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Abstract. We study a 2D semi-linear equation in a domain with a small Dirichlet obstacle of size δ. Using the method of matched asymptotic expansions, we compute an asymptotic expansion of the solution as δ tends to zero. Its relevance is justified by proving a rigorous error estimate. Then we construct an approximate model, based on an equation set in the limit domain without the small obstacle, which provides a good approximation of the far field of the solution of the original problem. The interest of this approximate model lies in the fact that it leads to a variational formulation which is very simple to discretize. We present numerical experiments to illustrate the analysis.

Key words. Small obstacle, semi-linear convex problem, asymptotic analysis, singular perturbation.

1 Introduction

Problems involving structures which are small compared to a given characteristic size arise in many applications. For example, they appear in electrical engineering when one wishes to model the propagation of electromagnetic waves in presence of thin wires [12, 4, 27] or in geophysics when one is interested in the study of flow transport around wells [8, 25]. The modelling of such physical phenomena leads to study partial differential equations (PDEs) depending on a small parameter δ through the geometry. An interesting question it to compute an asymptotic expansion of the solution of the PDE with respect to δ. Usually, this expansion is made of functions defined on geometries which are independent from δ multiplied by some gauge functions having an explicit dependence with respect to δ and to the features of the small structure.

There are many reasons for asymptotic calculus. One of them, that we have in mind, concerns numerical analysis and more precisely the use of techniques, like classical Galerkin methods, based on a mesh of the physical domain. In practice, it can be rather complicated to create a mesh adapted to the small obstacle. Moreover, the solving of the resulting algebraic system may be very expensive due to the high numbers of degrees of freedom. Of course now, computational power has much improved and strategies based on local refinement of the mesh near the obstacle where the solution is expected to exhibit rapid variations, offers a reasonable solution in many settings. However, these methods still require an extra effort. In particular, when there are several obstacles, constructing well-suited meshes can be tricky. Alternatively, since the terms of the asymptotic expansion are defined in geometries which are independent of δ, one can compute them easily to obtain a good approximation of the solution in the domain with the small obstacle.

Literature concerning the development of techniques of asymptotic analysis for linear elliptic PDEs is rich. Among others, let us cite the classical monographs [30, 16, 14, 20]. For non linear PDEs, it seems that there are less references, especially if one looks for error estimate. In this article, we consider a semi-linear convex problem (see (1)) in presence of a (small) Dirichlet obstacle of size δ. This problem can be seen as a non linear perturbation of the Dirichlet-Laplacian for which the asymptotic expansion of the solution with respect to δ is well-known (see, e.g., [14, 19, 20, 6, 21, 23, 24, 7]). We work in a 2D setting, the reason being that, in this case, it is known for the Dirichlet-Laplacian that neglecting the small obstacle, that is considering the solution of the problem without the obstacle, is a very crude approximation of order $|\ln\delta|^{-1}$. Note that $|\ln\delta|^{-1}\approx 0.0434$ for $\delta = 10^{-10}$. Such an estimate also holds for our problem (see Section 5). Therefore, it is essential to compute a corrective term. The problem we consider has been studied in [2] in the context of topological sensitivity analysis. In [2], the author provides an error estimate in $o(|\ln\delta|)$. The first outcome of the present work is an improvement of this result. More precisely, we will show an estimate in $o(\delta^{1-\varepsilon})$ for all $\varepsilon > 0$.

Let us mention that in [13] (see also the related papers [29, 10]), the authors study the same equation as ours, but in 3D for a Dirichlet obstacle and in 2D for a Neumann obstacle. We emphasize that the asymptotic expansion is completely different for the 2D Dirichlet obstacle (with $\ln\delta$ appearing in the gauge functions). Moreover, in [13], the compound method is used and error estimates are proved in Hölder spaces. In the present article, we work with different tools, employing the method of matched asymptotic expansions described for example in [30, 14, 20] and showing error estimates in Sobolev spaces. In literature, it seems that people prefer to use the compound method to derive asymptotic expansions for non linear PDEs. The second objective of the paper is to investigate the method of matched asymptotic expansions for non linear PDEs. For this technique, the general scheme is the following. First we compute an expansion of the solution with respect to δ far from the obstacle. Then we construct an expansion closed to the obstacle where we expect a rapid variation of the field. Finally, we match the two expansions in an intermediate region to define completely the unknown terms of the far and near field expansions. The main novelty appearing in the study of the non linear PDE (1) occurs during the matching step (see §3.3) which leads us to solve a non linear (and non explicit) equation to define the gauge functions of the expansion.

In the future, it would be interesting to consider other types of non linear PDEs that could be more pertinent for applications. Note that in [5, 3], stronger non linearities are studied in the field of topological sensitivity analysis. On the other hand, here we focus only on the first two terms of the asymptotics. One possible direction to continue this work is to compute higher order terms.

The outline is the following. In the next section, we present the geometry and the problem we wish to consider. We also recall how to prove that it admits a unique solution u_δ in a classical framework, where δ denotes the size of the Dirichlet obstacle. Section 3 is dedicated to the construction of a formal asymptotic expansion of u_δ with respect to δ. We use the method of matched asymptotic expansions as described above. In Proposition 3.1, we state an error estimate showing that this expansion yields a good approximation of u_δ. In Section 4, we construct an approximate model which admits a solution corresponding to the first two terms of the far field expansion of u_δ (note that for many applications, only the field far from the obstacle matters). From a practical point of view, it is much more simpler to discretize this approximate model than to compute separately each of the terms appearing in the expansion of u_δ. The derivation of this problem is the third main outcome of the paper. In Section 5, we provide numerical experiments, based on finite element methods, illustrating the analysis. Finally, in Section 6, we present the details of the proof of the error estimate stated in Proposition 3.1.

2 Problem under consideration

Consider ω, $\Omega \subset \mathbb{R}^2$ two bounded Lipschitz domains such that $\overline{\omega} \subset \Omega$ and $O \in \omega$. For $\delta \in (0;1]$, set $\omega_\delta := \{\boldsymbol{x} \in \mathbb{R}^2,\ \boldsymbol{x}/\delta \in \omega\}$ (the small obstacle) and $\Omega_\delta := \Omega \setminus \overline{\omega_\delta}$. Let f be a given source term in $\mathrm{L}^2(\Omega) := \{g : \Omega \to \mathbb{R} \mid \int_\Omega |g(\boldsymbol{x})|^2 d\boldsymbol{x} < +\infty\}$. Note that all through this article, we shall systematically work with functions that are real valued. For some fixed integer $m \in \mathbb{N}$, we are interested in the following semi-linear problem with Dirichlet boundary condition

$$\left|\begin{array}{l}\text{Find } u_\delta \in \mathrm{H}^1_0(\Omega_\delta) \text{ such that}\\ -\Delta u_\delta + (u_\delta)^{2m+1} = f \quad \text{in } \Omega_\delta.\end{array}\right. \tag{1}$$

In (1), $\mathrm{H}^1_0(\Omega_\delta)$ denotes the subspace of the elements of the Sobolev space $\mathrm{H}^1(\Omega_\delta)$ vanishing on $\partial\Omega_\delta$. First, we remind the reader how to prove that (1) admits a unique solution. For any Lipschitz domain $\mathcal{O} \subset \mathbb{R}^2$, the Rellich-Kondrachov embedding theorem ensures that $\mathrm{H}^1(\mathcal{O}) \subset \mathrm{L}^p(\mathcal{O})$ for all $p \geq 1$. As a consequence, the bilinear form $(u,v) \mapsto \int_{\Omega_\delta}(\nabla u \cdot \nabla v + (u)^{2m} uv)\, d\boldsymbol{x}$ is well-defined and continuous in $\mathrm{H}^1_0(\Omega_\delta) \times \mathrm{H}^1_0(\Omega_\delta)$. Introduce the continuous operator $\mathrm{A}_\delta : \mathrm{H}^1_0(\Omega_\delta) \to \mathrm{H}^{-1}(\Omega_\delta) := \mathrm{H}^1_0(\Omega_\delta)^*$ such that

$$\langle \mathrm{A}_\delta(u), v\rangle = \int_{\Omega_\delta} \nabla u \cdot \nabla v + (u)^{2m} uv\, d\boldsymbol{x}, \qquad \forall u, v \in \mathrm{H}^1_0(\Omega_\delta). \tag{2}$$

Classicaly, one finds that $u_\delta \in \mathrm{H}^1_0(\Omega_\delta)$ verifies (1) if and only if it is a solution to the problem

$$\left|\begin{array}{l}\text{Find } u_\delta \in \mathrm{H}^1_0(\Omega_\delta) \text{ such that}\\ \langle \mathrm{A}_\delta(u_\delta), v\rangle = \displaystyle\int_{\Omega_\delta} fv\, d\boldsymbol{x}, \qquad \forall v \in \mathrm{H}^1_0(\Omega_\delta).\end{array}\right. \tag{3}$$

Simple differential calculus shows that $\mathrm{A}_\delta(u) - f$ is the Fréchet differential evaluated at u of the cost functional $\mathrm{J}_\delta : \mathrm{H}^1_0(\Omega_\delta) \to \mathbb{R}$ defined by

$$\mathrm{J}_\delta(\varphi) = \frac{1}{2}\, \|\nabla\varphi\|^2_{\mathrm{L}^2(\Omega_\delta)} + \frac{1}{2m+2}\, \|\varphi\|^{2m+2}_{\mathrm{L}^{2m+2}(\Omega_\delta)} - \int_{\Omega_\delta} f\varphi\, d\boldsymbol{x}, \qquad \forall \varphi \in \mathrm{H}^1_0(\Omega_\delta).$$

One can check that u_δ is a minimum of J_δ if and only it satisfies Problem (3). Since J_δ is continuous, strictly convex and coercive, it admits a unique minimum (see e.g. [15, Chap. 3]). Therefore (3) (and so (1)) has a unique solution. From the convexity of the map $t \mapsto t^{2m+2}$ on $\mathbb{R}$, we obtain $(s^{2m+1} - t^{2m+1})(s-t) \geq 0$ for all $s, t \in \mathbb{R}$. This implies $\langle \mathrm{A}_\delta(u) - \mathrm{A}_\delta(v), u - v\rangle \geq \|u - v\|^2_{\mathrm{H}^1_0(\Omega_\delta)}$ for all $u, v \in \mathrm{H}^1_0(\Omega_\delta)$ and leads to the stability estimate

$$\|u - v\|_{\mathrm{H}^1_0(\Omega_\delta)} \leq \|\mathrm{A}_\delta(u) - \mathrm{A}_\delta(v)\|_{\mathrm{H}^{-1}(\Omega_\delta)}, \qquad \forall u, v \in \mathrm{H}^1_0(\Omega_\delta). \tag{4}$$

Here we use the notation $\|\cdot\|_{\mathrm{H}^1_0(\Omega_\delta)} := \|\nabla \cdot\|_{\mathrm{L}^2(\Omega_\delta)^2}$. To sum up, for all $\delta \in (0;1]$, the operator $\mathrm{A}_\delta : \mathrm{H}^1_0(\Omega_\delta) \to \mathrm{H}^{-1}(\Omega_\delta)$ is a uniformly continuous bijection, and its inverse $\mathrm{A}_\delta^{-1} : \mathrm{H}^1_0(\Omega_\delta) \to \mathrm{H}^{-1}(\Omega_\delta)$ is also uniformly continuous. The same conclusion holds for the operator $\mathrm{A}_0 : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ defined by

$$\langle \mathrm{A}_0(u), v\rangle = \int_{\Omega} \nabla u \cdot \nabla v + (u)^{2m} uv\, d\boldsymbol{x}, \qquad \forall u, v \in \mathrm{H}^1_0(\Omega). \tag{5}$$

Finally, for all $\delta \in [0;1]$, the Rellich-Kondrachov embedding theorem guarantees that the solution of (1) verifies $(u_\delta)^{2m+1} \in \mathrm{L}^2(\Omega_\delta)$. We shall assume that the boundary $\partial\Omega_\delta$ is regular enough to guarantee $u_\delta \in \mathscr{C}^0(\overline{\Omega_\delta})$. Note that this is true for example if $\partial\Omega_\delta$ is smooth or if $\partial\Omega_\delta$ is polygonal (see e.g. [11, 22]).

In the next section, we construct an asymptotic expansion of the solution u_δ to Problem (1) as $\delta \to 0$.

3 Asymptotic expansion of the solution

We will look for an asymptotic expansion of u_δ, the unique solution to (1), which decomposes into two components: a far field expansion which is a good approximation of u_δ far from ω_δ and a near field expansion which approximates correctly u_δ in a close neighbourhood of ω_δ.

3.1 Far field expansion

Far from ω_δ, we search for an approximation of u_δ under the form $v_\delta := u_0 + \lambda(\delta)u_{1,\delta}$ where the functions u_0, $u_{1,\delta}$ have to be determined and where the gauge function $\lambda(\delta)$ (independent of $\boldsymbol{x}$) is supposed to go to zero as $\delta \to 0$. Formally plugging this expansion into Equation (1) and retaining the first order terms as $\delta \to 0$, we find that u_0 must be a solution to the problem

$$\left|\begin{array}{l} \text{Find } u_0 \in \mathrm{H}^1_0(\Omega) \text{ such that} \\ -\Delta u_0 + (u_0)^{2m+1} = f \quad \text{in } \Omega. \end{array}\right. \tag{6}$$

According to the analysis of the preceding paragraph, we know that this problem admits a unique solution (the operator $\mathrm{A}_0 : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ introduced in (5) is a continuous bijection). Therefore, u_0 can be defined as $u_0 := \mathrm{A}_0^{-1}(f)$. In other words and quite naturally, we postulate that at first order, far from the origin, u_δ does not see the small obstacle.

To derive the correction term $u_{1,\delta}$ we impose that

$$\left|\begin{array}{rcll} -\Delta(u_0 + \lambda(\delta)u_{1,\delta}) + (u_0 + \lambda(\delta)u_{1,\delta})^{2m+1} &=& f & \text{in } \Omega \setminus \{O\} \\ u_0 + \lambda(\delta)u_{1,\delta} &=& 0 & \text{on } \partial\Omega. \end{array}\right. \tag{7}$$

Since $\mathrm{A}_0 : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ is a bijective map, we have to look for $u_{1,\delta}$ in a larger space than $\mathrm{H}^1_0(\Omega)$ (otherwise, we would conclude $u_{1,\delta} = 0$ which is not interesting). Denote $\mathcal{G}$ the Green's function for the Laplace operator with homogeneous Dirichlet boundary condition such that $-\Delta\mathcal{G} = \delta_O$ in Ω, $\mathcal{G} = 0$ on $\partial\Omega$ (δ_O is the Dirac distribution centered at O). By analogy with the more classical asymptotics for the 2D Dirichlet Laplace problem with a small obstacle (see for example [20, Chap. 2]), we impose the condition $u_{1,\delta} - \mathcal{G} \in \mathrm{H}^1(\Omega)$ (as a consequence, $u_{1,\delta}$ admits a logarithmic singularity at the origin[1]). Expanding (7), we find that $u_{1,\delta}$ must be solution to the problem

$$\left|\begin{array}{rcll} \multicolumn{4}{l}{\text{Find } u_{1,\delta} \text{ such that } u_{1,\delta} - \mathcal{G} \in \mathrm{H}^1(\Omega) \text{ and}} \\ -\Delta u_{1,\delta} + (2m+1)(u_0)^{2m}u_{1,\delta} &&& \\ +\lambda(\delta)\displaystyle\sum_{k=2}^{2m+1}\binom{2m+1}{k}(u_0)^{2m+1-k}\lambda(\delta)^{k-2}(u_{1,\delta})^k &=& 0 & \text{in } \Omega \setminus \{O\} \\ u_{1,\delta} &=& 0 & \text{on } \partial\Omega. \end{array}\right. \tag{8}$$

Let us prove that Problem (8) admits a unique solution for $\lambda(\delta) \in \mathbb{R}$ such that $|\lambda(\delta)|$ is small enough. Considering the change of unknown $w_{1,\delta} = u_{1,\delta} - \mathcal{G}$, Problem (8) can be recast into

$$\left|\begin{array}{l} \text{Find } w_{1,\delta} \in \mathrm{H}^1_0(\Omega) \text{ such that} \\ \Phi(\lambda(\delta), w_{1,\delta}) = 0, \end{array}\right. \tag{9}$$

where $\quad \Phi(\lambda, w) := -\Delta w + (2m+1)(u_0)^{2m}(w + \mathcal{G})$

$$+\,\lambda\sum_{k=2}^{2m+1}\binom{2m+1}{k}(u_0)^{2m+1-k}\lambda^{k-2}(w+\mathcal{G})^k.$$

The Green's function $\mathcal{G}$ admits the expansion

$$\mathcal{G}(\boldsymbol{x}) = \frac{1}{2\pi}\ln\Big(\frac{1}{|\boldsymbol{x}|}\Big) + \gamma_{\mathcal{G}} + \tilde{\mathcal{G}}, \tag{10}$$

[1] See also the beginning of §3.2 for an explanation of this choice.

where $\gamma_{\mathcal{G}}$ is a constant and where $\tilde{\mathcal{G}} \in \mathrm{H}^1(\Omega) \cap \mathscr{C}^\infty(\Omega)$ vanishes at the origin. Thus, for all $k \geq 1$, we have $(\mathcal{G})^k \in \mathrm{L}^2(\Omega)$. We infer that for all $w \in \mathrm{H}^1_0(\Omega)$, $k \geq 1$, the function $(w + \mathcal{G})^k$ belongs to $\mathrm{L}^2(\Omega)$. In addition, according to the assumptions made on the geometry (see the end of Section 2), there holds $u_0 \in \mathscr{C}^0(\overline{\Omega})$. Therefore, for all $\lambda \in \mathbb{R}$, $w \in \mathrm{H}^1_0(\Omega)$, we deduce that $\Phi(\lambda, w)$ is an element of $\mathrm{H}^{-1}(\Omega)$. And more precisely, one can show that $\Phi : \mathbb{R} \times \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ is a map of class $\mathscr{C}^1$. For $\lambda = 0$, we have

$$\Phi(0, w) = -\Delta w + (2m+1)(u_0)^{2m} w + (2m+1)(u_0)^{2m}\mathcal{G}.$$

Since $(u_0)^{2m} \geq 0$, the Lax-Milgram lemma ensures that there is a unique $w_{1,0} \in \mathrm{H}^1_0(\Omega)$ such that $\Phi(0, w_{1,0}) = 0$. On the other hand, if we denote $\partial_w \Phi(0, w_{1,0})$ the differential of Φ with respect to its second argument at the point $(0, w_{1,0})$, for all $\varphi \in \mathrm{H}^1_0(\Omega)$, we find $\partial_w \Phi(0, w_{1,0})\varphi = -\Delta\varphi + (2m + 1)(u_0)^{2m}\varphi$. Again, from Lax-Milgram lemma, we infer that $\partial_w \Phi(0, w_{1,0}) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ is an isomorphism.

We can apply the implicit function theorem, see e.g. [1, Thm. 2.5.7], which yields the existence of $\lambda_\star > 0$ and a continuous function $\lambda \mapsto w[\lambda] \in \mathrm{H}^1_0(\Omega)$ such that $\Phi(\lambda, w[\lambda]) = 0$ for all $\lambda \in (-\lambda_\star; \lambda_\star)$. Then, assuming that $\lambda(\delta)$ tends to zero as δ goes to zero so that $\lambda(\delta) \in (-\lambda_\star; \lambda_\star)$, we set $w_{1,\delta} := w[\lambda(\delta)]$ and

$$u_{1,\delta} = w_{1,\delta} + \mathcal{G}. \tag{11}$$

The function $u_{1,\delta}$ is a solution to Problem (8). Let us describe the behaviour of $u_{1,\delta}(\boldsymbol{x})$ as $\boldsymbol{x} \to 0$. To proceed, we introduce adapted weighted spaces. We define the Kondratiev space $\mathrm{V}^k_\beta(\Omega)$ [17] as the closure of $\mathscr{C}^\infty(\overline{\Omega} \setminus \{O\}) = \{\varphi \in \mathscr{C}^\infty(\overline{\Omega}) \,|\, \varphi = 0 \text{ in a neighbourhood of } O\}$ with respect to the weighted norm

$$\|\varphi\|_{\mathrm{V}^k_\beta(\Omega)} := \Big(\sum_{|\alpha| \leq k} \int_\Omega r^{2(\beta + |\alpha| - k)} |\partial^\alpha_{\boldsymbol{x}} \varphi|^2 \, d\boldsymbol{x}\Big)^{1/2}.$$

From Equation (9), we see that $\Delta w_{1,\delta} \in \mathrm{L}^2(\Omega)$. Kondratiev's theory then shows that there exists a constant $c_{\lambda(\delta)} \in \mathbb{R}$ such that $w_{1,\delta} - c_{\lambda(\delta)} \in \mathrm{V}^1_{-1+\varepsilon}(\Omega)$ for all $\varepsilon > 0$, see for example [22, Thm.5.6]. Therefore from (11), (10), we deduce that we have the decomposition

$$u_{1,\delta}(\boldsymbol{x}) = \frac{1}{2\pi} \ln\Big(\frac{1}{|\boldsymbol{x}|}\Big) \; + \; \gamma_{\mathrm{F},\lambda(\delta)} \; + \; \tilde{u}_{1,\delta}(\boldsymbol{x}), \qquad \text{with } \gamma_{\mathrm{F},\lambda(\delta)} \in \mathbb{R},\; \tilde{u}_{1,\delta} \in \mathrm{V}^1_{-1+\varepsilon}(\Omega). \tag{12}$$

Moreover, Kondratiev's theory [22, Thm.5.6] ensures that $c_{\lambda(\delta)}$ depends continuously on $\Delta w_{1,\delta} \in \mathrm{L}^2(\Omega)$. Since

- $\Delta w_{1,\delta}$ (for the L^2-norm) depends continuously on $\lambda(\delta)$ and $w_{1,\delta}$ (for the H^1-norm);
- $w_{1,\delta}$ (for the H^1-norm) depends continuously on $\lambda(\delta)$;

we conclude that $\lambda(\delta) \mapsto \gamma_{\mathrm{F},\lambda(\delta)}$ is continuous.

Remark 3.1. *We have not yet defined the gauge function $\lambda(\delta)$. However, we have assumed that $\delta \mapsto \lambda(\delta)$ tends to zero as $\delta \to 0$. We will see that the expression of $\lambda(\delta)$ obtained as an outcome of the matching procedure satisfies this requirement (see (23)).*

Finally, for the far field approximation $v_\delta = u_0 + \lambda(\delta) u_{1,\delta}$, we obtain the decomposition

$$v_\delta = u_0(0) + \lambda(\delta)\Big(\frac{1}{2\pi}\ln\Big(\frac{1}{|\boldsymbol{x}|}\Big) + \gamma_{\mathrm{F},\lambda(\delta)}\Big) + \tilde{v}_\delta, \qquad \text{with} \quad \tilde{v}_\delta \in \mathrm{V}^1_{-1+\varepsilon}(\Omega), \quad \forall \varepsilon > 0. \tag{13}$$

Therefore, as $|\boldsymbol{x}| \to 0$, there holds

$$v_\delta(\boldsymbol{x}) = u_0(0) + \lambda(\delta)\Big(\frac{1}{2\pi}\ln\Big(\frac{1}{|\boldsymbol{x}|}\Big) + \gamma_{\mathrm{F},\lambda(\delta)}\Big) + \dots, \tag{14}$$

where the dots stand for terms irrelevant for the formal procedure.

3.2 Near field expansion

In the region close to the small obstacle, because we anticipate a boundary layer phenomenon, we consider another ansatz. We introduce the change of variable $\boldsymbol{\xi} := \boldsymbol{x}/\delta$. As $\delta \to 0$, the region $\{\boldsymbol{\xi} = \boldsymbol{x}/\delta \in \mathbb{R}^2 \,|\, \boldsymbol{x} \in \Omega_\delta\}$ converts into $\mathbb{R}^2 \setminus \overline{\omega}$. Define the function $\boldsymbol{\xi} \mapsto U_\delta(\boldsymbol{\xi})$ such that $U_\delta(\boldsymbol{\xi}) = u_\delta(\delta\boldsymbol{\xi})$. Writing Problem (1) in the $\boldsymbol{\xi}$-coordinates yields

$$-\Delta_{\boldsymbol{\xi}} U_\delta + \delta^2 (U_\delta)^{2m} U_\delta = \delta^2 f(\delta\boldsymbol{\xi}). \tag{15}$$

We look for an approximation of U_δ under the form $\mu(\delta)U_1$ where the function U_1 has to be determined and where the gauge function $\mu(\delta)$ (independent of $\boldsymbol{\xi}$) tends to zero as $\delta \to 0$. Plugging this expansion in (15) and letting δ tends to zero yields $-\Delta U_1 = 0$ in $\mathbb{R}^2 \setminus \overline{\omega}$, $U_1 = 0$ on $\partial\omega$. Assuming that U_1 remains bounded as $|\boldsymbol{\xi}| \to +\infty$ would lead to $U_1 \equiv 0$. This is not reasonable since $\mu(\delta)U_1$ is assumed to be the predominant (non zero) behaviour of U_δ as $\delta \to 0$. Hence we allow U_1 to admit a logarithmic singularity at infinity. Multiplying the gauge function $\mu(\delta)$ by a multiplicative factor if necessary, this finally leads us to consider the following problem

$$\left|\begin{array}{l} \text{Find } U_1 \text{ such that } U_1(\boldsymbol{\xi}) + (2\pi)^{-1}\ln|\boldsymbol{\xi}| \in \mathcal{W}(\mathbb{R}^2 \setminus \overline{\omega}) \text{ and} \\ \begin{array}{rcll} -\Delta U_1 & = & 0 & \text{in } \mathbb{R}^2 \setminus \overline{\omega} \\ U_1 & = & 0 & \text{on } \partial\omega. \end{array}\end{array}\right. \tag{16}$$

In (16), $\mathcal{W}(\mathbb{R}^2 \setminus \overline{\omega})$ is defined as the completion of the set of $\mathscr{C}^\infty$-functions with bounded support for the norm

$$\|U\|^2_{\mathcal{W}(\mathbb{R}^2\setminus\overline{\omega})} := \int_{\mathbb{R}^2\setminus\overline{\omega}} \Big(|\nabla U|^2 + \frac{|U(\boldsymbol{\xi})|^2}{1+|\boldsymbol{\xi}|^2(\ln|\boldsymbol{\xi}|)^2} \Big) d\boldsymbol{\xi}.$$

This space contains functions which are locally in H^1 and bounded at infinity. Applying Kondratiev's analysis allows to prove that the linear problem (16) admits a unique solution U_1. Moreover, there exists a constant γ_{N} that only depends on ω such that there holds the decomposition

$$U_1(\boldsymbol{\xi}) = \frac{1}{2\pi}\ln\Big(\frac{1}{|\boldsymbol{\xi}|}\Big) \;+\; \gamma_{\mathrm{N}} \;+\; \tilde{U}_1(\boldsymbol{\xi}), \qquad \text{with} \quad \tilde{U}_1 \in \mathrm{W}^1_{1-\varepsilon}(\mathbb{R}^2 \setminus \overline{\omega}), \quad \forall \varepsilon > 0. \tag{17}$$

The coefficient γ_{N} is well-known and, commonly in the literature, $\exp(2\pi\gamma_{\mathrm{N}})$ is called the *logarithmic capacity* or the *external conformal radius* of the domain ω [18, 26]. Here, for $k \in \mathbb{N} := \{0, 1, \dots\}$, $\beta \in \mathbb{R}$, the space $\mathrm{W}^k_\beta(\mathbb{R}^2 \setminus \overline{\omega})$ is defined as the completion of the set $\mathscr{C}^\infty_0(\mathbb{R}^2 \setminus \overline{\omega})$ for the norm

$$\|\varphi\|_{\mathrm{W}^k_\beta(\mathbb{R}^2\setminus\overline{\omega})} := \Big(\sum_{|\alpha|\le k} \int_{\mathbb{R}^2\setminus\overline{\omega}} |\boldsymbol{\xi}|^{2(\beta+|\alpha|-k)} |\partial^\alpha_{\boldsymbol{\xi}}\varphi|^2 d\boldsymbol{\xi} \Big)^{1/2}.$$

Finally, the near field approximation V_δ defined by $V_\delta(\boldsymbol{x}) = \mu(\delta)U_1(\boldsymbol{x}/\delta)$ satisfies, as $|\boldsymbol{x}/\delta| \to +\infty$,

$$V_\delta(\boldsymbol{x}) = \mu(\delta)\Big(\frac{1}{2\pi}\ln\Big(\frac{\delta}{|\boldsymbol{x}|}\Big) + \gamma_{\mathrm{N}}\Big) + \dots, \tag{18}$$

where the dots stand for terms irrelevant for the formal procedure.

3.3 Matching principle

To determine the gauge functions $\lambda(\delta)$, $\mu(\delta)$ coming into play respectively in the far field and near field expansions, we apply the matching principle [30, 14]. In the present case, it consists in choosing the gauge functions so that the expansion of $v_\delta(\boldsymbol{x})$ for $\delta \to 0, \boldsymbol{x} \to 0$ coincides with the expansion of $V_\delta(\boldsymbol{x})$ for $\delta \to 0$, $|\boldsymbol{x}/\delta| \to +\infty$. Using expressions (14), (18) and matching only the first two terms of these expansions yields

$$u_0(0) + \lambda(\delta)\Big(\frac{1}{2\pi}\ln\Big(\frac{1}{|\boldsymbol{x}|}\Big) + \gamma_{\mathrm{F},\,\lambda(\delta)}\Big) = \mu(\delta)\Big(\frac{1}{2\pi}\ln\Big(\frac{\delta}{|\boldsymbol{x}|}\Big) + \gamma_{\mathrm{N}}\Big). \tag{19}$$

This leads to impose $\mu(\delta) = \lambda(\delta)$. On the other hand, we find that $\lambda(\delta)$ must satisfy the following fixed point equation

$$\lambda(\delta) = \varphi_\delta(\,\lambda(\delta)\,) \quad \text{where} \quad \varphi_\delta(\,\lambda\,) := \frac{2\pi\, u_0(0)}{\ln\delta + 2\pi(\gamma_{\mathrm{N}} - \gamma_{\mathrm{F},\,\lambda})}\,. \tag{20}$$

Let us emphasize that these types of non linear equations to define the gauge functions do not appear in the study of linear PDEs. Consider some given $\lambda_\star > 0$. For $\lambda,\ \lambda' \in [-\lambda_\star; \lambda_\star]$, we find

$$|\varphi_\delta(\,\lambda\,) - \varphi_\delta(\,\lambda'\,)| = (2\pi)^2 |u(0)|\, \frac{|\gamma_{\mathrm{F},\,\lambda} - \gamma_{\mathrm{F},\,\lambda'}|}{(\ln\delta + 2\pi(\gamma_{\mathrm{N}} - \gamma_{\mathrm{F},\,\lambda'}))(\ln\delta + 2\pi(\gamma_{\mathrm{N}} - \gamma_{\mathrm{F},\,\lambda}))}. \tag{21}$$

Since the map $\lambda \mapsto \gamma_{\mathrm{F},\,\lambda}$ is continuous on the compact set $[-\lambda_\star; \lambda_\star]$, there exists a constant $C > 0$ independent of δ such that $|\gamma_{\mathrm{F},\,\lambda} - \gamma_{\mathrm{F},\,\lambda'}| \le C\,|\lambda - \lambda'|$ for all $\lambda,\ \lambda' \in [-\lambda_\star; \lambda_\star]$. From (20), (21), we deduce that for δ small enough, we have

$$\sup_{\lambda\in[-\lambda_*;+\lambda_*]} |\varphi_\delta(\,\lambda\,)| \le \frac{C}{|\ln\delta|} \qquad \text{and} \qquad |\varphi_\delta(\,\lambda\,) - \varphi_\delta(\,\lambda'\,)| \le \frac{C}{(\ln\delta)^2}\,|\lambda - \lambda'|, \tag{22}$$

where C is independent of δ. The first estimate proves that there is some $\delta_0 > 0$ such that for all $\delta \in (0; \delta_0]$, we have $\varphi_\delta([-\lambda_\star; \lambda_\star]) \subset [-\lambda_\star; \lambda_\star]$. The second one shows that $\lambda \mapsto \varphi_\delta(\,\lambda\,)$ is a contraction mapping of $[-\lambda_\star; \lambda_\star]$. According to the Banach-fixed point theorem, we deduce that for all $\delta \in (0; \delta_0]$, the equation $\lambda = \varphi_\delta(\lambda)$ admits a unique solution in $[-\lambda_\star; \lambda_\star]$. We denote $\lambda(\delta)$ this solution. The relation $\lambda(\delta) = \varphi_\delta(\lambda(\delta))$ and the first estimate of (22) guarantee the existence of some $C > 0$ independent of $\delta \in (0; \delta_0]$ such that $|\lambda(\delta)| \le C\,/|\ln\delta|$. Using the continuous dependence of $\gamma_{\mathrm{F},\,\lambda}$ with respect to λ, we infer that $\gamma_{\mathrm{F},\,\lambda(\delta)}$ remains bounded for $\delta \in (0; \delta_0]$. From (20), we deduce that, as $\delta \to 0$,

$$\lambda(\delta) \;=\; \frac{2\pi u_0(0)}{\ln\delta} \;+\; O\left(\frac{1}{|\ln\delta|^2}\right). \tag{23}$$

With this construction, note that $\lambda(\delta)$ indeed tends to zero as δ goes to zero. In order to simplify notation in the following, we introduce the function m_δ such that

$$m_\delta(\boldsymbol{x}) = u_0(0) + \lambda(\delta)\left(\frac{1}{2\pi}\ln\left(\frac{1}{|\boldsymbol{x}|}\right) + \gamma_{\mathrm{F},\,\lambda(\delta)}\right) = \lambda(\delta)\Big(\frac{1}{2\pi}\ln\left(\frac{\delta}{|\boldsymbol{x}|}\right) + \gamma_{\mathrm{N}}\Big). \tag{24}$$

3.4 Error estimate

Now, we construct a global approximation $\hat{u}_\delta$ of u_δ as an interpolation between the far field and the near field contribution. To proceed and to prove later an error estimate, we use the trick of *overlapping cut-off functions* [20, Chap. 2], [21]. Let $\mathcal{R}^1$ denote a domain such that $\overline{\varpi} \subset \mathcal{R}^1$ and $\overline{\mathcal{R}^1} \subset \Omega$. We introduce $\chi \in \mathscr{C}_0^\infty(\Omega, [0; 1])$ a cut-off function such that $\chi = 1$ on $\mathcal{R}^1$. We set $\psi := 1 - \chi$ and for $t > 0$, we define the functions χ_t, ψ_t such that $\chi_t(\boldsymbol{x}) = \chi(\boldsymbol{x}/t)$, $\psi_t(\boldsymbol{x}) = \psi(\boldsymbol{x}/t)$.

In Ω, define $\hat{u}_\delta$ such that

$$\hat{u}_\delta(\boldsymbol{x}) \;=\; \psi_\delta(\boldsymbol{x})\,v_\delta(\boldsymbol{x}) + \chi(\boldsymbol{x})\,V_\delta(\boldsymbol{x}) - \psi_\delta(\boldsymbol{x})\chi(\boldsymbol{x})\,m_\delta(\boldsymbol{x}) \tag{25}$$

$$\text{where} \quad \left|\begin{array}{lcl} v_\delta(\boldsymbol{x}) & = & u_0(\boldsymbol{x}) + \lambda(\delta)\,u_{1,\,\delta}(\boldsymbol{x}) \\[2mm] V_\delta(\boldsymbol{x}) & = & \lambda(\delta)\,U_1(\boldsymbol{x}/\delta). \end{array}\right. \tag{26}$$

With the above definition for ψ_δ, χ, observe that there holds

$$\psi_\delta + \chi - \psi_\delta\chi = \psi_\delta + \chi - \psi_\delta(1-\psi) = \chi + \psi_\delta\psi = \chi + \psi = 1.$$

With the matching procedure we have enforced that the predominant behaviours of v_δ, V_δ coincide in the matching region. Remark that m_δ is nothing else than the main part of v_δ, V_δ in this region. The following proposition guarantees that $\hat{u}_\delta$ yields a good approximation of u_δ, the solution of Problem (1), as δ goes to zero. Its proof, which is a bit long, is postponed to Section 6.

Proposition 3.1. *Let f be a source term of $\mathrm{L}^2(\Omega)$. For all $\varepsilon > 0$, there exists $\delta_0 > 0$ such that the function $\hat{u}_\delta$ defined by (25) satisfies*

$$\|u_\delta - \hat{u}_\delta\|_{\mathrm{H}^1_0(\Omega_\delta)} \le C\,\delta^{1-\varepsilon}, \qquad \forall \delta \in (0;\delta_0]. \tag{27}$$

In this estimate, the constant $C > 0$ depends on ε, $\|f\|_{\mathrm{L}^2(\Omega)}$ but not on δ.

4 An approximate model for the far field expansion

In the previous section, we constructed and justified by means of an error estimate an asymptotic expansion for the solution u_δ to the original Problem (1). For many applications, only an approximation of the far field of u_δ is needed. Working as in the proof of Proposition 3.1 (Section 6), we can show that $v_\delta = u_0 + \lambda(\delta)u_{1,\delta}$ (see the definition of the terms in §3.1) yields a good approximation of the far field of u_δ. More precisely, for any given source term $f \in \mathrm{L}^2(\Omega)$, for any domain Ξ such that $O \in \Xi$, for all $\varepsilon > 0$, there exists $\delta_0 > 0$ such that there holds

$$\|u_\delta - v_\delta\|_{\mathrm{H}^1_0(\Omega\setminus\overline{\Xi})} \le C\,\delta^{1-\varepsilon}, \qquad \forall \delta \in (0;\delta_0]. \tag{28}$$

In this estimate, again, the constant $C > 0$ depends on ε, $\|f\|_{\mathrm{L}^2(\Omega)}$ but not on δ. From a numerical point of view, the computation of $v_\delta = u_0 + \lambda(\delta)u_{1,\delta}$ may appear reasonable since the functions u_0, $u_{1,\delta}$ are defined as the solutions of problems set in the limit geometry without the small obstacle. However, if one looks more carefully, one sees that the computation of the gauge function $\lambda(\delta)$, which is the solution of the fixed point equation (20), requires a bit of work. And then using $\lambda(\delta)$ in (8) to compute $u_{1,\delta}$ seems very laborious. In the present section, we propose an alternative method which is simpler to implement. More precisely, we derive a new problem set in Ω admitting a solution coinciding with v_δ that can be easily computed. To proceed, we follow the approach of [24, §4]. According to (7), we know that $v_\delta = u_0 + \lambda(\delta)u_{1,\delta}$ verifies the equations

$$\left|\begin{array}{rcll} -\Delta v_\delta + (v_\delta)^{2m+1} & = & f & \text{in } \Omega\setminus\{O\} \\ v_\delta & = & 0 & \text{on } \partial\Omega. \end{array}\right. \tag{29}$$

Moreover, according to (24), v_δ admits the expansion

$$\begin{array}{rcl} v_\delta & = & \lambda(\delta)\Big(\dfrac{1}{2\pi}\ln\Big(\dfrac{\delta}{|\boldsymbol{x}|}\Big) + \gamma_{\mathrm{N}}\Big) + \hat{v}_\delta \\ & = & \lambda(\delta)\Big(\mathcal{G} + \dfrac{1}{2\pi}\ln\delta + \gamma_{\mathrm{N}} - \gamma_{\mathcal{G}}\Big) + \tilde{v}_\delta, \end{array}$$

where $\tilde{v}_\delta \in \mathscr{C}^0(\Omega)$ verifies $\tilde{v}_\delta(O) = 0$. Define

$$\nu(\delta) = \Big|\frac{1}{2\pi}\ln\delta + \gamma_{\mathrm{N}} - \gamma_{\mathcal{G}}\Big|^{-1/2}. \tag{30}$$

Remark that $\nu(\delta)$ is well-defined for δ small enough. With this notation, we have the decomposition $v_\delta = \alpha\nu(\delta)\mathcal{G} + w$ where the pair (α, w) is a solution to the following problem

$$\left|\begin{array}{l} \text{Find } (\alpha, w) \in \mathbb{R}\times(\mathrm{H}^1_0(\Omega)\cap\mathscr{C}^0(\Omega)) \text{ such that} \\ \begin{array}{rcll} -\Delta w + (\alpha\nu(\delta)\mathcal{G} + w)^{2m+1} & = & f & \text{in } \Omega\setminus\{O\} \\ \alpha + \nu(\delta)w(O) & = & 0. & \end{array} \end{array}\right. \tag{31}$$

To obtain (31), we used that $\Delta\mathcal{G} = 0$ in $\Omega\setminus\{O\}$. Now for numerical purposes, we would like to obtain a variational formulation of Problem (31). If the pair (α, w) satisfies (31), then multiplying by $w' \in \mathrm{H}^1_0(\Omega)$ the first equation of (31) and integrating by parts, we find, for all $w' \in \mathrm{H}^1_0(\Omega)$,

$$\int_\Omega \nabla w\cdot\nabla w'\,d\boldsymbol{x} + \int_\Omega (\alpha\nu(\delta)\mathcal{G} + w)^{2m+1}w'\,d\boldsymbol{x} = \int_\Omega f w'\,d\boldsymbol{x}. \tag{32}$$

From the second equation of (31), multiplying by $\alpha' \in \mathbb{R}$, and using that $\mathcal{G}$ is the Green's function centered at the origin of the Dirichlet-Laplacian, we obtain

$$-\alpha\alpha' = \nu(\delta)w(O)\alpha' = -\int_\Omega \Delta w\, \alpha'\nu(\delta)\mathcal{G}\, d\boldsymbol{x} = \int_\Omega \left(f - (\alpha\nu(\delta)\mathcal{G} + w)^{2m+1}\right) \alpha'\nu(\delta)\mathcal{G}\, d\boldsymbol{x}. \tag{33}$$

Summing (32) and (33), we deduce that if the pair (α, w) verifies (31), then it is a solution to the following problem

$$\left|\begin{array}{l} \text{Find } (\alpha, w) \in \mathbb{R} \times \mathrm{H}^1_0(\Omega) \text{ such that} \\ \displaystyle\int_\Omega \nabla w \cdot \nabla w'\, d\boldsymbol{x} + \int_\Omega (\alpha\nu(\delta)\mathcal{G} + w)^{2m+1}(\alpha'\nu(\delta)\mathcal{G} + w')\, d\boldsymbol{x} - \alpha\alpha' \\ \displaystyle= \int_\Omega f(\alpha'\nu(\delta)\mathcal{G} + w')\, d\boldsymbol{x}, \qquad \forall(\alpha', w') \in \mathbb{R} \times \mathrm{H}^1_0(\Omega). \end{array}\right. \tag{34}$$

Routine calculus shows that the solutions to (34) correspond to the stationary points of the functional $\mathcal{J}_\delta : \mathbb{R} \times \mathrm{H}^1_0(\Omega) \to \mathbb{R}$ such that

$$\begin{aligned} \mathcal{J}_\delta(\alpha, w) := \frac{1}{2}\int_\Omega |\nabla w|^2\, d\boldsymbol{x} + \frac{1}{2m+2}\int_\Omega |\alpha\nu(\delta)\mathcal{G} + w|^{2m+2}\, d\boldsymbol{x} - \frac{\alpha^2}{2} \\ - \int_\Omega f(\alpha\nu(\delta)\mathcal{G} + w)\, d\boldsymbol{x}. \end{aligned} \tag{35}$$

Reciprocally, now assume that (α, w) is a solution to (34). As test function, take first $\alpha' = 0$ and $w' \in \mathrm{H}^1_0(\Omega)$ arbitrary. We find that (α, w) must verify $-\Delta w + (\alpha\nu(\delta)\mathcal{G} + w)^{2m+1} = f$ in Ω. From this we deduce that $w \in \mathrm{H}^1_0(\Omega) \cap \mathscr{C}^0(\Omega)$. We obtain in particular

$$w(O) = -\int_\Omega \Delta w\, \mathcal{G}\, d\boldsymbol{x} = \int_\Omega \left(f - (\alpha\nu(\delta)\mathcal{G} + w)^{2m+1}\right) \mathcal{G}\, d\boldsymbol{x}. \tag{36}$$

Now take $\alpha' = 1$ and $w' = 0$ in (34). This gives $-\alpha = \int_\Omega \left(f - (\alpha\nu(\delta)\mathcal{G} + w)^{2m+1}\right) \nu(\delta)\mathcal{G}\, d\boldsymbol{x}$ and, according to (36), proves that $\alpha + \nu(\delta)w(O) = 0$. We infer that the pair (α, w) verifies (31). Practically, one can solve Problem (34). Then defining $v_\delta = \alpha\nu(\delta)\mathcal{G} + w$ yields a good approximation of the far field of the original Problem (1). Note that numerically, this approach is much more simpler to implement than computing each term of the other decomposition $v_\delta = u_0 + \lambda(\delta)u_{1,\delta}$. This idea of modifying the expansion of v_δ to obtain something easy to compute was used in [9] for a Laplacian with a small Dirichlet obstacle. However, in the latter work, a different decomposition was employed. We end this section by proving a result of local well-posedness for Problem (34).

Proposition 4.1. *Consider some arbitrary $f \in \mathrm{L}^2(\Omega)$. There are some $\varepsilon, \delta_0 > 0$ such that, for all $\delta \in (0; \delta_0]$, there is a unique pair $(\alpha_\delta, w_\delta) \in \mathbb{R} \times \mathrm{H}^1_0(\Omega)$ solution to (34) satisfying $\|w_\delta - u_0\|_{\mathrm{H}^1(\Omega)} + |\alpha_\delta| < \varepsilon$. In this statement, u_0 refers to the unique solution to (6), the limit problem without the obstacle.*

Proof. Define $\Psi : \mathbb{R} \times (\mathbb{R} \times \mathrm{H}^1_0(\Omega)) \to (\mathbb{R} \times \mathrm{H}^1_0(\Omega))^*$ the map such that, for all $\nu \in \mathbb{R}$, $\mathfrak{u} = (\alpha, w)$, $\mathfrak{u}' = (\alpha', w') \mapsto \langle \Psi(\nu, \mathfrak{u}), \mathfrak{u}' \rangle$ is given by (see (34))

$$\langle \Psi(\nu, \mathfrak{u}), \mathfrak{u}' \rangle = \int_\Omega \nabla w \cdot \nabla w'\, d\boldsymbol{x} + \int_\Omega (\alpha\nu\mathcal{G} + w)^{2m+1}(\alpha'\nu\mathcal{G} + w')\, d\boldsymbol{x} - \alpha\alpha' - \int_\Omega f(\alpha'\nu\mathcal{G} + w')\, d\boldsymbol{x}.$$

Set $\mathfrak{u}_0 = (0, u_0)$. Observe that $\Psi(0, \mathfrak{u}_0) = 0$. The map Ψ is clearly of class $\mathscr{C}^1$. In addition, the partial differential $\partial_\mathfrak{u}\Psi(0, \mathfrak{u}_0) : \mathbb{R} \times \mathrm{H}^1_0(\Omega) \to (\mathbb{R} \times \mathrm{H}^1_0(\Omega))^*$ of Ψ evaluated in the direction $\tilde{\mathfrak{u}} = (\tilde{\alpha}, \tilde{w})$ is given by

$$\langle \partial_\mathfrak{u}\Psi(0, \mathfrak{u}_0)\, \tilde{\mathfrak{u}}, \mathfrak{u}' \rangle = \int_\Omega \nabla \tilde{w} \cdot \nabla w'\, d\boldsymbol{x} + (2m+1)\int_\Omega u_0^{2m}\tilde{w}w'\, d\boldsymbol{x} - \tilde{\alpha}\alpha'$$

for all $\mathfrak{u}' = (\alpha', w')$. Let us prove that $\partial_{\mathfrak{u}}\Psi(0, \mathfrak{u}_0)$ is a continuous isomorphism. Pick some $\ell \in (\mathbb{R} \times \mathrm{H}^1_0(\Omega))^*$. We wish to prove that there is a unique $\tilde{\mathfrak{u}} \in \mathbb{R} \times \mathrm{H}^1_0(\Omega)$ such that

$$\langle \partial_{\mathfrak{u}}\Psi(0, \mathfrak{u}_0)\, \tilde{\mathfrak{u}}, \mathfrak{u}' \rangle = \ell(\mathfrak{u}'), \qquad \forall \mathfrak{u}' \in \mathbb{R} \times \mathrm{H}^1_0(\Omega). \tag{37}$$

Define the map $\mathtt{T} : \mathbb{R} \times \mathrm{H}^1_0(\Omega) \to \mathbb{R} \times \mathrm{H}^1_0(\Omega)$ such that, for $\mathfrak{u}' = (\alpha', w')$, $\mathtt{T}\mathfrak{u}' = (-\alpha', w')$. Observe that $\mathtt{T}$ is an isomorphism. As a consequence, $\tilde{\mathfrak{u}} \in \mathbb{R} \times \mathrm{H}^1_0(\Omega)$ verifies (37) if and only if it satisfies

$$\langle \partial_{\mathfrak{u}}\Psi(0, \mathfrak{u}_0)\, \tilde{\mathfrak{u}}, \mathtt{T}\mathfrak{u}' \rangle = \ell(\mathtt{T}\mathfrak{u}'), \qquad \forall \mathfrak{u}' \in \mathbb{R} \times \mathrm{H}^1_0(\Omega). \tag{38}$$

Since the map $(\tilde{\mathfrak{u}}, \mathfrak{u}') \mapsto \langle \partial_{\mathfrak{u}}\Psi(0, \mathfrak{u}_0)\, \tilde{\mathfrak{u}}, \mathtt{T}\mathfrak{u}' \rangle$ is coercive, according to the Lax-Milgram lemma, we know that there is a unique $\tilde{\mathfrak{u}} \in \mathbb{R} \times \mathrm{H}^1_0(\Omega)$ satisfying (38). This shows that $\partial_{\mathfrak{u}}\Psi(0, \mathfrak{u}_0)$ is a continuous isomorphism. As a consequence, we can apply the implicit function theorem [1, Thm. 2.5.7] to obtain the result of Proposition 4.1. $\square$

5 Numerical experiments

In this section, we provide numerical results illustrating the interest of the approximate model. First, we describe the setting of the experiments. For $t > 0$, we denote $\mathrm{D}_t \subset \mathbb{R}^2$ the disk centered at O of radius t. We shall assume that $\Omega = \mathrm{D}_1$. In this geometry, the Green's function $\mathcal{G}$ such that $-\Delta\mathcal{G} = \delta_O$ in Ω, $\mathcal{G} = 0$ on $\partial\Omega$, is given by $\mathcal{G} = (2\pi)^{-1} \ln |\boldsymbol{x}|^{-1}$. Therefore, the constant $\gamma_{\mathcal{G}}$ appearing in the decomposition of $\mathcal{G}$ (10), and used in the definition of $\nu(\delta)$ (see (30)), is $\gamma_{\mathcal{G}} = 0$. For the small obstacle, we shall consider two situations:

- ω_δ is the disk centered at O of radius δ;
- ω_δ is the ellipse centered at O of semi-axes δ ((Ox) axis) and 2δ ((Oy) axis).

We consider the problem of finding $u_\delta \in \mathrm{H}^1_0(\Omega_\delta)$ such that

$$-\Delta u_\delta + (u_\delta)^{2m+1} = f \quad \text{in } \Omega_\delta = \Omega \setminus \overline{\omega_\delta}, \quad \text{and} \quad u_\delta = 0 \quad \text{on } \partial\Omega_\delta. \tag{39}$$

Let Ω^h be a polygonal approximation of the domain $\Omega = \Omega_0$. Introduce $(\mathcal{T}^h)_h$ a shape regular family of triangulations of $\overline{\Omega^h}$. Here, h denotes the average mesh size. Define the family of finite element spaces

$$\mathrm{V}^h := \left\{ \varphi \in \mathrm{H}^1_0(\Omega^h) \text{ such that } \varphi|_\tau \in \mathbb{P}_1(\tau) \text{ for all } \tau \in \mathcal{T}^h \right\},$$

where $\mathbb{P}_1(\tau)$ is the space of polynomials of degree at most 1 on the triangle τ. In what follows, the errors are expressed in the norm $\|\cdot\|_{\mathrm{H}^1(\Omega^h \setminus \mathrm{D}_\rho)}$ with $\rho = 0.15$. For the computations, we use the *FreeFem++*[2] software.

In the numerical experiments, we first approximate the solution u_0 to the simple limit problem (6). We remind the reader that this problem writes

$$\left| \begin{array}{l} \text{Find } u_0 \in \mathrm{H}^1_0(\Omega) \text{ such that} \\ \displaystyle\int_\Omega \nabla u_0 \cdot \nabla v + (u_0)^{2m} u_0 v \, d\boldsymbol{x} = \int_\Omega f v \, d\boldsymbol{x}, \qquad \forall v \in \mathrm{H}^1_0(\Omega). \end{array} \right. \tag{40}$$

To proceed, we use the following algorithm.

Algorithm 1. *Set $u_{0h}^{[0]} \equiv 0$ and select a stopping criterion $\eta > 0$. If $u_{0h}^{[n]}$ is known, define $u_{0h}^{[n+1]}$ as the solution to the (linear) problem*

$$\left| \begin{array}{l} \textit{Find } u_{0h}^{[n+1]} \in \mathrm{V}^h \textit{ such that} \\ \displaystyle\int_{\Omega^h} \nabla u_{0h}^{[n+1]} \cdot \nabla v + (u_{0h}^{[n]})^{2m} u_{0h}^{[n+1]} v \, d\boldsymbol{x} = \int_{\Omega^h} f v \, d\boldsymbol{x}, \qquad \forall v \in \mathrm{V}^h. \end{array} \right. \tag{41}$$

Run the procedure until the inequality $\|u_{0h}^{[n+1]} - u_{0h}^{[n]}\|_{\mathrm{H}^1(\Omega^h)} < \eta$ is satisfied.

[2] *FreeFem++*, http://www.freefem.org/ff++/.

We also discretize the model problem introduced in (34) which gives a better approximation of the far field of u_δ than u_0 does. This problem states

$$
\left|
\begin{array}{l}
\text{Find } (\alpha, w) \in \mathbb{R} \times \mathrm{H}^1_0(\Omega) \text{ such that} \\
\displaystyle\int_\Omega \nabla w \cdot \nabla w' \, d\boldsymbol{x} + \int_\Omega (\alpha \nu(\delta) \mathcal{G} + w)^{2m+1} (\alpha' \nu(\delta) \mathcal{G} + w') \, d\boldsymbol{x} - \alpha\alpha' \\
\qquad\qquad\qquad\qquad = \displaystyle\int_\Omega f(\alpha' \nu(\delta) \mathcal{G} + w') \, d\boldsymbol{x}, \qquad \forall (\alpha', w') \in \mathbb{R} \times \mathrm{H}^1_0(\Omega).
\end{array}
\right.
\tag{42}
$$

To compute an approximation of (α, w), we implement the following algorithm.

Algorithm 2. *Set $(\alpha_h^{[0]}, w_h^{[0]}) = (0, 0)$ and select a stopping criterion $\eta > 0$. If $(\alpha_h^{[n]}, w_h^{[n]})$ is known, define $(\alpha_h^{[n+1]}, w_h^{[n+1]})$ as the solution to the (linear) problem*

$$
\left|
\begin{array}{l}
\textit{Find } (\alpha_h^{[n+1]}, w_h^{[n+1]}) \in \mathbb{R} \times \mathrm{V}^h \textit{ such that} \\
\displaystyle\int_{\Omega^h} \nabla w_h^{[n+1]} \cdot \nabla w' \, d\boldsymbol{x} + \int_{\Omega^h} (\alpha_h^{[n]} \nu(\delta) \mathcal{G} + w_h^{[n]})^{2m} (\alpha_h^{[n+1]} \nu(\delta) \mathcal{G} + w_h^{[n+1]}) (\alpha' \nu(\delta) \mathcal{G} + w') \, d\boldsymbol{x} \\
\qquad\qquad\qquad\qquad - \alpha_h^{[n+1]} \alpha' = \displaystyle\int_{\Omega_h} f(\alpha' \nu(\delta) \mathcal{G} + w') \, d\boldsymbol{x}, \qquad \forall (\alpha', w') \in \mathbb{R} \times \mathrm{V}^h.
\end{array}
\right.
\tag{43}
$$

Run the procedure until the inequality $\|(\alpha_h^{[n+1]}, w_h^{[n+1]}) - (\alpha_h^{[n]}, w_h^{[n]})\|_{\mathbb{R} \times \mathrm{H}^1(\Omega^h)} < \eta$ is satisfied.

In the following, we denote u_{0h} (resp. (α_h, w_h)) the solution obtained at the end of Algorithm 1 (resp. Algorithm 2). In these iterative procedures, the stopping criterion η is set to $\eta = 10^{-8}$.

$\star$ **Disk shaped obstacle.** When the obstacle is a small disk centered at O, we can compute explicit solutions for special source terms. In Ω_δ, define the functions

$$
u_{\mathrm{e}} := \frac{\ln |\boldsymbol{x}|}{\ln \delta} - \frac{1 - |\boldsymbol{x}|^2}{1 - \delta^2} \qquad \text{and} \qquad f := -\Delta u_{\mathrm{e}} + (u_{\mathrm{e}})^{2m+1} = \frac{-4}{1 - \delta^2} + (u_{\mathrm{e}})^{2m+1}.
$$

Observe that we have $u_{\mathrm{e}} \in \mathrm{H}^1_0(\Omega_\delta)$ and that f is an element of $\mathrm{L}^2(\Omega_\delta)$. On the other hand, note that, with $\omega_\delta = \mathrm{D}_\delta$, we have $\omega = \omega_1 = \mathrm{D}_1$ so that the logarithmic capacity potential U_1 defined by (16) verifies $U_1(\boldsymbol{\xi}) = (2\pi)^{-1} \ln |\boldsymbol{\xi}|^{-1}$. As a consequence, the parameter γ_N appearing in the definition of $\nu(\delta)$ (see (30)) satisfies $\gamma_N = 0$. Therefore, for this configuration we have $\nu(\delta) = |(2\pi)^{-1} \ln \delta|^{-1/2}$. In Figures 1 and 2, we display the evolution of the errors $\|u_{0h} - u_{\mathrm{e}}\|_{\mathrm{H}^1(\Omega^h \setminus \mathrm{D}_\rho)}$, $\|\alpha_h \nu(\delta) \mathcal{G} + w_h - u_{\mathrm{e}}\|_{\mathrm{H}^1(\Omega^h \setminus \mathrm{D}_\rho)}$ for an index of non linearity respectively equal to 1 and 3. For both cases, we observe that $\alpha_h \nu(\delta) \mathcal{G} + w_h$ yields a better approximation of the far field of u_{e} than u_{0h}. Moreover, we notice that even for very small values of δ ($\delta = 10^{-8}$), u_{0h} is a relatively poor approximation of the far field of u_{e}. On the other hand, we remark that $\|\alpha_h \nu(\delta) \mathcal{G} + w_h - u_{\mathrm{e}}\|_{\mathrm{H}^1(\Omega^h \setminus \mathrm{D}_\rho)}$ does not change much for small values of δ. This is due to the fact that the main (singular) part of u_{e} (equal to $\ln |\boldsymbol{x}| / \ln \delta$) is correctly approximated and that the smooth part of u_{e} (equal to $(1 - |\boldsymbol{x}|^2)/(1 - \delta^2)$) does not depend much on δ for small δ. When the mesh is refined, that is when the number Nt of triangles defining the triangulation of $\overline{\Omega^h}$ increases, basically the error $\|u_{0h} - u_{\mathrm{e}}\|_{\mathrm{H}^1(\Omega^h \setminus \mathrm{D}_\rho)}$ stays the same. The reason is that, even for $\delta = 10^{-8}$, the error in the model (in δ) is predominant. On the contrary, (34) yields a good model for the far field of u_{e} which gets more and more accurate as δ tends to zero. Therefore, in this case, refining the mesh improves the quality of the approximation (especially when δ is small). Finally, we note that changing the index of non linearity m does not affect much the results.

$\star$ **Ellipse shaped obstacle.** For the second series of experiments, ω_δ is the ellipse centered at O of semi-axes δ ((Ox) axis) and 2δ ((Oy) axis). With such a choice $\omega = \omega_1$ is the ellipse centered at O of semi-axes 1 and 2. Therefore (see e.g. [28]), the coefficient γ_N appearing in the decomposition of the logarithmic capacity potential U_1 defined in (16) verifies $\gamma_N = \ln(3/2)/(2\pi)$. From definition (30), we infer that $\nu(\delta) = |(2\pi)^{-1} \ln \delta + \gamma_N|^{-1/2}$. We take a source term f such that for $\boldsymbol{x} = (x, y)$,

	$\ln(\|u_{0h} - u_{\mathrm{e}}\|_{\mathrm{H}^1(\Omega^h\setminus \mathrm{D}_\rho)})$			$\ln(\|\alpha_h \nu(\delta)\mathcal{G} + w_h - u_{\mathrm{e}}\|_{\mathrm{H}^1(\Omega^h\setminus \mathrm{D}_\rho)})$		
δ	Nt=3924	Nt=15508	Nt=35688	Nt=3924	Nt=15508	Nt=35688
10^{-1}	0.14577	0.146261	0.146236	−1.77798	−1.80248	−1.80248
10^{-2}	−0.176527	−0.176082	−0.176154	−2.18553	−2.46737	−2.72857
10^{-4}	−0.49295	−0.492738	−0.492857	−2.18525	−2.46742	−2.72878
10^{-6}	−0.674594	−0.674666	−0.674838	−2.18514	−2.46739	−2.72877
10^{-8}	−0.802204	−0.802597	−0.802832	−2.18507	−2.46738	−2.72876

Figure 1: Errors with respect to δ for an index of non linearity $m = 1$. The parameter Nt corresponds to the number of triangles defining the triangulation of $\overline{\Omega^h}$.

	$\ln(\|u_{0h} - u_{\mathrm{e}}\|_{\mathrm{H}^1(\Omega^h\setminus \mathrm{D}_\rho)})$			$\ln(\|\alpha_h \nu(\delta)\mathcal{G} + w_h - u_{\mathrm{e}}\|_{\mathrm{H}^1(\Omega^h\setminus \mathrm{D}_\rho)})$		
δ	Nt=3924	Nt=15508	Nt=35688	Nt=3924	Nt=15508	Nt=35688
10^{-1}	0.163072	0.162999	0.163442	−1.79011	−1.96852	−1.8197
10^{-2}	−0.163489	−0.163087	−0.163159	−2.18574	−2.46738	−2.72853
10^{-4}	−0.498904	−0.498748	−0.498882	−2.18555	−2.4675	−2.72882
10^{-6}	−0.693318	−0.693495	−0.6937	−2.1854	−2.46746	−2.7288
10^{-8}	−0.829154	−0.829727	−0.830013	−2.18529	−2.46744	−2.72879

Figure 2: Errors with respect to δ for an index of non linearity $m = 3$. The parameter Nt corresponds to the number of triangles defining the triangulation of $\overline{\Omega^h}$.

$f(\boldsymbol{x}) = 1+x+y$. In this setting, we cannot use separation of variables to compute the exact solution $u_\delta \in \mathrm{H}^1_0(\Omega_\delta)$ to problem (1). For that reason, in order to assess the quality of our model (34), we compute a direct approximation of u_δ meshing the domain Ω_δ with the small obstacle. Of course this is precisely the operation that we would like to avoid because it requires to use a very refined mesh close to the obstacle. We emphasize that here, it is just a way to validate our model. Now, we explain how to approximate u_δ.

Let Ω^h_δ be a polygonal approximation of the domain Ω_δ. Introduce $(\mathcal{T}^h_\delta)_h$ a shape regular family of triangulations of $\overline{\Omega^h_\delta}$. Here, h denotes the average mesh size. Define the family of finite element spaces

$$\mathrm{V}^h_\delta := \left\{\varphi \in \mathrm{H}^1_0(\Omega^h_\delta) \text{ such that } \varphi|_\tau \in \mathbb{P}_1(\tau) \text{ for all } \tau \in \mathcal{T}^h_\delta\right\},$$

Consider the following algorithm.

Algorithm 3. *Set $u^{[0]}_{\delta h} \equiv 0$ and select a stopping criterion $\eta > 0$. If $u^{[n]}_{\delta h}$ is known, define $u^{[n+1]}_{\delta h}$ as the solution to the (linear) problem*

$$\left|\begin{array}{l} \textit{Find } u^{[n+1]}_{\delta h} \in \mathrm{V}^h_\delta \textit{ such that} \\ \displaystyle\int_{\Omega^h_\delta} \nabla u^{[n+1]}_{\delta h} \cdot \nabla v + (u^{[n]}_{\delta h})^{2m} u^{[n+1]}_{\delta h} v \, d\boldsymbol{x} = \int_{\Omega^h_\delta} f v \, d\boldsymbol{x}, \qquad \forall v \in \mathrm{V}^h_\delta. \end{array}\right. \tag{44}$$

Run the procedure until the inequality $\|u^{[n+1]}_{\delta h} - u^{[n]}_{\delta h}\|_{\mathrm{H}^1(\Omega^h)} < \eta$ is satisfied.

In the following, we denote $u_{\delta h}$ the solution obtained at the end of Algorithm 3. In the procedure, the stopping criterion η is again set to $\eta = 10^{-8}$.

In Figures 3 and 4, we display the evolution of the errors $\|u_{0h} - u_{\delta h}\|_{\mathrm{H}^1(\Omega^h\setminus\mathrm{D}_\rho)}$, $\|\alpha_h\nu(\delta)\mathcal{G} + w_h - u_{\delta h}\|_{\mathrm{H}^1(\Omega^h\setminus\mathrm{D}_\rho)}$ for an index of non linearity respectively equal to 1 and 3. The conclusions are the same as for the case of the small obstacle being a disk. Note however that we cannot consider very small values of δ because we need to mesh the domain with the small obstacle Ω_δ to compute the reference solution $u_{\delta h}$.

	$\ln(\|u_{0h} - u_{\delta h}\|_{\mathrm{H}^1(\Omega^h\setminus\mathrm{D}_\rho)})$			$\ln(\|\alpha_h\nu(\delta)\mathcal{G} + w_h - u_{\delta h}\|_{\mathrm{H}^1(\Omega^h\setminus\mathrm{D}_\rho)})$		
δ	Nt=3700	Nt=10936	Nt=14892	Nt=3700	Nt=10936	Nt=14892
5.10^{-2}	-0.462198	-0.463417	-0.463421	-1.59618	-1.61939	-1.61931
10^{-2}	-0.668966	-0.672756	-0.672772	-2.47907	-2.88783	-2.90098
5.10^{-3}	-0.735467	-0.739402	-0.73942	-2.5395	-3.00765	-3.02612
10^{-3}	-0.859438	-0.863692	-0.863714	-2.64253	-3.12358	-3.14209
5.10^{-4}	-0.903445	-0.90787	-0.907893	-2.67254	-3.15314	-3.17106

Figure 3: Errors with respect to δ for an index of non linearity $m = 1$. The parameter Nt corresponds to the number of triangles defining the triangulation of $\overline{\Omega^h}$.

	$\ln(\|u_{0h} - u_{\delta h}\|_{\mathrm{H}^1(\Omega^h\setminus\mathrm{D}_\rho)})$			$\ln(\|\alpha_h\nu(\delta)\mathcal{G} + w_h - u_{\delta h}\|_{\mathrm{H}^1(\Omega^h\setminus\mathrm{D}_\rho)})$		
δ	Nt=3700	Nt=10936	Nt=14892	Nt=3700	Nt=10936	Nt=14892
5.10^{-2}	-0.457329	-0.458237	-0.458534	-1.59529	-1.61387	-1.61843
10^{-2}	-0.66236	-0.665653	-0.666128	-2.47666	-2.80524	-2.89894
5.10^{-3}	-0.728337	-0.731665	-0.732251	-2.53655	-2.88955	-3.02367
10^{-3}	-0.851417	-0.855016	-0.855654	-2.63857	-3.00151	-3.13891
5.10^{-4}	-0.895138	-0.898947	-0.899548	-2.66818	-3.0373	-3.16755

Figure 4: Errors with respect to δ for an index of non linearity $m = 3$. The parameter Nt corresponds to the number of triangles defining the triangulation of $\overline{\Omega^h}$.

6 Proof of Proposition 3.1

In this section, we show the error estimate of Proposition 3.1. We use the same notation as in §3.4. Additionally, we introduce $\mathcal{R}^2$ a domain such that $\mathrm{supp}(\chi) \subset \mathcal{R}^2$ and $\overline{\mathcal{R}^2} \subset \Omega$. Here $\mathrm{supp}(\chi)$ stands for the support of the function χ. For $t > 0$, we set

$$\mathcal{R}^1_t := \{\boldsymbol{x} \in \Omega\,|\,\boldsymbol{x}/t \in \mathcal{R}^1\} \qquad \text{and} \qquad \mathcal{R}^2_t := \{\boldsymbol{x} \in \Omega\,|\,\boldsymbol{x}/t \in \mathcal{R}^2\}.$$

Note that for all $\delta \in (0;1]$, we have

$$\omega_\delta \subset \overline{\omega_\delta} \subset \mathcal{R}^1_\delta \subset \overline{\mathcal{R}^1_\delta} \subset \mathcal{R}^2_\delta \subset \overline{\mathcal{R}^2_\delta} \subset \Omega.$$

The stability estimate (4) gives

$$\|u_\delta - \hat{u}_\delta\|_{\mathrm{H}^1_0(\Omega_\delta)} \le \|\mathrm{A}_\delta(u_\delta) - \mathrm{A}_\delta(\hat{u}_\delta)\|_{\mathrm{H}^{-1}(\Omega_\delta)}. \tag{45}$$

Let us compute the right hand side of this inequality. Consider some $\varphi \in \mathrm{H}^1_0(\Omega_\delta)$ such that $\|\varphi\|_{\mathrm{H}^1_0(\Omega_\delta)} = 1$. By definition of A_δ (see (2)), we have

$$\langle \mathrm{A}_\delta(u_\delta) - \mathrm{A}_\delta(\hat{u}_\delta), \varphi\rangle = \int_{\Omega_\delta} f\varphi\, d\boldsymbol{x} - \int_{\Omega_\delta} \nabla\hat{u}_\delta \cdot \nabla\varphi + (\hat{u}_\delta)^{2m+1}\varphi\, d\boldsymbol{x}. \tag{46}$$

A direct calculus yields

$$
\begin{aligned}
& \nabla \hat{u}_\delta \cdot \nabla \varphi \\
= \; & \nabla(\psi_\delta \, v_\delta + \chi V_\delta - \psi_\delta \, \chi \, m_\delta) \cdot \nabla \varphi \\
= \; & (\psi_\delta \nabla v_\delta + \chi \nabla V_\delta - \psi_\delta \, \chi \, \nabla m_\delta) \cdot \nabla \varphi + (v_\delta \nabla \psi_\delta + V_\delta \nabla \chi - m_\delta \nabla(\psi_\delta \, \chi)) \cdot \nabla \varphi \\
= \; & \nabla v_\delta \cdot \nabla(\psi_\delta \, \varphi) + \nabla V_\delta \cdot \nabla(\chi \, \varphi) - \nabla m_\delta \cdot \nabla(\psi_\delta \, \chi \, \varphi) \\
& + (v_\delta \nabla \varphi - \varphi \nabla v_\delta) \cdot \nabla \psi_\delta + (V_\delta \nabla \varphi - \varphi \nabla V_\delta) \cdot \nabla \chi - (m_\delta \nabla \varphi - \varphi \nabla m_\delta) \cdot \nabla(\psi_\delta \, \chi).
\end{aligned}
$$

Observing that there holds $\nabla(\psi_\delta \, \chi) = \nabla \psi_\delta + \nabla \chi$, we obtain

$$
\begin{aligned}
& \nabla \hat{u}_\delta \cdot \nabla \varphi \\
= \; & \nabla v_\delta \cdot \nabla(\psi_\delta \, \varphi) + \nabla V_\delta \cdot \nabla(\chi \, \varphi) - \nabla m_\delta \cdot \nabla(\psi_\delta \, \chi \, \varphi) \\
& + \Big((v_\delta - m_\delta) \nabla \varphi - \varphi \nabla (v_\delta - m_\delta) \Big) \cdot \nabla \psi_\delta + \Big((V_\delta - m_\delta) \nabla \varphi - \varphi \nabla (V_\delta - m_\delta) \Big) \cdot \nabla \chi.
\end{aligned}
$$

Integrating by parts and using that $\Delta V_\delta = \Delta m_\delta = 0$ in Ω_δ, we deduce that

$$
\begin{aligned}
& \int_{\Omega_\delta} \nabla \hat{u}_\delta \cdot \nabla \varphi \, d\boldsymbol{x} = \int_{\Omega_\delta} \nabla v_\delta \cdot \nabla(\psi_\delta \, \varphi) \, d\boldsymbol{x} \\
& + \int_{\Omega_\delta} \Big((v_\delta - m_\delta) \nabla \varphi - \varphi \nabla (v_\delta - m_\delta) \Big) \cdot \nabla \psi_\delta + \Big((V_\delta - m_\delta) \nabla \varphi - \varphi \nabla (V_\delta - m_\delta) \Big) \cdot \nabla \chi \, d\boldsymbol{x}.
\end{aligned}
\tag{47}
$$

The function v_δ satisfies Problem (7). Since the support of $\psi_\delta \, \varphi$ excludes a neighbourhood of the origin, we can integrate by parts in the first term of right hand side of (47) to obtain

$$
\int_{\Omega_\delta} \nabla v_\delta \cdot \nabla(\psi_\delta \, \varphi) \, d\boldsymbol{x} = \int_{\Omega_\delta} f \psi_\delta \, \varphi \, d\boldsymbol{x} - \int_{\Omega_\delta} v_\delta^{2m+1} \psi_\delta \, \varphi \, d\boldsymbol{x}. \tag{48}
$$

Plugging (48) in (47), observing that $1 - \psi_\delta = \chi_\delta$ and using (46), we find

$$
\begin{aligned}
& \langle \mathrm{A}_\delta(u_\delta) - \mathrm{A}_\delta(\hat{u}_\delta), \varphi \rangle = \int_{\Omega_\delta} f \chi_\delta \varphi \, d\boldsymbol{x} + \int_{\Omega_\delta} v_\delta^{2m+1} \psi_\delta \, \varphi - \hat{u}_\delta^{2m+1} \varphi \, d\boldsymbol{x} \\
& - \int_{\Omega_\delta} \Big((v_\delta - m_\delta) \nabla \varphi - \varphi \nabla (v_\delta - m_\delta) \Big) \cdot \nabla \psi_\delta + \Big((V_\delta - m_\delta) \nabla \varphi - \varphi \nabla (V_\delta - m_\delta) \Big) \cdot \nabla \chi \, d\boldsymbol{x}.
\end{aligned}
\tag{49}
$$

It remains to estimate each of the terms of the right hand side of (49).

$\star$ Let us consider the first one. Cauchy-Schwarz inequality implies

$$
\Big| \int_{\Omega_\delta} f \chi_\delta \varphi \, d\boldsymbol{x} \Big| \le \| f \|_{\mathrm{L}^2(\Omega)} \, \| \chi_\delta \varphi \|_{\mathrm{L}^2(\Omega_\delta)}. \tag{50}
$$

On the other hand, using the logarithmic Hardy inequality (see e.g. [22, p. 38]), one can prove the classical estimate

$$
\int_{\Omega} \frac{\zeta^2}{|\boldsymbol{x}|^2 (\ln |\boldsymbol{x}|)^2} \, d\boldsymbol{x} \le C \int_{\Omega} |\nabla \zeta|^2 \, d\boldsymbol{x}, \qquad \forall \zeta \in \mathrm{H}^1_0(\Omega). \tag{51}
$$

The support of $\chi_\delta \varphi$ is included in $\mathcal{R}^2_\delta$. Remarking that there is a $d_2 > 0$ (independent of δ) such that $\mathcal{R}^2_\delta \subset \mathrm{D}_{d_2 \delta}$ (for $t > 0$, $\mathrm{D}_{d_2 \delta}$ stands for the disk centered at O of radius t), we can write

$$
\begin{aligned}
\| \chi_\delta \varphi \|^2_{\mathrm{L}^2(\Omega_\delta)} & \le C \, \delta^2 (\ln \delta)^2 \int_{\Omega} \frac{\chi_\delta^2 \varphi^2}{|\boldsymbol{x}|^2 (\ln |\boldsymbol{x}|)^2} \, d\boldsymbol{x} \\
& \le C \, \delta^2 (\ln \delta)^2 \int_{\Omega} \frac{\varphi^2}{|\boldsymbol{x}|^2 (\ln |\boldsymbol{x}|)^2} \, d\boldsymbol{x} \le C \, \delta^2 (\ln \delta)^2 \| \varphi \|^2_{\mathrm{H}^1_0(\Omega_\delta)}.
\end{aligned}
\tag{52}
$$

Here and in the following $C>0$ denotes a constant independent of δ which may change from one occurrence to another. The last line of (52) has been obtained extending $\varphi\in\mathrm{H}^1_0(\Omega_\delta)$ by zero on ω_δ and using (51). Plugging (52) in (50), we deduce

$$\Big|\int_{\Omega_\delta}\chi_\delta f\varphi\,d\boldsymbol{x}\Big|\le C\,\delta\,|\ln\delta|\,\|f\|_{\mathrm{L}^2(\Omega)}\,\|\varphi\|_{\mathrm{H}^1_0(\Omega_\delta)}. \tag{53}$$

⋆ Now, we work on the second term of the right hand side of (49). We decompose it as

$$\int_{\Omega_\delta} v_\delta^{2m+1}\psi_\delta\,\varphi-\hat{u}_\delta^{2m+1}\varphi\,d\boldsymbol{x}=\int_{\Omega_\delta}(v_\delta^{2m+1}-\hat{u}_\delta^{2m+1})\psi_\delta\,\varphi\,d\boldsymbol{x}-\int_{\Omega_\delta}\hat{u}_\delta^{2m+1}\chi_\delta\varphi\,d\boldsymbol{x}. \tag{54}$$

From definition (25) and the asumption made on the geometry, we note that $\hat{u}_\delta$ belongs to $\mathscr{C}^0(\overline{\Omega_\delta})$. Moreover, Lemma 6.1 hereafter guarantees that there is some $C>0$ (independent of δ but depending on $\|f\|_{\mathrm{L}^2(\Omega)}$) such that there holds $\|\hat{u}_\delta\|_{\mathrm{L}^\infty(\Omega_\delta)}\le C$ for δ small enough. This result and Cauchy-Schwarz inequality yield

$$\Big|\int_{\Omega_\delta}\hat{u}_\delta^{2m+1}\chi_\delta\varphi\,d\boldsymbol{x}\Big|\le\|\hat{u}_\delta\|^{2m+1}_{\mathrm{L}^\infty(\Omega_\delta)}\int_{\Omega_\delta}|\chi_\delta\varphi|\,d\boldsymbol{x}\le C\,\|\chi_\delta\|_{\mathrm{L}^2(\Omega_\delta)}\,\|\varphi\|_{\mathrm{L}^2(\Omega_\delta)}\le C\,\delta\,\|\varphi\|_{\mathrm{L}^2(\Omega_\delta)}. \tag{55}$$

To obtain the last inequality, we used again that the support of χ_δ is included in the disk $\mathrm{D}_{d_2\delta}$ for some given $d_2>0$. Now, we focus our attention on the first term of the right hand side of (54). From the identity $t^{2n+1}-s^{2n+1}=(t-s)\sum_{k=0}^{2n}s^{2n-k}t^k$ for all $s,t\in\mathbb{R}$, and from the estimates $\|\hat{u}_\delta\|_{\mathrm{L}^\infty(\Omega_\delta)}\le C$, $\|v_\delta\|_{\mathrm{L}^\infty(\Omega_\delta)}\le C$ for δ small enough (see (75), (76)), we find

$$\Big|\int_{\Omega_\delta}(v_\delta^{2m+1}-\hat{u}_\delta^{2m+1})\psi_\delta\,\varphi\,d\boldsymbol{x}\Big|\le C\,\|(v_\delta-\hat{u}_\delta)\psi_\delta\|_{\mathrm{L}^2(\Omega_\delta)}\|\varphi\|_{\mathrm{L}^2(\Omega_\delta)}. \tag{56}$$

The function ψ_δ belongs to $\mathscr{C}^\infty_0(\overline{\Omega}\setminus\overline{\mathcal{R}^1_\delta},[0;1])$. Using the decomposition $\hat{u}_\delta=\psi_\delta\,v_\delta+\chi V_\delta-\psi_\delta\,\chi\,m_\delta$ as well as the identity $\chi_\delta\chi=\chi_\delta$ (for $\delta\in(0;1]$), we obtain

$$\begin{array}{rcl}\|(v_\delta-\hat{u}_\delta)\psi_\delta\|_{\mathrm{L}^2(\Omega_\delta)}&\le&\|v_\delta-\hat{u}_\delta\|_{\mathrm{L}^2(\Omega\setminus\overline{\mathcal{R}^1_\delta})}\\ &\le&\|\chi_\delta v_\delta\|_{\mathrm{L}^2(\Omega\setminus\overline{\mathcal{R}^1_\delta})}+\|\chi(V_\delta-m_\delta)\|_{\mathrm{L}^2(\Omega\setminus\overline{\mathcal{R}^1_\delta})}+\|\chi_\delta m_\delta\|_{\mathrm{L}^2(\Omega\setminus\overline{\mathcal{R}^1_\delta})}.\end{array} \tag{57}$$

Estimate (76) ensures that $\|v_\delta\|_{\mathrm{L}^\infty(\Omega\setminus\overline{\mathcal{R}^1_\delta})}\le C$. This implies

$$\|\chi_\delta v_\delta\|_{\mathrm{L}^2(\Omega\setminus\overline{\mathcal{R}^1_\delta})}\le\|v_\delta\|_{\mathrm{L}^\infty(\Omega\setminus\overline{\mathcal{R}^1_\delta}))}\,\|\chi_\delta\|_{\mathrm{L}^2(\Omega)}\le C\,\delta. \tag{58}$$

To deal with the second term of the right hand side of (57), we can write

$$\|\chi(V_\delta-m_\delta)\|_{\mathrm{L}^2(\Omega\setminus\overline{\mathcal{R}^1_\delta}))}\le\|V_\delta-m_\delta\|_{\mathrm{L}^2(\mathcal{R}^2\setminus\overline{\mathcal{R}^1_\delta})}. \tag{59}$$

Note that $(V_\delta-m_\delta)(\boldsymbol{x})=\lambda(\delta)\,\tilde{U}_1(\boldsymbol{x}/\delta)$ where $\tilde{U}_1\in\mathrm{W}^1_{1-\varepsilon}(\mathbb{R}^2\setminus\overline{\omega})$. Making the change of variable $\boldsymbol{\xi}=\boldsymbol{x}/\delta$, for $\varepsilon>0$, we find

$$\begin{array}{rcl}\|V_\delta-m_\delta\|^2_{\mathrm{L}^2(\mathcal{R}^2\setminus\overline{\mathcal{R}^1_\delta})}&=&\displaystyle\int_{\mathcal{R}^2_{1/\delta}\setminus\overline{\mathcal{R}^1}}|\lambda(\delta)|^2\,|\tilde{U}_1(\boldsymbol{\xi})|^2\,\delta^2\,d\boldsymbol{\xi}\\ &\le&\displaystyle C\,\delta^2\,(1/\delta)^{2\varepsilon}\int_{\mathcal{R}^2_{1/\delta}\setminus\overline{\mathcal{R}^1}}|\boldsymbol{\xi}|^{-2\varepsilon}\,|\tilde{U}_1(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi}\\ &\le&C\,\delta^{2-2\varepsilon}\,\|\tilde{U}_1\|^2_{\mathrm{W}^1_{1-\varepsilon}(\mathbb{R}^2\setminus\overline{\omega})}.\end{array} \tag{60}$$

From (59), (60) and Estimate (83) hereafter, we deduce the existence of a constant $C_\varepsilon>0$ independent of δ such that

$$\|\chi(V_\delta-m_\delta)\|_{\mathrm{L}^2(\Omega\setminus\overline{\mathcal{R}^1_\delta}))}\le C\,\delta^{1-\varepsilon}. \tag{61}$$

In this inequality, the constant C_ε does depend on ε, and this will be the case of other constants coming into play in subsequent inequalities. However, the main point of the present analysis is to provide estimates for $\delta \to 0$. Hence for the sake of conciseness, we shall simply denote "C" the constants appearing in these estimates. Dependency of these constants with respect to ε may be systematically assumed.

Now, let us focus on the term $\|\chi_\delta m_\delta\|_{\mathrm{L}^2(\Omega\setminus\overline{\mathcal{R}^1_\delta})}$ appearing in (57). Note that $|m_\delta| \le C$ in $\mathcal{R}^2_\delta \setminus \overline{\mathcal{R}^1_\delta}$ and $\mathrm{supp}(\chi_\delta) \subset \mathcal{R}^2_\delta$, where C is independent of δ. This allows us to write

$$\|\chi_\delta\, m_\delta\|_{\mathrm{L}^2(\Omega\setminus\overline{\mathcal{R}^1_\delta})} \le C\,\|1\|_{\mathrm{L}^2(\mathcal{R}^2_\delta\setminus\overline{\mathcal{R}^1_\delta})} \le C\,\delta. \tag{62}$$

Plugging (58), (61) and (62) in (57) yields

$$\|(v_\delta - \hat{u}_\delta)\psi_\delta\|_{\mathrm{L}^2(\Omega_\delta)} \le C\,\delta^{1-\varepsilon}.$$

Using this estimate in (56) leads to

$$\Big|\int_{\Omega_\delta} (v_\delta^{2m+1} - \hat{u}_\delta^{2m+1})\psi_\delta\,\varphi\,d\boldsymbol{x}\Big| \le C\,\delta^{1-\varepsilon}\|\varphi\|_{\mathrm{L}^2(\Omega_\delta)}. \tag{63}$$

Plugging (55) and (63) in (54), we arrive at

$$\Big|\int_{\Omega_\delta} v_\delta^{2m+1}\psi_\delta\,\varphi - \hat{u}_\delta^{2m+1}\varphi\,d\boldsymbol{x}\Big| \le C\,\delta^{1-\varepsilon}\|\varphi\|_{\mathrm{L}^2(\Omega_\delta)}. \tag{64}$$

$\star$ Finally, we estimate the last term of the right hand side of (49). Triangular inequality yields

$$\begin{array}{rl} & \Big|\displaystyle\int_{\Omega_\delta} \Big((v_\delta - m_\delta)\nabla\varphi - \varphi\nabla(v_\delta - m_\delta)\Big)\cdot\nabla\psi_\delta + \Big((V_\delta - m_\delta)\nabla\varphi - \varphi\nabla(V_\delta - m_\delta)\Big)\cdot\nabla\chi\,d\boldsymbol{x}\Big| \\ \le & \Big|\displaystyle\int_{\Omega_\delta} \Big((v_\delta - m_\delta)\nabla\varphi - \varphi\nabla(v_\delta - m_\delta)\Big)\cdot\nabla\psi_\delta\,d\boldsymbol{x}\Big| \\ & +\Big|\displaystyle\int_{\Omega_\delta} \Big((V_\delta - m_\delta)\nabla\varphi - \varphi\nabla(V_\delta - m_\delta)\Big)\cdot\nabla\chi\,d\boldsymbol{x}\Big| \end{array} \tag{65}$$

For $t > 0$, we define the region $\mathcal{Q}_t := \mathcal{R}^2_t \setminus \overline{\mathcal{R}^1_t}$. Observing that the support of $\nabla\psi_\delta$ is included in $\mathcal{Q}_\delta$ and that there holds $\|\nabla\psi_\delta\|_{\mathrm{L}^\infty(\Omega)} \le C\,\delta^{-1}$ for some $C > 0$ independent of δ, we can write

$$\begin{array}{rl} & \Big|\displaystyle\int_{\Omega_\delta} \Big((v_\delta - m_\delta)\nabla\varphi - \varphi\nabla(v_\delta - m_\delta)\Big)\cdot\nabla\psi_\delta\,d\boldsymbol{x}\Big| \\ \le & C\,\delta^{-1}(\|v_\delta - m_\delta\|_{\mathrm{L}^2(\mathcal{Q}_\delta)}\,\|\nabla\varphi\|_{\mathrm{L}^2(\mathcal{Q}_\delta)^2} + \|\nabla(v_\delta - m_\delta)\|_{\mathrm{L}^2(\mathcal{Q}_\delta)^2}\,\|\varphi\|_{\mathrm{L}^2(\mathcal{Q}_\delta)}). \end{array} \tag{66}$$

Since $v_\delta - m_\delta = \tilde{v}_\delta$ with $\tilde{v}_\delta \in \mathrm{V}^1_{-1+\varepsilon}(\Omega)$, according to (13), we have

$$\begin{array}{rcl} \|v_\delta - m_\delta\|_{\mathrm{L}^2(\mathcal{Q}_\delta)}\,\|\nabla\varphi\|_{\mathrm{L}^2(\mathcal{Q}_\delta)^2} & = & \|\tilde{v}_\delta\|_{\mathrm{L}^2(\mathcal{Q}_\delta)}\,\|\nabla\varphi\|_{\mathrm{L}^2(\mathcal{Q}_\delta)^2} \\ & \le & \delta^{2-\varepsilon}\,\|r^{-2+\varepsilon}\tilde{v}_\delta\|_{\mathrm{L}^2(\mathcal{Q}_\delta)}\,\|\nabla\varphi\|_{\mathrm{L}^2(\mathcal{Q}_\delta)^2} \\ & \le & \delta^{2-\varepsilon}\,\|\tilde{v}_\delta\|_{\mathrm{V}^1_{-1+\varepsilon}(\Omega_\delta)}\,\|\nabla\varphi\|_{\mathrm{L}^2(\mathcal{Q}_\delta)^2} \end{array} \tag{67}$$

and

$$\begin{array}{rcl} \|\nabla(v_\delta - m_\delta)\|_{\mathrm{L}^2(\mathcal{Q}_\delta)^2}\,\|\varphi\|_{\mathrm{L}^2(\mathcal{Q}_\delta)} & = & \|\nabla\tilde{v}_\delta\|_{\mathrm{L}^2(\mathcal{Q}_\delta)^2}\,\|\varphi\|_{\mathrm{L}^2(\mathcal{Q}_\delta)} \\ & \le & \delta^{1-\varepsilon}\,\|r^{-1+\varepsilon}\nabla\tilde{v}_\delta\|_{\mathrm{L}^2(\mathcal{Q}_\delta)^2}\,\|\varphi\|_{\mathrm{L}^2(\mathcal{Q}_\delta)} \\ & \le & C\,\delta^{2-\varepsilon}\,|\ln\delta|\,\|\tilde{v}_\delta\|^2_{\mathrm{V}^1_{-1+\varepsilon}(\Omega_\delta)}\,\|\nabla\varphi\|_{\mathrm{L}^2(\mathcal{Q}_\delta)^2}. \end{array} \tag{68}$$

The last inequality of (68) has been obtained proceeding like in (52) and using the Hardy's inequality (51). Since (68) is valid for all $\varepsilon > 0$, we can remove the factor $|\ln\delta|$. Taking account of (83) that will be established independently in Proposition 6.1, and plugging (67), (68) in (66) yields a constant $C > 0$ independent of δ (but depending on ε) such that

$$\Big| \int_{\Omega_\delta} \Big((v_\delta - m_\delta)\nabla\varphi - \varphi\nabla(v_\delta - m_\delta) \Big) \cdot \nabla\psi_\delta \, d\boldsymbol{x} \Big| \leq C\,\delta^{1-\varepsilon}\, \|\varphi\|_{\mathrm{H}^1_0(\Omega_\delta)}. \tag{69}$$

Now we consider the second term of the right hand side of (65). We can write

$$\begin{array}{rl} & \Big| \displaystyle\int_{\Omega_\delta} \Big((V_\delta - m_\delta)\nabla\varphi - \varphi\nabla(V_\delta - m_\delta) \Big) \cdot \nabla\chi \, d\boldsymbol{x} \Big| \\ \leq & C\,(\|V_\delta - m_\delta\|_{\mathrm{L}^2(\mathcal{Q}_1)}\, \|\nabla\varphi\|_{\mathrm{L}^2(\mathcal{Q}_1)^2} + \|\nabla(V_\delta - m_\delta)\|_{\mathrm{L}^2(\mathcal{Q}_1)^2}\, \|\varphi\|_{\mathrm{L}^2(\mathcal{Q}_1)}). \end{array} \tag{70}$$

We have $(V_\delta - m_\delta)(\boldsymbol{x}) = \lambda(\delta)\, \tilde{U}_1(\boldsymbol{x}/\delta)$ where $\tilde{U}_1 \in \mathrm{W}^1_{1-\varepsilon}(\mathbb{R}^2 \setminus \overline{\omega})$. Making the change of variable $\boldsymbol{\xi} = \boldsymbol{x}/\delta$, for $\varepsilon > 0$, we find

$$\begin{array}{rcl} \|V_\delta - m_\delta\|^2_{\mathrm{L}^2(\mathcal{Q}_1)} & = & \displaystyle\int_{\mathcal{Q}_{1/\delta}} |\lambda(\delta)|^2\, |\tilde{U}_1(\boldsymbol{\xi})|^2\, \delta^2\, d\boldsymbol{\xi} \\ & \leq & C\,\delta^2\,(1/\delta)^{2\varepsilon} \displaystyle\int_{\mathcal{Q}_{1/\delta}} |\boldsymbol{\xi}|^{-2\varepsilon}\, |\tilde{U}_1(\boldsymbol{\xi})|^2\, d\boldsymbol{\xi} \;\leq\; C\,\delta^{2-2\varepsilon}\, \|\tilde{U}_1\|^2_{\mathrm{W}^1_{1-\varepsilon}(\mathbb{R}^2\setminus\overline{\omega})}. \end{array} \tag{71}$$

Analogously, we obtain

$$\begin{array}{rl} \|\nabla(V_\delta - m_\delta)\|^2_{\mathrm{L}^2(\mathcal{Q}_1)^2} = & \displaystyle\int_{\mathcal{Q}_{1/\delta}} |\lambda(\delta)|^2\, |\nabla\tilde{U}_1(\boldsymbol{\xi})|^2\, d\boldsymbol{\xi} \\ & \leq C\,(1/\delta)^{-2+2\varepsilon} \displaystyle\int_{\mathcal{Q}_{1/\delta}} |\boldsymbol{\xi}|^{2-2\varepsilon}\, |\nabla\tilde{U}_1(\boldsymbol{\xi})|^2\, d\boldsymbol{\xi} \leq C\,\delta^{2-2\varepsilon}\, \|\tilde{U}_1\|^2_{\mathrm{W}^1_{1-\varepsilon}(\mathbb{R}^2\setminus\overline{\omega})}. \end{array} \tag{72}$$

Plugging (71) and (72) in (70), we deduce

$$\Big| \int_{\Omega_\delta} \Big((V_\delta - m_\delta)\nabla\varphi - \varphi\nabla(V_\delta - m_\delta) \Big) \cdot \nabla\chi \, d\boldsymbol{x} \Big| \leq C\,\delta^{1-\varepsilon}\, \|\varphi\|_{\mathrm{H}^1_0(\Omega_\delta)}. \tag{73}$$

Using (69) and (73) in (65), we arrive at

$$\begin{array}{l} \Big| \displaystyle\int_{\Omega_\delta} \Big((v_\delta - m_\delta)\nabla\varphi - \varphi\nabla(v_\delta - m_\delta) \Big) \cdot \nabla\psi_\delta + \Big((V_\delta - m_\delta)\nabla\varphi - \varphi\nabla(V_\delta - m_\delta) \Big) \cdot \nabla\chi \, d\boldsymbol{x} \Big| \\ \leq C\,\delta^{1-\varepsilon}\, \|\varphi\|_{\mathrm{H}^1_0(\Omega_\delta)}. \end{array} \tag{74}$$

$\star$ Conclusion. Gathering (53), (64) and (74) in (49) yields

$$|\langle \mathrm{A}_\delta(u_\delta) - \mathrm{A}_\delta(\hat{u}_\delta), \varphi\rangle| \leq C\,\delta^{1-\varepsilon}\, \|\varphi\|_{\mathrm{H}^1_0(\Omega_\delta)}.$$

Taking the sup over all $\varphi \in \mathrm{H}^1_0(\Omega_\delta)$ satisfying $\|\varphi\|_{\mathrm{H}^1_0(\Omega_\delta)} = 1$, and using this result in (45) leads to the desired estimate (27).

In the following of the section, we establish some intermediate results which were needed in the previous proof.

Lemma 6.1. *Let $\hat{u}_\delta$ refer to the function defined by (25). Then*

$$\limsup_{\delta\to 0}\, \|\hat{u}_\delta\|_{\mathrm{L}^\infty(\Omega_\delta)} < +\infty. \tag{75}$$

Proof. By definition, $\hat{u}_\delta$ verifies $\hat{u}_\delta \;=\; \psi_\delta\, v_\delta + \chi\, V_\delta - \psi_\delta \chi\, m_\delta$. As a consequence, it is sufficient to show that

$$\limsup_{\delta\to 0}\{\; \|v_\delta\|_{\mathrm{L}^\infty(\Omega\setminus\overline{\mathcal{R}^1_\delta})} + \|V_\delta\|_{\mathrm{L}^\infty(\Omega_\delta\cap\mathcal{R}^2)} + \|m_\delta\|_{\mathrm{L}^\infty(\mathcal{R}^2\setminus\overline{\mathcal{R}^1_\delta})} \;\} < +\infty. \tag{76}$$

The function v_δ is given by $v_\delta = u_0 + \lambda(\delta) u_{1,\delta}$ where u_0 is defined by (6) and where $u_{1,\delta} = w_{1,\delta} + \mathcal{G}$. In the latter decomposition, $w_{1,\delta}$ is solution of (6) and $\mathcal{G}$ is Green's function for the Laplace operator with homogeneous Dirichlet boundary condition. Since $u_0 \in \mathscr{C}^0(\overline{\Omega})$ is independent of δ, there holds

$$\|u_0\|_{\mathrm{L}^\infty(\Omega\setminus\overline{\mathcal{R}^1_\delta})} \leq C. \tag{77}$$

Here and in the following, $C > 0$ denotes a constant, which may change from one occurrence to another, but which is independent of δ. Observing that there is some $d_1 > 0$ such that $(\Omega \setminus \overline{\mathcal{R}^1_\delta}) \cap \mathrm{D}_{d_1\delta} = \emptyset$, we obtain $\|\mathcal{G}\|_{\mathrm{L}^\infty(\Omega\setminus\overline{\mathcal{R}^1_\delta})} \leq C\,|\ln\delta|$. Since $|\lambda(\delta)| \leq C\,|\ln\delta|^{-1}$, this implies

$$\|\lambda(\delta)\,\mathcal{G}\|_{\mathrm{L}^\infty(\Omega\setminus\overline{\mathcal{R}^1_\delta})} \leq C. \tag{78}$$

We know that $w_{1,\delta}$ converges to $w_{1,0}$ for the H^1-norm as δ goes to zero (this is a result of the implicit function theorem). As a consequence of (9), for δ small enough, there is a constant $C > 0$ such that $\|\Delta w_{1,\delta}\|_{\mathrm{L}^2(\Omega_\delta)} \leq C$. This implies

$$\|\lambda(\delta)\, w_{1,\delta}\|_{\mathrm{L}^\infty(\Omega\setminus\overline{\mathcal{R}^1_\delta})} \leq C. \tag{79}$$

Gathering (77), (78) and (79) leads to

$$\|v_\delta\|_{\mathrm{L}^\infty(\Omega\setminus\overline{\mathcal{R}^1_\delta})} \leq C. \tag{80}$$

Now, we estimate the second term in the left hand side of (76). The function V_δ is defined by $V_\delta(\boldsymbol{x}) = \lambda(\delta)\, U_1(\boldsymbol{x}/\delta)$, where U_1 is the sum of $-(2\pi)^{-1}\ln|\boldsymbol{\xi}|$ and a function which remains bounded at infinity. Since $|\lambda(\delta)| \leq C\,|\ln\delta|^{-1}$, we deduce that

$$\|V_\delta\|_{\mathrm{L}^\infty(\Omega_\delta\cap\mathcal{R}^2_\delta)} \leq C. \tag{81}$$

In (76), it remains to bound the term involving m_δ. By definition (see (24)), we have $m_\delta = \lambda(\delta)\Big(\frac{1}{2\pi}\ln\Big(\frac{\delta}{|\boldsymbol{x}|}\Big) + \gamma_{\mathrm{N}}\Big)$. Using again the relation $|\lambda(\delta)| \leq C\,|\ln\delta|^{-1}$, one can check that

$$\|m_\delta\|_{\mathrm{L}^\infty(\mathbb{R}^2\setminus\overline{\mathcal{R}^1_\delta})} \leq C. \tag{82}$$

Plugging (80), (81) and (82) in (76) yields Estimate (75). $\square$

Proposition 6.1. *The far field v_δ and the near field V_δ respectively admit the decompositions $v_\delta(\boldsymbol{x}) = m_\delta(\boldsymbol{x}) + \tilde{v}_\delta(\boldsymbol{x})$, $V_\delta(\boldsymbol{x}) = m_\delta(\boldsymbol{x}) + \tilde{V}_\delta(\boldsymbol{x}/\delta)$ where m_δ is given by (24) and where $\tilde{v}_\delta$, $\tilde{V}_\delta$ satisfy*

$$\limsup_{\delta\to 0}\{\ \|\tilde{v}_\delta\|_{\mathrm{V}^1_{-1+\varepsilon}(\Omega)} + \|\tilde{V}_\delta\|_{\mathrm{W}^1_{1-\varepsilon}(\mathbb{R}^2\setminus\overline{\omega})}\ \} < +\infty \qquad \forall\varepsilon > 0. \tag{83}$$

Proof. First, we prove the estimate

$$\|\tilde{v}_\delta\|_{\mathrm{V}^1_{-1+\varepsilon}(\Omega)} \leq C_\varepsilon, \qquad \forall\varepsilon > 0. \tag{84}$$

The function v_δ is given by $v_\delta = u_0 + \lambda(\delta) u_{1,\delta}$ where u_0 is defined by (6) and where $u_{1,\delta} = w_{1,\delta} + \mathcal{G}$. In the latter equality, $w_{1,\delta}$ is solution of (6) and $\mathcal{G}$ is Green's function for the Laplace operator with homogeneous Dirichlet boundary condition. The function u_0 admits the decomposition $u_0 = u_0(0) + \tilde{u}_0$ with $\tilde{u}_0 \in \mathrm{V}^1_{-1+\varepsilon}(\Omega)$ independent of δ. As a consequence, to obtain (84), it is sufficient to show that there holds

$$\|\tilde{w}_\delta\|_{\mathrm{V}^1_{-1+\varepsilon}(\Omega)} \leq C_\varepsilon, \qquad \forall\varepsilon > 0, \tag{85}$$

where $\tilde{w}_\delta \in \mathrm{V}^1_{-1+\varepsilon}(\Omega)$ is the function such that $w_{1,\delta} = w_{1,\delta}(0) + \tilde{w}_\delta$. We know that $w_{1,\delta}$ converges to $w_{1,0}$ for the H^1-norm as δ goes to zero (this is a result of the implicit function theorem). As a consequence of (9), for δ small enough, there is a constant $C_\varepsilon > 0$ such that $\|\Delta w_\delta\|_{\mathrm{L}^2(\Omega_\delta)} \leq C_\varepsilon$. According to the Kondratiev's theory, this implies (85). Therefore (84) is established.

On the other hand, we have $(V_\delta - m_\delta)(\boldsymbol{x}) = \lambda(\delta)\,\tilde{U}_1(\boldsymbol{x}/\delta)$ where $\tilde{U}_1 \in \mathrm{W}^1_{1-\varepsilon}(\mathbb{R}^2 \setminus \overline{\omega})$ is independent of δ. Since $|\lambda(\delta)| \le C\,|\ln\delta|^{-1}$, we deduce that $\tilde{V}_\delta = \lambda(\delta)\,\tilde{U}_1$ verifies

$$\|\tilde{V}_\delta\|_{\mathrm{W}^1_{1-\varepsilon}(\mathbb{R}^2\setminus\overline{\omega})} \le C_\varepsilon, \qquad \forall\varepsilon > 0. \tag{86}$$

Finally, from (84), (86), we obtain (83). $\square$

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