq \prod_{i=1}^{n+1} \left(\int_{[-1,1]} e^{2f_i} d\nu_n \right)^{1/2},$$

which is the result. \square

Remark. Barthe, Cordero–Erausquin and Maurey in [3] and together with Ledoux in [4] gave several extensions of Theorem 4. The method exposed here also allows to recover most of their results. We chose to stick to the original statement of Carlen, Lieb and Loss for simplicity.

5 The dual formula

In [13], we established a dual version of Borell's formula (1). It states as follows: If μ is an absolutely continuous measure on \mathbb{R}^n satisfying some (reasonable) technical assumptions, the relative entropy of μ with respect to the Gaussian measure is given by the following formula

$$H(\mu \mid \gamma_n) = \inf \left\{ \frac{1}{2} \mathbb{E} [\|U\|_{\mathbb{H}}^2] \right\} \quad (12)$$

where the infimum is taken over all drifts U such that $B_1 + U_1$ has law μ . Informally this says that the minimal energy needed to constrain the Brownian motion to have a prescribed law at time 1 coincides with the relative entropy of this law with respect to γ_n . The infimum is actually a minimum and the optimal drift can be described as follows. We let f be the density of μ with respect to γ_n and (P_t) be the heat semigroup on \mathbb{R}^n . The following stochastic differential equation

$$\begin{cases} X_0 = 0 \\ dX_t = dB_t + \nabla \log P_{1-t} f(X_t) dt, \quad t \leq 1. \end{cases} \quad (13)$$

has a unique strong solution. The solution satisfies $X_1 = \mu$ in law and is optimal in the sense that there is equality in (12) for the drift U given by $\dot{U}_t = \nabla \log P_{1-t} f(X_t)$. The purpose of this section is to describe the Riemannian counterpart of (12). There is also a version for diffusions such as the ones considered in section 2 but we shall omit it for the sake of brevity.

The setting of this section is the same as that of section 3: (M, g) is a complete Riemannian manifold of dimension n whose Ricci curvature is bounded from below and (B_t) is a standard Brownian motion on \mathbb{R}^n . We denote the heat semigroup on M by (P_t) .

Theorem 6. *Fix $x \in M$ and a time horizon T . Let μ be a probability measure on M , assume that μ is absolutely continuous with respect to $\delta_x P_T$ and let f be its density. If f is Lipschitz and bounded away from 0 then*

$$H(\mu \mid \delta_x P_T) = \inf \left\{ \frac{1}{2} \mathbb{E} [\|U\|_{\mathbb{H}}^2] \right\}$$

where the infimum is taken all drifts U such that the stochastic development of $B + U$ starting from x has law μ at time T .

Proving that any drift satisfying the constraint has energy at least as large as the relative entropy of μ is a straightforward adaptation of Proposition 1 from [13], and we shall leave this to the reader. Alternatively, one can use Theorem 3 and combine it with the following variational formula for the entropy:

$$H(\mu \mid \delta_x P_T) = \sup_f \left\{ \int_M f d\mu - \log P_T(e^f)(x) \right\}$$

(again details are left to the reader). Besides, as in the Euclidean case, there is actually an optimal drift, whose energy is exactly the relative entropy of μ . This is the purpose of the next result.

Theorem 7. *Let ϕ be a fixed element of $\mathcal{O}(M)$. Let $x = \pi(\phi)$ and let T be a time horizon. Let μ have density f with respect to $\delta_x P_T$, and assume that f is Lipschitz and bounded away from 0. The stochastic differential equation*

$$\begin{cases} \Phi_0 = \phi \\ d\Phi_t = \mathcal{H}(\Phi_t) \circ (dB_t + \Phi_t^* \nabla \log P_{T-t} f(Y_t)) \\ Y_t = \pi(\Phi_t) \end{cases} \quad (14)$$

has a unique strong solution on $[0, T]$. The law of the process (Y_t) is given by the following formula: For every functional $H: \mathcal{C}([0, T]; M) \rightarrow \mathbb{R}$ we have

$$\mathbb{E}[H(Y)] = \mathbb{E}[H(X)f(X_1)], \quad (15)$$

where (X_t) is a Brownian motion on M starting from x . In other words Y_T has law μ and the bridges of Y equal those of the Brownian motion on M , in law. Moreover, letting U be the drift given by

$$U_t = \int_0^t \Phi_s^* \nabla \log P_{T-s} f(Y_s) ds, \quad t \leq T, \quad (16)$$

we have

$$\mathbb{H}(\mu \mid \delta_x P_T) = \frac{1}{2} \mathbb{E}[\|U\|_{\mathbb{H}}^2].$$

Proof. Since $\text{Ric} \geq -\lambda g$, we have the following estimate for the Lipschitz norm of f :

$$\|P_t f\|_{\text{Lip}} \leq e^{\lambda t/2} \|f\|_{\text{Lip}}.$$

One way to see this is to use Kendall's coupling for Brownian motions on a manifold, see for instance [10, section 6.5]. Alternatively, it is easily derived from the commutation property $|\nabla P_t f|^2 \leq e^{\lambda t} P_t(|\nabla f|^2)$ which, in turn, follows from Bochner's formula, see [2, Theorem 3.2.3]. Recall that f is assumed to be bounded away from 0, and for every $t \leq T$ let $F_t = \log P_{T-t} f$. Then $(t, x) \mapsto \nabla F_t(x)$ is smooth and bounded on $[0, T] \times M$, which is enough to insure the existence of a unique strong solution to (14). Besides an easy computation shows that

$$\partial_t F_t = -\frac{1}{2}(\Delta F_t + |\nabla F_t|^2). \quad (17)$$

Then using (14) and Itô's formula we get

$$\begin{aligned} dF(t, Y_t) &= \langle \nabla F_t(Y_t), \Phi_t dB_t \rangle + \frac{1}{2} |\nabla F_t(Y_t)|^2 dt. \\ &= \langle \dot{U}_t, dB_t \rangle + \frac{1}{2} |\dot{U}_t|^2 dt \end{aligned}$$

(recall the definition (16) of U). Therefore

$$\frac{1}{f(Y_T)} = e^{-F_T(Y_T)} = \exp\left(-\int_0^T \langle \dot{U}_t, dB_t \rangle - \frac{1}{2} \|U\|_{\mathbb{H}}^2\right). \quad (18)$$

Observe that the variable $\|U\|_{\mathbb{H}}$ is bounded (just because ∇F is bounded). So Girsanov's formula applies: $1/f(Y_T)$ has expectation 1 and under the measure \mathbb{Q} given by $d\mathbb{Q} = (1/f(Y_T)) d\mathbb{P}$ the process $B+U$ is a standard Brownian motion on \mathbb{R}^n . Since Y is the stochastic development of $B+U$ starting from x , this shows that under \mathbb{Q} the process Y is a Brownian motion on M starting from x . This is a mere reformulation of (15). For the entropy equality observe that since Y_T has law μ , we have

$$\mathbb{H}(\mu \mid \delta_x P_T) = \mathbb{E}[\log f(Y_T)].$$

Using (18) again and the fact that $\int \langle \dot{U}_t, dB_t \rangle$ is a martingale we get the desired equality. \square

To conclude this article, let us derive from this formula the log-Sobolev inequality for a manifold having a positive lower bound on its Ricci curvature. This is of course well-known, but our point is only to illustrate how the previous theorem can be used to prove inequalities. Recall the definition of the Fisher information: if μ is a probability measure on M having Lipschitz and positive density f with respect to some reference measure m , the relative Fisher information of μ with respect to m is defined by

$$\mathbb{I}(\mu \mid m) = \int_M \frac{|\nabla f|^2}{f} dm = \int_M |\nabla \log f|^2 d\mu.$$

By Bishop's Theorem, if $\text{Ric} \geq \kappa g$ pointwise for some positive κ then the volume measure on M is finite. We let m be the volume measure normalized to be a probability measure.

Theorem 8. *If $\text{Ric} \geq \kappa g$ pointwise for some $\kappa > 0$, then for any probability measure μ on M having a Lipschitz and positive density with respect to m we have*

$$\mathbb{H}(\mu \mid m) \leq \frac{n}{2} \log\left(1 + \frac{\mathbb{I}(\mu \mid m)}{n\kappa}\right). \quad (19)$$

Remarks. Since $\log(1+x) \leq x$ this inequality is a dimensional improvement of the more familiar inequality

$$\mathbb{H}(\mu \mid m) \leq \frac{1}{\kappa} \mathbb{I}(\mu \mid m). \quad (20)$$

It is known (see for instance [1, section 5.6]) that (20) admits yet another sharp form that takes the dimension into account, namely

$$\mathbb{H}(\mu \mid m) \leq \frac{n-1}{\kappa n} \mathbb{I}(\mu \mid m), \quad (21)$$

but we were not able to recover this one with our method. Note also that depending on the measure μ , the right hand side of (21) can be smaller than that of (19), or the other way around.

Proof. By the Bonnet–Myers theorem M is compact. Fix $x \in M$ and a time horizon T . Let $p_T(x, \cdot)$ be the density of the measure $\delta_x P_T$ with respect to m (in other words let (p_t) be the heat kernel on M). If $d\mu = \rho dm$ then μ has density $f = \rho/p_T(x, \cdot)$ with respect to $\delta_x P_T$. Since $p_T(x, \cdot)$ is smooth and positive (see for instance [8, chapter 6]) f satisfies the technical assumptions of the previous theorem. Let $F_t = \log P_{T-t} f$ and let (Y_t) be the process given by (14). We know from the previous theorem that

$$\mathbb{H}(\mu \mid \delta_x P_T) = \frac{1}{2} \mathbb{E} \left[\int_0^T |\nabla F_t(Y_t)|^2 dt \right]. \quad (22)$$

Using (17) we easily get

$$\partial_t (|\nabla F|^2) = -\langle \nabla \Delta F, \nabla F \rangle - \langle \nabla |\nabla F|^2, \nabla F \rangle.$$

Applying Itô's formula we obtain after some computations (omitting variables in the right hand side)

$$d|\nabla F(t, Y_t)|^2 = \langle \nabla |\nabla F|^2, \Phi_t dB_t \rangle - \langle \nabla \Delta F, \nabla F \rangle dt + \frac{1}{2} \Delta |\nabla F|^2 dt.$$

Now recall Bochner's formula

$$\frac{1}{2} \Delta |\nabla F|^2 = \langle \nabla \Delta F, \nabla F \rangle + \|\nabla^2 F\|_{HS}^2 + \text{Ric}(\nabla F, \nabla F).$$

So that

$$d|\nabla F(t, Y_t)|^2 = \langle \nabla |\nabla F|^2, \Phi_t dB_t \rangle + \|\nabla^2 F\|_{HS}^2 dt + \text{Ric}(\nabla F, \nabla F) dt. \quad (23)$$

Since ∇F is bounded and the Ricci curvature non negative, the local martingale part in the above equation is bounded from above. So by Fatou's lemma it is a sub-martingale, and its expectation is non decreasing. So taking expectation in (23) and using the hypothesis $\text{Ric} \geq \kappa g$ we get

$$\frac{d}{dt} \mathbb{E} [|\nabla F_t(Y_t)|^2] \geq \mathbb{E} [\|\nabla^2 F_t(Y_t)\|_{HS}^2] + \kappa \mathbb{E} [|\nabla F_t(Y_t)|^2]. \quad (24)$$

Throwing away the Hessian term would lead us to the inequality (20). Let us exploit this term instead. Using Cauchy-Schwartz and Jensen's inequality we get

$$\mathbb{E} [\|\nabla^2 F_t(Y_t)\|_{HS}^2] \geq \frac{1}{n} \mathbb{E} [\Delta F_t(Y_t)^2] \geq \frac{1}{n} \mathbb{E} [\Delta F_t(Y_t)]^2.$$

Also, by (15) and recalling that (P_t) is the heat semigroup on M we obtain

$$\begin{aligned} \mathbb{E} [\Delta F_t(Y_t) + |\nabla F_t(Y_t)|^2] &= \mathbb{E} \left[\frac{\Delta P_{T-t} f(Y_t)}{P_{T-t} f(Y_t)} \right] = \mathbb{E} \left[\frac{\Delta P_{T-t} f(X_t)}{P_{T-t} f(X_t)} f(X_T) \right] \\ &= \mathbb{E} [\Delta P_{T-t} f(X_t)] = \Delta P_T(f)(x). \end{aligned}$$

Letting $\alpha(t) = \mathbb{E} [|\nabla F_t(Y_t)|^2]$ and $C_T = \Delta P_T f(x)$ we thus get from (24)

$$\begin{aligned} \alpha'(t) &\geq \kappa \alpha(t) + \frac{1}{n} (\alpha(t) - C_T)^2 \\ &\geq \frac{1}{n} \alpha(t) (\alpha(t) + n \kappa - 2C_T) \end{aligned}$$

Since $P_T f$ tends to a constant function as T tends to $+\infty$, C_T tends to 0. So if T is large enough $n\kappa - 2C_T$ is positive and the differential inequality above yields

$$\alpha(t) \leq \frac{n \kappa(T) \alpha(T)}{e^{\kappa(T)(T-t)} (n \kappa(T) + \alpha(T)) - \alpha(T)}, \quad t \leq T$$

where $\kappa(T) = \kappa - 2C_T/n$. Integrating this between 0 and T we get

$$\int_0^T \alpha(t) dt \leq n \log \left(1 + \frac{\alpha(T)(1 - e^{-\kappa(T)T})}{n \kappa(T)} \right). \quad (25)$$

Observe that $\kappa(T) \rightarrow \kappa$ as T tends to $+\infty$. By (22) and since $(\delta_x P_t)$ converges to m measure on M we have

$$\int_0^T \alpha(t) dt \rightarrow 2 \mathbb{H}(\mu | m),$$

as T tends to $+\infty$. Also, since Y_T has law μ

$$\alpha(T) = \mathbb{E} [|\nabla \log f(Y_T)|^2] = \mathbb{I}(\mu | \delta_x P_T) \rightarrow \mathbb{I}(\mu | m).$$

Therefore, letting T tend to $+\infty$ in (25) yields

$$\mathbb{H}(\mu | m) \leq \frac{n}{2} \log \left(1 + \frac{\mathbb{I}(\mu | m)}{n \kappa} \right),$$

which is the result. \square

Let us give an open problem to finish this article. We already mentioned that Borell recovered the Prékopa–Leindler inequality from (1). It is natural to ask whether there is probabilistic proof of the Riemannian Prékopa–Leindler inequality of Cordero, McCann and Schmuckenschlger [9] based on Theorem 3. Copying naively Borell’s argument, we soon face the following difficulty: If X and Y are two Brownian motions on a manifold coupled by parallel transport, then unless the manifold is flat, the midpoint of X and Y is not a Brownian motion. We believe that there is a way around this but we could not find it so far.

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