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BIRATIONAL GEOMETRY OF FOLIATIONS ASSOCIATED TO SIMPLE DERIVATIONS AND GENERALIZATIONS

GAËL COUSIN, LUÍS GUSTAVO MENDES, AND IVÁN PAN

ABSTRACT. We propose a study of the foliations of the projective plane induced by simple derivations of the polynomial ring in two indeterminates over the complex field. These correspond to foliations which have no invariant algebraic curve nor singularities in the complement of a line. We establish the position of these foliations in the birational classification of foliations and prove the finiteness of their birational symmetries. Most of the results apply to wider classes of foliations.

1. INTRODUCTION AND RESULTS

A *simple derivation* (of the ring $\mathbb{C}[x, y]$) is a polynomial vector field of \mathbb{C}^2 without zeroes and without algebraic solutions.

The study of simple derivations is an active area of research in Algebra (*e.g.* [Sha77, Jor81, Now94, MMON01, Cou03, BLL03, Now08, GL12, Kou12]). Most of these papers are dedicated to the proof of simplicity of (families of) examples.

For any derivation D , the isotropy group $\text{Aut}(D)$ is composed by the \mathbb{C} -automorphisms $\rho : \mathbb{C}[x, y] \rightarrow \mathbb{C}[x, y]$ which verify

$$\rho D = D\rho.$$

Although there exist derivations with infinite group $\text{Aut}(D)$, the main result of [MP16] is that $\text{Aut}(D)$ is *trivial for any simple derivation*.

Take $\rho \in \text{Aut}(D)$, $R : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ the polynomial automorphism associated to ρ and let ω_D be the dual 1-form to the vector field $D = f\partial_x + g\partial_y$ (*i.e.* $\omega_D = gdx - fdy$). Then $\rho D = D\rho$ is equivalent to

$$R^*(\omega_D) = \text{Jac}(R) \cdot \omega_D$$

where $\text{Jac}(R) \in \mathbb{C}^*$ is the Jacobian determinant of R . A less restrictive condition is that

$$R^*(\omega_D) = c \cdot \omega_D,$$

for some $c \in \mathbb{C}^*$ (depending on R). This means that R preserves *the algebraic foliation* \mathcal{F}_D of \mathbb{C}^2 associated to D (or to ω_D), see Remark 2.1.

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We denote $Pol(\mathcal{F}_D)$ the group consisting of polynomial automorphisms of \mathbb{C}^2 which preserve the foliation \mathcal{F}_D . There is a natural homomorphism

$$\text{Aut}(D) \hookrightarrow Pol(\mathcal{F}_D).$$

In Section 8 we show that *for each $n \geq 2$ and $B > 0$ there are foliations \mathcal{F}_D of \mathbb{C}^2 associated to simple derivations with an element $T \in Pol(\mathcal{F}_D)$ of order n and degree greater or equal than B .*

Let us denote \mathcal{F} the foliation of the projective plane $\mathbb{P}^2 = \mathbb{C}^2 \cup L_\infty$ which is the extension of \mathcal{F}_D in \mathbb{C}^2 . All along the paper, if D is a simple derivation, both \mathcal{F}_D in \mathbb{C}^2 and its extension \mathcal{F} in \mathbb{P}^2 are called *foliations associated to simple derivations*.

But the reader must be warned that, even if \mathcal{F}_D has no singularity, some singularity of \mathcal{F} along the line at infinity L_∞ is unavoidable, see [Bru00, Prop. 1 p. 21]. Also beware that the line at infinity L_∞ may be invariant by \mathcal{F} .

Denote $Bir(\mathcal{F})$ the group of birational transformations of \mathbb{P}^2 which preserve a foliation \mathcal{F} . If \mathcal{F} extends a foliation \mathcal{F}_D of \mathbb{C}^2 , then there is a natural homomorphism

$$Pol(\mathcal{F}_D) \hookrightarrow Bir(\mathcal{F})$$

whose meaning is that a (non-linear) polynomial automorphism of \mathbb{C}^2 extends to a special type of birational map of \mathbb{P}^2 . Namely, a birational map with a unique (proper) point of indeterminacy $p \in L_\infty$, whose net effect in \mathbb{P}^2 is to replace L_∞ by the strict transform of the last exceptional curve introduced in the elimination of the indeterminacy point.

Our first result is the following generalization of [MP16, Thm. 1].

Theorem A. *Let \mathcal{F} be a foliation associated to a simple derivation. The group $Bir(\mathcal{F})$ is finite.*

It is actually derived from next result which determines, in particular, the positions that foliations associated to simple derivations may occupy in the birational classification of foliations, cf. [Bru00, McQ01, Men00]. This classification is based on the birational invariant of foliations called *Kodaira dimension*, denoted $\kappa(\mathcal{F})$, whose range is $\kappa(\mathcal{F}) \in \{-\infty, 0, 1, 2\}$, see Section 2.

All along the paper we use the expression *reduced singularity* in the sense of Seidenberg's reduction of singularities, cf. Section 2. By a *rational curve* of \mathbb{C}^2 we mean an algebraic curve whose projective closure has geometric genus zero. And by a *Riccati foliation* on \mathbb{P}^2 we mean a foliation which, up to a birational modification of \mathbb{P}^2 , is everywhere transverse to the general fiber of a rational fibration; one says the fibration is *adapted* to such a foliation.

Theorem B. *Let \mathcal{F} be a foliation of the projective plane such that the restriction $\mathcal{F}|_{\mathbb{C}^2}$ has no invariant rational curve.*

- i) Then $\kappa(\mathcal{F}) \geq 0$;*
- ii) If $\mathcal{F}|_{\mathbb{C}^2}$ has at most reduced singularities, then $\kappa(\mathcal{F}) \geq 1$;*
- iii) If $\mathcal{F}|_{\mathbb{C}^2}$ has no invariant algebraic curve, then $\kappa(\mathcal{F}) = 1$ if and only if \mathcal{F} is a Riccati foliation.*
- iv) The cases $\kappa(\mathcal{F}) \in \{1, 2\}$ are realized by foliations associated to simple derivations.*

Remark that case $B - iii$) includes the foliations associated to *Shamsuddin derivations*. Note also that Theorem B applies to a class of foliations which is larger than the one of foliations associated to simple derivations. In Section 6 we study the foliations associated to examples of simple derivations found throughout the literature and discuss their birational equivalence. In Section 8, we propose a construction of simple derivations D with arbitrary large (finite) $Pol(\mathcal{F}_D)$.

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2. PRELIMINARIES ON FOLIATIONS

The paper relies on concepts and results of the theory of singularities and birational geometry of foliations on algebraic complex surfaces. We present the basic facts in this preliminary section but along the paper, when necessary, we refer the reader to the corresponding sections of [Bru00] or [Bru03], where the theory is masterfully explained by Marco Brunella.

First definitions. On a smooth complex surface X , a foliation \mathcal{F} is given by an open covering (U_i) of X and local vector fields $v_i \in H^0(U_i, TX)$ with *isolated zeroes* such that there exist non vanishing holomorphic functions (g_{ij}) on the intersections $U_i \cap U_j$ satisfying

$$v_i = g_{ij}v_j. \quad (1)$$

The cocycle (g_{ij}) defines a line bundle $T^*\mathcal{F}$ on X , its dual is denoted $T\mathcal{F}$. Relation (1) means that the family (v_i) defines a section of $T^*\mathcal{F} \otimes TX$ and hence a sheaf map $T\mathcal{F} \rightarrow TX$. Two data $((U_i), (v_i)), ((U'_j), (v'_j))$ are said to define the same foliation if the images of the associated sheaf maps are the same. The line bundle $T\mathcal{F}$ is called the *tangent bundle* of the foliation and its dual $T^*\mathcal{F}$ is the *cotangent bundle* of \mathcal{F} . The locus defined by the vanishing of the local vector fields (v_i) is called the *singular locus* of \mathcal{F} , denoted $\text{Sing}(\mathcal{F})$. As defined, the line bundle $T\mathcal{F}$ is not canonically attached to \mathcal{F} , but only its isomorphism class in $\text{Pic}(X)$.

One may also consider foliations on normal singular complex surfaces. They are defined by the datum of a foliation on the complement of the singular locus of the surface.

Rational vector fields and 1-forms. If X is smooth projective, $T\mathcal{F}$ possesses a non trivial rational section and \mathcal{F} can be given by a rational vector field \mathcal{X} , hence in $\text{Pic}(X)$ we have

$$T\mathcal{F} = \mathcal{O}_X(\text{div}(\mathcal{X})).$$

On a suitable (Zariski) open covering the local vector fields v_i are obtained by chasing the zeroes and poles of \mathcal{X} : $v_i = h\mathcal{X}|_{U_i}$, for a well chosen regular function h on U_i . This is how we associate a foliation to a simple derivation: we have a *preferred projective compactification* of \mathbb{C}^2 , namely $\mathbb{P}^2 = \mathbb{C}^2 \cup L_\infty, (x, y) \mapsto [1 : x : y]$, and a polynomial vector field on \mathbb{C}^2 extends to a rational vector field on \mathbb{P}^2 .

One can also define a foliation by local holomorphic 1-forms with isolated zeroes (ω_i) that vanish on the local vector fields (v_i) . If X is projective, such a family (ω_i) is obtained by locally chasing the zeroes and poles of a non trivial rational 1-form. Hence, on a smooth projective surface, a foliation may be defined by either a non trivial rational 1-form or a a non trivial rational vector field.

Curves and foliations. A curve C is termed *invariant by \mathcal{F}* or *\mathcal{F} -invariant* if it is tangent to the local vector fields defining \mathcal{F} . When a compact curve $C \subset X$ is *not \mathcal{F} -invariant* we have the very useful *formula*

$$T^*\mathcal{F} \cdot C = \text{tang}(\mathcal{F}, C) - C \cdot C,$$

where $\text{tang}(\mathcal{F}, C)$ is the sum of orders of tangency between \mathcal{F} and C , *cf.* [Bru00, Prop 2 p. 23].

If $X = \mathbb{P}^2$ is the projective plane, the *degree* of \mathcal{F} is $\deg(\mathcal{F}) \in \mathbb{Z}_{\geq 0}$ defined as the number of tangencies of \mathcal{F} with a general projective line. In this case

$$T^*\mathcal{F} = \mathcal{O}_{\mathbb{P}^2}(\deg(\mathcal{F}) - 1).$$

Birational maps and foliations. Let X and Y be projective surface with at most normal singularities and $\phi : X \dashrightarrow Y$ is a birational map. If we have a foliation \mathcal{F} on X given by the rational vector field \mathcal{X} , we can define a foliation $\phi_*\mathcal{F}$ on Y as the one defined by the rational vector field $\phi_*\mathcal{X}$. Conversely, from a foliation \mathcal{G} on Y , one defines $\phi^*\mathcal{G} := (\phi^{-1})_*\mathcal{G}$. We say that the foliations \mathcal{F} and $\phi_*\mathcal{F}$ are *birationally equivalent* and that $\phi_*\mathcal{F}$ is a (*birational*) *model of \mathcal{F}* .

Remark 2.1. In the case $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is induced by a polynomial R automorphism of \mathbb{C}^2 , if \mathcal{F} is given by a polynomial vector field \mathcal{X} on \mathbb{C}^2 , with isolated zeroes, the condition $\phi^*\mathcal{F} = \mathcal{F}$ is tantamount to $R_*\mathcal{X} = h\mathcal{X}$; for a suitable rational function h . However, as R is a polynomial automorphism, the vector $R_*\mathcal{X}$ is a polynomial vector field on \mathbb{C}^2 , with isolated zeroes. In particular, the factor h is a constant $c \in \mathbb{C}^*$.

Singularities. In a neighborhood of a singular point $p \in X$ with local centered coordinates x, y the foliation \mathcal{F} is defined by a holomorphic vector field $v = f(z, w)\frac{\partial}{\partial z} + g(z, w)\frac{\partial}{\partial w}$; with $f(0, 0) = g(0, 0) = 0$. Denote λ_1, λ_2 the eigenvalues of the linear part (first jet) of $(z, w) \mapsto (f(z, w), g(z, w))$. We say p is a *reduced* singularity of \mathcal{F} if at least one of them, say λ_2 is not zero and if $\lambda := \lambda_1/\lambda_2 \notin \mathbb{Q}_{>0}$; otherwise the singularity is *non-reduced*. A special case of non-reduced singularity occurs when the linear part is the identity, in this case $\lambda = 1$, and we say p is a *radial point*.

If $\lambda \neq 0$ we say that the singularity is *non degenerate*; otherwise we call it a *saddle-node*.

We say p is a *Morse point* if it is non degenerate and, in suitable coordinates, admits a local holomorphic first integral of the form $\phi(z, w) = z^2 + w^2 + \text{h.o.t.}$; note that for Morse points $\lambda = -1$.

Reduction of singularities, relatively minimal models. A foliation \mathcal{F} on a smooth surface is said to be *reduced* if all its singularities are reduced. After Seidenberg [Sei68], foliations on smooth projective surfaces always admit a *reduction of singularities*: a birational morphism $\Sigma : M \rightarrow X$ obtained as a composition of blowing-ups such that $\overline{\mathcal{F}} := \Sigma^*\mathcal{F}$ is a reduced foliation. Such a *reduced model $\overline{\mathcal{F}}$* is not unique. Indeed, by performing a blowing-up at either a non-singular point or a reduced singularity the transformed foliation remains reduced. Doing such an “unnecessary” blowing-up creates a *foliated exceptional curve* or *$\overline{\mathcal{F}}$ -exceptional curve*: a rational curve of self-intersection -1 whose contraction to a point q yields a foliated surface with at most a reduced singularity at q . A reduced model $\overline{\mathcal{F}}$ is called a *relatively minimal model* when it is free of *$\overline{\mathcal{F}}$ -exceptional curves*.

Kodaira dimension. The *Kodaira dimension* $\kappa(\mathcal{F})$ of \mathcal{F} is defined by

$$\kappa(\mathcal{F}) := \limsup_{n \rightarrow +\infty} \frac{1}{\log n} \log h^0((T^*\overline{\mathcal{F}})^{\otimes n}).$$

This is a birational invariant with values in $\{-\infty, 0, 1, 2\}$ which is independent on the particular reduced model. If $\kappa(\mathcal{F}) = 2$, we say \mathcal{F} is of general type. The birational classification of foliations concerns the cases $\kappa(\mathcal{F}) \in \{-\infty, 0, 1\}$.

Zariski decomposition. If \mathcal{F} is not birationally equivalent to a rational fibration, Miyaoka and Fujita's results assure that the cotangent line bundle $T^*\mathcal{F}$ admits a so-called *Zariski decomposition* [Bru00, p. 100]

$$T^*\mathcal{F}' \equiv \mathbf{N} + \mathbf{P},$$

where

- \equiv means numerical equivalence,
- the *positive part* \mathbf{P} is a *nef* \mathbb{Q} -divisor (i.e. $\mathbf{P} \cdot C \geq 0$ for every curve C)
- the *negative part* $\mathbf{N} = \sum_j \alpha_j \mathbf{N}_j$ is a \mathbb{Q}^+ -divisor ($\alpha_j \in \mathbb{Q}^+$) and each connected component of $\cup_j \mathbf{N}_j$ is contractible to a normal singularity.
- $\mathbf{P} \cdot \mathbf{N}_j = 0, \forall j$.

Nef model. If $\overline{\mathcal{F}}$ is a relatively minimal (reduced) model of \mathcal{F} and if $\overline{\mathcal{F}}$ is not a rational fibration, McQuillan's theorem [Bru00, Thm.1, Chap. 8] assures that the support of \mathbf{N} in $T^*\overline{\mathcal{F}}' \equiv \mathbf{N} + \mathbf{P}$ is a union of the so-called *maximal $\overline{\mathcal{F}}$ -chains*.

A $\overline{\mathcal{F}}$ -chain is a chain of invariant rational $(-n)$ -curves, with $n \geq 2$, which starts with a curve containing just one singularity of $\overline{\mathcal{F}}$ and where other components, if it has more than one, contain two singularities, all singularities being reduced non-degenerate. The contraction of a $\overline{\mathcal{F}}$ -chain produces a rational surface singularity, more precisely, a cyclic quotient singularity. The induced foliation on the resulting singular surface is called a *nef model* of $\overline{\mathcal{F}}$.

3. PROOF OF THEOREM B – i)

The birational classification of foliations with $\kappa(\mathcal{F}) = -\infty$ ([Bru03] or [McQ01]) asserts that this class is composed by *rational fibrations* and by foliations birationally equivalent to the so-called *Hilbert modular foliations*. Hence, in order to prove the part (i) of Theorem B, we only need to exclude Hilbert modular foliations.

We recall this notion. A *Hilbert modular surface* is defined (following [Bru03, p. 25]) as a (possibly singular) projective surface Y containing a (possibly empty) curve $C \subset Y \setminus \text{Sing}(Y)$ such that:

- i) each connected component of C is a cycle of smooth rational curves, contractible to a normal singularity;

ii) $Y \setminus C$ is uniformised by the bidisc $\mathbb{H} \times \mathbb{H}$, i.e. we have an isomorphism of analytic spaces

$$Y \setminus C \simeq Y_\Gamma := (\mathbb{H} \times \mathbb{H})/\Gamma$$

where Γ is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R}) \subset \mathrm{Aut}(\mathbb{H} \times \mathbb{H})$;

iii) Γ is irreducible (i.e. does not contain a finite index subgroup of the form $\Gamma_1 \times \Gamma_2$ with $\Gamma_j \subset \mathrm{PSL}(2, \mathbb{R})$, $j = 1, 2$).

The natural singular foliations of the Hilbert modular surface Y which come from the horizontal and vertical foliations by discs of $\mathbb{H} \times \mathbb{H}$ are called *Hilbert modular foliations*. Both foliations leave invariant the curve C and, in the desingularization of the surface, they leave invariant the exceptional divisors.

Theorem $B - i$) follows from the following result.

Proposition 3.1. *Let \mathcal{F} be an algebraic foliation of \mathbb{P}^2 such that $\mathcal{F}|_{\mathbb{C}^2}$ has no invariant rational curve. Then \mathcal{F} is not birationally equivalent to a Hilbert modular foliation.*

Proof. Suppose by contradiction that \mathcal{F} is birationally equivalent to a Hilbert modular foliation. Let $\overline{\mathcal{F}}$ be a relatively minimal model of \mathcal{F} on a suitable smooth projective surface M . Then M is the minimal desingularization of a Hilbert modular surface Y (cf. [Bru00, Th. 1 p. 75]) and $\overline{\mathcal{F}}$ is the transform of a Hilbert modular foliation on Y .

The hypothesis that $\mathcal{F}|_{\mathbb{C}^2}$ has no invariant rational curve implies that there are no cycles of $\overline{\mathcal{F}}$ -invariant rational curves on M . Then the Hilbert modular surface is

$$Y = Y_\Gamma = \mathbb{H} \times \mathbb{H}/\Gamma,$$

for a discrete cocompact (hence lattice) irreducible subgroup Γ of $(\mathrm{PSL}(2, \mathbb{R}))^2$.

Let

$$U := \mathbb{P}^2 \setminus (L_\infty \cup \mathrm{Sing}(\mathcal{F})),$$

where $\mathrm{Sing}(\mathcal{F})$ is the singular locus of \mathcal{F} . Then U is isomorphic to a non-empty Zariski open set of M' and also U is isomorphic to a non-empty Zariski open set of \dot{Y}_Γ , the complement of the quotient singularities of Y_Γ . The morphism

$$\pi_1(U) \rightarrow \pi_1(\dot{Y}_\Gamma)$$

induced by the injection is surjective. As U is simply connected, so is \dot{Y}_Γ .

By irreducibility of Γ , the subset $F \subset \mathbb{H} \times \mathbb{H}$ given by the points that have non trivial stabilizer under Γ is discrete (cf. [Shi63]). The action of Γ on $\mathbb{H} \times \mathbb{H}$ can be restricted to an action on $(\mathbb{H} \times \mathbb{H}) \setminus F$. The quotient map

$$(\mathbb{H} \times \mathbb{H}) \setminus F \rightarrow [(\mathbb{H} \times \mathbb{H}) \setminus F]/\Gamma = \dot{Y}_\Gamma$$

is then a covering, in the strict sense of topologists. As $(\mathbb{H} \times \mathbb{H}) \setminus F$ is simply connected, this proves that the fundamental group of \dot{Y}_Γ is isomorphic to Γ . But \dot{Y}_Γ is simply connected, so Γ is trivial, a contradiction. \square

This result is quite sharp, a Hilbert modular foliation with exactly one invariant rational curve in \mathbb{C}^2 is given in [MP05].

4. PROOF OF THEOREM $B - ii)$

Now \mathcal{F} denotes a foliation of $\mathbb{P}^2 = \mathbb{C}^2 \cup L_\infty$ such that its restriction $\mathcal{F}|_{\mathbb{C}^2}$ to \mathbb{C}^2 does not admit invariant rational curves and has at most reduced singularities.

Remark 4.1. Recall that a degree zero foliation of the projective plane is a pencil of straight lines. Also recall that a degree one foliation of the projective plane has at least two invariant lines [Bru00, 2) p. 27], so at least one invariant affine line.

After Theorem $B - i)$, it suffices to exclude the possibility $\kappa(\mathcal{F}) = 0$. By contradiction we suppose

$$\kappa(\mathcal{F}) = 0.$$

If \mathcal{F} is a reduced foliation of the projective plane, then $\kappa(\mathcal{F}) = 0$ implies $T_{\mathcal{F}}^* = \mathcal{O}$ and $\deg(\mathcal{F}) = 1$, which contradicts Remark 4.1. Therefore, we have a non-reduced singularity $p \in L_\infty$. Denote $p_1, \dots, p_k \in L_\infty$ the non reduced singularities of \mathcal{F} .

Let $\Sigma : M \rightarrow \mathbb{P}^2$ be a reduction of singularities of \mathcal{F} composed by blowing-ups of points *affecting only non-reduced singularities* (in order to not introduce unnecessary foliated exceptional curves). Denote the reduced foliation in M by $\overline{\mathcal{F}}$. Denote by E_{p_i} the exceptional line of the *first blowing-up* affecting p_i , $i = 1, \dots, k$, and by \overline{E}_p the corresponding strict transform in M . Note that the reduction of singularities Σ may include additional blowing-ups.

Now let us denote by

$$q : M \rightarrow M'$$

a (finite, possibly trivial) sequence of blowing downs of foliated exceptional curves (and only them) and by

$$\overline{\mathcal{F}}' := q_* \overline{\mathcal{F}}$$

the foliation obtained in M' , which is a relatively minimal model of \mathcal{F} in M' .

Claim 4.2. *There exists a non reduced singularity $p \in L_\infty$ of \mathcal{F} such that the pencil of lines of \mathbb{P}^2 passing through p defines a fibration $\pi' : M' \rightarrow \mathbb{P}^1$.*

Proof. If $q : M \rightarrow M'$ is the identity, then any p_i verifies the required property. Then we assume $q \neq id$. We assert that q starts by contracting the strict transform $\overline{L}_\infty \subset M$ of the line at infinity L_∞ . To justify this, we first remark that \overline{E}_{p_i} cannot be a $\overline{\mathcal{F}}$ -exceptional curve. Indeed, if the effect of Σ on p_i is just one blowing-up, say σ , then $\overline{E}_{p_i} = E_{p_i}$ and this is not $\sigma^* \mathcal{F}$ -exceptional (remember that p_i is not reduced). If the effect of Σ on p_i includes extra blowing-ups, then the self-intersection of the (-1) -curve E_{p_i} decreases to ≤ -2 and so it cannot correspond to a $\overline{\mathcal{F}}$ -exceptional curve.

In particular L_∞ is \mathcal{F} -invariant and that line is affected by exactly two blowing-ups. After the first contraction is done, possibly new foliated exceptional curves are created and contracted by q in a domino effect (compare Example 6.5).

We distinguish two cases:

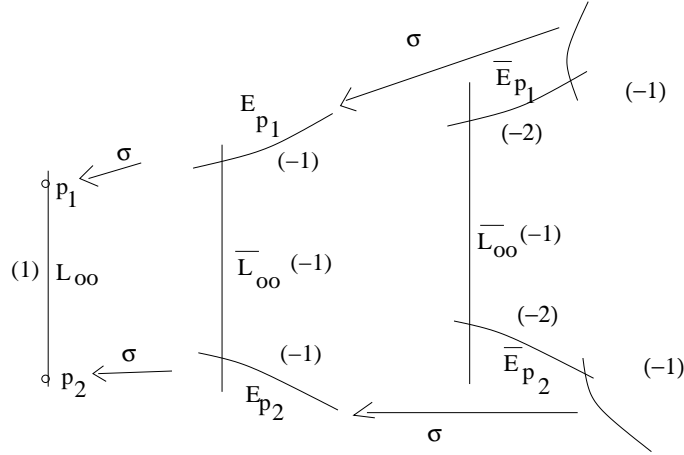
- Case 1: there are exactly two non-reduced points of \mathcal{F} in L_∞ , *i.e.* $k = 2$;
- Case 2: there is exactly one non-reduced point of \mathcal{F} in L_∞ , *i.e.* $k = 1$.

In both cases, if some $\overline{E_{p_i}}$ (resp. $\overline{E_{p_1}}$) is not transformed into a foliated exceptional curve, and so contracted by q , then $p = p_i$ is the required point.

Finally, we assume that every $\overline{E_{p_i}}$ is contracted and show it eventually gives a contradiction.

On Case 1:

After E_{p_1} and E_{p_2} have been introduced no extra blowing-up composing Σ affects the line at infinity. Then $\overline{E_{p_1}}$ and $\overline{E_{p_2}}$ become (-2) -curves after blowing up points in E_{p_1} and E_{p_2} . Next figure illustrates the situation.



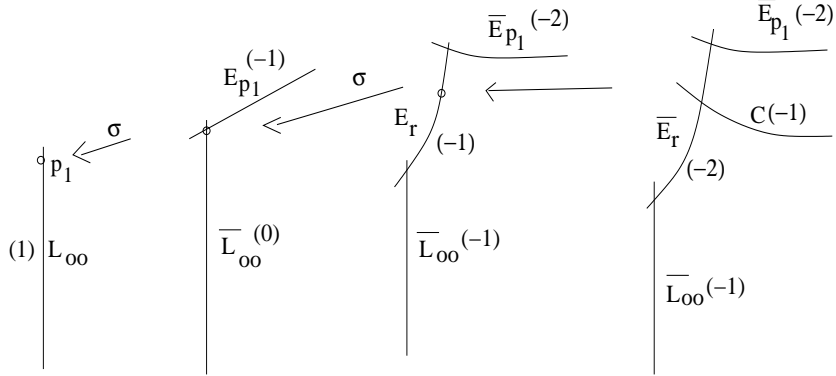
By contracting first $\overline{L_\infty}$ and then one of the $\overline{E_{p_i}}$, we see the other one becomes a (0) -curve: contradiction.

On Case 2:

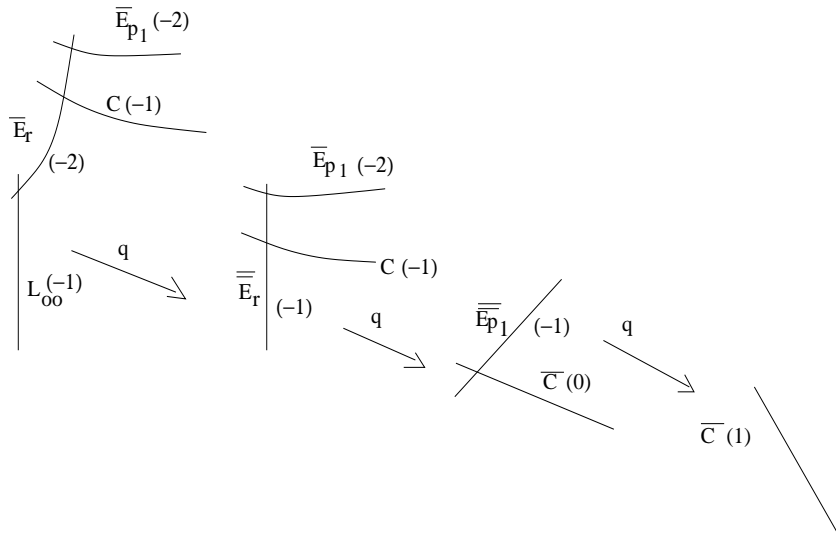
In this case one must blow up an infinitely near point of L_∞ in E_{p_1} , *r* say. Then $\overline{E_{p_1}}$ is a $(-n)$ -curve with $n \geq 2$ which intersects the strict transform $\overline{E_r}$ in M of the exceptional line E_r associates to r .

For q to contract $\overline{E_{p_1}}$, we need $n = 2$ and E_r to be $\overline{\mathcal{F}}$ -invariant. Moreover, since q contracts $\overline{L_\infty}$ then at least one of the blowing-ups composing Σ is done at a point $t \in E_r$. So $\overline{E_r}$ is a (-2) -curve which is part of a chain of curves contracted by q .

Next Figure illustrates the situation.



Hence q contracts \overline{L}_∞ , \overline{E}_r and \overline{E}_{p_1} , and transforms C into a curve \overline{C} of self-intersection 1 (see next figure).



The birational map

$$\phi := (q \circ \sigma^{-1}) : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$$

is a quadratic Cremona transformation with a unique “proper” indeterminacy point in L_∞ . Since $\kappa(\mathcal{F}) = \kappa(\overline{\mathcal{F}}') = 0$ and $\overline{\mathcal{F}}'$ is reduced we get $\deg(\overline{\mathcal{F}}') = 1$. In particular $\overline{\mathcal{F}}'$ has two invariant projective lines, by Remark 4.1. But the restriction

$$\phi|_{\mathbb{C}^2} = \phi|_{\mathbb{P}^2 \setminus L_\infty}$$

is an isomorphism sending $\mathcal{F}|_{\mathbb{C}^2}$ to $\overline{\mathcal{F}}'|_{\mathbb{P}^2 \setminus \overline{C}}$, which contradicts that $\mathcal{F}|_{\mathbb{C}^2}$ has no invariant rational curve. \square

Consider the fibration $\pi' : M' \rightarrow \mathbb{P}^1$ given by Claim 4.2.

Claim 4.3. *The foliation $\overline{\mathcal{F}}'$ is a Riccati foliation, with π' an adapted fibration.*

Proof. Denote p the point associated to π' in Claim 4.2. Let L be a general fiber of π' . We must prove $\text{tang}(L, \overline{\mathcal{F}}') = 0$. By contradiction, we suppose $\text{tang}(L, \overline{\mathcal{F}}') > 0$.

On one hand, we know $\overline{\mathcal{F}}'$ is not a rational fibration. Hence we have a Zariski decomposition (cf. Section 2)

$$T^*\overline{\mathcal{F}}' \equiv \mathbf{N} + \mathbf{P}.$$

Since $\overline{\mathcal{F}}'$ is a relatively minimal model we know that the support of $\mathbf{N} = \sum \alpha_j \mathbf{N}_j$ is a union of maximal $\overline{\mathcal{F}}$ -chains. Since we have $\kappa(\mathcal{F}) = 0$, from McQuillan's theorem [Bru00, Thm.2, Chap. 9] we deduce

$$\mathbf{P} \equiv 0.$$

On the other hand, every $\overline{\mathcal{F}}'$ -invariant rational curve either coincides with the strict transform of L_∞ in M' (if this curve was not collapsed by q) or is the strict transform of some exceptional component of Σ . Note that such a component intersects L if and only if it equals $q(\overline{E_p})$.

Since $T^*\overline{\mathcal{F}}' \cdot L = \mathbf{N} \cdot L \geq 1$ then $\mathbf{N}_{j_0} = q(\overline{E_p})$ for some j_0 , so $\mathbf{N} \cdot L = \alpha_{j_0}$. But according to [Bru00, pp. 109 – 110], one has $0 < \alpha_{j_0} < 1$; yielding $0 < \text{tang}(\overline{\mathcal{F}}', L) < 1$. This contradicts the integrality of tang . \square

Let $\sigma : (X, \mathcal{R}) \rightarrow (\mathbb{P}^2, \mathcal{F})$ be the blowing-up of the point given by Claim 4.2, denote $\pi : X \rightarrow \mathbb{P}^1$ the ruling of X , E the exceptional divisor and L_∞^σ the strict transform of L_∞ . For convenience, we normalize π to have $\pi^{-1}([0 : 1]) = L_\infty^\sigma$.

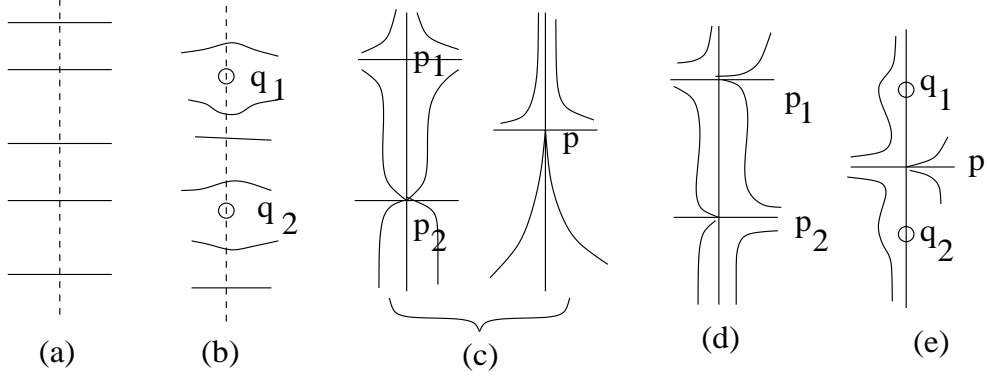
Claim 4.4. *The foliation \mathcal{R} is a Riccati foliation, the fibration π is adapted to \mathcal{R} .*

Proof. We have a factorization $\Sigma = \sigma \circ \Sigma_0$. Let F be a fiber of π . If L is sufficiently general, it possesses a neighborhood on which $q \circ \Sigma_0^{-1}$ is a biholomorphic map. By Claim 4.3, we conclude $\text{tang}(L, \mathcal{R}) = 0$. \square

Claim 4.5. *The only possible \mathcal{R} -invariant rational curves in X are E and L_∞^σ .*

Proof. Recall that $X \setminus (L_\infty^\sigma \cup E)$ is isomorphic to \mathbb{C}^2 via σ , so that no other rational curve is \mathcal{R} -invariant. \square

After Brunella [Bru03], by a rational \mathbb{P}^1 -bundle map (over the identity of the basis), biholomorphic outside the fiber L_∞^σ , we transform the Riccati foliation (X, \mathcal{R}, π) into a nef Riccati foliation $(X_0, \mathcal{R}_0, \pi_0)$. The wording “nef Riccati foliation” means the local structure of \mathcal{R}_0 around any fiber of π_0 is of one of the types (a), (b), (c), (d), (e) below (Figure adapted from [Bru03, p. 20]).



- (a) the foliation is transverse to π and S is smooth;
- (b) the surface S admits two quotient singular points q_1, q_2 of the same order $k \geq 2$ and the fiber is not invariant by the foliation;
- (c) there are two possibilities: either p_1, p_2 are non degenerate singularities, or p is an unique saddle-node with Milnor number 2 whose strong separatrix is transverse to the vertical fiber, S being smooth in both cases;
- (d) there are two saddle-nodes with Milnor number m whose strong separatrix is given by the vertical fiber, S being smooth;
- (e) the surface S admits two quotient singularities with same order equal to 2 and \mathcal{F}_{nef} admits a saddle node with Milnor number m whose strong separatrix is in the vertical fiber.

Claim 4.6. *i) The fiber $\pi_0^{-1}([0 : 1])$ is an \mathcal{R}_0 -invariant fiber, of type (d), with multiplicity $m = 2$.
 ii) The bundle $T^*\mathcal{R}_0$ is trivial.*

Proof. By Claim 4.5, $\pi_0^{-1}([0 : 1])$ is the only possibly invariant fiber. We review the possible types for $F := \pi_0^{-1}([0 : 1])$. If F is of type (a), the foliation \mathcal{R}_0 is everywhere transversal to the rational fibration and trivializes the corresponding \mathbb{P}^1 -bundle. The basis of the fibration being \mathbb{P}^1 , \mathcal{R}_0 should be a rational pencil, which is impossible.

As all the other fiber are of type (a) and $\mathbb{P}^1 \setminus \{[0 : 1]\}$ is simply connected, the fiber F has trivial local monodromy. The fibers of type (b) or (c) have non-trivial monodromy. Consequently F cannot be of these types.

To consider the last two cases (d) and (e), we recall the following formula for the degree of the \mathbb{Q} -divisor $\pi_{0*}(T^*\mathcal{R}_0)$ (see [Bru03, p. 20]):

$$\deg(\pi_{0*}(T^*\mathcal{R}_0)) = \begin{cases} -2 + \frac{m+1}{2} & \text{in case (e)} \\ -2 + m & \text{in case (d)} \end{cases} \quad (2)$$

There is a correspondence between $\deg(\pi_{0*}(T^*\mathcal{R}_0))$ being negative, zero or positive and $\kappa(\mathcal{F}) = -\infty, 0, 1$, respectively.

Since in our case $\deg(\pi_*(T^*\mathcal{R}_0)) = 0$, in the case (e) we obtain $m = 3$, an odd number, contradicting [Bru00, p. 56].

In case (d) we obtain Milnor number $m = 2$ for both saddle-nodes. We have just proved i). For ii), we use the formula for the cotangent bundle in [Bru00, p. 57] and obtain

$$T^*\mathcal{R}_0 = \pi^*\mathcal{O}_{\mathbb{P}^1}(-2) + \mathcal{O}_{X_0}(2F) = \mathcal{O}_{X_0}.$$

□

The surface X_0 is a smooth rational surface equipped with a regular ruling π_0 , it is therefore a Hirzebruch surface $X_0 = \mathbb{F}_n$ with $n \geq 0$.

Claim 4.7. i) *The fibration π_0 is not the trivial \mathbb{P}^1 -bundle over \mathbb{P}^1 .*
ii) *The negative section is \mathcal{R}_0 invariant, it is the transform of $E_p \subset X$ in X_0 .*

Proof. i). Suppose by contradiction $X_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Call vertical the fibration π_0 adapted to the Riccati foliation, the second ruling is termed horizontal. Denote by H_1, H_2 the two horizontal lines passing through the two saddle-nodes in $\pi_0^{-1}([0 : 1])$. One of them, say H_1 , is not \mathcal{R}_0 -invariant. Passing by a singularity of a foliation produces a tangency, so

$$\begin{aligned} 1 &\leq \text{tang}(\mathcal{R}_0, H_1) \\ &= \text{tang}(\mathcal{R}_0, H_1) - H_1 \cdot H_1 \\ &= T^*\mathcal{R}_0 \cdot H_1, \end{aligned}$$

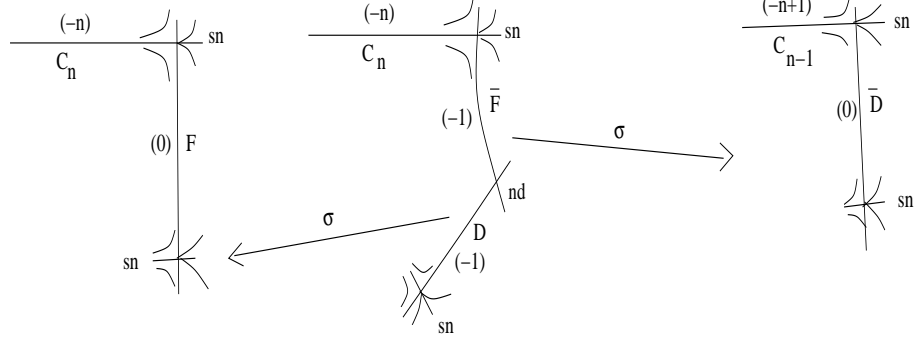
contradicting the triviality of $T^*\mathcal{R}_0$.

ii). If the negative section s is not invariant, using the tangency formula as above, we get $0 \leq \text{tang}(\mathcal{R}_0, s) = s \cdot s$; contradiction. Claim 4.5 says that the only possible invariant section is the transform of E_p in X_0 , whence the conclusion. □

Claim 4.8. *By rational \mathbb{P}^1 -bundle maps (over the identity of the basis), biholomorphic outside the fiber over $[0 : 1]$, we can transform $(X_0, \mathcal{R}_0, \pi_0)$ in $(X_1, \mathcal{R}_1, \pi_1)$, a nef Riccati foliation with $T^*\mathcal{R}_1 = \mathcal{O}_{X_1}$ and X_1 a Hirzebruch surface with a (-1) -section invariant by \mathcal{R}_1 .*

Proof. This is performed by induction, using the following flip: start by blowing up the saddle-node on F which lies outside the negative section; denote by D the exceptional curve produced by this. The intersection point of D with the strict transform \overline{F} of F is a Morse point. Then we contract \overline{F} obtaining a Hirzebruch surface on which D becomes an invariant fiber, again with two saddle-nodes of Milnor number 2, exactly as before, see the picture below. The self-intersection of the invariant section diminishes of 1 in this process. As the multiplicity of the

fiber of type (d) does not change, the tangent bundle remains trivial.



□

Now denote by \mathcal{F}_0 the foliation of \mathbb{P}^2 obtained from \mathcal{R}_1 by contracting the (-1) -curve in X_1 to a point $r \in \mathbb{P}^2$. The sequence of transformations $\mathbb{P}^2 \xrightarrow{\sigma^{-1}} X \rightarrow X_0 \rightarrow X_1 \rightarrow \mathbb{P}^2$, extends an automorphism of \mathbb{C}^2 , so that \mathcal{F}_0 should have no invariant rational curve in \mathbb{C}^2 . We reach a contradiction and conclude the proof of Theorem $B - ii)$ by the conjunction of Remark 4.1 and next claim.

Claim 4.9. *The foliation \mathcal{F}_0 is a degree one foliation of \mathbb{P}^2 .*

Proof. Let δ be a general line in \mathbb{P}^2 , denote $\bar{\delta}$ its strict transform in X_1 . If δ does not pass through r , we have $\text{tang}(\mathcal{R}_1, \bar{\delta}) = \text{tang}(\mathcal{F}_0, \delta)$. The first member is

$$\text{tang}(\mathcal{R}_1, \bar{\delta}) = \bar{\delta} \cdot \bar{\delta} + T^*\mathcal{R}_1 \cdot \bar{\delta} = 1 + 0 = 1$$

and the second is $\text{deg}(\mathcal{F}_0)$. Whence the conclusion. □

5. PROOF OF THEOREM $B - iii)$

In this section we assume $\kappa(\mathcal{F}) = 1$. According to the birational classification of foliations [Bru00, Th. 1. p. 118], and taking into account that \mathcal{F} is not birationally conjugate to a fibration, we know that \mathcal{F} is either a Riccati foliation or a *turbulent foliation*; recall that the definition of turbulent foliation is obtained from the one of Riccati foliation by replacing “rational fibration” with “elliptic fibration” (see page 2).

Therefore the proof of Theorem $B - iii)$ is equivalent to the exclusion of the turbulent case.

This will be done in Proposition 5.3, using the notion of a *transversely affine* foliation. Consider a foliation \mathcal{F} on a surface X given by $\omega = 0$ where ω is a rational 1-form on X . We say \mathcal{F} is *transversely affine* if there exists a *closed* rational 1-form η such that

$$d\omega = \omega \wedge \eta.$$

Remark 5.1. If $\tilde{\omega} = g\omega$ is another 1-form defining \mathcal{F} , then $\tilde{\eta} := \eta - dg/g$ is closed and satisfies $d\tilde{\omega} = \tilde{\omega} \wedge \tilde{\eta}$, so that this definition is independent of the defining 1-form ω .

The following has already been noticed in [Per03, Prop. 22], we give a slightly different proof.

Proposition 5.2. *Every turbulent foliation is a transversely affine foliation.*

Proof. As transverse affine structures may be transported by birational transformations, stable reduction [Bru00, Prop. 6 p. 69] and [CLNL⁺07, Th. 2.21 p. 37] reduce the proof to the case where the foliation \mathcal{F} is transverse to the general fiber of an elliptic fiber bundle $\pi : X \rightarrow C$.

Let X^* be the complement of the set of invariant fibers, and $F = \pi^{-1}(b) \subset X^*$ a fiber. Using the foliation to identify nearby fibers, we obtain a multiform submersion $\widetilde{X}^* \rightarrow F$ which defines the foliation; it lifts to a submersion $f : \widetilde{X}^* \rightarrow \mathbb{C}$ to the universal cover of $F \simeq \mathbb{C}/\Lambda$. By construction the monodromy group of f fixes the lattice Λ and must lie in $\text{Aff}(\mathbb{C})$. Hence the monodromy of df is linear (contained in \mathbb{C}^*). In particular, if v is a rational vector field on X which is not tangent to \mathcal{F} , the meromorphic function

$$g = df(v) : \widetilde{X}^* \rightarrow \mathbb{C}$$

has the same monodromy as df and

$$\omega := \frac{df}{g}$$

is a well defined meromorphic 1-form on X^* , tangent to \mathcal{F} . We have

$$d\omega = -\frac{df \wedge dg}{g^2} = \omega \wedge \eta,$$

with $\eta = -\frac{dg}{g}$ a well defined closed meromorphic one form on X^* .

It remains to show that the pair (ω, η) extends meromorphically in the neighborhood of any \mathcal{F} -invariant fiber of π . Let $U \simeq \mathbb{D} \times F$ be such a neighborhood, \mathbb{D} a disc. Let $(z, w) \in \mathbb{D} \times \mathbb{C}$ represent the elements of U , $z = 0$ corresponding to the invariant fiber. We have a local equation of the form

$$dw = \frac{dz}{A(z)},$$

for \mathcal{F} , with $A(z)$ holomorphic in \mathbb{D} . Let b be a point in \mathbb{D} , if the coordinate w is well chosen, in $\mathbb{D}^* \times F$, the submersion f expresses as

$$f(z, w) = w - \int_b^z \frac{ds}{A(s)}$$

and

$$df = dw - \frac{dz}{A(z)}$$

is meromorphic at $z = 0$, so as g ; we have the required extension property. \square

Proposition 5.3. *Let \mathcal{F} be a foliation on $\mathbb{P}^2 \simeq \mathbb{C}^2 \cup L_\infty$. If \mathcal{F} is a turbulent foliation of Kodaira dimension 1, then it possesses an invariant curve outside L_∞ .*

Proof. By contradiction, suppose \mathcal{F} possesses no invariant algebraic curve in $\mathbb{C}^2 = \mathbb{P}^2 \setminus L_\infty$.

By Proposition 5.2, \mathcal{F} is transversely affine. The assumption on non existence of invariant curves allows to use [CP14, Corollary B], to infer that \mathcal{F} is given by the pullback ω of a 1-form

$$\omega_0 = dy + (a(x) + b(x)y) dx, \quad a, b \in \mathbb{C}[x]$$

under a polynomial map $\mathbb{C}^2 \rightarrow \mathbb{C}^2$, which extends as a rational map $H_0 : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$.

Denote \mathcal{G} the foliation of \mathbb{P}^2 induced by ω_0 . From its equation, observe that $\mathcal{G}|_{\mathbb{C}^2}$ has no singularities. As $\mathcal{F}|_{\mathbb{C}^2}$, $\mathcal{G}|_{\mathbb{C}^2}$ possesses no algebraic invariant curves.

There exist sequences of blowing-ups $\Sigma_X : X \rightarrow \mathbb{P}^2$, $\Sigma_Y : Y \rightarrow \mathbb{P}^2$ in the source and the target of H_0 such that

- The foliations $\mathcal{F}_{red} := \Sigma_X^* \mathcal{F}$ and $\mathcal{R} := \Sigma_Y^* \mathcal{G}$ have at most reduced singularities.
- There exists an elliptic fibration $f_X : X \rightarrow \mathbb{P}^1$ adapted to the turbulent foliation \mathcal{F}_{red} .
- There exists a rational fibration $f_Y : Y \rightarrow \mathbb{P}^1$ adapted to the Riccati foliation \mathcal{R} .
- The rational map $H : X \dashrightarrow Y$ such that $\Sigma_Y \circ H = H_0 \circ \Sigma_X$ is actually a morphism (*i.e.* holomorphic).

By the already proved items *i)* and *ii)* of Theorem B, we must have $\kappa(\mathcal{R}) \geq 1$. As for every Riccati foliation $\kappa(\mathcal{R}) \leq 1$, we have

$$\kappa(\mathcal{R}) = 1.$$

By Lemma 5.4 below, q_Y is the Iitaka fibration ([Bru00, p.116]) of the cotangent divisor $T^*\mathcal{R}$. Similarly q_X is the Iitaka fibration of the cotangent divisor $T^*\mathcal{F}_{red}$.

From the remark in [Bru00, p. 29] it follows $T^*\mathcal{F}_{red} = H^*(T^*\mathcal{R}) \otimes \mathcal{O}_X(D)$ for D an effective divisor on X (see also the proof of [Tou03, Lemme 3.2.8])

Then, Lemma 5.5 below yields that H maps the fibers of f_X in the fibers of f_Y : for general $c \in \mathbb{P}^1$, there exists $r(c) \in \mathbb{P}^1$ such that $H(f_X^{-1}(c)) \subset f_Y^{-1}(r(c))$.

Consider, for general c , the following restriction of H ,

$$H_c : f_X^{-1}(c) \rightarrow f_Y^{-1}(r(c)).$$

Denote $R \subset X$ and $B \subset Y$ the ramification and branching curves of H , namely

$$R := \{x \in X; \det(d_x H) = 0\}, \quad B := H(R).$$

The map H_c is étale outside R .

Let \overline{L}_∞ be the strict transform of L_∞ in the sequence of blowing-ups p_Y . If B has \mathcal{R} -invariant components, they must be contained in \overline{L}_∞ or in the exceptional divisor of p_Y , because \mathcal{G} possesses no invariant curve in \mathbb{C}^2 . Denote B_{inv} the union of these components. Notice that the general fiber of f_Y intersects B_{inv} at most once, because f_Y is induced by the coordinate fibration x on \mathbb{C}^2 .

We Assert that *for a general c , the curve $f_Y^{-1}(r(c))$ does not intersect $B \setminus B_{inv}$.*

After proving this assertion we obtain that, for general c , the map H_c ramifies at most over one point of $f_Y^{-1}(r(c)) \simeq \mathbb{P}^1$, contradicting that $f_X^{-1}(c)$ is elliptic.

We conclude by proving this Assertion. First, remark that any non \mathcal{R} -invariant component of B is a curve transverse to the general fiber of f_Y and has a finite number of tangencies with \mathcal{R} . Therefore we have, for general c :

- (1) For any point $p' \in (f_Y^{-1}(r(c)) \setminus B_{inv}) \cap B$, B is transverse to both $f_Y^{-1}(r(c))$ and \mathcal{R} at p' ;
- (2) for any point $p \in f_X^{-1}(c) \cap R$, H writes as $(s, t) \mapsto (S, T) = (s^\ell, t)$, with $\ell > 1$, in suitable local coordinates centered at p .

Take c such that we have (1) and (2). Suppose we have a point $p' = H(p)$ in $f_Y^{-1}(r(c)) \cap B \setminus B_{inv}$. In the adapted coordinates (S, T) of (2), the leaf of \mathcal{R} through p' has a local equation $T = \lambda_1 S + o(S)$, $\lambda_1 \in \mathbb{C}$ and the fiber $f_Y^{-1}(r(c))$ passing through p' expresses as $T = \lambda_2 S + o(S)$, $\lambda_2 \in \mathbb{C}$. Thus in the neighborhood of p , their pull-backs have equation $t = \lambda_i s^\ell + o(s^\ell)$ and are tangent at $(s, t) = (0, 0)$ because $\ell > 1$. Meaning p is a tangency point between \mathcal{F}_{red} and $f_X^{-1}(c)$.

As f_X is an adapted fibration for \mathcal{F}_{red} , this cannot happen for c general enough. \square

For the reader's convenience, we prove two facts that belong to the birational theory of foliations and varieties.

Lemma 5.4. *Let \mathcal{F} be a reduced foliation on a projective manifold X , with $\kappa(\mathcal{F}) = 1$. Suppose \mathcal{F} is a Riccati or a turbulent foliation, with adapted fibration $f : X \rightarrow C$. Then f is the Iitaka fibration of $T^*\mathcal{F}$.*

Proof. Let F be the general fiber for a fibration f adapted to \mathcal{F} . Lemma 5.5 shows that the Iitaka fibration associated to F is the fibration f . The proof of [Bru00, Theorem 1 p. 118] shows $T^*\mathcal{F}^{\otimes m} = \mathcal{O}(nF + D)$ for an effective divisor D and suitable integers $m, n > 0$. Lemma 5.5 (with $\mathcal{L} = id_X$) allows to deduce that both divisors F and $T^*\mathcal{F}$ have the same Iitaka fibration, yielding the conclusion. \square

In our context, next Lemma should be applied in the case of (foliated) Kodaira dimension 1.

Lemma 5.5. *Let $\mathcal{L} : X_1 \rightarrow X_2$ be a morphism between projective manifolds. Let D_1, D_2 be divisors on X_1 and X_2 , respectively. Suppose these divisors have equal positive Iitaka dimension. Take $k > 0$ big enough so that $p_i : X_i \dashrightarrow \mathbb{P}\Gamma(X_i, \mathcal{O}(D_i)^{\otimes k})^*$ is the Iitaka fibration of D_i , $i = 1, 2$. Suppose $D_1 = \mathcal{L}^*D_2 + D$ with D effective. Let $r \circ q$ be the Stein factorization of $p_2 \circ \mathcal{L}$. Then q is the Iitaka fibration of D_1 .*

Proof. Choosing a nontrivial global section $s \in \Gamma(X_1, \mathcal{O}(D))$ we have an injection

$$\phi_k : \Gamma(X_2, \mathcal{O}(D_2)^{\otimes k}) \rightarrow \Gamma(X_1, \mathcal{O}(D_1)^{\otimes k})$$

$$\sigma \mapsto (\mathcal{L}^*\sigma) \otimes s^{\otimes k}$$

and the following diagram commutes, with ϕ_k^* onto.

$$\begin{array}{ccc} X_1 - \frac{p_1}{-} \succ \mathbb{P}\Gamma(X_1, \mathcal{O}(D_1)^{\otimes k})^* & & \\ \mathcal{L} \downarrow & & \downarrow \phi_k^* \\ X_2 - \frac{p_2}{-} \succ \mathbb{P}\Gamma(X_2, \mathcal{O}(D_2)^{\otimes k})^* & & \end{array}$$

Restricting the maps, with $S = p_1(X_1)$, $T = p_2(X_2)$, we get the following.

$$\begin{array}{ccc} X_1 - \frac{p_1}{-} \succ S & & \\ \mathcal{L} \downarrow & & \downarrow \\ X_2 - \frac{p_2}{-} \succ T & & \end{array}$$

The map $S \rightarrow T$ is onto. As $\dim S = \dim T$, it must be a generically finite map. Uniqueness in Stein factorization yields $q = g \circ p_1$ for some birational map $g : S \rightarrow S'$. This yields the conclusion, because the Iitaka fibration is defined only up to birational transforms in the target. \square

To explain the limits of Proposition 5.3, we present a turbulent foliation with $\kappa(\mathcal{F}) = 1$ having exactly one rational invariant curve in \mathbb{C}^2 .

Example 5.6. Consider the pencil of cubics \mathcal{E} in the projective plane generated by the cuspidal cubic

$$C : y^2 + x^3 = 0.$$

and the line at infinity L_∞ taken with multiplicity 3, whose general element is an elliptic curve. Also consider the pencil of rational cuspidal cubics (\mathcal{C}_λ) in the projective plane

$$\mathcal{C}_\lambda : y^2 = \lambda x^3.$$

For \mathcal{E} there is a unique indeterminacy point $q \in L_\infty$, which is an inflexion point for the general elements of the cubic pencil. After nine blowing-ups at q and at suitable infinitely near points we obtain a minimal elliptic fibration $\pi : M \rightarrow \mathbb{P}^1$, having exactly two singular fibers. One fiber is the strict transform of the cuspidal cubic (type *II* in Kodaira's notation) and the other fiber has nine rational components (type *II**): the strict transforms \overline{L}_∞ of L_∞ and $\overline{E}_1, \overline{E}_2, \dots, \overline{E}_8$ of the exceptional lines of the first eight blowing-ups. Denote by M' the surface obtained from M after three blowing-ups, first at the point coming from $(0, 0) \in \mathbb{C}^2$, which produces the strict transform \overline{C} of C , and the other ones in order to separate \overline{C} from its tangent line.

For (\mathcal{C}_λ) there are two base points, one (also) at $q \in L_\infty$ and the other at $(0, 0) \in \mathbb{C}^2$. Three blowing-ups at infinity are enough to separate the cuspidal cubics of (\mathcal{C}_λ) at infinity. And two additional blowing-ups at points infinitely near to $(0, 0)$ are enough to produce a rational fibration. Therefore the surface M' is endowed with both: *i*) a rational fibration obtained from (\mathcal{C}_λ) and *ii*) a non-minimal elliptic fibration.

Consider now the degree 4 foliation \mathcal{F} on \mathbb{P}^2 associated to the 1-form on \mathbb{C}^2 given by

$$\omega = d(y^2 + x^3) + (y^2 + x^3)(3ydx - 2xdy).$$

Note that \mathcal{F} leaves invariant L_∞ . From [Lor94, Lemme IV.2] it follows that the unique algebraic leaves of \mathcal{F} are C and L_∞ . We have

$$\omega \wedge d(y^2 + x^3) = 6(y^2 + x^3)^2 dx \wedge dy,$$

which means \mathcal{F} and \mathcal{E} are tangent exactly along $C \cup L_\infty$. The transform \mathcal{F}' of \mathcal{F} in M' is then a turbulent foliation whose adapted elliptic fibration is obtained from \mathcal{E} .

On the other hand, an explicit computation shows that \mathcal{F}' is tangent to the rational fibration of M' along the (-2) -curve $\overline{E}_3 \subset M'$. Hence, if F_r is a general fiber of the rational fibration,

$$T^*\mathcal{F}' \cdot F_r = \text{tang}(\mathcal{F}, F_r) - F_r \cdot F_r = \text{tang}(\mathcal{F}', F_r) > 0.$$

We assert that $\kappa(\mathcal{F}) = 1$. Otherwise, $T^*\mathcal{F}' \equiv \mathbf{N}$ is supported on the union of \mathcal{F}' -chains. More precisely, there are two \mathcal{F}' -chains consisting of the following two chains of (-2) -curves:

$$\overline{L}_\infty, \overline{E}_1, \overline{E}_2 \text{ and } \overline{E}_4, \dots, \overline{E}_8.$$

We then check $T^*\mathcal{F}' \cdot F_r = \mathbf{N} \cdot F_r = 0$: contradiction.

6. PROOF OF THEOREM $B - iv$) AND BIRATIONAL GEOMETRY OF EXAMPLES

In this section we give examples of foliations \mathcal{F} of \mathbb{P}^2 which are associated to simple derivations and whose Kodaira dimension satisfies $\kappa(\mathcal{F}) \in \{1, 2\}$. In every case we describe the corresponding reduction of singularities and give a nef model. Moreover, we show some birational (non-)equivalences between examples.

We will provide diagrams to illustrate the reduction of singularities and nef models. The following conventions are used in the examples.

- The affine coordinates $(x, y) \in \mathbb{C}^2$ correspond to the point $(x_0 : x_1 : x_2) = (1 : x : y) \in \mathbb{P}^2$.
- We denote as (σ_i) the sequence of blowing-ups of points composing of a given foliation \mathcal{F} and (p_j) the sequence of contractions composing the morphism $\rho : M \rightarrow S$ to a nef model \mathcal{F}_{nef} of \mathcal{F} .
- In the figures, we use *nd*, *sn*, *m*, *r*, *nil* for *non-degenerate*, *saddle-node*, *Morse*, *radial* and *nilpotent* singularities, respectively (cf. Section 2).
- The line at infinity $x_0 = 0$ will be denoted by L and any strict transform always by \overline{L} , except in the figures, see next point.
- In the figures, we use the same symbol for each exceptional curve and its strict transforms under other blowing-ups, but the self-intersection numbers indicated in parentheses (n) will help to avoid confusions.
- The bracket $[m]$ denotes the multiplicity of the fiber ($z = 0$) of the Riccati foliation locally defined by $z^m dw + (a(z)w^2 + b(z)w + c(z)) dz$, with a, b, c holomorphic at $z = 0$.

We start with examples having $\kappa(\mathcal{F}) = 1$. Part of these examples are associated to *Shamsuddin derivations*, see [Sha77]. These are derivations of the form

$$D = \partial_x + (a(x)y + b(x))\partial_y, \quad a, b \in \mathbb{C}[x].$$

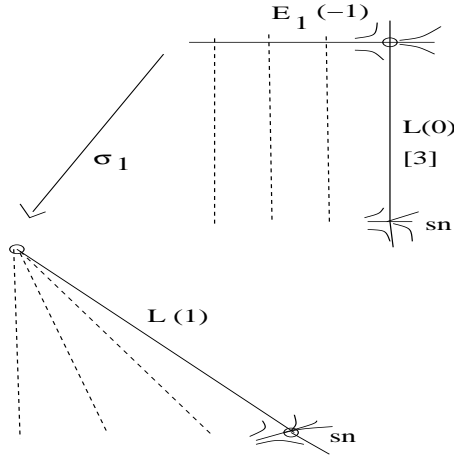
The associated foliation is given by $\omega = dy - (a(x)y + b(x)) dx$. It is a special Riccati foliation.

Example 6.1. Consider the foliation \mathcal{F} of the projective plane associated to

$$\omega = (xy + 1) dx - dy,$$

called Bergman's example in [Cou03]. The extended foliation \mathcal{F} of \mathbb{P}^2 has degree 2. The point at infinity $(1 : 0 : 0)$ is a saddle-node with Milnor number $m = 3$ whose strong separatrix is the line at infinity $L : x_0 = 0$; in particular L is \mathcal{F} -invariant. At $(0 : 1 : 0)$ there is a quadratic singularity: the blowing-up at this point produces a Riccati foliation on \mathbb{F}_1 , leaving invariant the exceptional curve E_1 . There is just one singular point along E_1 , a saddle-node with Milnor number $m = 3$ with strong separatrix \bar{L} and weak separatrix E_1 . This is already a nef model in \mathbb{F}_1 . The multiplicity of \bar{L} as an invariant fiber and the formula for the cotangent bundle of [Bru00, p. 57] enable us to compute

$$T^*\mathcal{F}_{nef} = -2\bar{L} + 3\bar{L} = \mathcal{O}(\bar{L}) \quad \text{and} \quad \kappa(\mathcal{F}) = 1.$$



Example 6.2. From [GL12] we consider the foliation of degree 2 in the projective plane associated to

$$\omega = (1 + x(2x + y))dy + 2x(2x + y)dx = 0$$

The singularities along L are a saddle-node at $(0 : 1 : 0)$ and a non-reduced (quadratic) singularity at $(-1 : 2 : 0)$. The affine lines $2x + y = c$ are completely transverse to the foliation. One blowing-up at $(-1 : 2 : 0)$ is enough to reduce the singularity and produces a Riccati foliation. The exceptional line is invariant and has a saddle-node. \bar{L} is the unique invariant fiber. This is already a nef model. The multiplicity of \bar{L} as an invariant fiber is 3 and we compute

$$T^*\mathcal{F}_{nef} = \mathcal{O}(-2\bar{L}) + 3\bar{L} = \mathcal{O}(\bar{L}), \quad \kappa(\mathcal{F}) = 1.$$

Example 6.3. From [Now94] we consider the foliation of degree 3 in the projective plane of Shamsuddin type given by

$$\omega = (yx^2 + xy + x^2)dx - dy = 0.$$

At $(1 : 0 : 0)$ there is a saddle-node with Milnor number $m = 4$ whose strong separatrix is L . At the vertical infinity there is a cubic singularity. The blowing-up at this point produces a Riccati foliation relatively to the vertical lines. The exceptional line E_1 is invariant and the unique singularity along E_1 is a saddle-node with Milnor number $m = 4$ with weak separatrix E_1 and strong separatrix \bar{L} . This curve is the unique invariant fiber and its multiplicity as

invariant fiber is 4. The foliation on \mathbb{F}_1 is already a nef model. The cotangent line bundle can be computed as above,

$$T^* \mathcal{F}_{nef} = \mathcal{O}(-2\bar{L}) + 4\bar{L} = \mathcal{O}(2\bar{L}) \quad \text{and} \quad \kappa(\mathcal{F}) = 1.$$

Example 6.4. Consider the Shamsuddin type foliation of degree 4 in the plane associated to

$$\omega = ((x^3 + 1)y + 5x^4 - x^3 - 2x^2 + 4x)dx - dy = 0.$$

At $(1 : -5 : 0)$ there is a saddle node with Milnor number $m = 5$. At $(0 : 1 : 0)$ there is a quartic singular point (algebraic multiplicity = 4). The foliation obtained after blowing up this point is Riccati, having just one singular point along E_1 which is saddle-node with Milnor number $m = 5$, with weak separatrix E_1 and strong separatrix \bar{L} .

The foliation on \mathbb{F}_1 is already a nef model. The multiplicity of \bar{L} as invariant fiber is 5 and again the cotangent line bundle can be computed as

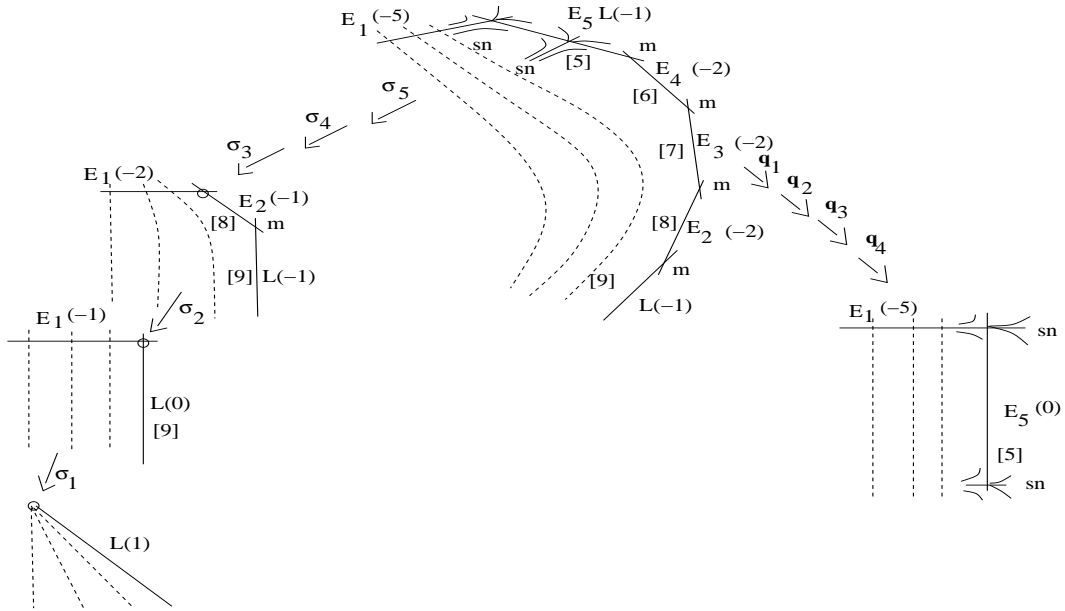
$$T^* \mathcal{F}_{nef} = \mathcal{O}(-2\bar{L}) + 5\bar{L} = \mathcal{O}(3\bar{L}) \quad \text{and} \quad \kappa(\mathcal{F}) = 1.$$

Example 6.5. From [Now94, Ex 13.3.7 p. 154] we have a foliation of degree 8 of Shamsuddin type

$$\omega = ((x^3 + 1)y + x^8 + 3x^5 + 1)dx - dy = 0.$$

At $(0 : 1 : 0)$ there is a highly degenerate singularity (with algebraic multiplicity = 8). The blow up of the foliation at this point produces a Riccati foliation, but the reduction of singularities is not completed yet. It needs four additional blowing-ups. From the second blown up point to the fifth the algebraic multiplicity is = 2. Along the fifth exceptional line E_5 there are three singular points: two saddle-nodes with Milnor number $m = 5$ and one Morse point. The foliation obtained is reduced but not a relatively minimal model.

To obtain a relatively minimal model we contract \bar{L}, E_2, E_3 and E_4 , in this order.



As we contract foliated exceptional curves with Morse points, the strict transform of E_5 contains just two saddle-nodes. In this example the relatively minimal model is already a nef model (on the Hirzebruch surface \mathbb{F}_5).

The multiplicity of E_5 as invariant fiber and the formula for the cotangent bundle gives

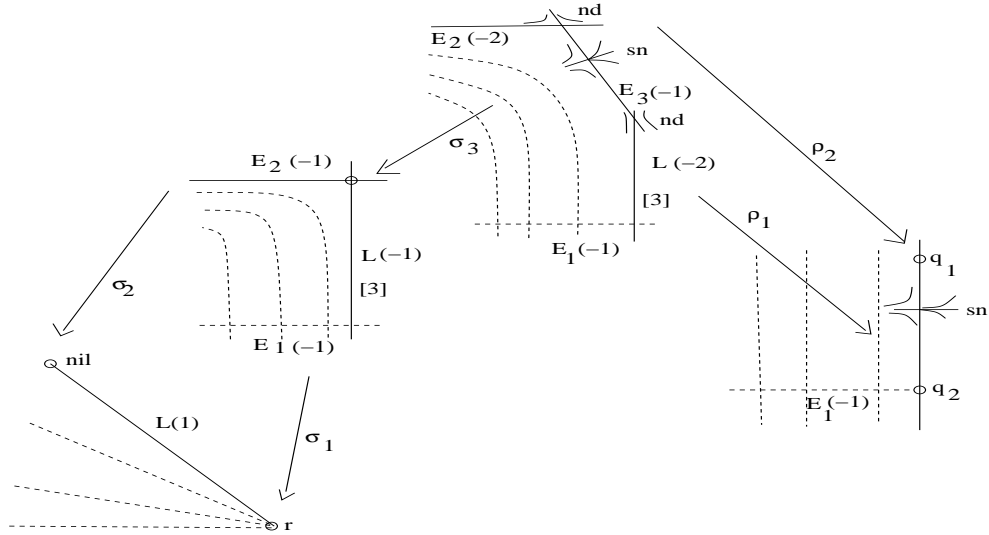
$$T^*\mathcal{F}_{nef} = \mathcal{O}(-2) + 5\overline{E}_5 = \mathcal{O}(3\overline{L}) \quad \text{and} \quad \kappa(\mathcal{F}) = 1.$$

Example 6.6. This example is not of Shamsuddin type but is of Riccati type. From [MMON01] (see also [Cou08]) we consider the foliation of degree 2 in the projective plane associated to

$$\omega = dx - (x^2 + y)dy = 0.$$

At the horizontal infinity point there is a radial point (Milnor number $m = 1$), whose blowing-up produces a Riccati foliation completely transverse to the exceptional line E_1 . There is a nilpotent singularity at the vertical infinity point whose Milnor number is $m = 6$ (thanks to Darboux's formula in the plane). The blowing-up at the nilpotent point produces an invariant exceptional curve E_2 having just one quadratic singularity, at the intersection with \overline{L} at infinity. The blowing-up at this quadratic singularity produces three singularities along E_3 : two of them being non-degenerate and reduced singularities, placed at the intersections of E_3 with \overline{L} and E_2 , and a third one being a saddle-node, with strong separatrix E_3 . We assert that the Milnor number of this saddle-node is $m = 4$: indeed, it follows from the diagram on the top of [Bru00, p. 56] and the fact that a nilpotent point in \mathbb{P}^2 has $m = 6$.

The foliation obtained is not a nef model. The morphism $\rho = \rho_2 \circ \rho_1$ contracts two (-2) -curves and produces a singular surface with two quotient singularities q_1, q_2 along the strict transform of E_3 (where there is also a saddle-node with Milnor number $m = 4$).



In this nef model of the foliation, the computation of the degree of $\pi_*(T^*\mathcal{F}) \in \text{Pic}(\mathbb{P}^1) \otimes \mathbb{Q}$ gives:

$$\deg(\pi_*(T^*\mathcal{F}_{nef})) = -2 + \frac{4+1}{2} = \frac{1}{2} > 0 \quad \text{and} \quad \kappa(\mathcal{F}) = 1.$$

All the remaining examples have $\kappa(\mathcal{F}) = 2$.

Example 6.7. Consider the foliation \mathcal{F} of degree $\deg(\mathcal{F}) = 2$ in the projective plane associated to the following equation in \mathbb{C}^2 :

$$\omega = x(1 + xy)dx - (1 + xy + x^3)dy = 0$$

which was taken from [CDGBM10, Prop. 1.3]. The line at infinity is *not* invariant and there is just one saddle-node with Milnor number $m = 7$ at infinity. So the singularity of \mathcal{F} is reduced. The cotangent line bundle is $T^*\mathcal{F} = \mathcal{O}(1)$ and $\kappa(\mathcal{F}) = 2$. As there is no curve with negative self intersection in \mathbb{P}^2 , \mathcal{F} is its own nef model. The group $Pol(\mathcal{F}|_{\mathbb{C}^2})$ contains the linear automorphism $L_j(x, y) \mapsto (j \cdot x, j^2 \cdot y)$ where j is a primitive cubic root of the unity and $L_j^*(\omega) = j^2 \cdot \omega$. The affine line $x = 0$ is transverse to the foliation and, for $c \neq 0$, the affine lines $x = c$ have one movable tangency.

Example 6.8. From [Now08], we have a family of foliations \mathcal{F}_k of degree $k \geq 2$ in the plane associated to simple derivations. It is defined by the family of 1-forms

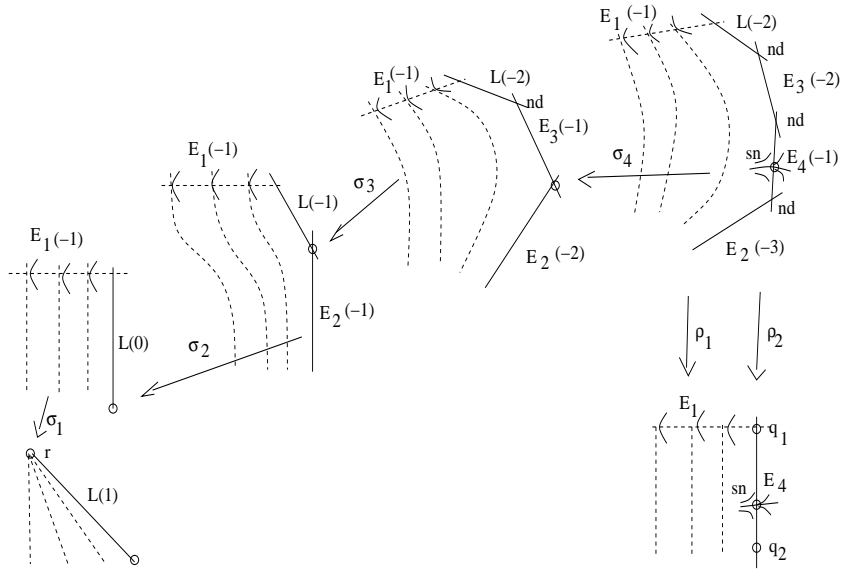
$$\omega_k = (y^k + x)dx - dy, \quad k \in \mathbb{N}^*.$$

For $k = 2$ this coincides with Example 6.6, up to permutation of (x, y) .

We assert that $\kappa(\mathcal{F}_k) = 2$ for all $k \geq 3$. At the vertical infinity point, each \mathcal{F}_k has a radial point p . The exceptional line of the blow up at p belongs to the contact divisor between the transformed foliation and the rational fibration of $\Sigma_1(p)$. For simplicity, let's focus on the case $k = 3$. The reduction of singularities of \mathcal{F}_3 is made up of 4 blowing-ups at quadratic singularities of the foliation. The fourth blowing-up introduces E_4 having a saddle-node and 2 extra non-degenerated reduced points (at the intersections of E_4 with the strict transforms of E_2 and E_3). The Zariski decomposition of the cotangent line bundle is

$$T^*\overline{\mathcal{F}}_3 \equiv P + \frac{1}{4}\overline{E}_3 + \frac{1}{2}\overline{L} + \frac{1}{3}\overline{E}_2.$$

The nef model is obtained after contraction of the support of \mathbf{N} and introduces two quotient singularities of the surface q_1, q_2 .



Since the 4 blown up singularities were quadratic ($l(\mathcal{F}) = 2$), we can compute

$$T^*\overline{\mathcal{F}}_3 \cdot T^*\overline{\mathcal{F}}_3 = (\deg(\mathcal{F}_3) - 1)^2 - \sum_{i=1}^4 (l_{p_i}(\mathcal{F}_3) - 1)^2 = 4 - 4 = 0$$

Combining this with

$$\mathbf{N} \cdot \mathbf{N} = \left(\frac{1}{4}\overline{E}_3 + \frac{1}{2}\overline{L} + \frac{1}{3}\overline{E}_2\right)^2 = -\frac{3}{8} - \frac{1}{3} < 0$$

and the fact that $\mathbf{P} \cdot \mathbf{N} = 0$ in the Zariski decomposition, the conclusion is that $\mathbf{P} \cdot \mathbf{P} > 0$. Therefore the so called numerical Kodaira dimension is 2 and also $\kappa(\mathcal{F}_3) = 2$.

Example 6.9. From [Jor81] we have a foliation \mathcal{F} of degree 3 in the projective plane associated to

$$\omega = y^3 dy - (1 - xy) dx.$$

At $(0 : 1 : 0)$, there is a quadratic singularity whose reduction is composed by 3 blow ups. The foliation has $\kappa(\mathcal{F}) = 2$ and its nef model is not much different than the one of Example 6.8. At last, we remark that the affine lines $y = c$, $c \neq 0$, exhibit one movable tangency point with the foliation (at the intersection of $y = c$ and $y = \frac{1}{x}$). The affine line $y = 0$ is completely transverse to the foliation, a property that will be useful in Section 8.

Example 6.10. From [Kou12] we have examples of foliations $\mathcal{F}_{r,s,g}$ in the projective plane with degrees $\deg(\mathcal{F}_{r,s,g}) = s + 1$, defined for $r, s \in \mathbb{N}$ such that $r + 2 \leq s$ and $g \in \mathbb{C}^*$ by

$$\omega_D := (xy^s + g) dx - y^r dy.$$

We have a singular point at $(0 : 1 : 0)$ with algebraic multiplicity $\ell(\mathcal{F}_{r,s,g}, p) = s$. At $(0 : 0 : 1)$, there is a singular point with algebraic multiplicity 3. Except for $y = 0$, all horizontal affine lines exhibit one movable contact point with the restricted foliation $\mathcal{F}_{r,s,g}|_{\mathbb{C}^2}$. However, the affine line $y = 0$ is completely transversal to the foliation.

Example 6.11. According to [Oda95], the foliations in the plane associated to

$$\omega := (f(x) \cdot y + g(x)) dx + y dy = 0, \quad f, g \in \mathbb{C}[x]$$

do *not* have algebraic solutions in \mathbb{C}^2 if three conditions are satisfied: *i*) $f, g \neq 0$, *ii*) $\deg(f) \geq \deg(g)$ and *iii*) $\frac{f}{g}$ is not constant. These foliations do not have singularities in \mathbb{C}^2 exactly when $g(x) = c \in \mathbb{C}^*$. Therefore, to produce foliations associated to simple derivations it suffices to take $g(x) = c \in \mathbb{C}^*$ and $f \in \mathbb{C}[x]$. In this case, the affine line $y = 0$ is everywhere transverse to the foliation while general horizontal lines $y = c$ exhibit $\deg(f)$ tangencies with the foliation. And the line at infinity is invariant by the extended foliation.

Now we establish some birational (non)-equivalences among the Examples.

Proposition 6.12. *i*) Example 6.2 is linearly equivalent to a foliation of Shamsuddin type.
ii) Examples 6.4 and 6.5 are equivalent by a polynomial automorphism of degree five.
iii) The foliations of Example 6.1 and Example 6.3 are not birationally equivalent.

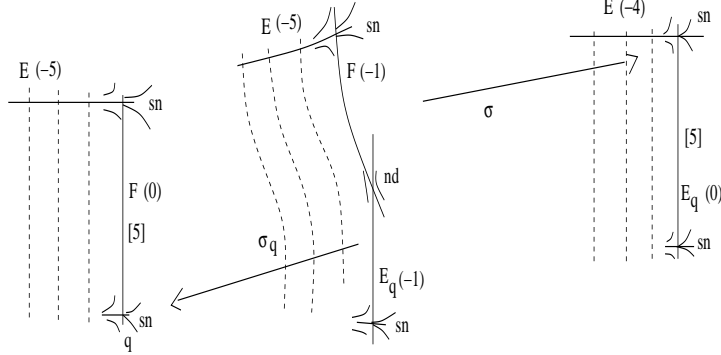
Proof.

i). With the linear change of variables $y = u - 2v$, $x = v$, from the equation of Example 6.2 we obtain

$$\eta = 2dv - (1 + vu)du$$

which is of Shamsuddin type.

ii). We start with the nef model of Example 6.5 in the Hirzebruch surface \mathbb{F}_5 . After an elementary transformation we pass to \mathbb{F}_4 keeping the saddle-nodes and the multiplicity [5] of the unique invariant fiber. This is shown in next figure.



After this we perform three more elementary transformations to obtain a foliation of \mathbb{F}_1 and then, after contraction of the (-1) -section, we arrive in the projective plane. The pair of saddle-nodes on the invariant vertical fiber of \mathbb{F}_5 is transformed in a pair of singularities along an invariant projective line of a degree 4 foliation.

The net effect in the plane can be described concretely by means of a polynomial isomorphism of \mathbb{C}^2 . Indeed,

$$\eta := ((x^3 + 1)y + 5x^4 - x^3 - 2x^2 + 4x)dx - dy$$

defines the foliation of Example 6.4 and, if $R : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is

$$(x, y) \mapsto (x, y + x^5 + 2x^2 + 1)$$

then $R^*(\eta)$ is the 1-form of Example 6.5.

iii). A way to see this is to use the birational invariant $g(\mathcal{F})$ of [Men00, p. 139]. In Example 6.1 and Example 6.3 the invariants are $g(\mathcal{F}) = 2$ and $g(\mathcal{F}) = 3$, respectively. \square

Proposition 6.12 – *ii*) illustrates the general issue of finding the simplest (of least degree) plane birational model for foliations of the plane or derivations. Algorithmic procedures toward this objective would be of great utility.

7. PROOF OF THEOREM A

In the language of foliations Theorem A takes the following form.

Theorem 7.1. *Let \mathcal{F} be a foliation of \mathbb{P}^2 that the restriction $\mathcal{F}|_{\mathbb{C}^2}$ has neither algebraic invariant curves nor non-reduced singularities. The group $Bir(\mathcal{F})$ is finite.*

Proof. We proceed by contradiction: suppose $Bir(\mathcal{F})$ is infinite. From our assumption on algebraic invariant curves, \mathcal{F} cannot have a rational first integral. We have two cases: either

- (1) there exists a birational model $(X, \tilde{\mathcal{F}})$ of $(\mathbb{P}^2, \mathcal{F})$ such that $\text{Aut}(\tilde{\mathcal{F}}) = \text{Bir}(\tilde{\mathcal{F}})$ or
- (2) for every birational model $(X, \tilde{\mathcal{F}})$, the inclusion $\text{Aut}(\tilde{\mathcal{F}}) \subset \text{Bir}(\tilde{\mathcal{F}})$ is strict.

In case (2), we may apply [CF03, Th. 1.1] to $(X, \tilde{\mathcal{F}})$, as fibrations are excluded we have one of the following situations.

- (i) There exists a non trivial holomorphic vector field on \tilde{X} defining a one parameter subgroup of $\text{Aut}(\tilde{\mathcal{F}})$ or
- (ii) the surface \tilde{X} is generalized Kummer surface (see [CF03, Ex. 1.1]), \tilde{X} is a quotient of an abelian surface A and $\tilde{\mathcal{F}}$ lifts to A as a linear foliation \mathcal{G} .

In case (1), we may apply [CF03, Th. 1.2]. Since fibrations are excluded, we are in the situation of [CF03, Ex. 1.3], in particular up to passing to a birational model $(X, \tilde{\mathcal{F}})$,

- (iii) the surface X is a finite quotient of $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $\tilde{\mathcal{F}}$ lifts to a foliation \mathcal{G} of S given by a differential form $\alpha wdz + \beta zdw$, for some $\alpha, \beta \in \mathbb{C}$.

In cases (ii) and (iii) the foliation \mathcal{G} has Kodaira dimension 0. By the remark in [Bru00, p. 29] or the proof of [Tou03, Lemme 3.2.8], this forces $\kappa(\tilde{\mathcal{F}}) \leq 0$ and contradicts Theorem B.

So we only need to derive a contradiction from situation (i) to complete the proof. Moreover, if the vector field is tangent to $\tilde{\mathcal{F}}$, we may use [Bru00, Prop. 6.6 iii)] to see that $\tilde{\mathcal{F}}$ is a Riccati foliation with two distinct adapted fibrations and consequently $\kappa(\mathcal{F}) = 0$, contradicting Theorem B.

The conclusion is given by the following argument, proposed by Jorge Pereira. If the vector field is not tangent to $\tilde{\mathcal{F}}$ consider its image \tilde{X} in \mathbb{P}^2 , consider also a rational 1-form ω defining the foliation. The form $\eta = \frac{\omega}{\omega(\tilde{X})}$ is closed, by the computation [PS02, Proof of Cor. 2] inherited from [CM82]. The poles of η give \mathcal{F} -invariant algebraic curves, so that η has no poles in \mathbb{C}^2 . Subsequently, the first integral $\int_{\star}^x \eta$ has no monodromy and gives a rational first integral for \mathcal{F} , contradiction. \square

8. POLYNOMIAL SYMMETRIES OF FOLIATIONS ASSOCIATED TO SIMPLE DERIVATIONS

Consider a foliation \mathcal{F} of the projective plane $\mathbb{P}^2 = \mathbb{C}^2 \cup L_{\infty}$ with two properties:

- a) \mathcal{F} is associated to a simple derivation,
- b) there exists some affine straight line completely transversal to the foliation $\mathcal{F}|_{\mathbb{C}^2}$.

Proposition 8.1. *Given $n \geq 2$ and $B > 0$, there exists a foliation \mathcal{G} associated to a simple derivation and an element in $\text{Pol}(\mathcal{G}|_{\mathbb{C}^2})$ of order n and degree greater than B .*

Actually the construction below starts from any foliation \mathcal{F} satisfying a) and b). These properties are verified in all examples of Section 6, so that we have plenty of examples.

Proof. Let \mathcal{F} be a foliation satisfying a) and b). Up to an affine transformation, we can suppose that the line in b) is $x = 0$. Consider the n to 1 rational map $\phi_n : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ given by $\phi_n(x : y) =$

(x^n, y) . The foliation \mathcal{F} is defined by the (rational extension to \mathbb{P}^2 of the) polynomial 1-form $\omega = a(x, y)dx + b(x, y)dy$. The map ϕ_n extends to a birational map $\widehat{\phi}_n : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$.

Let $\mathcal{G}_0 = \widehat{\phi}_n^* \mathcal{F}$. It is defined by (the extension to \mathbb{P}^2 of) $\phi_n^*(\omega) = a(x^n, y)d(x^n) + b(x^n, y)dy$. Since the affine line $x = 0$ is supposed to be completely transverse to \mathcal{F} , we know \mathcal{G} has no singularity along $x = 0$. Note also that in $\mathbb{C}^2 \setminus \{x = 0\}$ the map $\phi_n(x, y) = (x^n, y)$ is a local isomorphism, so the pull-back does not introduce any singularity. Any algebraic $\mathcal{G}_0|_{\mathbb{C}^2}$ -invariant curve would descend to an $\mathcal{F}|_{\mathbb{C}^2}$ -invariant curve, so that no such curve exists.

Hence \mathcal{G}_0 is associated to a simple derivation.

Let ξ be a primitive n -th root of unity. The linear automorphism $T_\xi(x, y) = (\xi \cdot x, y)$ clearly preserves \mathcal{G}_0 and has order n . Note however that $T_\xi \notin \text{Aut}(D_n)$ for the derivation D_n dual to $\phi_n^*(\omega)$, because $T_\xi^*(\phi_n^*(\omega)) = \phi_n^*(\omega)$ and $\text{Jac}(T_\xi) = \xi \neq 1$.

Now, for any polynomial $\tau \in \mathbb{C}[x]$, consider the automorphism of \mathbb{C}^2 defined by

$$P_\tau(x, y) = (x, y + \tau(x))$$

and set $\mathcal{G} := P_\tau^* \mathcal{G}_0$. If the polynomial $\tau(\xi \cdot x) - \tau(x)$ has degree $d \geq 1$, (e.g. if $\deg_x(\tau) = d$ and $(d, n) = 1$) the map

$$\Gamma_{\xi, \tau}(x, y) := P_\tau^{-1} \circ T_\xi \circ P_\tau(x, y) = (\xi \cdot x, y + \tau(\xi \cdot x) - \tau(x))$$

defines a polynomial automorphisms of degree d and order n in $\text{Aut}(\mathcal{G})$, which completes the proof. \square

Example 8.2. We exemplify the construction given in the proof of Proposition 8.1. Starting from Example 6.1

$$\omega = (xy + 1)dx - dy$$

and taking $n = 2$, so $\phi_2(x, y) = (x^2, y)$. We obtain a foliation \mathcal{G}_0 defined by the 1-form $\phi_2^*(\omega) = (2x^3y + 2x)dx - dy$ and admitting the linear automorphism

$$T(x, y) = (-x, y).$$

Now, we consider the automorphism $P_\tau(x, y) = (x, y + x^3)$ and define

$$\Omega := P_\tau^*(\phi_2^*(\omega)) = (2x^3y - 2x^6 + 3x^2 + 2x)dx - dy.$$

Then the involution $\Gamma_{\xi, \tau} = \Gamma_{-1, x^3} : (x, y) \mapsto (-x, y - 2x^3)$ preserves Ω .

At last, note that the foliations we have constructed in the proof of Proposition 8.1 do not have minimal degree in their birational classes, due to the fact that the automorphism P_τ has positive degree. In other terms, such foliations are not *primitive* in the sense of [CD15]. On the other side, the foliation of Example 6.7 is primitive but the exhibited automorphism is linear.

This raises the following question.

Are there non-linear polynomial automorphisms of primitive foliations associated to simple derivations ?

By [Bru99, Cor. p 293], such automorphisms would necessarily be conjugated to automorphisms of the form $(x, y) \mapsto (ax + P(y), by + c)$ with $a, b \in \mathbb{C}^*$, $c \in \mathbb{C}$ and $P \in \mathbb{C}[y]$.

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