An Antisymmetric Effective Hall Matrix

Marc Briane, Graeme W Milton

An antisymmetric effective Hall matrix

Marc BRIANE
IRMAR & INSA de Rennes
mbriane@insa-rennes.fr

Graeme W. MILTON
Dept. of Math., University of Utah
milton@math.utah.edu

January 9, 2013

Abstract

A periodic composite of four isotropic phases with an almost antisymmetric effective Hall matrix is constructed. This is done through the use of a microstructure, with high contrast conductivity, which acts to twist the direction of the induced Hall field. For an applied current along the y_3 axis and a magnetic field h in the y_1-y_2 plane the Hall field will be essentially parallel, rather than perpendicular, to h. It is also shown how to obtain composites with an effective Hall matrix matching any given symmetric positive definite matrix.

Keywords: Hall effect, homogenization, Hall matrix

AMS Classification: 35B27, 74Q15

1 Introduction

To maintain a specific current-flow direction in an isotropic homogeneous conductor in the presence of a perpendicular magnetic field, it is necessary to have an electric field with a component in the direction transverse to both the current and magnetic field. This transverse electric field, in the classical physics interpretation, is necessary to balance the magnetic force acting on the moving charge carriers. The ratio between the component of the transverse electric field and the current defines the Hall coefficient (see, e.g., [4, 29]). For an anisotropic material it is natural to introduce, analogously, a Hall matrix [10]. The question naturally arises as to what properties this Hall matrix can have. In particular can it be an antisymmetric matrix?

To address this question we consider anisotropic periodic composite materials and ask what effective Hall matrices the composite can have, and in particular if they can be antisymmetric. At the microscale the composite will be assumed to be locally isotropic, characterized by a periodic isotropic local conductivity $\sigma(y)\, I_3$ and a periodic isotropic local Hall matrix $r(y)\, I_3$. Then, in the presence of a small magnetic field $h \in \mathbb{R}^3$, the (assumed periodic) local electric field $e(y)$ has the expansion

$$e(y) = \rho_h(y)\, j(y) = \sigma(y)^{-1} j(y) + \underbrace{r(y)\, j(y) \times h}_{\text{Hall field}} + o(h). \tag{1.1}$$

where $j(y)$ is the periodic local current (see, e.g., [21] for details) and $\rho_h(y) = \sigma_h(y)^{-1}$ is the perturbed local (non-symmetric matrix-valued) resistivity. We call the transverse field

$r(y)\, j(y) \times h$ the Hall field as it generates the Hall voltage. The average electric field in the composite derived from (1.1) has the expansion

$$\langle e \rangle = \rho_h^* \langle j \rangle = (\sigma^*)^{-1} \langle j \rangle + \underbrace{\langle j \rangle \times (R^* h)}_{\substack{\text{Macroscopic} \\ \text{Hall field}}} + o(h), \tag{1.2}$$

in which $\langle \cdot \rangle$ denotes the average value over the period cell, ρ_h^* is the perturbed effective resistivity tensor, σ^* is the (matrix-valued) effective conductivity tensor in the absence of the magnetic field, and R^* is the effective Hall matrix. We will call $\langle j \rangle \times (R^* h)$ the macroscopic Hall field.

It was discovered by Bergman [5] that the effective Hall coefficient in a composite can be obtained from the local Hall coefficient and the knowledge of several current fields that solve the conductivity equations in the absence of magnetic field. His arguments extend directly to the anisotropic case and provide a formula for the effective Hall matrix [10]. First note that since the relation between $e(y)$ and $\langle e \rangle$ is linear we can write, in the absence of any magnetic field,

$$e(y) = P(y)\langle e \rangle \qquad (h = 0) \tag{1.3}$$

which following Tartar [32] defines the matrix-valued unperturbed electric field $P(y) = DU(y)$. Then, upon multiplying this equation by $\sigma(y)$ and averaging over the unit cell, one sees that the effective matrix conductivity is given by the classical formula

$$\sigma^* = \langle \sigma P \rangle. \tag{1.4}$$

By contrast the effective Hall matrix is given by the more intricate formula ([10] Theorem 3)

$$\mathrm{Cof}\,(\sigma^*)\, R^* = \left\langle r\, \mathrm{Cof}\,(\sigma P)^T \right\rangle, \tag{1.5}$$

where Cof denotes the cofactor matrix.

The relation between $\rho_h(y)$ and ρ_h^* is still poorly understood. In two dimensions, one can shift $\rho_h(y)$ (or $\rho_h(y)^{-1}$) by a constant antisymmetric matrix and then ρ_h^* (respectively $(\rho_h^*)^{-1}$) will shift in exactly the same way [15]. For a fixed magnetic field h this observation led to a complete characterization, in two dimensions, of the possible tensors ρ_h^* that are obtainable from mixtures of two given, possibly anisotropic, phases with prescribed orientations and mixed in fixed proportions [26]. In three dimensions one can add a periodic divergence free antisymmetric matrix field to $\rho_h(y)^{-1}$ and then $(\rho_h^*)^{-1}$ will shift by the average of this tensor field (see [31] and section 4.4 of [27]). For fibre-reinforced composites (with structure independent of one coordinate) an amazing plethora of microstructure independent exact relations and links between effective tensors have been derived [7, 8, 30, 9, 17, 18]. For strong magnetic fields h the tensor ρ_h^* can have a very sensitive dependence on the orientation of h relative to the microstructure [6].

It would be interesting to know the range of values the matrix pair (σ^*, R^*), or at least the matrix R^*, can take given some information about the pair of functions $(\sigma(y), r(y))$, with $\sigma(y) > 0$. Even for isotropic composites, with $R^* = r^* I_3$, there are some surprises: contrary to the common belief that the sign of the Hall coefficient determines the sign of the charge carrier, r^* in a certain geometry of cubic chain mail is negative even though $r(y)$ is non-negative everywhere [10]. Physically such an unusual effect arises essentially because the inclusions with appreciable Hall coefficient (situated between the links of the chain mail) have an current field flowing through them which is in the opposite direction of the applied current field. Also the matrix elements of R^* can be orders of magnitude greater that the maximum value of $r(y)$ in certain microstructures [11], unlike in the two-dimensional case where the Hall coefficient is bounded above by the maximum value of $r(y)$ [12].

The present paper provides a partial answer to the following question: What are the possible effective Hall matrices, R^*, when $\sigma(y)$ and $r(y)$ are allowed to be any pair of positive functions? Starting from a nonnegative isotropic Hall matrix, we first note (see the preliminary Proposition 2.1) that any positive definite symmetric matrix is an effective Hall matrix. More surprisingly, and this is the main result of the paper (see Theorem 2.2) which adds to the menagerie of pathologies arising in the homogenization of the Hall effect, there exists a four-phase composite the effective Hall matrix of which is (asymptotically) antisymmetric!

Physically this effect arises because for an applied current in the y_3 direction, a portion of this current flows through the cylinders with non-zero Hall coefficient and the induced macroscopic Hall field is essentially a 90° degree rotation of the Hall field across these inclusions (see figure 1). Thus if the magnetic field h is parallel to the y_1-y_2 plane, then the direction of the macroscopic Hall field will be essentially parallel to h. To achieve this change of direction we use a highly conductivity phase. This is reminiscent of the use of high contrast materials for achieving the reversal of sign of Hall coefficient in the cubic chain mail of [10], but the geometry is completely different.

2 Statement and proof of the main results

In dimension three the expression of the effective the Hall matrix incorporates the cofactor matrix of the local electric field $DU(y)$ (see formula (1.5)). Therefore it should not be surprising that our analysis uses the quasi-affine property satisfied by the minors of the matrix-valued gradient $DU(y)$. We refer to [13] (Section 4.2) for a complete presentation of the quasi-affinity. In the present context we will specifically use the quasi-affine property of the quadratic minors:

- In dimension two the determinant is quasi-affine. This means that for any function $V : \mathbb{R}^2 \to \mathbb{R}^2$ such that DV is a Y-periodic matrix-valued function in $L^2(Y)^{2\times 2}$,

$$\int_Y \det(DV)\, dy = \det\left(\int_Y DV\, dy\right). \tag{2.1}$$

- In dimension three the cofactor matrix is quasi-affine. This means that for any function $V : \mathbb{R}^3 \to \mathbb{R}^3$ such that DV is a Y-periodic matrix-valued function in $L^2(Y)^{3\times 3}$,

$$\int_Y \mathrm{Cof}\,(DV)\, dy = \mathrm{Cof}\left(\int_Y DV\, dy\right). \tag{2.2}$$

We start with the preliminary result of Proposition 2.1. This proposition uses the fact that any symmetric positive definite matrix is the effective matrix of a periodic isotropic conductivity, at least asymptotically. Indeed, it suffices to consider composites of two isotropic phases and use the multiple rank lamination due to Maxwell [25] and its extension to dimension greater than three by Tartar [33] (see also [24]), or the coated ellipsoid assemblage (see chapter 7 of [27] and references therein, and its extension to higher dimension by Tartar [33]), or the periodic constructions of Vigdergauz [34] and Liu, James, and Leo [23]. Then, letting the free parameters in the microstructure vary, any symmetric positive definite matrix can be obtained in any dimension as the effective matrix of a composite of two isotropic phases with suitably extreme conductivities, one close to zero and one large. Even though multiple rank laminates are not periodic, they can be approximated arbitrarily closely by periodic constructions (see, e.g., [2] Theorem 1.3.23). Also it is easily seen that the coated ellipsoid assemblage can be made periodic.

Proposition 2.1. *Any positive definite symmetric* 3×3 *matrix* A *is the effective Hall matrix of a suitable periodic structure.*

Proof. Consider a Y-periodic isotropic conductivity $\sigma(y)\, I_3$ the homogenized conductivity of which is $\sigma^* = A/\sqrt{\det A}$, and the Y-periodic Hall coefficient is $r := \sigma^{-2}$. By (1.5) the effective Hall matrix R^* associated with this periodic structure is given by the formula

$$\int_Y r\, \mathrm{Cof}\,(\sigma DU)^T\, dy = \mathrm{Cof}\,(\sigma^*)\, R^*, \tag{2.3}$$

where DU is the periodic local matrix-valued electric field solving

$$\mathrm{Div}\,(\sigma DU) = 0 \;\text{ in } \mathbb{R}^3, \quad \text{with} \quad \int_Y DU\, dy = I_3, \tag{2.4}$$

(the j-th column of the matrix DU is the gradient of the j-th coordinate u_j of U, and the j-th coordinate of the vector $\mathrm{Div}\,(\sigma DU)$ is the divergence of $\sigma \nabla u_j$) and σ^* is the homogenized conductivity given by the classical formula

$$\sigma^* = \int_Y \sigma DU\, dy. \tag{2.5}$$

Hence, by the quasi-affinity of the cofactor matrix (2.2) and the equality $\mathrm{Cof}\,(\sigma^*) = A^{-1}$, we obtain

$$I_3 = \int_Y \mathrm{Cof}\,(DU)^T\, dy = \int_Y r\, \mathrm{Cof}\,(\sigma DU)^T\, dy = A^{-1}\, R^*, \tag{2.6}$$

which implies that $R^* = A$. $\square$

On the contrary the derivation of an antisymmetric effective Hall matrix is much more delicate. To this end we have the following asymptotic result:

Theorem 2.2. *There exists a periodic conductivity associated with an isotropic Hall coefficient parametrized by a large number* $\kappa \gg 1$*, such that the homogenized Hall matrix* $R^{*,\kappa}$ *satisfies*

$$\lim_{\kappa \to \infty} R^{*,\kappa} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.7}$$

Proof. Let $Y := (-\frac{1}{2}, \frac{1}{2})^3$. We consider a columnar Y-periodic structure along the y_3-axis, the period cell of which is represented in figure 1. On the one hand, the Y-periodic local conductivity σ^κ, for $\kappa > 0$, is defined by

$$\sigma^\kappa(y) := \begin{cases} \mathrm{diag}\,(\kappa, \kappa, 1) & \text{if } y \in Q_s = Q_1 \cup Q_2 \cup Q_3 \cup Q_4 \\ I_3 & \text{if } y \in Y \setminus Q_s, \end{cases} \tag{2.8}$$

where Q_s is the grey region in figure 1. The conductivity $\sigma^\kappa(y)$ in the y_3 direction is chosen to be 1 in the region Q_s, rather than κ, to ensure that a significant fraction of the current flows

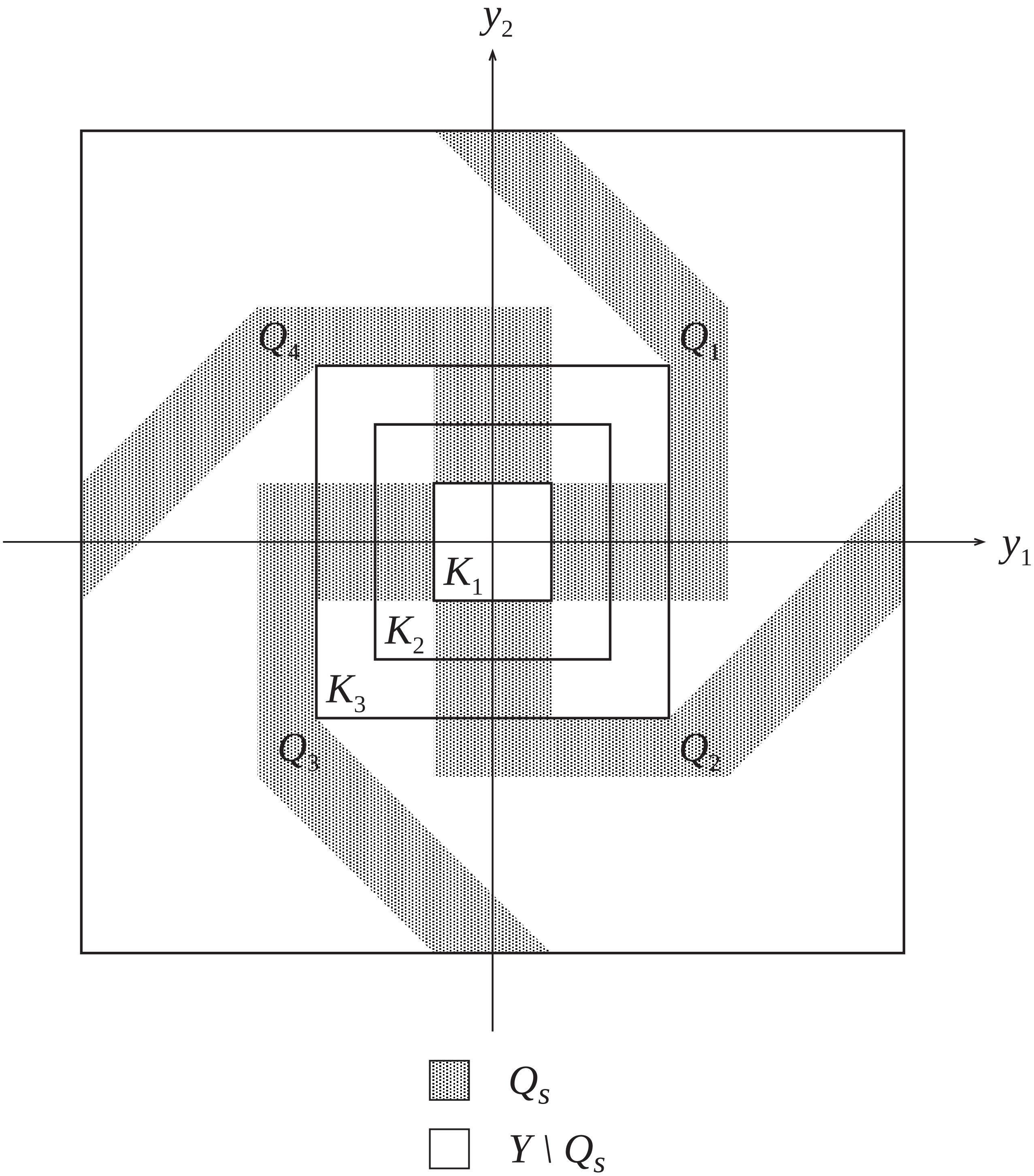

Figure 1: The cross section of the columnar period cell. The regions Q_1, Q_2, Q_3, Q_4 are highly conducting in the plane, and only the central square cylinder K_1 has a non-zero Hall coefficient.

through the material with non-zero Hall coefficient (which will be the square cylinder at the center) when the applied current is in the y_3 direction.

Although σ^κ is anisotropic it can be replaced, if desired, when $\kappa > 1$ by a laminate of two isotropic phases, finely layered in the y_3-direction so its local effective conductivity is σ^κ. The conductivity matrix σ^κ is associated with the corrector DU^κ, where $U^\kappa = (u_1^\kappa, u_2^\kappa, u_3^\kappa)$ is the unique function with zero Y-average which is the solution of

$$\operatorname{Div}\left(\sigma^\kappa DU^\kappa\right)=0 \text{ in } \mathbb{R}^3, \quad y\mapsto U^\kappa(y)-y \text{ is } Y\text{-periodic}, \quad \int_Y DU^\kappa\,dy=I_3, \quad \nabla u_3^\kappa=e_3. \tag{2.9}$$

Note that the components u_1, u_2 of U^κ do not depend on the y_3-coordinate due to the columnar geometry. Then, the cofactor matrix of DU^κ satisfies the equality

$$\operatorname{Cof}\left(DU^\kappa\right)^T=\begin{pmatrix} \frac{\partial u_2^\kappa}{\partial y_2} & -\frac{\partial u_2^\kappa}{\partial y_1} & 0 \\ -\frac{\partial u_1^\kappa}{\partial y_2} & \frac{\partial u_1^\kappa}{\partial y_1} & 0 \\ 0 & 0 & \frac{\partial u_1^\kappa}{\partial y_1}\frac{\partial u_2^\kappa}{\partial y_2}-\frac{\partial u_1^\kappa}{\partial y_2}\frac{\partial u_2^\kappa}{\partial y_1}\end{pmatrix}. \tag{2.10}$$

On the other hand, the Y-periodic local Hall matrix is defined by

$$R(y):=r_\kappa\,1_{K_1}(y)\,I_3, \quad \text{for } y\in Y, \tag{2.11}$$

where $r_\kappa>0$ will be chosen later and 1_{K_1} is the characteristic function of the central square K_1 of side $\ell<\frac{1}{3}$ in figure 1.

Due to the invariance of figure 1 by a rotation of 90°, the homogenized conductivity $\sigma^{*,\kappa}$ reads as

$$\sigma^{*,\kappa}=\begin{pmatrix} a_\kappa & 0 & 0 \\ 0 & a_\kappa & 0 \\ 0 & 0 & 1\end{pmatrix}, \quad \text{where } a_\kappa>0. \tag{2.12}$$

By analogy with a two-phase $(1,\kappa)$ checkerboard structure we have the following result proved at the end of the paper:

Lemma 2.3. *The coefficient a_κ of the homogenized matrix $\sigma^{*,\kappa}$ satifies the estimates*

$$\forall\,\kappa>0, \quad c_1\,\kappa^{\frac{1}{2}}\le a_\kappa\le c_2\,\kappa^{\frac{1}{2}}, \tag{2.13}$$

where c_1, c_2 are two positive constants.

Remark 2.4. The factor of $\kappa^{\frac{1}{2}}$ in (2.13) arises because of the resistance of the corners where highly conducting regions meet.

From now on u_1, u_2 will be regarded as functions of the two variables y_1, y_2, and the sets Y, Q_s, Q_i, K_i will be identified with their cross sections in the transverse plane y_1-y_2 as shown in figure 1.

According to the variational principle (3.1) we have

$$a_\kappa=\sigma_{11}^{*,\kappa}=\int_Y \sigma^\kappa\nabla u_1^\kappa\cdot\nabla u_1^\kappa\,dy\ge\kappa\int_{Q_s}|\nabla u_1^\kappa|^2\,dy, \tag{2.14}$$

which combined with estimate (2.13) implies that ∇u_1^κ strongly converges to zero in $L^2(Q_s)$. That is expected because the electric field should be close to zero in the highly conducting phase, except near the corner contact points. Then, the Poincaré-Wirtinger inequality (see, e.g., [16] p. 164) in the regular connected open set Q_i, for $i=1,\ldots,4$, yields

$$⨍_{\partial K_1\cap\partial Q_i} u_1^\kappa\,dy-⨍_{\partial Y\cap\partial Q_i} u_1^\kappa\,dy=o(1), \quad \text{as } \kappa\to\infty, \tag{2.15}$$

where the integral with a bar through it denotes an average over the interval. In other words, the average electric potential along the boundary $\partial K_1\cap\partial Q_i$ should be close to that along the boundary $\partial Y \cap \partial Q_i$, as expected because the region Q_i is highly conducting in the plane. Moreover, since $u_1^\kappa(y) - y_1$ is Y-periodic, we have (see figure 1)

$$\int_{\partial Y \cap \partial Q_1} u_1^\kappa \, dy = \int_{\partial Y \cap \partial Q_3} u_1^\kappa \, dy \qquad \text{and} \qquad \int_{\partial Y \cap \partial Q_2} u_1^\kappa \, dy = \int_{\partial Y \cap \partial Q_4} u_1^\kappa \, dy + \ell. \tag{2.16}$$

Therefore, it follows from (2.15) and (2.16) that

$$\begin{cases} \displaystyle\int_{K_1} \frac{\partial u_1^\kappa}{\partial y_1} \, dy & = \displaystyle\int_{\partial K_1 \cap \partial Q_1} u_1^\kappa \, dy - \int_{\partial K_1 \cap \partial Q_3} u_1^\kappa \, dy = o(1) \\ \displaystyle\int_{K_1} \frac{\partial u_1^\kappa}{\partial y_2} \, dy & = \displaystyle\int_{\partial K_1 \cap \partial Q_4} u_1^\kappa \, dy - \int_{\partial K_1 \cap \partial Q_2} u_1^\kappa \, dy = -\ell + o(1), \end{cases} \quad \text{as } \kappa \to \infty. \tag{2.17}$$

Similarly, with a change of sign we get for the function u_2^κ

$$\begin{cases} \displaystyle\int_{K_1} \frac{\partial u_2^\kappa}{\partial y_1} \, dy & = \displaystyle\int_{\partial K_1 \cap \partial Q_1} u_2^\kappa \, dy - \int_{\partial K_1 \cap \partial Q_3} u_2^\kappa \, dy = \ell + o(1) \\ \displaystyle\int_{K_1} \frac{\partial u_2^\kappa}{\partial y_2} \, dy & = \displaystyle\int_{\partial K_1 \cap \partial Q_4} u_2^\kappa \, dy - \int_{\partial K_1 \cap \partial Q_2} u_2^\kappa \, dy = o(1). \end{cases} \quad \text{as } \kappa \to \infty. \tag{2.18}$$

Finally, taking into account (2.8), (2.11) the formula (1.5) for the homogenized Hall matrix reads as

$$R^{*,\kappa} = \frac{\sigma^{*,\kappa}}{\det \sigma^{*,\kappa}} \int_{K_1} r_\kappa \, \mathrm{Cof} \left(DU^\kappa \right)^T dy. \tag{2.19}$$

This combined with (2.10), (2.12), (2.17), (2.18) yields

$$R^{*,\kappa} = \frac{r_\kappa}{a_\kappa^2} \begin{pmatrix} a_\kappa & 0 & 0 \\ 0 & a_\kappa & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} o(1) & -\ell + o(1) & 0 \\ \ell + o(1) & o(1) & 0 \\ 0 & 0 & c_\kappa \end{pmatrix}, \tag{2.20}$$

where

$$c_\kappa := \int_{K_1} \left(\frac{\partial u_1^\kappa}{\partial y_1} \frac{\partial u_2^\kappa}{\partial y_2} - \frac{\partial u_1^\kappa}{\partial y_2} \frac{\partial u_2^\kappa}{\partial y_1} \right) dy = \int_{K_1} \det \left(\nabla u_1^\kappa, \nabla u_2^\kappa \right) dy. \tag{2.21}$$

Moreover, by the Alessandrini, Nesi [1] positivity result we have $\det \left(\nabla u_1^\kappa, \nabla u_2^\kappa \right) > 0$ in Y. This combined with the quasi-affinity of the determinant (2.1) yields

$$0 < c_\kappa \leq \int_Y \det \left(\nabla u_1^\kappa, \nabla u_2^\kappa \right) dy = \det \left(\int_Y \left(\nabla u_1^\kappa, \nabla u_2^\kappa \right) dy \right) = 1. \tag{2.22}$$

Therefore, choosing

$$r_\kappa := \frac{a_\kappa}{\ell}, \tag{2.23}$$

the estimate from below of (2.13) and (2.22) give

$$\lim_{\kappa \to \infty} \frac{r_\kappa \, c_\kappa}{a_\kappa^2} = \lim_{\kappa \to \infty} \frac{c_\kappa}{\ell \, a_\kappa} = 0, \tag{2.24}$$

which implies the desired limit (2.7). □

3 Proof of Lemma 2.3.

This lemma basically states that the effective conductivity in the y_1-y_2 plane is dominated by the resistance of the corners where the highly conducting regions meet and has the asymptotic form expected from the analysis of Keller [20].

First of all, recall that for a given square Z of $\mathbb{R}^2$, the effective conductivity σ^* of a Z-periodic symmetric conductivity $\sigma(y)$ is given by the variational principal (see, e.g., [3])

$$\sigma^*\lambda\cdot\lambda = \min\left\{ \fint_Z \sigma\nabla v\cdot\nabla v\,dy \;:\; (y\mapsto v(y)-\lambda\cdot y)\ \ Z\text{-periodic}\right\} = \fint_Z \sigma\nabla w^\lambda\cdot\nabla w^\lambda\,dy, \quad (3.1)$$

where the minimizer w^λ, for $\lambda\in\mathbb{R}^2$, is the unique solution (up to an additive constant) in $H^1_{\rm loc}(\mathbb{R}^2)$ of the equation

$$\operatorname{div}\left(\sigma\nabla w^\lambda\right) = 0 \ \text{ in } \mathbb{R}^2, \quad \text{with} \quad \left(y\mapsto w^\lambda(y)-\lambda\cdot y\right)\ Z\text{-periodic}. \quad (3.2)$$

Proof of the upper bound of (2.13): Let K_j, for $j=1,2,3$, be the square centered at the origin and of side $j\ell$. On the one hand, consider for $\kappa>0$ the potential w_κ associated with the two-phase $(1,\kappa)$ checkerboard of period K_2 (see figure 1) which is the solution of the problem

$$\operatorname{div}\left(\chi_\kappa\nabla w_\kappa\right) = 0 \ \text{ in } \mathbb{R}^2, \quad y\mapsto w_\kappa(y)-y_2 \ \text{ is } K_2\text{-periodic} \quad \text{and} \quad \int_{K_2} w_\kappa\,dy = 0, \quad (3.3)$$

where χ_κ is the K_2-periodic conductivity of the checkerboard defined by

$$\chi_\kappa(y) := 1 + (\kappa-1)\,1_{Q_s}(y) \quad \text{for } y\in K_2. \quad (3.4)$$

Note that χ_κ and the first entry σ^κ_{11} of σ^κ (2.8) agree in the square K_3 as shown in figure 1. The formula (3.1) for the effective conductivity of a checkerboard (see, e.g., [14]) reads as

$$\fint_{K_2} \chi_\kappa\,|\nabla w_\kappa|^2\,dy = \kappa^{\frac{1}{2}}, \quad (3.5)$$

which implies by the Poincaré-Wirtinger inequality in K_2,

$$\int_{K_2} w_\kappa^2\,dy + \int_{K_2} |\nabla w_\kappa|^2\,dy = O\big(\kappa^{\frac{1}{2}}\big). \quad (3.6)$$

On the other hand, let v_κ be function defined by

$$v_\kappa := \frac{1}{2\ell}\fint_{Q_i\cap\partial K_2} w_\kappa\,dy \quad \text{in } Q_i\cap(Y\setminus K_2), \quad \text{for } i=1,\dots,4, \quad (3.7)$$

and extended by interpolation in the whole set $Y\setminus K_2$. Since $y\mapsto w_\kappa(y)-y_2$ is K_2-periodic, we have

$$\begin{cases} v_\kappa\left(\frac{1}{2},y_2\right) = v_\kappa\left(-\frac{1}{2},y_2\right) - 1 & \text{for } y_2\in\left(-\frac{\ell}{2},\frac{\ell}{2}\right) \\ v_\kappa\left(y_1,\frac{1}{2}\right) = v_\kappa\left(y_1,-\frac{1}{2}\right) & \text{for } y_1\in\left(-\frac{\ell}{2},\frac{\ell}{2}\right). \end{cases} \quad (3.8)$$

Moreover, by the trace inequality in K_2 (see, e.g., [22]):

$$\exists\, C>0,\ \forall\, v\in H^1(K_2), \quad \int_{\partial K_2} v^2\,ds \le C\int_{K_2}\left(v^2+|\nabla v|^2\right)dx, \quad (3.9)$$

combined with the Cauchy-Schwarz inequality in $Q_i \cap \partial K_2$ and estimate (3.6), we have

$$\int_{Q_i \cap \partial K_2} w_\kappa \, dy = O\big(\kappa^{\frac{1}{4}}\big), \quad \text{for } i = 1, \ldots, 4. \tag{3.10}$$

Then, we can construct v_κ in $Y \setminus K_2$ in such a way that $y \mapsto v_\kappa(y) + y_1$ is 1-periodic on ∂Y and

$$\int_{Y \setminus K_2} v_\kappa^2 \, dy + \int_{Y \setminus K_2} |\nabla v_\kappa|^2 \, dy = O\big(\kappa^{\frac{1}{2}}\big). \tag{3.11}$$

Let φ be a smooth Y-periodic function such that $\varphi = 1$ in $Y \setminus K_3$ and $\varphi = 0$ in K_2. Define the function

$$u_\kappa := \varphi \, v_\kappa + (1 - \varphi) \, \frac{w_\kappa}{2\ell}. \tag{3.12}$$

Since $y \mapsto u_\kappa(y) + y_1$ is Y-periodic, the function $-u_\kappa$ can be used as a test function in the variational principle giving a_κ, hence

$$a_\kappa \leq \int_Y \big(1 + (\kappa - 1) \, 1_{Q_s}\big) \, |\nabla u_\kappa|^2 \, dy. \tag{3.13}$$

By (3.6) and (3.11) we have

$$\begin{aligned} &\int_{(Y \setminus K_3) \cup K_2} \big(1 + (\kappa - 1) \, 1_{Q_s}\big) \, |\nabla u_\kappa|^2 \, dy \\ &= \int_{Y \setminus K_3} |\nabla v_\kappa|^2 \, dy + \frac{1}{(2\ell)^2} \int_{K_2} \chi_\kappa \, |\nabla w_\kappa|^2 \, dy = O\big(\kappa^{\frac{1}{2}}\big). \end{aligned} \tag{3.14}$$

It remains to estimate the conduction energy of u_κ in the transition region $K_3 \setminus K_2$ where

$$\nabla u_\kappa := \nabla \varphi \left(v_\kappa - \frac{w_\kappa}{2\ell}\right) + \varphi \left(\nabla v_\kappa - \frac{\nabla w_\kappa}{2\ell}\right).$$

We easily deduce from (3.6), (3.11) and the K_2-periodicity of ∇w_κ the estimate

$$\int_{K_3 \setminus K_2} \big(1 + (\kappa - 1) \, 1_{Q_s}\big) \, \varphi^2 \left|\nabla v_\kappa - \frac{\nabla w_\kappa}{2\ell}\right|^2 dy = O\big(\kappa^{\frac{1}{2}}\big), \tag{3.15}$$

and out of the region of high conductivity Q_s the estimate

$$\int_{(K_3 \setminus K_2) \setminus Q_s} |\nabla \varphi|^2 \left(v_\kappa - \frac{w_\kappa}{2\ell}\right)^2 dy = O\big(\kappa^{\frac{1}{2}}\big). \tag{3.16}$$

Finally, in each high-conductivity set $Q_i \cap (K_3 \setminus K_2)$, for $i = 1, \ldots 4$, the Poincaré-Wirtinger inequality yields

$$\begin{aligned} \int_{Q_i \cap (K_3 \setminus K_2)} |\nabla \varphi|^2 \left(v_\kappa - \frac{w_\kappa}{2\ell}\right)^2 dy \; &= \frac{1}{(2\ell)^2} \int_{Q_i \cap (K_3 \setminus K_2)} |\nabla \varphi|^2 \left(w_\kappa - \fint_{Q_i \cap \partial K_2} w_\kappa \, dy\right)^2 dy \\ &\leq c \int_{Q_i \cap (K_3 \setminus K_2)} |\nabla w_\kappa|^2 \, dy, \end{aligned} \tag{3.17}$$

which multiplying by κ gives

$$\int_{Q_i \cap (K_3 \setminus K_2)} \kappa \, |\nabla \varphi|^2 \left(v_\kappa - \frac{w_\kappa}{2\ell}\right)^2 dy \leq c \int_{K_3} \chi_\kappa |\nabla w_\kappa|^2 \, dy = O\big(\kappa^{\frac{1}{2}}\big). \tag{3.18}$$

The estimates (3.14)-(3.18) combined with (3.13) imply the upper bound of (2.13).

Proof of the lower bound of (2.13): The Keller [19] duality (see also [14]) and the upper bound proved above give the equalities $a_\kappa^{-1} = a_{\kappa^{-1}} = O\big(\kappa^{-\frac{1}{2}}\big)$. This implies the lower bound of (2.13) and concludes the proof of Lemma 2.3. $\square$

Acknowledgments: The authors wish to thank the referees for suggestions which lead to improvements. They thank David Tanner for a helpful comment. Graeme Milton is grateful for support from the National Science Foundation through grant DMS-0707978.

References

[1] G. Alessandrini & V. Nesi: "Univalent σ-harmonic mappings", *Arch. Rat. Mech. Anal.*, **158** (2001), 155-171.

[2] G. Allaire: *Shape Optimization by the Homogenization Method*, Applied Mathematical Sciences **146**, Springer-Verlag, New-York, 2002.

[3] A. Bensoussan, J.L. Lions & G. Papanicolaou: *Asymptotic Analysis for Periodic Structures*, North-Holland, 1978.

[4] N.W. Ashcroft & N.D. Mermin: *Solid State Physics*, Saunders College, Philadelphia, 1976.

[5] D.J. Bergman: "Self duality and the low field Hall effect in 2D and 3D metal-insulator composites", in G. Deutscher, R. Zallen, and J. Adler eds., *Percolation Structures and Processes*, 1983, 297-321.

[6] D.J. Bergman, Y.M. Strelniker & A.K. Sarychev: "Recent advances in strong field magneto-transport in a composite medium", *Physica A*, **241** (1997), 278-283.

[7] D.J. Bergman & Y.M. Strelniker: "Duality transformation in a three dimensional conducting medium with two dimensional heterogeneity and an in-plane magnetic field", *Phys. Rev. Lett.*, **80** (1998), 3356-3359.

[8] D.J. Bergman & Y.M. Strelniker: "Magnetotransport in conducting composite films with a disordered columnar microstructure and an in-plane magnetic field", *Phys. Rev. B*, **60** (1999), 13016-13027.

[9] D.J. Bergman, X. Li & Y.M. Strelniker: "Macroscopic conductivity tensor with a one- or two-dimensional microstructure", *Phys. Rev. B*, **71** (2005), 035120.

[10] M. Briane & G.W. Milton: "Homogenization of the three-dimensional Hall effect and change of sign of the Hall coefficient", *Arch. Rat. Mech. Anal.*, **193** (3) (2009), 715-736.

[11] M. Briane & G.W. Milton: "Giant Hall effect in composites", *Multiscale Model. Simul.*, **7** (2009), 1405-1427.

[12] M. Briane, D. Manceau & G.W. Milton: "Homogenization of the two-dimensional Hall effect", *J. Math. Anal. Appl.*, **339** (2008), 1468-1484.

[13] B. Dacorogna: *Direct Methods in the Calculus of Variations*, in Applied Mathematical Sciences, **78**, Springer-Verlag, Berlin Heidelberg, 1989.

[14] A.M. Dykhne: "Conductivity of a two-dimensional two-phase system", *Zh. Eksp. Teor. Fiz.*, **59** (1970), 110-115. English translation in *Soviet Physics JETP*, **32** (1971), 63-65.

[15] A.M. Dykhne: "Anomalous plasma resistance in a strong magnetic field", *Zh. Eksp. Teor. Fiz.*, **59** (1970), 641-647. English translation in *Soviet Physics JETP*, **32** (1971), 348-351.

[16] D. Gilbarg & N. Trudinger: *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 1998.

[17] Y. Grabovsky: "An application of the general theory of exact relations to fiber-reinforced conducting composites with Hall effect", *Mech. Materials*, **41** (2009), 456-462.

[18] Y. Grabovsky: "Exact relations for effective conductivity of fiber reinforced conducting composites with Hall effect via a general theory", to appear in *SIAM J. Math. Anal.*

[19] J.B. Keller: "A theorem on the conductivity of a composite medium", *J. Math. Phys.*, **5** (4) (1964), 548-549.

[20] J.B. Keller: "Effective conductivity of periodic composites composed of two very unequal conductors", *J. Math. Phys.*, **28** (10) (1987), 2516-2520.

[21] L. Landau & E. Lifshitz: *Électrodynamique des Milieux Continus*, Éditions Mir, Moscow, 1969.

[22] J.-L. Lions & E. Magenes: *Problèmes aux limites non homogènes*, Vol I, Dunod, 1968. English translation in: *Non-homogeneous boundary value problems and applications*, Vol. I, Die Grundlehren der Mathematischen Wissenschaften, Band 181, Springer-Verlag, New-York Heidelberg, 1972.

[23] L.P. Liu, R. James & P. Leo: "New extremal inclusions and their applications to two-phase composites", *Arch. Rat. Mech. Anal.* to appear.

[24] K.A. Lurie & A.V. Cherkaev: "Exact estimates of the conductivity of a binary mixture of isotropic materials", *Proc. Roy. Soc. Edinburgh Sect. A*, **104** (1-2) (1986), 21-38.

[25] J.C. Maxwell: *A Treatise on Electricity and Magnetism*, Article 322, 371-372, Clarendon Press, Oxford, United Kingdom, 1873.

[26] G.W. Milton: "Classical Hall effect in two-dimensional composites: A characterization of the set of realizable effective conductivity tensors", *Physical Review B (Solid State)* **38** (16) (1988), 11296-11303.

[27] G.W. Milton: *The Theory of Composites*, Cambridge, United Kingdom, 2002.

[28] F. Murat: "H-convergence", *Séminaire d'Analyse Fonctionnelle et Numérique*, 1977-78, Université d'Alger, multicopied, 34 pp. English translation in: F. Murat & L. Tartar, "H-convergence", *Topics in the Mathematical Modelling of Composite Materials*, ed. by L. Cherkaev & R.V. Kohn, in Progress in Nonlinear Differential Equations and their Applications, **31**, Birkaüser, Boston, 1998, 21-43.

[29] M. Ali Omar: *Elementary Solid State Physics*, Addison Wesley, Reading Massachusetts, World Student Series Edition, 1975.

[30] Y.M. Strelniker & D.J. Bergman: "Exact relations between magnetoresistivity tensor components of conducting composites with a columnar microstructure", *Phys. Rev. B*, **61** (2000), 6288-6297.

[31] D. Stroud & D.J. Bergman: "New exact results for the Hall-coefficient and magnetoresistance of inhomogeneous two-dimensional metals", *Phys. Rev. B*, **30** (1984), 447-449.

[32] L. Tartar: *Cours Peccot, Collège de France*, 1977, unpublished, partly written in [28].

[33] L. Tartar: *Estimations fines des coefficients homogénéisés*, in: Ennio De Giorgi's Colloquium, Paris 1983, ed. P. Kree, Pitman, Boston 1985, 168-187.

[34] S.B. Vigdergauz: "Effective elastic parameters of a plate with a regular system of equal-strength holes", *Inzhenernyi Zhurnal. Mekhanika Tverdogo Tela: MTT*, **21** (1986), 165-169.