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Lattice-ordered groups generated by ordered groups and regular systems of ideals

Thierry Coquand Henri Lombardi Stefan Neuwirth

11th January 2017

Abstract

Unbounded entailment relations, introduced by Paul Lorenzen (1951), are a slight variant of a notion which plays a fundamental rôle in logic (see Scott 1974) and in algebra (see Lombardi and Quitté 2015). We propose to define systems of ideals for a commutative ordered monoid G as unbounded single-conclusion entailment relations that preserve its order and are equivariant: they describe all morphisms from G to meet-semilattice-ordered monoids generated by (the image of) G . Taking an article by Lorenzen (1953) as a starting point, we also describe all morphisms from a commutative ordered group G to lattice-ordered groups generated by G through unbounded entailment relations that preserve its order, are equivariant, and satisfy a “regularity” property invented by Lorenzen (1950); we call them *regular systems of ideals*. In particular, the free lattice-ordered group generated by G is described by the finest regular system of ideals for G , and we provide an explicit description for it; it is order-reflecting if and only if the morphism is injective, so that the Lorenzen-Clifford-Dieudonné theorem fits in our framework. In fact, Lorenzen’s research in algebra is motivated by the system of Dedekind ideals for the divisibility group of an integral domain R ; in particular, we provide an explicit description of the lattice-ordered group granted by Wolfgang Krull’s “Fundamentalsatz” if (and only if) R is integrally closed as the “regularisation” of the Dedekind system of ideals.

Keywords. Ordered monoid · Unbounded single-conclusion entailment relation · System of ideals · Morphism from an ordered monoid to a meet-semilattice-ordered monoid · Ordered group · Regular system of ideals · Unbounded entailment relation · Morphism from an ordered group to a lattice-ordered group · Lorenzen-Clifford-Dieudonné theorem · Fundamentalsatz for integral domains · Grothendieck ℓ -group.

MSC. 06F20 · 06F05 · 13A15 · 13B22.

1 Introduction

In this article, all monoids and groups are supposed to be commutative.

The idea of describing an unbounded semilattice by an unbounded single-conclusion entailment relation, and an unbounded distributive lattice by an unbounded entailment relation, dates back to Lorenzen (1951, §2) and is motivated there by ideal theory, which provides formal gcds, i.e., formal meets for elements of an integral domain.

An *unbounded meet-semilattice* is by definition a purely equational algebraic structure with a unique law \wedge that is idempotent, commutative and associative. We are dropping the axiom of meet-semilattices providing a greatest element (i.e., meets are only supposed to exist for *nonempty* finitely enumerated sets).

Notation I (see Lorenzen 1951, Satz 1). Let $P_{fe}^*(G)$ be the set of nonempty finitely enumerated subsets of a set G . For an unbounded meet-semilattice S we denote by $A \triangleright x$ or $A \triangleright_S x$ the relation defined between the sets $P_{fe}^*(S)$ and S in the following way:

$$A \triangleright x \stackrel{\text{def}}{\iff} \bigwedge A \leq x \stackrel{\text{def}}{\iff} x \wedge \bigwedge A = \bigwedge A.$$

This relation is reflexive, monotone (a property also called “thinning”) and transitive (a property also called “cut” because it cuts x) in the following sense:

$$\begin{aligned} a &\triangleright a && \text{(reflexivity);} \\ \text{if } A \triangleright b, & \text{ then } A, A' \triangleright b && \text{(monotonicity);} \\ \text{if } A \triangleright x \text{ and } A, x \triangleright b, & \text{ then } A \triangleright b && \text{(transitivity).} \end{aligned}$$

In the context of relations, we shall make the following abuses of notation for finitely enumerated sets: we write a for the singleton consisting of a , and A, A' for the union of the sets A and A' .

These three properties correspond respectively to the “tautologic assertions”, the “immediate deductions”, and to an elementary form of the “syllogisms” of the systems of axioms introduced by Paul Hertz (1923, § 1), so that the following notion may be attributed to him¹; see also Gerhard Gentzen (1933, § 2), who coined the terms “thinning” and “cut”.

Definition II. Let G be an arbitrary set.

¹See Jean-Yves Béziau (2006, § 6) for a discussion on its relationship with Alfred Tarski’s consequence operation, which may be compared to the relationship of our Definition III of a system of ideals with the set-theoretic star-operation: see Item 2 of Remarks 2.4.

1. A relation between $P_{fe}^*(G)$ and G which is reflexive, monotone and transitive is called an *unbounded single-conclusion entailment relation* for G .

2. The unbounded single-conclusion entailment relation \triangleright_2 is *coarser* than the unbounded single-conclusion entailment relation \triangleright_1 if $A \triangleright_1 y$ implies $A \triangleright_2 y$. One says also that \triangleright_1 is *finer* than \triangleright_2 .

Now suppose that (G, \leq_G) is an ordered monoid², (M, \leq_M) a meet-semilattice-ordered monoid³, a *meet-monoid* for short, and $\psi: G \rightarrow M$ a morphism of ordered monoids. The relation

$$\bigwedge_{i \in \llbracket 1..n \rrbracket} \psi(x_i) \leq_M \psi(y)$$

defines an unbounded single-conclusion entailment relation for G that satisfies furthermore the following properties:

- S1* if $a \leq_G b$, then $a \triangleright b$ (preservation of order);
S2 if $A \triangleright b$, then $x + A \triangleright x + b$ ($x \in G$) (equivariance).

We may thus introduce systems of ideals in a purely algebraic way (i.e., as entailment relations that require only a naive set theory for finitely enumerated sets): we propose the following definition, that we extracted from [Lorenzen \(1939, Definition 1\)](#) (or [Jaffard 1960, I, § 3, 1](#)).

Definition III. A *system of ideals* for an ordered monoid G is an unbounded single-conclusion entailment relation for G satisfying Properties *S1* and *S2*.

Theorem I. Let \triangleright be a system of ideals for an ordered monoid G . Let S be the unbounded meet-semilattice generated by the unbounded single-conclusion entailment relation \triangleright . There is a unique law $+$ on S that makes S a meet-monoid and for which the natural morphism of ordered sets $G \rightarrow S$ is a monoid morphism.

[Lorenzen \(1950, page 486\)](#) emphasises the transparency of this approach with respect to the set-theoretic ideals: “But if one removes this set-theoretic clothing, then the concept of ideal may be defined quite simply: a system of ideals of a pre-ordered set is nothing other than an embedding into a semilattice.” See [Remarks 2.4](#) for more on this.

An *unbounded distributive lattice* is by definition a purely equational algebraic structure with two laws \wedge and \vee satisfying the axioms of distributive lattices, except the two axioms providing a greatest and a least element.

²I.e., a monoid $(G, +, 0)$ endowed with an order relation \leq_G compatible with addition: $x \leq_G y \Rightarrow x + z \leq_G y + z$.

³I.e., a monoid endowed with an unbounded meet-semilattice law \wedge inducing \leq_M and compatible with addition: the equality $x + (y \wedge z) = (x + y) \wedge (x + z)$ holds.

Notation IV (see [Lorenzen 1951](#), Satz 5). Let L be an unbounded distributive lattice and let us define the relation $A \vdash B$ or $A \vdash_L B$ on the set $P_{fe}^*(L)$ in the following way:

$$A \vdash B \stackrel{\text{def}}{\iff} \bigwedge A \leq \bigvee B.$$

This relation is reflexive, monotone (a property also called “thinning”) and transitive (a property also called “cut”) in the following sense.

$$\begin{aligned} a \vdash a & \quad (\text{reflexivity}); \\ \text{if } A \vdash B, \text{ then } A, A' \vdash B, B' & \quad (\text{monotonicity}); \\ \text{if } A \vdash B, x \text{ and } A, x \vdash B, \text{ then } A \vdash B & \quad (\text{transitivity}). \end{aligned}$$

We insist on the fact that A and B must be nonempty.

The following definition is a variant of a notion whose name has been coined by Dana [Scott \(1974, page 417\)](#). It is introduced as a description of an unbounded distributive lattice (see [Theorem 3.1](#)) in [Lorenzen \(1951, § 2\)](#).

Definition V. 1. For an arbitrary set G , a binary relation on $P_{fe}^*(G)$ which is reflexive, monotone and transitive is called an *unbounded entailment relation*.

2. An unbounded entailment relation \vdash_2 is *coarser* than an unbounded entailment relation \vdash_1 if $A \vdash_1 B$ implies $A \vdash_2 B$. One says also that \vdash_1 is *finer* than \vdash_2 .

Now suppose that (G, \leq_G) is an ordered group⁴, (H, \leq_H) a lattice-ordered group⁵, an ℓ -group for short, and $\varphi: G \rightarrow H$ a morphism of ordered groups.

The laws \wedge and \vee on an ℓ -group provide an unbounded distributive lattice structure, and the relation

$$\bigwedge_{i \in [1..n]} \varphi(x_i) \leq_H \bigvee_{j \in [1..m]} \varphi(y_j)$$

defines typically an unbounded entailment relation for G that satisfies furthermore the following properties:

$$\begin{aligned} R1 \quad \text{if } a \leq_G b, \text{ then } a \vdash b & \quad (\text{preservation of order}); \\ R2 \quad x + a, y + b \vdash x + b, y + a & \quad (\text{regularity}); \\ R3 \quad \text{if } A \vdash B, \text{ then } x + A \vdash x + B \quad (x \in G) & \quad (\text{equivariance}). \end{aligned}$$

Properties $R1$ and $R3$ are straightforward, and the property $R2$ of regularity follows from the fact that if x', a', y', b' are elements of H , then the inequality

$$(x' + a') \wedge (y' + b') \leq_H (x' + b') \vee (y' + a')$$

⁴I.e., a group that is an ordered monoid.

⁵I.e., an ordered group that is a semilattice: this is enough to ensure that it is a meet-monoid, that any two elements have a join, and that the distributivity laws hold.

reduces successively to

$$\begin{aligned} 0 &\leq_H ((-x' - a') \vee (-y' - b')) + ((x' + b') \vee (y' + a')) \\ 0 &\leq_H (b' - a') \vee (y' - x') \vee (x' - y') \vee (a' - b') \\ 0 &\leq_H |b' - a'| \vee |y' - x'|. \end{aligned}$$

We hence propose the following new definition (compare [Lorenzen 1953](#), § 1).

Definition VI. Let G be an ordered group.

1. A *regular system of ideals* for G is an unbounded entailment relation for G satisfying Properties *R1*, *R2* and *R3*.
2. A system of ideals for G is *regular* if it is the restriction of a regular system of ideals to $P_{fe}^*(G) \times G$.

The ambiguity introduced by these two definitions is harmless because it turns out that a regular system of ideals is determined by its restriction to $P_{fe}^*(G) \times G$ (see [Theorem 3.9](#)).

A system of ideals gives rise to a regular system of ideals if one affords to suppose that elements occurring in a computation are comparable, in the following way.

Definition VII (see [Lorenzen 1953](#), (2.2) and page 23). Let \triangleright be a system of ideals for an ordered group G .

1. For every element x of G , consider the system of ideals \triangleright_x coarser than \triangleright obtained by forcing the property $x \geq 0$. The *regularisation* of \triangleright is the relation on $P_{fe}^*(G)$ defined by

$$A \vdash_{\triangleright} B \stackrel{\text{def}}{\iff} \begin{cases} \text{there are } x_1, \dots, x_\ell \text{ such that for every choice of signs } \pm \\ A - B \triangleright_{\pm x_1, \dots, \pm x_\ell} 0 \text{ holds.} \end{cases}$$

2. The group G is \triangleright -closed if $a \vdash_{\triangleright} b \Rightarrow a \leq_G b$ holds for all $a, b \in G$.

Theorem II (see [Lorenzen 1953](#), § 1). *Let \triangleright be a system of ideals for an ordered group G . The regularisation $A \vdash_{\triangleright} B$ given in [Definition VII](#) is the finest regular system of ideals for G whose restriction to $P_{fe}^*(G) \times G$ is coarser than \triangleright .*

This enhances the first part of the proof of the remarkable [Satz 1](#) of [Lorenzen \(1953\)](#). In place of its second part, we propose the new [Theorem IV](#) below: regular systems of ideals provide a description of all morphisms from an ordered group G to ℓ -groups generated by (the image of) G .

Underway, we provide the following constructive version of a key observation concerning the ℓ -group freely generated by an ordered group.

Theorem III. *Let ν be the morphism from an ordered group G to the ℓ -group H that it freely generates. Let $u_1, \dots, u_k \in G$. We have $\bigvee_{j=1}^k \nu(u_j) \geq_H 0$ if and only if there exist integers $m_j \geq 0$ not all zero such that $\sum_{j=1}^k m_j u_j \geq_G 0$.*

Theorem IV. *Let \vdash be a regular system of ideals for an ordered group G . Let H be the unbounded distributive lattice generated by the unbounded entailment relation \vdash . Then H has a (unique) group law which is compatible with its lattice structure and such that the morphism (of ordered sets) $\varphi: G \rightarrow H$ is a group morphism.*

These results give rise to the following construction and corollary, that one can find in [Lorenzen \(1953, § 2 and page 23\)](#).

Definition VIII. Let \triangleright be a system of ideals for an ordered group G . The *Lorenzen group* associated to \triangleright is the ℓ -group provided by Theorems [II](#) and [IV](#).

Corollary to Theorem IV. *Let \triangleright be a system of ideals for an ordered group G . If G is \triangleright -closed, then G embeds into the Lorenzen group associated to \triangleright .*

In this paper, our aim is to give a precise account of the approach by regular systems of ideals; we are directly inspired by [Lorenzen \(1953\)](#). The literature on ℓ -groups seems not to have taken notice of these results. In Lorenzen’s work, this approach supersedes another, based on the Grothendieck ℓ -group of the meet-monoid obtained by forcing cancellativity of the system of ideals, ideated by Heinz [Prüfer \(1932\)](#) and generalised to the setting of ordered monoids in the Ph.D. thesis [Lorenzen \(1939\)](#). We also provide an account for that approach, which yields a construction of an ℓ -group from a system of ideals which turns out to be the associated Lorenzen group.

The motivating example for Lorenzen’s analysis of the concept of ideal is Wolfgang Krull’s “Fundamentalsatz” that an integral domain is an intersection of valuation rings if and only if it is integrally closed. As [Krull \(1935, page 111\)](#) himself emphasises, “Its main defect, that one must not overlook, lies in that it is a purely existential theorem”, resulting from a well-ordering argument. Lorenzen’s goal is to unveil its constructive content, i.e., to express it without reference to valuations. He shows that the well-ordering argument may be replaced by the right to compute as if the divisibility group was linearly ordered (see [Definition VII](#) above)⁶, that integral

⁶In a letter to Heinrich Scholz dated 18th April 1953 (Scholz-Archiv, Westfälische Wilhelms-Universität Münster, <http://www.uni-muenster.de/IVV5WS/ScholzWiki/doku.php?id=scans:blogs:ko-05-06> accessed 21st September 2016), Krull writes: “At working with the uncountable, in particular with the well-ordering theorem, I always had the feeling that one uses fictions there that need to be replaced some day by more reasonable concepts. But I was not getting upset over it, because I was convinced that at a careful application of the common “fictions” nothing false comes out, and because I was firmly counting on the man who would some day put all in order. Lorenzen has now found according to my conviction the right way [...]”.

closedness guarantees that such computations do not add new relations of divisibility to the integral domain, and that this generates an ℓ -group. The corollary to Theorem IV is in fact an abstract version of the following theorem (see Theorem 5.5).

Theorem. *Let R be an integral domain, K its field of fractions, and $G = K^\times/R^\times$ its divisibility group. Consider the Dedekind system of ideals for G defined by*

$$A \triangleright_d b \stackrel{\text{def}}{\iff} b \in \langle A \rangle_R,$$

where $\langle A \rangle_R$ is the fractional ideal generated by A over R in K . Then G embeds into an ℓ -group that contains the Dedekind system of ideals if and only if R is integrally closed.

Let us now briefly describe the structure of this article.

Section 2 deals with unbounded meet-semilattices as generated by unbounded single-conclusion entailment relations, discusses systems of ideals for an ordered monoid and the meet-monoid they generate (Theorem I), and describes the case in which the system of ideals for an ordered group is in fact a group: then G is a *divisorial group*, a notion tightly connected to Weil divisor groups.

Section 3 deals with unbounded distributive lattices as generated by unbounded entailment relations, discusses regular systems of ideals and provides the proof of Theorem II. It also provides two applications: a description of the finest regular system of ideals and Lorenzen's theory of divisibility for integral domains.

Section 4 provides a constructive proof of Theorem III based on the Positivstellensatz for ordered groups.

Section 5 proves the main theorem of the paper, Theorem IV, that regular systems of ideals for an ordered group generate in fact an ℓ -group. Some consequences for Lorenzen's theory of divisibility for integral domains are stated.

Section 6 reminds us of an important theorem by Prüfer which leads to the historically first approach to the Lorenzen group associated to a system of ideals.

A more elaborate study of Lorenzen's work will be the subject of another article that will provide a detailed analysis of Lorenzen (1950, 1952, 1953). These works, all published in *Mathematische Zeitschrift*, are written with careful attention to the possibility of constructive formulations for abstract existence theorems.

The paper is written in Errett Bishop's style of constructive mathematics (Bishop 1967; Bridges and Richman 1987; Lombardi and Quitté 2015; Mines, Richman and Ruitenburg 1988): all theorems can be viewed as providing an algorithm that constructs the conclusion from the hypotheses.

2 Unbounded meet-semilattices and systems of ideals

2.1 Unbounded meet-semilattices

Let us first discuss the notion of single-conclusion entailment relations.

Remarks 2.1 (for Definition II). 1. If instead of nonempty subsets, we had considered nonempty multisets, we would have had to add a contraction rule, and if we had considered nonempty lists, we would have had to add also a permutation rule.

2. The terminology “coarser than” has the following explanation. The nonempty finitely enumerated set A to the left of \triangleright represents a formal meet of A for the preorder \leq_{\triangleright} on $P_{fe}^*(G)$ associated to the unbounded single-conclusion entailment relation \triangleright (and defined by the equivalence (*) below). To say that the relation \triangleright_2 is coarser than the relation \triangleright_1 is to say this for the associated preorders, i.e., that $A \leq_{\triangleright_1} B$ implies $A \leq_{\triangleright_2} B$, and this corresponds to the usual meaning of “coarser than” for preorders, since $A =_{\triangleright_1} B$ implies then $A =_{\triangleright_2} B$, i.e., the equivalence relation $=_{\triangleright_2}$ is coarser than $=_{\triangleright_1}$. ■

A fundamental theorem holds for an unbounded single-conclusion entailment relation for a given set G : it states that it generates an unbounded meet-semilattice S which defines in turn an unbounded single-conclusion entailment relation that reflects the original one. This is the single-conclusion analogue of the better known Theorem 3.1.

Theorem and definition 2.2 (Fundamental theorem of unbounded single-conclusion entailment relations, see Lorenzen 1951, Satz 3). *Let G be a set and \triangleright_G an unbounded single-conclusion entailment relation between $P_{fe}^*(G)$ and G . Let us consider the unbounded meet-semilattice S defined by generators and relations in the following way: the generators are the elements of G and the relations are the*

$$A \triangleright_S x \text{ whenever } A \triangleright_G x.$$

Then, for all (A, x) in $P_{fe}^(G) \times G$, we have the reflection of entailment*

$$\text{if } A \triangleright_S x, \text{ then } A \triangleright_G x.$$

In fact, S can be defined as the ordered set obtained by descending to the quotient of $(P_{fe}^(G), \leq_{\triangleright})$, where \leq_{\triangleright} is the preorder defined by*

$$A \leq_{\triangleright} B \stackrel{\text{def}}{\iff} A \triangleright b \text{ for all } b \in B. \quad (*)$$

Proof. One sees easily that \leq_{\triangleright} is a preorder on $P_{fe}^*(G)$ that endows the quotient by $=_{\triangleright}$ with a meet-semilattice structure, where the law \wedge_{\triangleright} is obtained by descending

the law $(A, B) \mapsto A \cup B$ to the quotient. The reader will prove that S can also be defined by generators and relations as in the statement. \square

Note that the preorder $x \triangleright y$ on G makes its quotient a subobject of S in the category of ordered sets.

Remarks 2.3. 1. Suppose that (G, \leq_G) is an ordered set. The “finite Dedekind-MacNeille completion” that adds formal finite meets to G in a minimal way corresponds to the construction of an unbounded semilattice from (G, \triangleright_v) , where \triangleright_v is the coarsest order-reflecting unbounded single-conclusion entailment relation for G :

$$A \triangleright_v b \stackrel{\text{def}}{\iff} \forall z \in G \text{ if } z \leq_G A, \text{ then } z \leq_G b, \quad (\dagger)$$

where $z \leq_G A$ means $z \leq_G a$ for all $a \in A$.

2. The relation $x \triangleright_G y$ is a priori just a preorder relation for G , not an order relation. Let us denote the element x viewed in the ordered set \overline{G} associated to this preorder by \overline{x} , and let $\overline{A} = \{\overline{x} \mid x \in A\}$ for a subset A of G . In Theorem 2.2, we consider a meet-semilattice S yielding the same single-conclusion entailment relation for G as \triangleright_G ; for the sake of rigour, we should have written $\overline{A} \triangleright_S \overline{x}$ rather than $A \triangleright_S x$ in order to deal with the fact that the equality of S is coarser than the equality of G . In particular, it is \overline{G} rather than G which can be identified with a subset of S . \blacksquare

2.2 Systems of ideals for an ordered monoid

Let us now discuss the definition of a system of ideals à la Lorenzen for an ordered monoid, Definition III, given in the language of single-conclusion entailment relations.

Remarks 2.4 (for Definition III). 1. We find that it is more natural to state a direct implication rather than an equivalence in Item *S1*; we deviate here from Lorenzen and Paul Jaffard (1960, page 16). The reverse implication expresses the supplementary property that the system of ideals is order-reflecting.

2. Lorenzen (1939), following Prüfer (1932, § 2) and Hilbert who subordinated algebra to set theory, describes a (finite) “ r -system” of ideals through a set-theoretical map (sometimes called star-operation)

$$P_{\text{fe}}^*(G) \longrightarrow P(G), \quad A \longmapsto \{x \in G \mid A \triangleright x\} \stackrel{\text{def}}{=} A_r$$

(here $P(G)$ stands for all subsets of G , and r is just a variable name for distinguishing

different systems) that satisfies:

- $I1$ $A_r \supseteq A$;
- $I2$ $A_r \supseteq B \implies A_r \supseteq B_r$;
- $I3$ $\{a\}_r = \{x \in G \mid a \leq x\}$ (preservation and reflection of order);
- $I4$ $(x + A)_r = x + A_r$ (equivariance).

Let us note that the containment $A_r \supseteq B_r$ corresponds to the inequality $A \leq_{\triangleright} B$ in the meet-semilattice associated to the single-conclusion entailment relation \triangleright by Theorem 2.2.

As previously indicated, in contradistinction to Lorenzen and Jaffard, we find it more natural to relax the equality in $I3$ to a containment: if we do so, the reader can prove that the definition of the star-operation⁷ is equivalent to Definition III. Items $I1$ and $I2$ correspond to the definition of a single-conclusion entailment relation, and Items $I3$ (relaxed) and $I4$ correspond to Items $S1$ and $S2$ in Definition III. Compare Lorenzen (1950, pages 504-505).

3. In the set-theoretic framework of the previous item, the r_2 -system is coarser than the r_1 -system exactly if $A_{r_2} \supseteq A_{r_1}$ holds for all $A \in \mathsf{P}_{\text{fe}}^*(G)$ (see Jaffard 1960, I, § 3, Proposition 2). ■

In the case that G is an ordered group, we may state an apparently simpler definition for systems of ideals.

Proposition 2.5 (Variant for the definition of a system of ideals for an ordered group). *Let us consider a predicate $\cdot \triangleright 0$ on $\mathsf{P}_{\text{fe}}^*(G)$ for an ordered group G and let us define a relation between the sets $\mathsf{P}_{\text{fe}}^*(G)$ and G by*

$$A \triangleright b \stackrel{\text{def}}{\iff} A - b \triangleright 0.$$

In order for this relation to be a system of ideals, it is necessary and sufficient that the following properties be fulfilled:

- $T1$ if $a \leq_G 0$, then $a \triangleright 0$ (preservation of order);
- $T2$ if $A \triangleright 0$, then $A, A' \triangleright 0$ (monotonicity);
- $T3$ if $A - x \triangleright 0$ and $A, x \triangleright 0$, then $A \triangleright 0$ (transitivity).

⁷Lorenzen unveiled the lattice theory behind multiplicative ideal theory step by step, the decisive one being dated back by him to 1940. In a footnote to his definition, Lorenzen (1939, page 536) writes: “If one understood hence by a system of ideals every [semi]lattice that contains the principal ideals and satisfies Property $[I4]$, then this definition would be only unessentially broader”. In a letter to Krull dated 13th March 1944 (Philosophisches Archiv, Universität Konstanz, PL-1-1-131), he writes: “For example, the insight that a system of ideals is actually nothing more than a supersemilattice, and a valuation nothing more than a linear order, strikes me as the most essential result of my effort”.

Proof. Left to the reader. □

The finest and the coarsest system of ideals admit the following descriptions.

Proposition 2.6 (Lorenzen 1950, Satz 14, Satz 15, Footnote 26). *Let G be an ordered monoid.*

1. *The finest system of ideals for G is defined by*

$$A \triangleright_s b \stackrel{\text{def}}{\iff} a \leq_G b \text{ for some } a \in A.$$

Note that \triangleright_s is order-reflecting: $x \triangleright_s y$ iff $x \leq_G y$.

2. *The coarsest order-reflecting system of ideals for G is defined by*

$$A \triangleright_v b \stackrel{\text{def}}{\iff} \forall z, w \in G \text{ if } z \leq_G A + w, \text{ then } z \leq_G b + w,$$

where $z \leq_G A + w$ means $z \leq_G a + w$ for all $a \in A$.

3. *If G is an ordered group, this simplifies to the definitional equivalence (†) on page 9.*

Remarks 2.7. 1. As noted in Item 3 for \triangleright_v , the definition of \triangleright_s could be stated verbatim in the framework of ordered sets and single-conclusion entailment relations.

2. The system of ideals \triangleright_v was introduced independently by Bartel Leendert van der Waerden (see van der Waerden 1931, § 103)⁸ and Prüfer (1932) (“v” like “Vielfache”, “multiples” of gcds). The system of ideals \triangleright_s appears first in Lorenzen (1939) (“s” standing perhaps for “sum”). ■

Proof. 1. Left to the reader.

2. One sees easily that \triangleright_v is a single-conclusion entailment relation for G .

• *S1.* Let $y \in G$. Suppose $a \leq_G b$: then $a + y \leq_G b + y$, i.e., if $x \leq_G a + y$, then $x \leq_G b + y$; hence $a \triangleright_v b$.

• *Reflection of order.* Conversely, suppose $a \triangleright_v b$: taking $x = a$ and $y = 0$ in the definition of $a \triangleright_v b$, we get $a \leq_G b$.

• *S2.* Let us suppose $A \triangleright_v b$ and prove $A + x \triangleright_v b + x$ for $x \in G$. Let $z, w \in G$; if $z \leq_G (A + x) + w$, then $z \leq_G A + (x + w)$, and by hypothesis $z \leq_G b + (x + w)$, i.e., $z \leq_G (b + x) + w$.

Now let \triangleright be an order-reflecting system of ideals for G and suppose that $A \triangleright b$. Let us prove that $A \triangleright_v b$. Let $z, w \in G$ and suppose that $z \leq_G A + w$; by the definition of \leq_{\triangleright} and because $A + w \triangleright b + w$, we have $z \leq_{\triangleright} A + w \leq_{\triangleright} b + w$. Since \leq_{\triangleright} reflects the order on G , $z \leq_G b + w$. □

⁸Or the translation van der Waerden (1950, § 105) of its second edition.

2.3 Proof of Theorem I

Proof of Theorem I. We define $A + B = \{a + b \mid a \in A, b \in B\}$ in $\mathbf{P}_{\text{fe}}^*(G)$. We have to check that this law descends to the quotient S . It suffices to show that $B \leq_{\triangleright} C$ implies $A + B \leq_{\triangleright} A + C$: in fact, $B \leq_{\triangleright} C$ implies $x + B \leq_{\triangleright} x + C$ by equivariance, and $A + B \leq_{\triangleright} x + C$ for every $x \in A$ by monotonicity. Finally, let us verify the compatibility of \wedge_{\triangleright} with addition: we note that already in $\mathbf{P}_{\text{fe}}^*(G)$ we have $A + (B \cup C) = (A + B) \cup (A + C)$. \square

2.4 The classical (Weil) divisor group in commutative algebra

Prüfer (1932, § 3) introduces a property for a system of ideals, “Property B ”, expressing that the associated meet-monoid is in fact a group (and hence an ℓ -group). The next proposition shows that this is essentially a property of the ordered monoid itself.

Proposition 2.8 (Lorenzen 1950, Satz 16). *Let G be an ordered monoid and \triangleright an order-reflecting system of ideals for G . If the associated meet-monoid is a group, then \triangleright coincides with the coarsest system of ideals \triangleright_v for G .*

Proof. Suppose that $A \triangleright_v b$, i.e., that $A \leq_{\triangleright_v} b$. We need to prove that $A \triangleright b$, i.e., that $A \leq_{\triangleright} b$. Since \leq_{\triangleright_v} and \leq_{\triangleright} reflect \leq_G , we know that

$$0 \leq_{\triangleright_v} B \iff 0 \leq_{\triangleright} B \iff 0 \leq_G B.$$

Let $C \in \mathbf{P}_{\text{fe}}^*(G)$ such that $A + C =_{\triangleright} 0$. We have $0 \leq_{\triangleright} A + C$ and hence $0 \leq_{\triangleright_v} A + C$. We get $0 \leq_{\triangleright_v} A + C \leq_{\triangleright_v} b + C$. Therefore $0 \leq_{\triangleright} b + C$ and $A \leq_{\triangleright} b + C + A =_{\triangleright} b$. \square

In the rest of this section, we shall only consider the case where G is a group because of the lack of applications, and because it avoids a more involved definition of divisorial opposites below.

Proposition 2.11 shows that “Property B ” may be caught by the following definitions.

Definitions 2.9. Let G be an ordered group.

1. Two nonempty finitely enumerated subsets A, B of G are *divisorially opposite* if 0 is meet for $A + B$ in G .
2. The group G is *divisorial* if every nonempty finitely enumerated subset admits a divisorial opposite.

Remarks 2.10. 1. The notion of divisorially opposite sets coincides with the notion of *divisorially inverse lists* in (the multiplicative notation of) [Coquand and Lombardi \(2016\)](#). ■

2. Formally, in Item 1, we think of $\bigwedge B = \bigvee -A$ as of $\bigwedge(A + B) = 0$, so that the join of $-A$ is given by the meet of B . It remains to show that this intuition works. ■

Proposition 2.11. *Let G be an ordered group. T.f.a.e.*

1. *The meet-monoid associated to the system of ideals \triangleright_v is a group.*
2. *The group G is divisorial.*

Proof. 1 \Rightarrow 2. Consider $A \in P_{fe}^*(G)$. Then the opposite of A in the meet-monoid associated to \triangleright_v writes B for some $B \in P_{fe}^*(G)$, i.e., $A + B =_{\triangleright_v} 0$. But $A + B \leq_{\triangleright_v} 0$ means that $x \leq_G A + B \Rightarrow x \leq_G 0$, and $0 \leq_{\triangleright_v} A + B$ means that every element of $A + B$ is $\geq_G 0$.

2 \Rightarrow 1. It suffices to check that if $A \in P_{fe}^*(G)$, then a divisorial opposite B of A satisfies $A + B =_{\triangleright_v} 0$. First $0 \leq_G A + B$, so that $0 \leq_{\triangleright_v} A + B$. Second, let $x \in G$ such that $x \leq_{\triangleright_v} A + B$. We have $x \leq_G A + B$ (\triangleright_v is order-reflecting), so $x \leq_G 0$ and $x \leq_{\triangleright_v} 0$. Thus $A + B \leq_{\triangleright_v} 0$. □

Divisorial groups will provide natural examples of the Lorenzen group associated to a system of ideals, i.e., the meet-monoid associated to \triangleright_v .

Remarks 2.12. 1. Proposition 2.11 can be seen as a variant of [Jaffard \(1960, II, § 3, Corollaire du théorème 3, page 55\)](#).

2. Divisorial groups are tightly connected to Weil divisor groups in commutative algebra. [Coquand and Lombardi \(2016\)](#) give a constructive presentation of “rings with divisors” (in French, “anneaux à diviseurs”), which they define as integral domains whose divisibility group is divisorial. Rings with divisors with an additional condition of noetherianity are called *Krull domains*. H. M. [Edwards \(1990\)](#) describes in his *Divisor theory* an approach à la Kronecker to rings with divisors in the case where they are constructed as integral closures of finite extensions of “Kronecker natural rings”. See also in the same spirit Hermann [Weyl \(1940\)](#). Rings with divisors are called “pseudo-Prüferian integral domains” by Nicolas [Bourbaki \(1972, VII.2.Ex.19\)](#), and “Prüfer- v -multiplication domains (PvMD)” in the English literature (one can also find the terminology “rings with a theory of divisors”). The main examples are the gcd domains (for which the divisor group coincides with the divisibility group) and the coherent normal domains (especially in algebraic geometry). In case of noetherian coherent normal domains, the divisor group is usually called the Weil divisor group. For a ring with divisors R , the Weil divisor group $\text{Div}(R)$ is a quotient of the Lorenzen group $\text{Lor}(R)$ as defined in Definition 5.4, with equality in the case of Prüfer domains. We expand on this topic in Remark 5.8. ■

3 Unbounded distributive lattices and regular systems of ideals

3.1 Unbounded distributive lattices

References: Grätzer (2011, Chapter 2), Cederquist and Coquand (2000); Lombardi and Quitté (2015); Lorenzen (1951).

Note that Item 1 of Remark 2.1 applies again verbatim for Definition V.

Let us adapt Theorem 2.2 to the setting of unbounded entailment relations: this yields Theorem 3.1, an unbounded variant of the fundamental theorem of entailment relations (Cederquist and Coquand 2000, Theorem 1), which dates back to Lorenzen (1951, Satz 7). It states that an unbounded entailment relation for a set G generates an unbounded distributive lattice L which defines an unbounded entailment relation that reflects the original one. The proof is essentially the same as in Cederquist and Coquand (2000) or in Lombardi and Quitté (2015, Theorem XI-5.3).

Theorem 3.1 (Fundamental theorem of unbounded entailment relations, see Lorenzen 1951, Satz 7). *Let G be a set and \vdash_G an unbounded entailment relation on $P_{\text{fe}}^*(G)$. Let us consider the unbounded distributive lattice L defined by generators and relations in the following way: the generators are the elements of G and the relations are the*

$$A \vdash_L B \text{ whenever } A \vdash_G B.$$

Then, for all A, B in $P_{\text{fe}}^(G)$, we have the reflection of entailment*

$$\text{if } A \vdash_L B, \text{ then } A \vdash_G B.$$

Item 2 of Remark 2.3 applies again mutatis mutandis.

3.2 Regular systems of ideals for an ordered group

Let us now undertake an investigation of Definition VI.

Comment 3.2 (for Definition VI). The property of regularity arises in Lorenzen's analysis of the rôle played by the commutativity of the group: he isolates an inequality which is trivially verified in a commutative ℓ -group (see page 5), but not in a noncommutative one: that $(x + a) \wedge (b + y) \leq (x + b) \vee (a + y)$. Lorenzen (1950, Satz 13) states that a (noncommutative) ℓ -group that is regular in this sense is a subdirect product of linearly preordered groups by a well-ordering argument. In the commutative setting, this corresponds to the theorem (in classical mathematics) stating that any commutative ℓ -group is a subdirect product of linearly preordered commutative groups. ■

When we assume Property *R1*, the following fact concerning entailment relations takes a flavour of “monotonicity for the order relation of G ”.

Fact 3.3. *Assume that $c \vdash d$.*

1. *If $A \vdash B, c$, then $A \vdash B, d$.*
2. *If $A, d \vdash B$, then $A, c \vdash B$.*

Proof. By monotonicity, $c \vdash d$ gives $A, c \vdash B, d$. 1. $A \vdash B, c$ gives $A \vdash B, c, d$. Cutting c , we get $A \vdash B, d$. 2. Symmetric argument. \square

Proposition 3.4. *Let \vdash be a regular system of ideals for an ordered group G . The following properties (of which *R2* is a particular case) are valid for each integer $n \geq 1$:*

$R2_n$ if $x_1 + \cdots + x_n =_G y_1 + \cdots + y_n$, then $x_1, \dots, x_n \vdash y_1, \dots, y_n$.

Note that if we have $x_1 + \cdots + x_n =_G y_1 + \cdots + y_m$ with $m \neq n$, we may add 0s to the shorter list in order to apply the lemma. E.g., if $a =_G b + c$, then $0, a \vdash b, c$. In this way, we get⁹ $0 \vdash a, -a$ and $a, -a \vdash 0$.

Proof. *Case $n = 2$.* This is Property *R2*: if x, y, a, b are given, let $x_1 = x + a$, $x_2 = y + b$, $y_1 = x + b$ and $y_2 = y + a$; conversely, if $x_1 + x_2 =_G y_1 + y_2$ are given, let $x = x_1$, $a = 0$, $y = y_2$ and $b = x_2 - y_2 =_G y_1 - x_1$.

Case $n > 2$. By induction. Assume that $x_1 + \cdots + x_n =_G y_1 + \cdots + y_n$. We remark that it is sufficient to prove $x_1, \dots, x_n \vdash y_1, \dots, y_n$ in the case $y_1 =_G 0$, as we get the general case by a translation (Property *R3*). Here we use the fact that the same number of terms are being added on both sides.

So, assume $x_1 + \cdots + x_n =_G y_2 + \cdots + y_n$. We need to prove

$$x_1, \dots, x_n \vdash 0, y_2, \dots, y_n. \quad (\ddagger)$$

By the induction hypothesis we have on the one hand

$$x_3, \dots, x_n, (x_1 + x_2) \vdash y_2, \dots, y_n,$$

which gives by monotonicity

$$x_1, x_2, x_3, \dots, x_n, (x_1 + x_2) \vdash 0, y_2, \dots, y_n. \quad (\S)$$

On the other hand, we have $x_1, x_2 \vdash (x_1 + x_2), 0$ which gives by monotonicity

$$x_1, x_2, x_3, \dots, x_n \vdash (x_1 + x_2), 0, y_2, \dots, y_n. \quad (\parallel)$$

Finally, cutting $x_1 + x_2$ in (\S) and (\parallel) , we get (\ddagger) . \square

⁹Precisely, we get $0, 0 \vdash a, -a$, which contracts to $0 \vdash a, -a$.

When $x_1 + \dots + x_n \leq_G y_1 + \dots + y_n$, we have $x_1 + \dots + x_n =_G y_1 + \dots + y_{n-1} + y'_n$ for some $y'_n \leq_G y_n$. So $y'_n \vdash y_n$ and $x_1, \dots, x_n \vdash y_1, \dots, y_{n-1}, y'_n$, and Fact 3.3 yields again $x_1, \dots, x_n \vdash y_1, \dots, y_n$. In particular, the following holds.

Corollary 3.5. *Let n_i be integers ≥ 0 not all zero. If $0 \leq_G \sum_{i=1}^p n_i u_i$, then we have $0 \vdash u_1, \dots, u_p$. Similarly, if $\sum_{i=1}^p n_i u_i \leq_G 0$, then $u_1, \dots, u_p \vdash 0$.*

Proof. Assume, e.g., that $0 \leq_G 2u_1 + 3u_2$; then

$$0 + 0 + 0 + 0 + 0 \leq_G u_1 + u_1 + u_2 + u_2 + u_2.$$

Proposition 3.4 gives $0, 0, 0, 0, 0 \vdash u_1, u_1, u_2, u_2, u_2$. By contraction and monotonicity $0 \vdash u_1, u_2, \dots, u_p$ holds. \square

Lemma 3.6 (Lorenzen’s inequality, Lorenzen 1953, (2.11)). *Let \vdash be a regular system of ideals. Let $x_1, \dots, x_n, y_1, \dots, y_m \in G$, let τ be a map $\llbracket 1..n \rrbracket \rightarrow \llbracket 1..m \rrbracket$, and σ a map $\llbracket 1..m \rrbracket \rightarrow \llbracket 1..n \rrbracket$. Then*

$$x_1 + y_{\tau_1}, \dots, x_n + y_{\tau_n} \vdash x_{\sigma_1} + y_1, \dots, x_{\sigma_m} + y_m$$

holds¹⁰.

Proof. Consider the sequence defined by $\lambda_1 = 1$ and $\lambda_{k+1} = \sigma_{\tau_{\lambda_k}}$. Then this sequence “contains a cycle”: there are $i \leq j$ such that $\lambda_i = \lambda_{j+1}$. Therefore

$$(x_{\lambda_i} + y_{\tau_{\lambda_i}}) - (x_{\sigma_{\tau_{\lambda_i}}} + y_{\tau_{\lambda_i}}) + \dots + (x_{\lambda_j} + y_{\tau_{\lambda_j}}) - (x_{\sigma_{\tau_{\lambda_j}}} + y_{\tau_{\lambda_j}})$$

is a telescopic sum and

$$(x_{\lambda_i} + y_{\tau_{\lambda_i}}) + \dots + (x_{\lambda_j} + y_{\tau_{\lambda_j}}) =_G (x_{\sigma_{\tau_{\lambda_i}}} + y_{\tau_{\lambda_i}}) + \dots + (x_{\sigma_{\tau_{\lambda_j}}} + y_{\tau_{\lambda_j}}).$$

By Proposition 3.4,

$$x_{\lambda_i} + y_{\tau_{\lambda_i}}, \dots, x_{\lambda_j} + y_{\tau_{\lambda_j}} \vdash x_{\sigma_{\tau_{\lambda_i}}} + y_{\tau_{\lambda_i}}, \dots, x_{\sigma_{\tau_{\lambda_j}}} + y_{\tau_{\lambda_j}}.$$

The result follows by monotonicity. \square

¹⁰Note that $\bigvee_{\tau} \bigwedge_i (x_i + y_{\tau_i}) =_H \bigwedge_i \bigvee_j (x_i + y_j)$ and that $\bigwedge_{\sigma} \bigvee_i (x_{\sigma_i} + y_j) =_H \bigvee_j \bigwedge_i (x_i + y_j)$. If we already knew that $+$ is compatible with the lattice operations in H , then this entailment would follow from the simple observation that $\bigwedge_i \bigvee_j (x_i + y_j) \leq_H \bigvee_j \bigwedge_i (x_i + y_j)$ because both would be seen to be equal to $(\bigwedge_i x_i) + (\bigvee_j y_j)$.

Comment 3.7. **Lorenzen (1953)** proceeds in the following way for the proof of his Satz 1: he starts by proving the key facts that (for a noncommutative group, in multiplicative notation)

$$\begin{aligned} &\text{if } c, c_1, \dots, c_n \vdash 1, \text{ then } xc x^{-1}, c_1, \dots, c_n \vdash 1 \\ &\quad c_1 c_2^{-1}, c_2 c_3^{-1}, \dots, c_{n-1} c_n^{-1}, c_n c_1^{-1} \vdash 1 \end{aligned}$$

(the second of which corresponds to Corollary 3.5) and deduces from these Property *R2* only as the basic ingredient for proving that the distributive lattice generated by \vdash is regular. The main use of these facts is for establishing Lemma 3.6 as a tool for endowing the distributive lattice with a compatible group operation as in our Main Theorem IV. \blacksquare

Scholion 3.8. *In an ℓ -group (H, \leq_H) , the inequality $x_1 \wedge \dots \wedge x_n \leq_H y_1 \vee \dots \vee y_m$ is equivalent to*

$$\bigwedge_{i \in \llbracket 1..n \rrbracket, j \in \llbracket 1..m \rrbracket} (x_i - y_j) \leq_H 0.$$

Proof. The inequality $x_1 \wedge \dots \wedge x_n \leq_H y_1 \vee \dots \vee y_m$ is equivalent to

$$(x_1 \wedge \dots \wedge x_n) - (y_1 \vee \dots \vee y_m) \leq_H 0$$

and also, by distributivity, to the stated inequality. \square

This scholion explains why the following theorem is decisive.

Theorem 3.9. *Let \vdash be a regular system of ideals. We have*

$$x_1, \dots, x_n \vdash y_1, \dots, y_m \tag{\#}$$

if and only if

$$0 \vdash (y_j - x_i)_{i \in \llbracket 1..n \rrbracket, j \in \llbracket 1..m \rrbracket} \tag{\P}$$

if and only if

$$(x_i - y_j)_{i \in \llbracket 1..n \rrbracket, j \in \llbracket 1..m \rrbracket} \vdash 0. \tag{\%$$

Proof. $\{\#\} \Rightarrow \{\P\}$. Let C be the right-hand side of $\{\P\}$. The hypothesis gives by equivariance for each $k \in \llbracket 1..n \rrbracket$

$$L_k \vdash y_1 - x_k, \dots, y_m - x_k$$

with $L_k = \{x_1 - x_k, \dots, x_n - x_k\}$, and by monotonicity holds $L_k \vdash C$. By Theorem 3.1 we have for each $k \in \llbracket 1..n \rrbracket$ an inequality $\bigwedge L_k \leq_H \bigvee C$. So

$$\bigvee_{k \in \llbracket 1..n \rrbracket} \bigwedge L_k \leq_H \bigvee C.$$

By distributivity we get

$$\bigvee_{k \in \llbracket 1..n \rrbracket} \bigwedge L_k = \bigwedge_{\sigma: \llbracket 1..n \rrbracket \rightarrow \llbracket 1..n \rrbracket} \bigvee_{k \in \llbracket 1..n \rrbracket} x_{\sigma_k} - x_k.$$

Let $\sigma: \llbracket 1..n \rrbracket \rightarrow \llbracket 1..n \rrbracket$. By Lemma 3.6 with $\tau_k = k$,

$$x_1 - x_1, \dots, x_n - x_n \vdash x_{\sigma_1} - x_1, \dots, x_{\sigma_n} - x_n,$$

so that $0 \leq_H \bigvee_{k \in \llbracket 1..n \rrbracket} x_{\sigma_k} - x_k$ and by transitivity $0 \leq_H \bigvee C$.

(**¶**) \Rightarrow (**#**). Let $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$. By a translation, we have for each k that $x_k \vdash (Y - x_i + x_k)_{i \in \llbracket 1..n \rrbracket}$. Thus $X \vdash (Y - x_i + x_k)_{i \in \llbracket 1..n \rrbracket}$. By Theorem 3.1 we have for each $k \in \llbracket 1..n \rrbracket$ an inequality

$$\bigwedge X \leq_H \bigvee_{i \in \llbracket 1..n \rrbracket} \bigvee Y - x_i + x_k,$$

so that

$$\bigwedge X \leq_H \bigwedge_{k \in \llbracket 1..n \rrbracket} \bigvee_{i \in \llbracket 1..n \rrbracket} \bigvee Y - x_i + x_k,$$

By distributivity we get

$$\bigwedge_{k \in \llbracket 1..n \rrbracket} \bigvee_{i \in \llbracket 1..n \rrbracket} \bigvee Y - x_i + x_k = \bigvee_{v: \llbracket 1..n \rrbracket \rightarrow \llbracket 1..m \rrbracket} \bigvee_{\tau: \llbracket 1..n \rrbracket \rightarrow \llbracket 1..n \rrbracket} \bigwedge_{k \in \llbracket 1..n \rrbracket} y_{v_k} - x_{\tau_k} + x_k.$$

Let $v: \llbracket 1..n \rrbracket \rightarrow \llbracket 1..m \rrbracket$ and $\tau: \llbracket 1..n \rrbracket \rightarrow \llbracket 1..n \rrbracket$. By Lemma 3.6 with $\sigma_k = k$,

$$(x_1 + y_{v_1}) - x_{\tau_1}, \dots, (x_n + y_{v_n}) - x_{\tau_n} \vdash (x_1 + y_{v_1}) - x_1, \dots, (x_n + y_{v_n}) - x_n,$$

so that by monotonicity $\bigwedge_{k \in \llbracket 1..n \rrbracket} y_{v_k} - x_{\tau_k} + x_k \leq_H \bigvee Y$, and by transitivity $X \vdash Y$.

Finally (**#**) \Leftrightarrow (**¶**) shows that $u_1, \dots, u_\ell \vdash 0$ is equivalent to $0 \vdash -u_1, \dots, -u_\ell$, and this yields (**¶**) \Leftrightarrow (**%**). \square

In particular, this theorem asserts that a regular system of ideals is determined by its restriction to $\text{P}_{\text{fe}}^*(G) \times G$. However, given an unbounded single-conclusion entailment relation \triangleright , there are several unbounded entailment relations that reflect \triangleright , and the coarsest one admits a simple description, given in Lorenzen (1952, § 3):

$$A \vdash_{\triangleright}^v B \stackrel{\text{def}}{\Leftrightarrow} \forall C \in \text{P}_{\text{fe}}(G) \forall z \in G \text{ if } C, b \triangleright z \text{ for all } b \text{ in } B, \text{ then } C, A \triangleright z \quad (||)$$

(here $\text{P}_{\text{fe}}(G)$ stands for the set of finitely enumerated subsets of the set G ; see Scott (1974, Theorem 1.2) for a proof; $\vdash_{\triangleright}^v$ is $\vdash_{\triangleright}^{\text{max}}$ in Rinaldi, Schuster and Wessel (2016,

§ 3.1)). This definition is “dual” to the definitional equivalence (†) on page 9 for the coarsest single-conclusion entailment relation; the presence of the C in (||) is needed for proving transitivity of $\vdash_{\triangleright}^v$. The following corollary tells us that if a system of ideals \triangleright is regular, then the unique regular system of ideals extending it coincides with the coarsest unbounded entailment relation $\vdash_{\triangleright}^v$ (see Lorenzen 1950, page 509).

Corollary 3.10. *Let G be an ordered group and \vdash a regular system of ideals for G . Let \triangleright be the system of ideals given as the restriction of the relation \vdash to $\mathbf{P}_{\text{fe}}^*(G) \times G$. Then \vdash coincides with the coarsest unbounded entailment relation $\vdash_{\triangleright}^v$ that reflects \triangleright , defined in (||).*

Proof. It suffices to prove that $\vdash_{\triangleright}^v$ is a regular system of ideals, because then Theorem 3.9 yields that it is determined by its restriction to $\mathbf{P}_{\text{fe}}^*(G) \times G$.

R1. Suppose that $a \leq_G b$, so that $a \triangleright b$. If $C, b \triangleright z$, then $C, a \triangleright z$ by transitivity. Therefore $a \vdash_{\triangleright}^v b$.

R2. As \vdash is regular, we have $x + a, y + b \vdash x + b, y + a$. As $\vdash_{\triangleright}^v$ is coarser than \vdash , we have $x + a, y + b \vdash_{\triangleright}^v x + b, y + a$.

R3. Just note that if $C, b + x \triangleright z$, then $C - x, b \triangleright z - x$, and that if $C - x, A \triangleright z - x$, then $C, A + x \triangleright z$. □

Now we are also able to give the analogue of Proposition 2.5 for regular systems of ideals.

Corollary 3.11 (Variant for the definition of a regular system of ideals).

Let us consider a predicate $\cdot \triangleright 0$ on $\mathbf{P}_{\text{fe}}^(G)$ for an ordered group G and let us define a binary relation on $\mathbf{P}_{\text{fe}}^*(G)$ by*

$$x_1, \dots, x_n \vdash y_1, \dots, y_m \stackrel{\text{def}}{\iff} (x_i - y_j)_{i \in [1..n], j \in [1..m]} \triangleright 0 \quad (^\circ)$$

($n, m \geq 1$). In order for this relation to be a regular system of ideals, it is necessary and sufficient that the following properties be fulfilled:

- G1* if $\sum_{i=1}^n x_i \leq_G 0$, then $x_1, \dots, x_n \triangleright 0$ (preservation of order);
- G2* if $A \triangleright 0$, then $A, A' \triangleright 0$ (monotonicity);
- G3* if $B + C, B \triangleright 0$ and $B + C, C \triangleright 0$, then $B + C \triangleright 0$ (transitivity).

Proof. Using the definitional equivalence ($^\circ$), Property *G3* is a direct translation of the cut of 0 in $B \vdash 0, -C$ and $B, 0 \vdash -C$. For the other properties, use Theorem 3.9 and Corollary 3.5. The details are left to the reader. □

3.3 The regularisation of a system of ideals for an ordered group

Let us now discuss Definition VII, and prove Theorem II.

The precise description of the system \triangleright_x obtained by forcing the property $x \geq 0$ given in Proposition 3.12 is the counterpart for single-conclusion entailment relations to the cone generated by adding an element to a cone in an ordered monoid (see Lorenzen 1950, page 518).

Proposition 3.12. *Let \triangleright be a system of ideals for an ordered monoid G . Let us denote by \triangleright_x the finest system of ideals coarser than \triangleright and satisfying the property $x \geq 0$. Then we have the equivalence*

$$A \triangleright_x b \iff \text{there exists a } p \geq 0 \text{ such that } A, A+x, \dots, A+px \triangleright b.$$

Proof. Let us denote by $A \triangleright' b$ the right-hand side in the equivalence above. In any meet-monoid, $x \geq 0$ implies $\bigwedge(A, A+x, \dots, A+px) = \bigwedge A$, so that $A \triangleright' b$ implies $A \tilde{\triangleright} b$ for any system of ideals $\tilde{\triangleright}$ coarser than \triangleright and satisfying $0 \tilde{\triangleright} x$.

It remains to prove that $A \triangleright' b$ defines a system of ideals for G (clearly $0 \triangleright' x$ and \triangleright' is coarser than \triangleright). Reflexivity, preservation of order, equivariance and monotonicity are straightforward. It remains to prove transitivity. Assume, e.g., that $A \triangleright' z$ and $A, z \triangleright' y$. We have to show that $A \triangleright' y$. E.g., we have

$$A, A+x, A+2x, A+3x \triangleright z, \tag{*}$$

$$A, A+x, A+2x, z, z+x, z+2x \triangleright y. \tag{†}$$

(*) gives by a translation $A+2x, A+3x, A+4x, A+5x \triangleright z+2x$, and by monotonicity

$$A, A+x, A+2x, A+3x, A+4x, A+5x, z, z+x \triangleright z+2x. \tag{**}$$

(†) gives by monotonicity

$$A, A+x, A+2x, A+3x, A+4x, A+5x, z, z+x, z+2x \triangleright y. \tag{††}$$

By transitivity we get from (**) and (††)

$$A, A+x, A+2x, A+3x, A+4x, A+5x, z, z+x \triangleright y.$$

So we have cancelled $z+2x$ out of the left-hand side of (†). A similar trick allows us to cancel out successively $z+x$ and z . \square

Let us go through a simple example that shows how regularisation catches the content of Corollary 3.5.

Example 3.13 (an illustration of Definition VII). Let us apply a case by case reasoning in order to prove that in a linearly ordered group, if $n_1u_1 + \dots + n_ku_k \leq 0$ for some integers $n_i \geq 0$ not all zero, then $u_j \leq 0$ for some j . If $u_j \leq 0$ for some j , everything is all right. If $u_j \geq 0$ for all j , take i such that $n_i \geq 1$: then $u_i \leq n_iu_i \leq n_1u_1 + \dots + n_ku_k \leq 0$. The conclusion holds in each case.

Similarly, assume that $n_1u_1 + \dots + n_ku_k > 0$ with $n_i \geq 0$ not all zero. We have $u_j \triangleright_{-u_j} 0$ for each j . By monotonicity,

$$u_1, \dots, u_k \triangleright_{\epsilon_1u_1, \dots, \epsilon_ku_k} 0$$

if at least one ϵ_j is equal to -1 . Suppose that $0 \triangleright u_j$ for each j ; take i such that $n_i \geq 1$: then $u_i \leq \leq_{\triangleright} n_iu_i \leq_{\triangleright} n_1u_1 + \dots + n_ku_k \leq_{\triangleright} 0$. This proves that $u_1, \dots, u_k \triangleright_{+u_1, \dots, +u_k} 0$. We conclude that

$$u_1, \dots, u_k \vdash_{\triangleright} 0. \quad \blacksquare$$

Comment 3.14 (for Definition VII). Lorenzen (1950, 1952, 1953) considers a pre-ordered commutative or noncommutative group (G, \preceq_G) and a meet-monoid H_r (“ H ” like “Halbverband”, semilattice, r a variable name for distinguishing different monoids) given by a system of ideals \triangleright for G . The monoid H_r gives rise to another meet-monoid, H_{r_a} (“ a ” like “algebraically representable”), given by a system of ideals \triangleright_a that is not defined as in Section 6 by forcing cancellativity, but so as to catch the classical definition of integral dependence of an element b over a nonempty finitely enumerated set A , i.e.,

$$A \triangleright_a b \stackrel{\text{def}}{\iff} \min A \leq b \text{ holds for every linear order } \leq \text{ that is coarser than } \triangleright.$$

Lorenzen’s analysis of the constructive content of this definition results in the system of ideals \vdash_{\triangleright} of Definition VII with B a single conclusion, i.e., in the system \triangleright while affording to suppose that elements occurring in a computation are comparable. Lorenzen (1950, Satz 24) proves that $A \vdash_{\triangleright} b$ holds if and only if $A \triangleright_a b$: more precisely, it is straightforward that every linear order coarser than \triangleright is also coarser than \vdash_{\triangleright} , so that $A \vdash_{\triangleright} b \Rightarrow A \triangleright_a b$; conversely, he considers a maximal order without $A \vdash_{\triangleright} b$ holding (granted by a well-ordering argument) and shows that it cannot be other than linear. He defines that G is r -closed if one recovers its preorder when restricting \triangleright to G , i.e., if $a \vdash_{\triangleright} b$ implies $a \preceq_G b$. \blacksquare

Proof of Theorem II

Comment 3.15. Lemma 3.16 corresponds to the first part of the proof of Satz 1 in Lorenzen (1953). In our analysis of Lorenzen’s proof, we separate the construction of the regularisation from the investigation of its relationship with the group law. In doing so, we make the regularity property (Property $R2$) the lever for sending G homomorphically into an ℓ -group. \blacksquare

Lemma 3.16. *Let \triangleright be a system of ideals for an ordered group G . Its regularisation \vdash_{\triangleright} is a regular system of ideals for G .*

Proof. The regularisation is clearly reflexive and monotone, and satisfies Properties *R1* and *R3*.

Let us prove that the regularisation is transitive. Suppose that $A, 0 \vdash_{\triangleright} B$ and $A \vdash_{\triangleright} 0, B$ with $A = \{a_1, \dots, a_m\}$ and $B = \{b_1, \dots, b_n\}$: there are $x_1, \dots, x_k, y_1, \dots, y_\ell$ such that for every choice of signs \pm holds

$$A - B, -B \triangleright_{\pm x_1, \dots, \pm x_k} 0 \quad \text{and} \quad A, A - B \triangleright_{\pm y_1, \dots, \pm y_\ell} 0.$$

If $a_i \triangleright 0$ for some i , then $A \leq_{\triangleright} A, 0$ and $A - B \leq_{\triangleright} A - B, -B$. Therefore

$$A - B \leq_{\triangleright_{-a_i, \pm x_1, \dots, \pm x_k}} A - B, -B \leq_{\triangleright_{-a_i, \pm x_1, \dots, \pm x_k}} 0 \quad \text{for } i = 1, \dots, m.$$

If $0 \triangleright b_j$ for some j , then $-b_j \triangleright 0$ and $-B \leq_{\triangleright} 0, -B$ and $A - B \leq_{\triangleright} A, A - B$. Therefore

$$A - B \leq_{\triangleright_{b_j, \pm y_1, \dots, \pm y_\ell}} A, A - B \leq_{\triangleright_{b_j, \pm y_1, \dots, \pm y_\ell}} 0 \quad \text{for } j = 1, \dots, n.$$

If $0 \triangleright a_1, \dots, 0 \triangleright a_m$, then we have $0 \leq_{\triangleright} A, 0$ and $-B \leq_{\triangleright} A - B, -B$. Therefore

$$-B \leq_{\triangleright_{a_1, \dots, a_m, \pm x_1, \dots, \pm x_k}} 0.$$

If $b_1 \triangleright 0, \dots, b_n \triangleright 0$, then $0 \triangleright -b_1, \dots, 0 \triangleright -b_n$, and we have $0 \leq_{\triangleright} 0, -B$ and $A \leq_{\triangleright} A, A - B$. Therefore successively

$$\begin{aligned} A &\leq_{\triangleright_{-b_1, \dots, -b_n, \pm y_1, \dots, \pm y_\ell}} 0, \\ A - B &\leq_{\triangleright_{-b_1, \dots, -b_n, \pm y_1, \dots, \pm y_\ell}} -B, \\ \text{and } A - B &\leq_{\triangleright_{a_1, \dots, a_m, -b_1, \dots, -b_n, \pm x_1, \dots, \pm x_k, \pm y_1, \dots, \pm y_\ell}} 0. \end{aligned}$$

We conclude that

$$A - B \triangleright_{\pm a_1, \dots, \pm a_m, \pm b_1, \dots, \pm b_n, \pm x_1, \dots, \pm x_k, \pm y_1, \dots, \pm y_\ell} 0.$$

Let us prove that the regularisation is regular, i.e., that $x + a, y + b \vdash_{\triangleright} x + b, y + a$ holds for all $a, b, x, y \in G$: it suffices to note that

$$\begin{aligned} \text{if } a - b \triangleright 0, & \text{ then } a - b, x - y, y - x, b - a \triangleright 0; \\ \text{if } b - a \triangleright 0, & \text{ then } a - b, x - y, y - x, b - a \triangleright 0. \end{aligned} \quad \square$$

The following lemma justifies the terminology of Definition VIII: with the ambiguity introduced by the two items of Definition VI, one may say that regularisation leaves a regular system of ideals unchanged.

Lemma 3.17. *Let G be an ordered group and \vdash a regular system of ideals for G . Let \triangleright_{\vdash} be the system of ideals given as the restriction of \vdash to $P_{\text{fe}}^*(G) \times G$. Then \vdash coincides with the regularisation of \triangleright_{\vdash} .*

Proof. Let $p, q \geq 0$ be integers. It suffices to prove that if $A, A+x, \dots, A+px \triangleright_{\vdash} 0$ and $A, A-x, \dots, A-qx \triangleright_{\vdash} 0$, then $A \triangleright_{\vdash} 0$. By Theorem 3.9, the hypotheses are

$$A \vdash 0, -x, \dots, -px \text{ and } A \vdash 0, x, \dots, qx.$$

If $p = 0$ or $q = 0$, we are done. Otherwise, since $q \times (-p) + p \times q = 0$, Corollary 3.5 gives $-px, qx \vdash 0$. Cutting $-px$ yields $A, qx \vdash 0, -x, \dots, -px$; cutting qx yields $A \vdash -(p-1)x, \dots, -x, 0, x, \dots, (q-1)x$. If $p = 1$, we may iterate this and obtain that $A \vdash 0$. Otherwise, first acknowledge that $A' \vdash 0, -x$ and $A' \vdash 0, x, \dots, (q-1)x$ imply $A' \vdash 0$; with A' equal to $A, A+x, \dots, A+(p-1)x$ these hypotheses turn out to be

$$A \vdash 0, -x, \dots, -px \text{ and } A \vdash -(p-1)x, \dots, -x, 0, x, \dots, (q-1)x$$

and do therefore hold. We may iterate this and obtain that $A \vdash 0$. \square

Proof of Theorem II. Lemma 3.16 tells that \vdash_{\triangleright} is a regular system of ideals, and it is clear from the definition that its restriction to $P_{\text{fe}}^*(G) \times G$ is coarser than \triangleright . Now let \vdash be a regular system of ideals whose restriction \triangleright_{\vdash} to $P_{\text{fe}}^*(G) \times G$ is coarser than \triangleright . Then the same holds for their regularisation, i.e., by Lemma 3.17, \vdash is coarser than \vdash_{\triangleright} . \square

3.4 The finest regular system of ideals

We shall now give a precise description of the regularisation $\vdash_{\triangleright_s}$ of the finest system of ideals.

Lemma 3.18. *Let G be an ordered group. For $u_1, \dots, u_k \in G$, t.f.a.e.*

1. $u_1, \dots, u_k \vdash_{\triangleright_s} 0$.
2. *There exist integers $n_i \geq 0$ not all zero such that we have*

$$n_1 u_1 + \dots + n_k u_k \leq_G 0.$$

Proof. Let us denote Item 2 by $\varrho(u_1, \dots, u_k)$.

$1 \Rightarrow 2$. First it is clear that $u_1, \dots, u_k \triangleright_s 0$ implies $\varrho(u_1, \dots, u_k)$. Thus it is enough to prove that if one supposes that for some p and q ,

$$\begin{aligned} &\varrho(u_1, \dots, u_k, u_1 + x, \dots, u_k + x, \dots, u_1 + px, \dots, u_k + px) \text{ and} \\ &\varrho(u_1, \dots, u_k, u_1 - x, \dots, u_k - x, \dots, u_1 - qx, \dots, u_k - qx), \end{aligned}$$

then $\varrho(u_1, \dots, u_k)$. The hypothesis implies that there are integers $n_i, n \geq 0$, at least one n_i nonzero, such that $n_1 u_1 + \dots + n_k u_k + n x \leq_G 0$, and integers $m_j, m \geq 0$, at least one m_j nonzero, such that $m_1 u_1 + \dots + m_k u_k - m x \leq_G 0$. If $n = 0$ or if $m = 0$, then we are done; otherwise, $(mn_1 + nm_1)u_1 + \dots + (mn_k + nm_k)u_k \leq_G 0$ with at least one $mn_i + nm_i > 0$.

2 \Rightarrow 1. Consequence of Theorem II and Corollary 3.5. \square

Theorem 3.19. *Let (G, \leq_G) be an ordered group.*

1. *The finest regular system of ideals for G is the regularisation $\vdash_{\triangleright_s}$ of the finest system of ideals \triangleright_s .*

2. *The group G is \triangleright_s -closed if and only if*

$$nx \geq_G 0 \text{ implies } x \geq_G 0 \quad (x \in G, n > 1).$$

Proof. Theorem 3.9 shows that a regular system of ideals for G is determined by the unbounded single-conclusion entailment relation that it defines by restriction to $P_{\text{fe}}^*(G) \times G$. Thus every regular system of ideals for G is coarser than $\vdash_{\triangleright_s}$ by Lemma 3.18 and Corollary 3.5. \square

3.5 The regularisation of the Dedekind system of ideals

Let R be an integral domain, K its field of fractions and $G = K^\times / R^\times$ its divisibility group (where, in multiplicative notation, $1 \leq_G x$ when $x \in R$). One defines the *Dedekind system of ideals* \triangleright_d for G by letting

$$A \triangleright_d b \stackrel{\text{def}}{\iff} b \in \langle A \rangle_R,$$

where $\langle A \rangle_R$ is the (fractional) ideal generated by A over R in K : if a_1, \dots, a_n are the elements of A , then $\langle A \rangle_R = a_1 R + \dots + a_n R$. Note that if A contains nonintegral elements, i.e., elements not in R , then $\langle A \rangle_R^2$ is not contained in $\langle A \rangle_R$.

Adding the constraint $x \geq 1$ for an $x \in K^\times$ amounts to replacing R by $R[x]$ since we get by Proposition 3.12 that for the new system of ideals

$$A (\triangleright_d)_x b \iff \text{there is a } p \geq 0 \text{ such that } A, Ax, \dots, Ax^p \triangleright_d b$$

which means that $b \in \langle A \rangle_{R[x]}$ (where A and b are in K^\times).

An element $b \in K$ is said to be *integral over the ideal* $\langle A \rangle_R$ when one has an integral dependence relation $b^m = \sum_{k=1}^m c_k b^{m-k}$ with $c_k \in \langle A \rangle_R^k$. If $A = \{1\}$, then this reduces to the same integral dependence relation with $c_k \in R$, i.e., to b being integral over R .

Lemma 3.20. *One has $A \vdash_{\triangleright_d} 1$ if and only if $1 \in \langle A \rangle_{R[A]}$.*

Proof. Suppose that $A \vdash_{\triangleright_d} 1$, i.e., that there are elements $x_1, \dots, x_\ell \in G$ such that $1 \in \langle A \rangle_{R[x_1^{\pm 1}, \dots, x_\ell^{\pm 1}]}$. It suffices to prove the following fact and to use it in an induction argument: suppose that $1 \in \langle A \rangle_{R[A, x]}$ and $1 \in \langle A \rangle_{R[A, x^{-1}]}$; then $1 \in \langle A \rangle_{R[A]}$. In fact, the hypothesis means that $1 \in \langle A, Ax, \dots, Ax^p \rangle_{R[A]}$ and $1 \in \langle A, Ax^{-1}, \dots, Ax^{-p} \rangle_{R[A]}$ for some p , which implies that

$$\forall i \in \llbracket -p..p \rrbracket \quad x^i \in \langle Ax^{-p}, \dots, Ax^{-1}, A, Ax, \dots, Ax^p \rangle_{R[A]},$$

i.e., that there is a matrix M with coefficients in $\langle A \rangle_{R[A]}$ such that $(x^i)_{-p}^p = M(x^i)_{-p}^p$, i.e., $(1 - M)(x^i)_{-p}^p = 0$. Let us now apply the determinant trick: multiplying $1 - M$ by the matrix of its cofactors and expanding yields that $1 \in \langle A \rangle_{R[A]}$.

Conversely, let a_1, \dots, a_n be the elements of A . For each j , $1 = a_j a_j^{-1}$, so that $1 \in \langle A \rangle_{R[a_j^{-1}]}$ and $A \triangleright_d a_1^{\pm 1}, \dots, a_n^{\pm 1} 1$ for every choice of signs with at least one negative sign: the only missing choice of signs consists in the hypothesis $1 \in \langle A \rangle_{R[A]}$. \square

Theorem 3.21 (Lorenzen 1953, Satz 2). *Let R be an integral domain and \triangleright_d the Dedekind system of ideals.*

1. *One has $A \vdash_{\triangleright_d} b$, i.e., there are x_1, \dots, x_ℓ such that for every choice of signs holds $b \in \langle A \rangle_{R[x_1^{\pm 1}, \dots, x_\ell^{\pm 1}]}$, if and only if b is integral over the ideal $\langle A \rangle_R$.*

2. *One has $A \vdash_{\triangleright_d} B$, i.e., there are x_1, \dots, x_ℓ such that for every choice of signs holds $1 \in \langle AB^{-1} \rangle_{R[x_1^{\pm 1}, \dots, x_\ell^{\pm 1}]}$, if and only if $1 \in \sum_{k=1}^m \langle AB^{-1} \rangle_R^k$, i.e., there is an equality $1 = \sum_{k=1}^m f_k$ with each f_k a homogeneous polynomial of degree k in the elements of AB^{-1} with coefficients in R .*

3. *The divisibility group G is \triangleright_d -closed, i.e., the equivalence $x \vdash_{\triangleright_d} y \Leftrightarrow x$ divides y holds, if and only if R is integrally closed.*

Proof. 1 and 2. This follows from the previous lemma because

$$\begin{aligned} A \vdash_{\triangleright_d} b &\iff Ab^{-1} \vdash_{\triangleright_d} 1, \\ b^m = \sum_{k=1}^m c_k b^{m-k} \text{ with } c_k \in \langle A \rangle_R^k &\iff 1 \in \sum_{k=1}^m \langle Ab^{-1} \rangle_R^k, \\ 1 \in \langle A \rangle_{R[A]} &\iff \exists m \ 1 \in \sum_{k=1}^m \langle A \rangle_R^k. \end{aligned}$$

3. \triangleright_d -closedness is equivalent to $1 \vdash_{\triangleright_d} b \Rightarrow b \in R$; by Item 1, $1 \vdash_{\triangleright_d} b$ holds if and only if b is integral over R . \square

4 The lattice-ordered group freely generated by a finitely presented ordered group

4.1 A Positivstellensatz for ordered groups

Reference: [Coste, Lombardi and Roy \(2001, Section 5\)](#).

In the article we refer to, Theorem 5.7 can be seen as a generalisation of results concerning rational linear programming (e.g., the Farkas lemma).

If G is a commutative group and x_1, \dots, x_m are indeterminates, let $G\{\mathbf{x}\} = G\{x_1, \dots, x_m\}$ be the group of \mathbb{Z} -affine forms on G , i.e., of polynomials $g + \sum_{\mu=1}^m z_\mu x_\mu$ with g in G and the z_μ s in \mathbb{Z} . We may consider G as the subgroup of $G\{\mathbf{x}\}$ consisting of the constant forms.

Theorem 4.1 (Positivstellensatz: algebraic certificates for ordered groups, see [Coste, Lombardi and Roy 2001](#)). *Let $(G, \cdot, +, \cdot, -, 0, \cdot = 0, \cdot \geq 0, \cdot > 0)$ be a discrete divisible linearly ordered group. Let x_1, \dots, x_m be indeterminates and $R_{=0}, R_{\geq 0}, R_{>0}$ three finitely enumerated subsets of $G\{x_1, \dots, x_m\}$. Consider the associated system \mathcal{S} of sign conditions*

$$\mathcal{S} : \quad z(\boldsymbol{\xi}) = 0 \text{ if } z \in R_{=0}, \quad p(\boldsymbol{\xi}) \geq 0 \text{ if } p \in R_{\geq 0}, \quad s(\boldsymbol{\xi}) > 0 \text{ if } s \in R_{>0}.$$

There is an algorithm giving the following answer:

1. either an algebraic certificate telling that the system \mathcal{S} is impossible in G (and in every linearly ordered group extending G),
2. or a point $(\boldsymbol{\xi}) = (\xi_1, \dots, \xi_m) \in G^m$ realising the system \mathcal{S} .

An algebraic certificate is an algebraic identity

$$s + p + z = 0 \text{ in } G\{x_1, \dots, x_m\},$$

where s is a (nonempty) sum of elements of $R_{>0} \cup G_{>0}$, p is a (possibly empty) sum of elements of $R_{\geq 0} \cup G_{\geq 0}$, and z is a \mathbb{Z} -linear combination of elements of $R_{=0}$.

4.2 A concrete construction

A finitely presented ordered group G is given by a finite system of generators e_1, \dots, e_m with a finite set of relations $R = R_{=0} \cup R_{\geq 0}$. The relations in $R_{=0}$ have the form $z = 0$, and those in $R_{\geq 0}$ have the form $p \geq 0$, where $z, p \in \mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_m$.

Since a relation $q = 0$ is equivalent to the two relations $q \geq 0$ and $-q \geq 0$, we may assume that the presentation of G as an ordered group is given by a finite subset $R_{\geq 0} = \{p_1, \dots, p_\ell\}$ only. Let us work with this new presentation.

Let $\text{LGOG}(G)$ be the ℓ -group freely generated by the ordered group G . We shall give a description of an ℓ -group $\text{Lgog}(G)$, and prove that it is naturally isomorphic to $\text{LGOG}(G)$.

Let \mathbb{Z}' be the group \mathbb{Z} with the usual linear order, and let $\text{Lo}(G, \mathbb{Z}')$ be the set of order morphisms from G to \mathbb{Z}' that are linear for the \mathbb{Z} -module structure of G . This is an additive monoid whose natural order relation is compatible with addition.

We define $\text{Lgog}(G)$ as the sub- ℓ -group of

$$\text{Set}(\text{Lo}(G, \mathbb{Z}'), \mathbb{Z}')$$

generated by the join-semilattice-ordered monoid $j(G)$, where j is the bidual morphism of ordered groups $G \rightarrow j(G) \subseteq \text{Lgog}(G) \subseteq \text{Set}(\text{Lo}(G, \mathbb{Z}'), \mathbb{Z}')$:

$$j(z) \text{ is the map } \alpha \mapsto \alpha(z).$$

This \mathbb{Z} -linear map is a morphism of ordered groups since, if $z \geq 0$ in G and $\alpha \in \text{Lo}(G, \mathbb{Z}')$, then one has $\alpha(z) \geq 0$ in \mathbb{Z}' . Let us denote the element $j(z)$ viewed in $\text{Lgog}(G)$ by \bar{z} .

We shall use the following principle ([Lombardi and Quitté 2015](#), Principle XI-2.10).

Principle of covering by quotients (for ℓ -groups). *In order to prove an equality $u = v$ or an inequality $u \leq v$ in an ℓ -group H , we can always suppose that the (finite number of) elements which occur in a computation for a proof are comparable.*

In fact, we shall need the following easy consequence of this principle.

Lemma 4.2. *In an ℓ -group H , if $\sum_{i=1}^k u_i \geq 0$ holds (with an integer $k > 0$), then one has $\bigvee_{i=1}^k u_i \geq 0$.*

Let us now consider the canonical morphism $\iota: G \rightarrow \text{LGOG}(G)$ and the unique (surjective) morphism $\vartheta: \text{LGOG}(G) \rightarrow \text{Lgog}(G)$ factorising j (i.e., such that $\vartheta \circ \iota = j$). In order to show that ϑ is an isomorphism, it suffices to show that $\vartheta(y) \geq 0$ implies $y \geq 0$ for all $y \in \text{LGOG}(G)$.

Let us write the element $y \in \text{LGOG}(G)$ as $y = \bigwedge y_j = \bigwedge_j (\bigvee_i \iota(y_{ji}))$ with the y_{ji} s in G . The hypothesis is that $\bigwedge_j (\bigvee_i \bar{y}_{ji}) \geq 0$, i.e., that for each j one has $\vartheta(y_j) = \bigvee_i \bar{y}_{ji} \geq 0$. In order to show that $\bigwedge y_j \geq 0$, it is thus sufficient to show that if $\bigvee \bar{u}_i \geq 0$ with u_1, \dots, u_k in G , then $\bigvee \iota(u_i) \geq 0$ in $\text{LGOG}(G)$.

Let us write $u_i = \sum_{\mu=1}^m u_{i\mu} e_\mu$, $i = 1, \dots, k$, and $p_j = \sum_{\mu=1}^m p_{j\mu} e_\mu$, $j = 1, \dots, \ell$, and introduce indeterminates x_1, \dots, x_m and linear forms

$$\lambda_i(x_1, \dots, x_m) = \sum_{\mu=1}^m u_{i\mu} x_\mu \quad \text{and} \quad \rho_j(x_1, \dots, x_m) = \sum_{\mu=1}^m p_{j\mu} x_\mu.$$

Let us consider, on the divisible linearly ordered group $(\mathbb{Q}, \leq_{\mathbb{Q}})$, the following system of sign conditions w.r.t. the indeterminates x_1, \dots, x_m :

- $\lambda_i(x_1, \dots, x_m) < 0$ for $i = 1, \dots, k$;
- $\rho_j(x_1, \dots, x_m) \geq 0$ for $j = 1, \dots, \ell$.

Theorem 4.1 says that we are in one of the two following cases.

1. The system is incompatible and this implies an algebraic identity $\sum n_i \lambda_i = P$ for integers $n_i \geq 0$ not all zero and P in the additive monoid generated by the ρ_j s. When one substitutes the x_μ s with the e_μ s, one gets $P(e_1, \dots, e_m) \geq 0$ in G because each $\rho_j(e_1, \dots, e_m) = p_j$ is ≥ 0 in G , and therefore $\sum n_i u_i \geq 0$ in G and $\sum n_i \iota(u_i) \geq 0$ in $\text{LGOG}(G)$, and $\sum n_i \bar{u}_i \geq 0$ in $\text{Lgog}(G)$. Lemma 4.2 implies that we have $\bigvee \iota(u_i) \geq 0$ as well as $\bigvee \bar{u}_i \geq 0$.

2. One can find $(\xi_1, \dots, \xi_m) \in \mathbb{Q}^m$ such that the $\lambda_i(\xi)$ s are all < 0 and the $\rho_j(\xi)$ s are all ≥ 0 . Multiplying by a convenient positive rational number, we may assume that $(\xi_1, \dots, \xi_m) \in \mathbb{Z}^m$. Let $\alpha: G \rightarrow \mathbb{Z}'$ be the linear form such that $e_\mu \mapsto \xi_\mu$: as $\alpha(p_j) = \rho_j(\xi) \geq 0$ for $j = 1, \dots, \ell$, we have that α belongs to $\text{Lo}(G, \mathbb{Z}')$; let us note that $\bar{u}_i(\alpha) = \alpha(u_i) = \lambda_i(\xi)$. We deduce that $v = \bigvee \bar{u}_i$ is not ≥ 0 , as $v \geq 0$ implies that for all $\beta \in \text{Lo}(G, \mathbb{Z}')$, one has $v(\beta) \geq 0$; but $\alpha \in \text{Lo}(G, \mathbb{Z}')$ and $v(\alpha) = \bigvee \bar{u}_i(\alpha) = \bigvee \lambda_i(\xi) < 0$.

In brief, we have proved that $\bigvee \bar{u}_i \not\geq 0$ and $\bigvee \iota(u_i) \geq 0$ are exclusive of each other. The case distinction above shows more precisely the following theorem.

Theorem 4.3. *Let G be a finitely presented ordered group.*

1. *The canonical morphism $\text{LGOG}(G) \rightarrow \text{Lgog}(G)$ is an isomorphism.*
2. *Let u_1, \dots, u_k be in the ℓ -group $\text{LGOG}(G)$. T.f.a.e.:*
 - $\bigvee \iota(u_i) \geq 0$;
 - *there exist integers $n_i \geq 0$ not all zero such that $\sum n_i u_i \geq 0$ in G .*

In particular, an element x of G is ≥ 0 in $\text{LGOG}(G)$ if and only if one has $nx \geq 0$ in G with an integer $n > 0$.

3. *The group $\text{LGOG}(G)$ is discrete (the order is decidable).*

4.3 Proof of Theorem III

Constructive proof of Theorem III. This follows from the preceding theorem, from the fact that any ordered group is a filtered colimit of finitely presented ordered groups, and from the fact that the functor LGOG preserves filtered colimits. \square

Theorem III may be seen as a generalisation of the classical Lorenzen-Clifford-Dieudonné theorem, Corollary 4.4 below.

Corollary 4.4 (Lorenzen-Clifford-Dieudonné, see Lorenzen 1939, Satz 14 for the s-system of ideals, Clifford 1940, Theorem 1, Dieudonné 1941, Section 1). *The ordered group (G, \leq_G) is embeddable into an ℓ -group if and only if*

$$nx \geq_G 0 \text{ implies } x \geq_G 0 \quad (x \in G, n > 1). \quad (\S)$$

Proof. The condition is clearly necessary. Theorem III shows that it yields the injectivity of the morphism $\iota: G \rightarrow H$ as well as the fact that $\iota(x) \leq_H \iota(y)$ implies $x \leq_G y$. \square

Comments 4.5. 1. The reader will recognise in Condition (§) the condition of \triangleright_s -closedness established in Item 2 of Theorem 3.19. In fact, in his Ph.D. thesis, Lorenzen (1939) proves Corollary 4.4 as a side-product of his enterprise of generalising the concepts of multiplicative ideal theory to preordered groups. More precisely, he follows there the Prüfer approach of Section 6, in which \triangleright_s -closedness is introduced according to Definition 6.4 and the equivalence with Condition (§) is easy to check (see Lorenzen 1939, page 358 or Jaffard 1960, I, § 4, Théorème 2).

2. In each of the three references given in Corollary 4.4, the authors invoke a maximality argument for showing that G embeds in fact into a direct product of linearly ordered groups. The goal of Lorenzen (1950, § 4) and of Lorenzen (1953) is to avoid the reference to linear orders in constructing embeddings into an ℓ -group, and this endeavour culminates in the Corollary to Theorem IV. But this goal may also be achieved in the Prüfer approach of Lorenzen (1939) and the sought-after ℓ -group may be constructed via Item 2 of Theorem 6.5. \blacksquare

5 The lattice-ordered group generated by a regular system of ideals

We shall now undertake the proof of the main theorem of this article, Theorem IV.

Comment 5.1. Lorenzen (1953, § 2) uses the heuristics of Scholion 3.8 to define a distributive lattice V_{r_a} (“V” like “Verband”, lattice) given by the regular system of ideals \vdash_{\triangleright} of Definition VII. Theorem IV is new and replaces the second step of the proof of Satz 1 in Lorenzen (1953), which establishes that V_{r_a} is in fact an ℓ -group. Its first step is the proof of Lemma 3.16, in which the entailment relation \vdash_{\triangleright} is constructed and shown to be regular (see Comment 3.15). Its second step is a construction “by hand” of group laws for V_{r_a} in which the rôle of regularity is not emphasised. A merit of our approach is to reveal its importance and to allow for more conceptual arguments, but with the price of resorting to Theorem III. \blacksquare

5.1 The free case

Theorem 5.2. *Let (G, \leq_G) be an ordered group. Let G_s be the unbounded distributive lattice generated by the finest regular system of ideals $\vdash_{\triangleright_s}$. Then G_s admits a (unique) group law that is compatible with the lattice structure and such that the morphism (of ordered sets) $G \rightarrow G_s$ is a group morphism. This defines the ℓ -group*

freely generated by the ordered group (G, \leq_G) (in the sense of the left adjoint functor of the forgetful functor).

Proof. Using the distributivity of $+$ over \wedge and \vee , there is no choice in defining the group laws $+$ and $-$ from those of G . The problem is to show that these laws are well-defined and are in fact group laws.

Let us consider the ℓ -group $\text{LGOG}(G)$ freely generated by G . It is generated as an unbounded distributive lattice by (the image of) G because any term constructed from G , $+$, $-$, \wedge , \vee can be rewritten as an \wedge - \vee combination of elements of G . Let us denote by \vdash_{free} the entailment relation thus defined for G . We know that $u_1, \dots, u_k \vdash_{\text{free}} 0$ is equivalent to $u_1, \dots, u_k \vdash_{\triangleright_s} 0$ (this follows from Theorem III and Theorem 3.19). Moreover $\text{LGOG}(G)$ satisfies the equivalent properties given in Theorem 3.9 simply because it is an ℓ -group. If we see it as an unbounded distributive lattice generated by G , $\text{LGOG}(G)$ is thus the distributive lattice which is defined by the unbounded entailment relation $\vdash_{\triangleright_s}$. Therefore the laws $+$ and $-$ on G_s are well-defined and G_s , endowed with these laws, becomes an ℓ -group for which we have a canonical isomorphism $\text{LGOG}(G) \rightarrow G_s$. \square

5.2 The general case: proof of Main Theorem IV

Proof of Theorem IV. Let G_s denote the ℓ -group freely generated by (G, \leq_G) constructed in Theorem 5.2 via the entailment relation $\vdash_{\triangleright_s}$. The relation \vdash is coarser than the relation $\vdash_{\triangleright_s}$, so that the distributive lattice H is a quotient lattice of G_s . It remains to see that the group law descends to the quotient.

Let $G_0 = \{x \in G_s \mid x =_H 0\}$. We have to show that

- i. G_0 is a subgroup of G_s ;
- ii. for $x, y, z \in G_s$ with $x =_H y$ holds $x + z =_H y + z$.

It is sufficient to show that

1. for $x \in G_s$, if $0 \leq_H x$, then $-x \leq_H 0$;
2. for $x, y \in G_s$, if $0 \leq_H x$ and $0 \leq_H y$, then $0 \leq_H x + y$;
3. for $x, y, z \in G_s$, if $x \leq_H y$, then $x + z \leq_H y + z$.

Item 1 is a particular case of Item 3 and Item 2 follows easily from Item 3.

Item 3. Let us write $x = \bigvee_i \bigwedge_j x_{ij}$, $y = \bigwedge_k \bigvee_\ell y_{k\ell}$ with the x_{ij} s and the $y_{k\ell}$ s in G . The hypothesis $x \leq_H y$ means that for each i and k we have $\bigwedge_j x_{ij} \leq_H \bigvee_\ell y_{k\ell}$, i.e.,

$$x_{i1}, \dots, x_{ip} \vdash y_{k1}, \dots, y_{kq}.$$

Using R3 one has

$$x_{i1} + z, \dots, x_{ip} + z \vdash y_{k1} + z, \dots, y_{kq} + z,$$

i.e., for each (i, k) ,

$$\bigwedge_j (x_{ij} + z) \leq_H \bigvee_\ell (y_{k\ell} + z),$$

from which we deduce that $x + z = \bigvee_i \bigwedge_j (x_{ij} + z) \leq_H \bigwedge_k \bigvee_\ell (y_{k\ell} + z) = y + z$. \square

Remark 5.3. **Lorenzen** (1939, § 4) and **Jaffard** (1960, II, § 2, 2) define the Lorenzen group associated to a system of ideals as in Definition 6.7, i.e., according to the Prüfer approach. The present approach leading to Definition VIII dates back to **Lorenzen** (1950, § 6). The two definitions are equivalent according to Proposition 6.8. \blacksquare

5.3 The Lorenzen divisor group of an integral domain

In this section, we draw the conclusions allowed by Theorem IV in Lorenzen's theory of divisibility presented in Section 3 on page 24.

Definition 5.4. Let R be an integral domain. The *Lorenzen divisor group* $\text{Lor}(R)$ of R is the Lorenzen group associated by Definition VIII to the Dedekind system of ideals $\triangleright_{\text{d}}$.

Theorem 5.5. *Let R be an integral domain with field of fractions K and divisibility group $G = K^\times/R^\times$. The entailment relation $\vdash_{\triangleright_{\text{d}}}$ generates the Lorenzen divisor group $\text{Lor}(R)$ together with a morphism of ordered groups $\varphi: G \rightarrow \text{Lor}(R)$ that satisfies the following properties.*

1. The “ideal Lorenzen gcd” of a family $(a_i)_{i \in [1..n]}$ in K^* is characterised by

$$\varphi(a_1) \wedge \dots \wedge \varphi(a_n) \leq \varphi(b) \iff b \text{ is integral over the ideal } \langle a_1, \dots, a_n \rangle_R. \quad (\#)$$

2. The morphism φ is an embedding if and only if R is integrally closed.

Proof. As the entailment relation $\vdash_{\triangleright_{\text{d}}}$ is a regular system of ideals (Theorem II), the corresponding distributive lattice H admits a unique group law such that the natural morphism $\varphi: G \rightarrow H$ is a morphism of ordered groups (by Theorem IV), which explains Definition 5.4 since here H is the distributive lattice underlying $\text{Lor}(R)$.

1. Theorem 3.21 states that $\varphi(a_1) \wedge \dots \wedge \varphi(a_n) \leq_{\vdash_{\triangleright_{\text{d}}}} \varphi(b)$ if and only if b is integral over $\langle A \rangle_R$. On the other hand $\varphi(a_1) \wedge \dots \wedge \varphi(a_n) \leq_{\vdash_{\triangleright_{\text{d}}}} \varphi(b_1) \wedge \dots \wedge \varphi(b_p)$ if and only if $\varphi(a_1) \wedge \dots \wedge \varphi(a_n) \leq_{\vdash_{\triangleright_{\text{d}}}} \varphi(b_j)$ for each j because $\varphi(b_1) \wedge \dots \wedge \varphi(b_p)$ is the meet of the b_j s in $\text{Lor}(R)$. This explains why Property (#) characterises the element $\varphi(a_1) \wedge \dots \wedge \varphi(a_n)$ of $\text{Lor}(R)$.

2. The morphism φ is an embedding if and only if $\varphi(a) \leq_{\vdash_{\triangleright_{\text{d}}}} \varphi(b)$ implies $a \triangleright_{\text{d}} b$, which means that R is integrally closed. \square

Corollary 5.6. *Let R be an integrally closed domain. When \mathfrak{a} is a finitely generated ideal, we let $\overline{\mathfrak{a}}$ be the integral closure of \mathfrak{a} . Then, if \mathfrak{a} , \mathfrak{b} and \mathfrak{c} are nonzero finitely generated ideals, we have the cancellation property*

$$\overline{\mathfrak{a}\mathfrak{b}} \subseteq \overline{\mathfrak{a}\mathfrak{c}} \implies \overline{\mathfrak{b}} \subseteq \overline{\mathfrak{c}}.$$

This corollary is considered by H. S. Macaulay (1916, pages 108-109) as a key result; he gives a proof based on the multivariate resultant. We may also deduce it as a consequence of Prüfer’s theorem 6.5 (see Remark 6.10, compare Prüfer 1932, § 6, Krull 1935, 46.).

In Items 2 and 4 below, we use the conventional additive notation for a “divisor group” of an integral domain.

Corollary 5.7. *Let R be an integral domain. The Lorenzen divisor group $\text{Lor}(R)$ can be described set-theoretically in the following way.*

1. *Basic nonnegative divisors are identified with integral closures $\text{Icl}(a_1, \dots, a_n)$ of (ordinary, i.e., integral) finitely generated ideals¹¹ $\langle a_1, \dots, a_n \rangle_R$ with $a_1, \dots, a_n \in R$.*
2. *The zero divisor is $\text{Icl}(1)$.*
3. *The meet of two basic nonnegative divisors is given by*

$$\text{Icl}(a_1, \dots, a_n) \wedge \text{Icl}(b_1, \dots, b_m) = \text{Icl}(a_1, \dots, a_n, b_1, \dots, b_m).$$

4. *The sum of two basic nonnegative divisors is given by*

$$\text{Icl}(a_1, \dots, a_n) + \text{Icl}(b_1, \dots, b_m) = \text{Icl}(a_1 b_1, \dots, a_n b_m).$$

5. *The order relation between basic nonnegative divisors is given by*

$$\text{Icl}(a_1, \dots, a_n) \leq \text{Icl}(b_1, \dots, b_m) \iff \text{Icl}(a_1, \dots, a_n) \supseteq \text{Icl}(b_1, \dots, b_m).$$

6. *General divisors are identified with formal differences of two basic nonnegative divisors.*

Proof. Item 1 is a rephrasing of Item 1 in Theorem 5.5. Items 2 to 5 are clear. Let us consider Item 6. $\text{Lor}(R)$ is generated by $\varphi(G)$ as an ℓ -group. An element of $\varphi(G)$ is written as $\varphi(a) - \varphi(b)$ with $a, b \in R^*$. It remains to verify that differences of basic nonnegative divisors are stable by the laws \wedge , $+$, and $-$ of an ℓ -group. Only the \wedge -stability requires a little trick: in order to compute $\delta = (\varphi(A) - \varphi(B)) \wedge (\varphi(C) - \varphi(D))$, it is sufficient to compute $\delta + \varphi(B) + \varphi(D)$, which is equal to $(\varphi(A) + \varphi(D)) \wedge (\varphi(C) + \varphi(B))$, which can be computed using the previous items. \square

¹¹If the integral domain is not integrally closed, $\text{Icl}(a_1, \dots, a_n)$ may contain elements not in R .

Remarks 5.8. 1. When R is a Prüfer domain, the Lorenzen divisor group $\text{Lor}(R)$ coincides with the usual divisor group, the group of finitely generated fractional ideals defined by Dedekind and Kronecker. In fact, the relation $\vdash_{\triangleright_d}$ is determined by its trace on $\text{P}_{\text{fe}}^*(R^*) \times R^*$, and in a Prüfer domain all finitely generated ideals are integrally closed, so that $A \vdash_{\triangleright_d} b$ simplifies to $b \in \langle A \rangle_R$ (see Item 1 of Theorem 3.21). For more general rings with divisors, the Weil divisor group (see Remark 2.12) is a strict quotient of the Lorenzen divisor group.

2. The integral domain $R = \mathbb{Q}[x, y]$ is a gcd domain of dimension ≥ 2 , so that its divisibility group G is an ℓ -group. The domain R is not Prüfer and the Lorenzen divisor group is much greater than G : e.g., the ideal gcd of x^3 and y^3 in $\text{Lor}(R)$ corresponds to the integrally closed ideal $\langle x^3, x^2y, xy^2, y^3 \rangle$, whereas their gcd in R^* is 1, corresponding to the ideal $\langle 1 \rangle$. In this case, we see that G is a proper quotient of $\text{Lor}(R)$. ■

6 Systems of ideals and Prüfer's theorem

In this section, we account for another way to obtain the Lorenzen group associated to a system of ideals for an ordered group (Definition VIII). This way has historical precedence, as it dates back to the Ph.D. thesis Lorenzen (1939), that builds on earlier work by Prüfer (1932). As a particular case this provides another access to understanding the Lorenzen divisor group of an integral domain.

6.1 The Grothendieck ℓ -group of a meet-semilattice-ordered monoid

The following easy construction, for which we did not locate a good reference, is particularly significant in the case where the meet-monoid associated to a system of ideals proves to be cancellative.

Theorem 6.1. *Let $(M, +, 0, \wedge)$ be a meet-monoid. Let H be the Grothendieck group of M with monoid morphism $\varphi: M \rightarrow H$.*

1. *There exists a unique meet-monoid structure on H such that φ is a morphism of ordered sets.*
2. *$(H, +, -, 0, \wedge)$ is an ℓ -group: it is the ℓ -group generated by $(M, +, 0, \wedge)$ in the usual meaning of adjoint functors, and called the Grothendieck ℓ -group of M .*
3. *Assume that M is cancellative, i.e., that $x + y = x + z$ implies $y = z$. Then φ is an embedding of meet-monoids.*

Proof. 1. The elements of H are written as $a - b$ for $a, b \in M$, with the equality $a - b = c - d$ holding if and only if there exists x such that $a + d + x = b + c + x$.

By transitivity and symmetry, every equality $a - b = c - d$ may be reduced to two “elementary” ones, i.e., of the form $e - f = (e + y) - (f + y)$:

$$a - b = (a + d + x) - (b + d + x) = (b + c + x) - (b + d + x) = c - d.$$

When trying to define $z = (e - f) \wedge (g - h)$ we need to ensure that

$$f + h + z = (e + h) \wedge (g + f).$$

So we may propose to set $(e - f) \wedge (g - h) \stackrel{\text{def}}{=} ((e + h) \wedge (g + f)) - (f + h)$.

Let us show first that the law \wedge is well-defined on H .

It suffices to show that $(e - f) \wedge (g - h) = ((e + y) - (f + y)) \wedge (g - h)$, which reduces successively to

$$((e + h) \wedge (g + f)) - (f + h) = ((e + h + y) \wedge (g + f + y)) - (f + h + y),$$

$$((e + h) \wedge (g + f)) + (f + h + y) = ((e + h + y) \wedge (g + f + y)) + (f + h).$$

Since \wedge is compatible with $+$ in M , both sides are equal to

$$(e + 2h + f + y) \wedge (g + 2f + h + y).$$

- The map $\varphi: M \rightarrow H$ preserves \wedge : in fact $\varphi(a) \stackrel{\text{def}}{=} a - 0$, and the checking is immediate.

- The law \wedge on H is idempotent, commutative and associative. This is easy to check and left to the reader.

- The law \wedge is compatible with $+$ on H . This is easy to check and left to the reader.

2. Left to the reader.

3. The meet-monoid structure is purely equational. So an injective morphism is always an embedding. \square

As an application of this construction, let us state a variant of Theorem IV.

Corollary 6.2 (to Theorem IV). *Let (G, \leq_G) be an ordered group and \triangleright a system of ideals for G . The following are equivalent:*

1. *The system of ideals \triangleright is regular, i.e., it is the restriction of a regular system of ideals \vdash .*

2. *The meet-monoid associated to the system of ideals \triangleright for G (Theorem I) is cancellative.*

When this is the case, let (H, \leq_H) be the unbounded distributive lattice generated by the regular system of ideals \vdash . Then the group law and the group morphism $\varphi: G \rightarrow H$ constructed by Theorem IV can also be obtained as the Grothendieck ℓ -group of the monoid in Item 2.

Proof. $1 \Rightarrow 2$. The subset $M \subseteq H$ of those elements that may be written $\varphi(x_1) \wedge \dots \wedge \varphi(x_n)$ for some x_1, \dots, x_n is the meet-semilattice associated to the system of ideals \triangleright obtained by restricting \vdash to $\text{P}_{\text{fe}}^*(G) \times G$. This subset M is stable by addition, so that the restriction of addition to M endows it with the structure of a cancellative meet-monoid. Thus H is necessarily (naturally isomorphic to) the Grothendieck ℓ -group of M .

$2 \Rightarrow 1$. If the monoid is cancellative, then it embeds into its Grothendieck ℓ -group H . So, using the observation on page 5 leading to Definition VI, we get Item 1. \square

6.2 Prüfer's properties Γ and Δ

Let us now express cancellativity of the meet-monoid as a property of the system of ideals itself, as in Prüfer (1932, § 3).

Lemma 6.3 (a version of Prüfer's Property Γ). *Let \triangleright be a system of ideals for an ordered group G . The corresponding meet-monoid M is cancellative, i.e., $a + b = a + c$ implies $b = c$ in M , if and only if the following property holds:*

$$A + B \leq_{\triangleright} x + B \Rightarrow A \triangleright x.$$

This holds if and only if

$$A + B \leq_{\triangleright} B \Rightarrow A \triangleright 0.$$

Proof. The second implication, a particular case of the first one, implies the first one by a translation. Let us work with the first implication.

Cancellativity means that the implication $A + B \leq_{\triangleright} C + B \Rightarrow A \leq_{\triangleright} C$ holds. The property is necessary: take $C = \{x\}$. Let us show that it is sufficient. Assume $A + B \leq_{\triangleright} C + B$ and let $x \in C$. As $C \triangleright x$, we get by equivariance $C + B \leq_{\triangleright} x + B$, whence $A + B \leq_{\triangleright} x + B$. So $A \triangleright x$. Since this holds for each $x \in C$, we get $A \leq_{\triangleright} C$. \square

Prüfer's theorem 6.5 will reveal the significance of the following definition. We shall check in Proposition 6.8 that it agrees with Definition VII.

Definition 6.4 (a version of Prüfer's Property Δ of integral closedness). Let \triangleright be a system of ideals for an ordered group G . The group G is \triangleright -closed if $B \leq_{\triangleright} x + B \Rightarrow 0 \leq_G x$.

6.3 Forcing cancellativity: Prüfer's theorem

When the monoid M in Theorem I is not cancellative, it is possible to adjust the system of ideals in order to straighten the situation. A priori, it suffices to consider the Grothendieck ℓ -group of M (Theorem 6.1). But we have to see that this corresponds to a system of ideals for G , and to provide a description for it.

The following theorem is a reformulation of Prüfer's theorem (Prüfer 1932, § 6). We follow the proofs in Jaffard (1960, pages 42-43). In fact, the language of single-conclusion entailment relations simplifies the proofs. We are adding Items 2 and 3 to Jaffard's statement, which corresponds to Items 1 and 4 of ours.

Theorem 6.5 (Prüfer's theorem). *Let \triangleright be a system of ideals for an ordered group G . We define the relation \triangleright_a between $P_{fe}^*(G)$ and G by*

$$A \triangleright_a y \stackrel{\text{def}}{\iff} \exists B \in P_{fe}^*(G) \quad A + B \leq_{\triangleright} y + B.$$

1. *The relation \triangleright_a is a system of ideals for G , and the associated meet-monoid M_a (Theorem I) is cancellative.*
2. *Therefore M_a embeds into its Grothendieck ℓ -group H_a .*
3. *The system \triangleright_a is the finest system of ideals that is coarser than \triangleright and satisfies Item 1.*
4. *We have that $a \triangleright_a b$ implies $a \leq_G b$ if (and only if) G is \triangleright -closed (Definition 6.4); in this case, G embeds into H_a .*

Proof. Note that if $A + B \leq_{\triangleright} y + B$, then $A + B + C \leq_{\triangleright} y + B + C$ for all C (see the proof of Theorem I on page 12). This makes the definition of \triangleright_a very easy to use. In the proof below, we have two preorder relations on $P_{fe}^*(G)$ (\leq_{\triangleright} and \leq_a), and we shall do as if they were order relations (i.e., we shall descend to the quotients).

1. • *Reflexivity and preservation of order* (of the relation \triangleright_a). Setting $B = \{0\}$ in the definition of \triangleright_a shows that $x \leq_G y$ implies $x \triangleright_a y$.
 - *Monotonicity.* It suffices to note that the elements $(A \cup A') + B$ and $(A + B) \cup (A' + B)$ of $P_{fe}^*(G)$ are the same: therefore, if $A + B \leq_{\triangleright} y + B$, then $(A, A') + B \leq_{\triangleright} y + B$.
 - *Transitivity.* Assume $A \triangleright_a x$ and $A, x \triangleright_a b$: we have a B such that $A + B \leq_{\triangleright} x + B$ and a C such that $(A, x) + C \leq_{\triangleright} b + C$; these inequalities imply respectively $A + B + C \leq_{\triangleright} x + B + C$ and $(A + B + C), (x + B + C) \leq_{\triangleright} b + B + C$; we deduce $A + B + C \leq_{\triangleright} b + B + C$, so that $A \triangleright_a b$.
 - *Equivariance.* If $A \triangleright_a y$, we have a B such that $A + B \leq_{\triangleright} y + B$, so that, since \leq_{\triangleright} is equivariant, $x + A + B \leq_{\triangleright} x + y + B$. This yields $x + A \triangleright_a x + y$.
 - *Cancellativity (of the meet-monoid M_a).* Let us denote by $A \leq_a B$ the order relation associated to \triangleright_a . By Lemma 6.3, it suffices to suppose that $A + B \leq_a A$ and to deduce that $B \triangleright_a 0$. But the hypothesis means that $A + B \triangleright_a y$ for each

$y \in A$, i.e., that for each $y \in A$ there is a C_y such that $A + B + C_y \leq_{\triangleright} y + C_y$. Let $C = \sum_{y \in A} C_y$; we have

$$A + B + C \leq_{\triangleright} y + C \leq_{\triangleright} y + z \text{ for each } y \in A \text{ and each } z \in C,$$

so that $A + B + C \leq_{\triangleright} A + C$. This yields $B \triangleright_a 0$ as desired.

2. Follows from Item 1 by Theorem 6.1.

3. This is immediate from the definition of \triangleright_a : it has been defined in a minimal way as coarser than \triangleright and forcing the cancellativity of the monoid M_a .

4. If $x \triangleright_a y$, then we have a B such that $x + B \leq_{\triangleright} y + B$, so that by a translation $B \leq_{\triangleright} (y - x) + B$. The hypothesis on G yields $0 \triangleright y - x$. By a translation, we get $x \triangleright y$. \square

Comment 6.6. This is the approach proposed in Lorenzen (1939, § 4). Lorenzen abandoned it in favour of Definition VII for the purpose of generalising his theory to noncommutative groups. See also Comments 3.14 and 4.5. \blacksquare

Definition 6.7. The ℓ -group in Item 2 of Theorem 6.5 is called the *Lorenzen group for the system of ideals \triangleright* .

Proposition 6.8 (Lorenzen 1950, Satz 27). *The definition of $A \triangleright_a 0$ in Theorem 6.5 agrees with Definition VII of $A \vdash_{\triangleright} 0$. So Definition 6.4 of \triangleright -closedness agrees with that of Definition VII, and Definition 6.7 of the Lorenzen group agrees with that of Definition VIII.*

Proof. This proposition expresses that, given a system of ideals \triangleright for an ordered group G and an $A \in \mathbf{P}_{\text{fe}}^*(G)$, we have $A \vdash_{\triangleright} 0$ (Definition VIII) if and only if $A + B \leq_{\triangleright} B$ for some $B \in \mathbf{P}_{\text{fe}}^*(G)$.

First, $A + B \leq_{\triangleright_x} B$ and $A + C \leq_{\triangleright_{-x}} C$ imply $A + D \leq_{\triangleright} D$ for some D . In fact, we have p and q such that

$$\begin{aligned} A + B, A + B + x, \dots, A + B + px &\leq_{\triangleright} B \text{ and} \\ A + C, A + C - x, \dots, A + C - qx &\leq_{\triangleright} C, \end{aligned}$$

which yield that for $c \in C$, $j \leq q$, $b \in B$ and $k \leq p$,

$$\begin{aligned} A + B + c - jx, \dots, A + B + c + (p - j)x &\leq_{\triangleright} B + c - jx \text{ and} \\ A + b + C + kx, \dots, A + b + C + (k - q)x &\leq_{\triangleright} b + C + kx, \end{aligned}$$

so that $A + D \leq_{\triangleright} D$ for $D = B + C + \{-qx, \dots, px\}$.

In the other direction assume that $A + B \triangleright b_i$ for each b_i in $B = \{b_1, \dots, b_m\}$. Let $c_{i,j} = b_i - b_j$ ($i < j \in \llbracket 1..m \rrbracket$) and let us prove that $A \triangleright_{\pm c_{1,2}, \dots, \pm c_{m-1,m}} 0$. In

fact, for any system of constraints $(\epsilon_{1,2}c_{1,2}, \dots, \epsilon_{m-1,m}c_{m-1,m})$ with $\epsilon_{i,j} = \pm 1$, the elements b_i in the corresponding meet-monoid $M_{\epsilon_{1,2}, \dots, \epsilon_{m-1,m}}$ are linearly ordered. E.g., $b_1 \leq b_2 \leq \dots \leq b_m$, in which case

$$\bigwedge(A + b_1, \dots, A + b_m) = \bigwedge(A + b_1) \leq b_1$$

holds in the monoid $M_{\epsilon_{1,2}, \dots, \epsilon_{m-1,m}}$, which yields $\bigwedge A \leq 0$ by a translation. \square

Remark 6.9. Informally the content of this proposition may be expressed as follows. By starting from \triangleright and by adding new pairs (A, b) such that $A \triangleright' b$, on the one side Prüfer forces the cancellativity of the meet-monoid M_a , and on the other side Lorenzen forces \triangleright to become the restriction of an entailment relation¹². In fact, each approach realises both aims, but each one realises its own aim in a minimal way. So they give the same result. \blacksquare

Remark 6.10. Theorem 6.5 enables to recover the results of Theorem 3.21 and of Theorem 5.5 in the Prüfer approach. In particular, one may check that $A (\triangleright_d)_a b$ holds if and only if b is integral over the fractional ideal $\langle A \rangle_R$, and that both hypotheses in Item 4 of Theorem 6.5 are fulfilled when R is integrally closed. Furthermore, elements ≥ 1 of the ℓ -group M_a in Item 2 of Theorem 6.5 can be identified with integrally closed ideals generated by nonempty finitely enumerated subsets A of R^* . Therefore Item 1 of Theorem 6.5 yields the cancellation property stated in Corollary 5.6. \blacksquare

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¹²The fact that it remains a system of ideals is trivial from Lorenzen's definition.

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