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# ON THE CAUCHY PROBLEM FOR THE NONLINEAR SEMI-RELATIVISTIC EQUATION IN SOBOLEV SPACES

VAN DUONG DINH

## Abstract

We proved the local well-posedness for the power-type nonlinear semi-relativistic or half-wave equation (NLHW) in Sobolev spaces. Our proofs mainly base on the contraction mapping argument using Strichartz estimate. We also apply the technique of Christ-Colliander-Tao in [7] to prove the ill-posedness for (NLHW) in some cases of the super-critical range.

## 1 Introduction and main results

We consider the Cauchy semi-relativistic or half-wave equation posed on  $\mathbb{R}^d, d \geq 1$ , namely

$$\begin{cases} i\partial_t u(t, x) + \Lambda u(t, x) &= -\mu |u|^{\nu-1} u(t, x), & (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0, x) &= u_0(x), & x \in \mathbb{R}^d, \end{cases} \quad (\text{NLHW})$$

where  $\nu > 1, \mu \in \{\pm 1\}$  and  $\Lambda = \sqrt{-\Delta}$  is the Fourier multiplier by  $|\xi|$ . The number  $\mu = 1$  (resp.  $\mu = -1$ ) corresponds to the defocusing case (resp. focusing case). The Cauchy problem problem such as (NLHW) arises in various physical contexts, such as water waves (see e.g. [19]), and gravitational collapse (see e.g. [11], [12]).

It is worth noticing that the (NLHW) is invariant under the scaling

$$u_\lambda(t, x) = \lambda^{-\frac{1}{\nu-1}} u(\lambda^{-1}t, \lambda^{-1}x).$$

That is, for  $T \in (0, +\infty]$ ,  $u$  solves (NLHW) on  $(-T, T)$ , which is equivalent to  $u_\lambda$  solves (NLHW) on  $(-\lambda T, \lambda T)$ . A direct computation gives

$$\|u_\lambda(0)\|_{\dot{H}^\gamma} = \lambda^{\frac{d}{2} - \frac{1}{\nu-1} - \gamma} \|u_0\|_{\dot{H}^\gamma}.$$

From this, we define the critical regularity exponent for (NLHW) by

$$\gamma_c = \frac{d}{2} - \frac{1}{\nu-1}. \quad (1.1)$$

One said that  $H^\gamma$  is sub-critical (critical, super-critical) if  $\gamma > \gamma_c$  ( $\gamma = \gamma_c$ ,  $\gamma < \gamma_c$ ) respectively. Another important property of (NLHW) is that the following mass and energy are formally conserved under the flow of the equation,

$$M(u(t)) = \int |u(t, x)|^2 dx, \quad E(u(t)) = \int \frac{1}{2} |\Lambda^{1/2} u(t, x)|^2 + \frac{\mu}{\nu+1} |u(t, x)|^{\nu+1} dx.$$

The nonlinear half-wave equation (NLHW) has attracted a lot of works in a past decay (see e.g. [12], [23], [14], [6], [13] and references therein). The main purpose of this note is to give the well-posedness and ill-posedness results for (NLHW) in Sobolev spaces. The proofs of the well-posedness base on Strichartz estimate and the standard contraction argument. We thus only focus on the case  $d \geq 2$  where Strichartz estimate appears, and just recall the known results in one dimensional case. Precisely, we prove the well-posedness in  $H^\gamma$  with

$$\begin{cases} \gamma > 1 - 1/\max(\nu-1, 4) & \text{when } d = 2, \\ \gamma > d/2 - 1/\max(\nu-1, 2) & \text{when } d \geq 3, \end{cases} \quad (1.2)$$

and of course with some regularity assumption on  $\nu$ . This remains a gap between  $\gamma_c$  and  $1 - 1/\max(\nu - 1, 4)$  when  $d = 2$  and  $d/2 - 1/\max(\nu - 1, 2)$  when  $d \geq 3$ . Next, we can apply successfully the argument of [18] (see also [10]) to prove the local well-posedness with small data scattering in the critical case provided  $\nu > 5$  for  $d = 2$  and  $\nu > 3$  for  $d \geq 3$ . The cases  $\nu \in (1, 5]$  when  $d = 2$  and  $\nu \in (1, 3]$  when  $d \geq 3$  still remain open. It requires another technique rather than just Strichartz estimate. Finally, using the technique of Christ-Colliander-Tao given in [7], we are able to prove the ill-posedness for (NLHW) in some cases of the super-critical range, precisely in  $H^\gamma$  with  $\gamma \in ((-\infty, -d/2] \cap (-\infty, \gamma_c)) \cup [0, \gamma_c)$ . We expect that the ill-posedness still holds in the range  $\gamma \in (-d/2, 0) \cap (-\infty, \gamma_c)$  as for the nonlinear Schrödinger equation (see [7]). But it is not clear to us how to prove it at the moment. Recently, Hong and Sire in [18] used the technique of [7] with the pseudo-Galilean transformation to get the ill-posedness for the nonlinear fractional Schrödinger equation with negative exponent. Unfortunately, it seem to be difficult to control the error of the pseudo-Galilean transformation in high Sobolev norms and so far only restricted in one dimension. Note also that one has a sharp ill-posed result for the cubic (NLHW) in 1D (see [6]). Specifically, one has the ill-posedness for  $\gamma < 1/2$  which is larger than  $\gamma_c$ . The proof of this result mainly bases on the relation with the cubic Szegö equation which can not extend easily to general nonlinearity.

Let us firstly recall some known results about the local existence of (NLHW) in 1D. It is well-known that (NLHW) is locally well-posed in  $H^\gamma(\mathbb{R})$  with  $\gamma > 1/2$  and of course with some regularity condition using the energy method and the contraction mapping argument. When  $\nu = 3$ , i.e. cubic nonlinearity, the (NLHW) is locally well-posed in  $H^\gamma(\mathbb{R})$  with  $\gamma \geq 1/2$  (see e.g. [23], [25]). This result is optimal in the sense that the (NLHW) is ill-posed in  $H^\gamma(\mathbb{R})$  provided  $\gamma < 1/2$  (see e.g. [6]). To our knowledge, the local well-posedness for the generalized (NLHW) in  $H^\gamma(\mathbb{R})$  with  $\gamma \leq 1/2$  seems to be an open question.

Before stating our results, let us introduce some notations (see the appendix of [15], Chapter 5 of [29] or Chapter 6 of [3]). Given  $\gamma \in \mathbb{R}$  and  $1 \leq q \leq \infty$ , the Sobolev and Besov spaces are defined by

$$H_q^\gamma := \left\{ u \in \mathcal{S}' \mid \|u\|_{H_q^\gamma} := \|\langle \Lambda \rangle^\gamma u\|_{L^q} < \infty \right\}, \quad \langle \Lambda \rangle := \sqrt{1 + \Lambda^2},$$

$$B_q^\gamma := \left\{ u \in \mathcal{S}' \mid \|u\|_{B_q^\gamma} := \|P_0 u\|_{L^q} + \left( \sum_{N \in 2^{\mathbb{Z}}} N^{2\gamma} \|P_N u\|_{L^q}^2 \right)^{1/2} < \infty \right\},$$

where  $\mathcal{S}'$  is the space of tempered distributions. The Littlewood-Paley projections  $P_0 := \varphi_0(D)$  and  $P_N := \varphi(N^{-1}D)$ ,  $N \in 2^{\mathbb{Z}}$  are the Fourier multipliers by  $\varphi_0(\xi)$  and  $\varphi(N^{-1}\xi)$  respectively, where  $\varphi_0 \in C_0^\infty(\mathbb{R}^d)$  is such that  $\varphi_0(\xi) = 1$  for  $|\xi| \leq 1$ ,  $\text{supp}(\varphi_0) \subset \{\xi \in \mathbb{R}^d, |\xi| \leq 2\}$  and  $\varphi(\xi) := \varphi_0(\xi) - \varphi_0(2\xi)$ . The homogeneous Sobolev and Besov spaces are defined by

$$\dot{H}_q^\gamma := \left\{ u \in \mathcal{S}'_0 \mid \|u\|_{\dot{H}_q^\gamma} := \|\Lambda^\gamma u\|_{L^q} < \infty \right\},$$

$$\dot{B}_q^\gamma := \left\{ u \in \mathcal{S}'_0 \mid \|u\|_{\dot{B}_q^\gamma} := \left( \sum_{N \in 2^{\mathbb{Z}}} N^{2\gamma} \|P_N u\|_{L^q}^2 \right)^{1/2} < \infty \right\},$$

where  $\mathcal{S}'_0$  is a subspace of the Schwartz space  $\mathcal{S}$  consisting of functions  $\phi$  satisfying  $D^\alpha \hat{\phi}(0) = 0$  for all  $\alpha \in \mathbb{N}^d$  where  $\hat{\cdot}$  is the Fourier transform on  $\mathcal{S}$  and  $\mathcal{S}'_0$  its topological dual space. Under these settings,  $H_q^\gamma, B_q^\gamma, \dot{H}_q^\gamma$  and  $\dot{B}_q^\gamma$  are Banach spaces with the norms  $\|u\|_{H_q^\gamma}, \|u\|_{B_q^\gamma}, \|u\|_{\dot{H}_q^\gamma}$  and  $\|u\|_{\dot{B}_q^\gamma}$  respectively. In this note, we shall use  $H^\gamma := H_2^\gamma$ ,  $\dot{H}^\gamma := \dot{H}_2^\gamma$ . We note that (see again Chapter 6 of [3] or the appendix of [15]) if  $2 \leq q < \infty$ , then  $\dot{B}_q^\gamma \subset \dot{H}_q^\gamma$ . The reverse inclusion holds for  $1 < q \leq 2$ . In particular,  $\dot{B}_2^\gamma = \dot{H}^\gamma$  and  $\dot{B}_2^0 = \dot{H}_2^0 = L^2$ . Moreover, if  $\gamma > 0$ , then  $H_q^\gamma = L^q \cap \dot{H}_q^\gamma$  and  $B_q^\gamma = L^q \cap \dot{B}_q^\gamma$ .

Throughout this sequel, a pair  $(p, q)$  is said to be admissible if

$$(p, q) \in [2, \infty]^2, \quad (p, q, d) \neq (2, \infty, 3), \quad \frac{2}{p} + \frac{d-1}{q} \leq \frac{d-1}{2}.$$

We also denote for  $(p, q) \in [1, \infty]^2$ ,

$$\gamma_{p,q} = \frac{d}{2} - \frac{d}{q} - \frac{1}{p}. \quad (1.3)$$

Since we are working in spaces of fractional order  $\gamma$  or  $\gamma_c$ , we need the nonlinearity  $F(z) = -\mu|z|^{\nu-1}z$  to have enough regularity. When  $\nu$  is an odd integer,  $F \in C^\infty(\mathbb{C}, \mathbb{C})$  (in the real sense). When  $\nu$  is not an odd integer, we need the following assumption

$$[\gamma] \text{ or } [\gamma_c] \leq \nu, \quad (1.4)$$

where  $[\gamma]$  is the smallest integer greater than or equal to  $\gamma$ , similarly for  $[\gamma_c]$ . Our first result concerns the local well-posedness of (NLHW) in the sub-critical case.

**Theorem 1.1.** *Let  $\gamma \geq 0$  and  $\nu > 1$  be such that (1.2), and also, if  $\nu$  is not an odd integer, (1.4). Then for all  $u_0 \in H^\gamma$ , there exist  $T^* \in (0, \infty]$  and a unique solution to (NLHW) satisfying*

$$u \in C([0, T^*), H^\gamma) \cap L_{\text{loc}}^p([0, T^*), L^\infty),$$

for some  $p > \max(\nu - 1, 4)$  when  $d = 2$  and some  $p > \max(\nu - 1, 2)$  when  $d \geq 3$ . Moreover, the following properties hold:

- (i) If  $T^* < \infty$ , then  $\|u(t)\|_{H^\gamma} \rightarrow \infty$  as  $t \rightarrow T^*$ .
- (ii)  $u$  depends continuously on  $u_0$  in the following sense. There exists  $0 < T < T^*$  such that if  $u_{0,n} \rightarrow u_0$  in  $H^\gamma$  and if  $u_n$  denotes the solution of (NLHW) with initial data  $u_{0,n}$ , then  $0 < T < T^*(u_{0,n})$  for all  $n$  sufficiently large and  $u_n$  is bounded in  $L^a([0, T], H_b^{\gamma-\gamma_{a,b}})$  for any admissible pair  $(a, b)$  with  $b < \infty$ . Moreover,  $u_n \rightarrow u$  in  $L^a([0, T], H_b^{-\gamma_{a,b}})$  as  $n \rightarrow \infty$ . In particular,  $u_n \rightarrow u$  in  $C([0, T], H^{\gamma-\epsilon})$  for all  $\epsilon > 0$ .
- (iii) Let  $\beta > \gamma$  be such that if  $\nu$  is not an odd integer,  $[\beta] \leq \nu$ . If  $u_0 \in H^\beta$ , then  $u \in C([0, T^*), H^\beta)$ .

The continuous dependence can be improved to hold in  $C([0, T], H^\gamma)$  if we assume that  $\nu > 1$  is an odd integer or  $[\gamma] \leq \nu - 1$  otherwise (see Remark 2.8).

**Theorem 1.2.** *Let*

$$\begin{cases} \nu > 5 & \text{when } d = 2, \\ \gamma > 3 & \text{when } d \geq 3, \end{cases} \quad (1.5)$$

and also, if  $\nu$  is not an odd integer, (1.4). Then for all  $u_0 \in H^{\gamma_c}$ , there exist  $T^* \in (0, \infty]$  and a unique solution to (NLHW) satisfying

$$u \in C([0, T^*), H^{\gamma_c}) \cap L_{\text{loc}}^p([0, T^*), B_q^{\gamma_c - \gamma_{p,q}}),$$

where  $p = 4, q = \infty$  when  $d = 2$ ;  $2 < p < \nu - 1, q = p^* = 2p/(p - 2)$  when  $d = 3$ ;  $p = 2, q = 2^* = 2(d - 1)/(d - 3)$  when  $d \geq 4$ . Moreover, if  $\|u_0\|_{\dot{H}^{\gamma_c}} < \epsilon$  for some  $\epsilon > 0$  small enough, then  $T^* = \infty$  and the solution is scattering in  $H^{\gamma_c}$ , i.e. there exists  $u_0^+ \in H^{\gamma_c}$  such that

$$\lim_{t \rightarrow +\infty} \|u(t) - e^{it\Lambda} u_0^+\|_{H^{\gamma_c}} = 0.$$

Our final result is the following ill-posedness for the (NLHW).

**Theorem 1.3.** *Let  $\nu > 1$  be such that if  $\nu$  is not an odd integer,  $\nu \geq k + 1$  for some integer  $k > d/2$ . Then (NLHW) is ill-posed in  $H^\gamma$  for  $\gamma \in ((-\infty, -d/2] \cap (-\infty, \gamma_c)) \cup [0, \gamma_c)$ . Precisely, if  $\gamma \in ((-\infty, -d/2] \cap (-\infty, \gamma_c)) \cup (0, \gamma_c)$ , then for any  $t > 0$  the solution map  $\mathcal{S} \ni u(0) \mapsto u(t)$  of (NLHW) fails to be continuous at 0 in the  $H^\gamma$  topology. Moreover, if  $\gamma_c > 0$ , the solution map fails to be uniformly continuous on  $L^2$ .*

## 2 Well-posedness

In this section, we will give the proofs of Theorem 1.1 and Theorem 1.2. Our proof is based on the standard contraction mapping argument using Strichartz estimate and nonlinear fractional derivatives estimates.

### 2.1 Linear estimates

In this subsection, we recall Strichartz estimate for the half-wave equation.

**Theorem 2.1** ([2], [21], [22]). *Let  $d \geq 2, \gamma \in \mathbb{R}$  and  $u$  be a (weak) solution to the linear half-wave equation, namely*

$$u(t) = e^{it\Lambda}u_0 + \int_0^t e^{i(t-s)\Lambda}F(s)ds,$$

for some data  $u_0, F$ . Then for all  $(p, q)$  and  $(a, b)$  admissible pairs,

$$\|u\|_{L^p(\mathbb{R}, \dot{B}_q^\gamma)} \lesssim \|u_0\|_{\dot{H}^{\gamma+\gamma_{p,q}}} + \|F\|_{L^{a'}(\mathbb{R}, \dot{B}_{q'}^{\gamma+\gamma_{p,q}-\gamma_{a',b'}-1})}, \quad (2.1)$$

where  $\gamma_{p,q}$  and  $\gamma_{a',b'}$  are as in (1.3). In particular,

$$\|u\|_{L^p(\mathbb{R}, \dot{B}_q^{\gamma-\gamma_{p,q}})} \lesssim \|u_0\|_{\dot{H}^\gamma} + \|F\|_{L^1(\mathbb{R}, \dot{H}^\gamma)}. \quad (2.2)$$

Here  $(a, a')$  and  $(b, b')$  are conjugate pairs.

The proof of the above result is based on the scaling argument and the Fourier transform of spherical measure.

**Corollary 2.2.** *Let  $d \geq 2$  and  $\gamma \in \mathbb{R}$ . If  $u$  is a (weak) solution to the linear half-wave equation for some data  $u_0, F$ , then for all  $(p, q)$  admissible satisfying  $q < \infty$ ,*

$$\|u\|_{L^p(\mathbb{R}, H_q^{\gamma-\gamma_{p,q}})} \lesssim \|u_0\|_{H^\gamma} + \|F\|_{L^1(\mathbb{R}, H^\gamma)}. \quad (2.3)$$

*Proof.* We firstly remark that (2.2) together with the Littlewood-Paley theorem yield for any  $(p, q)$  admissible satisfying  $q < \infty$ ,

$$\|u\|_{L^p(\mathbb{R}, \dot{H}_q^{\gamma-\gamma_{p,q}})} \lesssim \|u_0\|_{\dot{H}^\gamma} + \|F\|_{L^1(\mathbb{R}, \dot{H}^\gamma)}. \quad (2.4)$$

We next write  $\|u\|_{L^p(\mathbb{R}, H_q^{\gamma-\gamma_{p,q}})} = \|\langle \Lambda \rangle^{\gamma-\gamma_{p,q}} u\|_{L^p(\mathbb{R}, L^q)}$  and apply (2.4) with  $\gamma = \gamma_{p,q}$  to get

$$\|u\|_{L^p(\mathbb{R}, H_q^{\gamma-\gamma_{p,q}})} \lesssim \|\langle \Lambda \rangle^{\gamma-\gamma_{p,q}} u_0\|_{\dot{H}^{\gamma_{p,q}}} + \|\langle \Lambda \rangle^{\gamma-\gamma_{p,q}} F\|_{L^1(\mathbb{R}, \dot{H}^{\gamma_{p,q}})}.$$

The estimate (2.3) then follows by using the fact that  $\gamma_{p,q} > 0$  for all  $(p, q)$  is admissible satisfying  $q < \infty$ .  $\square$

### 2.2 Nonlinear estimates

In this subsection, we recall some nonlinear fractional derivatives estimates related to our purpose. Let us start with the following fractional Leibniz rule (or Kato-Ponce inequality). We refer the reader to [17] for the proof of a more general result.

**Proposition 2.3.** *Let  $\gamma \geq 0, 1 < r < \infty$  and  $1 < p_1, p_2, q_1, q_2 \leq \infty$  satisfying*

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}.$$

Then there exists  $C = C(d, \gamma, r, p_1, q_1, p_2, q_2) > 0$  such that for all  $u, v \in \mathcal{S}$ ,

$$\|\Lambda^\gamma(uv)\|_{L^r} \leq C \left( \|\Lambda^\gamma u\|_{L^{p_1}} \|v\|_{L^{q_1}} + \|u\|_{L^{p_2}} \|\Lambda^\gamma v\|_{L^{q_2}} \right), \quad (2.5)$$

$$\|\langle \Lambda \rangle^\gamma(uv)\|_{L^r} \leq C \left( \|\langle \Lambda \rangle^\gamma u\|_{L^{p_1}} \|v\|_{L^{q_1}} + \|u\|_{L^{p_2}} \|\langle \Lambda \rangle^\gamma v\|_{L^{q_2}} \right). \quad (2.6)$$

We also have the following fractional chain rule.

**Proposition 2.4.** *Let  $F \in C^1(\mathbb{C}, \mathbb{C})$  and  $G \in C(\mathbb{C}, \mathbb{R}^+)$  such that  $F(0) = 0$  and*

$$|F'(\theta z + (1 - \theta)\zeta)| \leq \mu(\theta)(G(z) + G(\zeta)), \quad z, \zeta \in \mathbb{C}, \quad 0 \leq \theta \leq 1,$$

where  $\mu \in L^1((0, 1))$ . Then for  $\gamma \in (0, 1)$  and  $1 < r, p < \infty$ ,  $1 < q \leq \infty$  satisfying

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q},$$

there exists  $C = C(d, \mu, \gamma, r, p, q) > 0$  such that for all  $u \in \mathcal{S}$ ,

$$\|\Lambda^\gamma F(u)\|_{L^r} \leq C \|F'(u)\|_{L^q} \|\Lambda^\gamma u\|_{L^p}, \quad (2.7)$$

$$\|\langle \Lambda \rangle^\gamma F(u)\|_{L^r} \leq C \|F'(u)\|_{L^q} \|\langle \Lambda \rangle^\gamma u\|_{L^p}. \quad (2.8)$$

We refer to [8] (see also [26]) for the proof of (2.7) and Proposition 5.1 of [28] for (2.8). Combining the fractional Leibniz rule and the fractional chain rule, one has the following result (see the appendix of [20]).

**Lemma 2.5.** *Let  $F \in C^k(\mathbb{C}, \mathbb{C})$ ,  $k \in \mathbb{N} \setminus \{0\}$ . Assume that there is  $\nu \geq k$  such that*

$$|D^i F(z)| \leq C |z|^{\nu-i}, \quad z \in \mathbb{C}, \quad i = 1, 2, \dots, k.$$

Then for  $\gamma \in [0, k]$  and  $1 < r, p < \infty$ ,  $1 < q \leq \infty$  satisfying  $\frac{1}{r} = \frac{1}{p} + \frac{\nu-1}{q}$ , there exists  $C = C(d, \nu, \gamma, r, p, q) > 0$  such that for all  $u \in \mathcal{S}$ ,

$$\|\Lambda^\gamma F(u)\|_{L^r} \leq C \|u\|_{L^q}^{\nu-1} \|\Lambda^\gamma u\|_{L^p}, \quad (2.9)$$

$$\|\langle \Lambda \rangle^\gamma F(u)\|_{L^r} \leq C \|u\|_{L^q}^{\nu-1} \|\langle \Lambda \rangle^\gamma u\|_{L^p}. \quad (2.10)$$

Moreover, if  $F$  is a homogeneous polynomial in  $u$  and  $\bar{u}$ , then (2.9) and (2.10) hold true for any  $\gamma \geq 0$ .

**Corollary 2.6.** *Let  $F(z) = |z|^{\nu-1} z$  with  $\nu > 1$ ,  $\gamma \geq 0$  and  $1 < r, p < \infty$ ,  $1 < q \leq \infty$  satisfying  $\frac{1}{r} = \frac{1}{p} + \frac{\nu-1}{q}$ .*

- (i) *If  $\nu$  is an odd integer or  $^1 \lceil \gamma \rceil \leq \nu$  otherwise, then there exists  $C = C(d, \nu, \gamma, r, p, q) > 0$  such that for all  $u \in \mathcal{S}$ ,*

$$\|F(u)\|_{\dot{H}^\gamma} \leq C \|u\|_{L^q}^{\nu-1} \|u\|_{\dot{H}^p}.$$

*A similar estimate holds with  $\dot{H}_r^\gamma, \dot{H}_p^\gamma$ -norms are replaced by  $H_r^\gamma, H_p^\gamma$ -norms respectively.*

- (ii) *If  $\nu$  is an odd integer or  $\lceil \gamma \rceil \leq \nu - 1$  otherwise, then there exists  $C = C(d, \nu, \gamma, r, p, q) > 0$  such that for all  $u, v \in \mathcal{S}$ ,*

$$\begin{aligned} \|F(u) - F(v)\|_{\dot{H}^\gamma} &\leq C \left( (\|u\|_{L^q}^{\nu-1} + \|v\|_{L^q}^{\nu-1}) \|u - v\|_{\dot{H}^p} \right. \\ &\quad \left. + (\|u\|_{L^q}^{\nu-2} + \|v\|_{L^q}^{\nu-2}) (\|u\|_{\dot{H}^p} + \|v\|_{\dot{H}^p}) \|u - v\|_{L^q} \right). \end{aligned}$$

*A similar estimate holds with  $\dot{H}_r^\gamma, \dot{H}_p^\gamma$ -norms are replaced by  $H_r^\gamma, H_p^\gamma$ -norms respectively.*

The next result will give a good control on the nonlinear term which allows us to use the contraction mapping argument.

**Lemma 2.7.** *Let  $\nu$  be as in Theorem 1.2 and  $\gamma_c$  as in (1.1). Then*

$$\|u\|_{L^{\nu-1}(\mathbb{R}, L^\infty)}^{\nu-1} \lesssim \begin{cases} \|u\|_{L^4(\mathbb{R}, \dot{B}_\infty^{\gamma_c - \gamma_4, \infty})}^4 \|u\|_{L^\infty(\mathbb{R}, \dot{B}_2^{\gamma_c})}^{\nu-5} & \text{when } d = 2, \\ \|u\|_{L^p(\mathbb{R}, \dot{B}_{p^*}^{\gamma_c - \gamma_{p, p^*}})}^p \|u\|_{L^\infty(\mathbb{R}, \dot{B}_2^{\gamma_c})}^{\nu-1-p} & \text{where } 2 < p < \nu - 1 \text{ when } d = 3, \\ \|u\|_{L^2(\mathbb{R}, \dot{B}_{2^*}^{\gamma_c - \gamma_{2, 2^*}})}^2 \|u\|_{L^\infty(\mathbb{R}, \dot{B}_2^{\gamma_c})}^{\nu-3} & \text{when } d \geq 4, \end{cases}$$

where  $p^* = 2p/(p-2)$  and  $2^* = 2(d-1)/(d-3)$ .

<sup>1</sup>see (1.4) for the definition of  $\lceil \cdot \rceil$ .

The above lemma follows the same spirit as Lemma 3.5 of [18] (see also [10]) using the argument of Lemma 3.1 of [9].

*Proof.* We only give a sketch of the proof in the case  $d \geq 4$ , the cases  $d = 2, 3$  are treated similarly. By interpolation, we can assume that  $\nu - 1 = m/n > 2$ ,  $m, n \in \mathbb{N}$  with  $\gcd(m, n) = 1$ . We proceed as in [18] and set

$$c_N(t) = N^{\gamma c - \gamma_{2,2^*}} \|P_N u(t)\|_{L^{2^*}(\mathbb{R}^d)}, \quad c'_N(t) = N^{\gamma c} \|P_N u(t)\|_{L^2(\mathbb{R}^d)}.$$

By Bernstein's inequality, we have

$$\begin{aligned} \|P_N u(t)\|_{L^\infty(\mathbb{R}^d)} &\lesssim N^{\frac{d}{2^*} - \gamma c + \gamma_{2,2^*}} c_N(t) = N^{\frac{n}{m} - \frac{1}{2}} c_N(t), \\ \|P_N u(t)\|_{L^\infty(\mathbb{R}^d)} &\lesssim N^{\frac{d}{2} - \gamma c} c'_N(t) = N^{\frac{n}{m}} c'_N(t). \end{aligned} \quad (2.11)$$

This implies that for  $\theta \in (0, 1)$  which will be chosen later,

$$\|P_N u(t)\|_{L^\infty(\mathbb{R}^d)} \lesssim N^{\frac{n}{m} - \frac{\theta}{2}} (c_N(t))^\theta (c'_N(t))^{1-\theta}. \quad (2.12)$$

We next use

$$A(t) := \left( \sum_{N \in 2^{\mathbb{Z}}} \|P_N u(t)\|_{L^\infty(\mathbb{R}^d)} \right)^m \lesssim \sum_{N_1 \geq \dots \geq N_m} \prod_{j=1}^m \|P_{N_j} u(t)\|_{L^\infty(\mathbb{R}^d)}.$$

Estimating the  $n$  highest frequencies by (2.11) and the rest by (2.12), we get

$$A(t) \lesssim \sum_{N_1 \geq \dots \geq N_m} \left( \prod_{j=1}^n N_j^{\frac{n}{m} - \frac{1}{2}} c_{N_j}(t) \right) \left( \prod_{j=n+1}^m N_j^{\frac{n}{m} - \frac{\theta}{2}} (c_{N_j}(t))^\theta (c'_{N_j}(t))^{1-\theta} \right).$$

For an arbitrary  $\delta > 0$ , we set

$$\tilde{c}_N(t) = \sum_{N' \in 2^{\mathbb{Z}}} \min(N/N', N'/N)^\delta c_{N'}(t), \quad \tilde{c}'_N(t) = \sum_{N' \in 2^{\mathbb{Z}}} \min(N/N', N'/N)^\delta c'_{N'}(t).$$

Using the fact that  $c_N(t) \leq \tilde{c}_N(t)$  and  $\tilde{c}_{N_j}(t) \lesssim (N_1/N_j)^\delta \tilde{c}_{N_1}(t)$  for  $j = 2, \dots, m$  and similarly for primes, we see that

$$A(t) \lesssim \sum_{N_1 \geq \dots \geq N_m} \left( \prod_{j=1}^n N_j^{\frac{n}{m} - \frac{1}{2}} (N_1/N_j)^\delta \tilde{c}_{N_1}(t) \right) \left( \prod_{j=n+1}^m N_j^{\frac{n}{m} - \frac{\theta}{2}} (N_1/N_j)^\delta (\tilde{c}_{N_1}(t))^\theta (\tilde{c}'_{N_1}(t))^{1-\theta} \right).$$

We can rewrite the above quantity in the right hand side as

$$\sum_{N_1 \geq \dots \geq N_m} \left( \prod_{j=n+1}^m N_j^{\frac{n}{m} - \frac{\theta}{2} - \delta} \right) \left( \prod_{j=2}^n N_j^{\frac{n}{m} - \frac{1}{2} - \delta} \right) N_1^{\frac{n}{m} - \frac{1}{2} + (m-1)\delta} (\tilde{c}_{N_1}(t))^{n+(m-n)\theta} (\tilde{c}'_{N_1}(t))^{(m-n)(1-\theta)}.$$

By choosing  $\theta = 1/(\nu - 2) \in (0, 1)$  and  $\delta > 0$  so that

$$\frac{n}{m} - \frac{\theta}{2} - \delta > 0, \quad \frac{n}{m} - \frac{1}{2} + (m-1)\delta < 0 \quad \text{or} \quad \delta < \frac{m-2n}{2m(m-1)}.$$

Here condition  $\nu > 3$  ensures that  $m - 2n > 0$ . Summing in  $N_m$ , then in  $N_{m-1}, \dots$ , then in  $N_2$ , we have

$$A(t) \lesssim \sum_{N_1 \in 2^{\mathbb{Z}}} (\tilde{c}_{N_1}(t))^{2n} (\tilde{c}'_{N_1}(t))^{(\nu-3)n}.$$

The Hölder inequality with the fact that  $(\nu - 3)n \geq 1$  implies

$$\begin{aligned} A(t) &\lesssim \|(\tilde{c}(t))^{2n}\|_{\ell^2(2^{\mathbb{Z}})} \|(\tilde{c}'(t))^{(\nu-3)n}\|_{\ell^2(2^{\mathbb{Z}})} \\ &= \|\tilde{c}(t)\|_{\ell^{4n}(2^{\mathbb{Z}})}^{2n} \|\tilde{c}'(t)\|_{\ell^{2(\nu-3)n}(2^{\mathbb{Z}})}^{(\nu-3)n} \leq \|\tilde{c}(t)\|_{\ell^2(2^{\mathbb{Z}})}^{2n} \|\tilde{c}'(t)\|_{\ell^2(2^{\mathbb{Z}})}^{(\nu-3)n}, \end{aligned}$$

where  $\|\tilde{c}(t)\|_{\ell^q(2^{\mathbb{Z}})} := \left(\sum_{N \in 2^{\mathbb{Z}}} |\tilde{c}_N(t)|^q\right)^{1/q}$  and similarly for prime. The Minkowski inequality then implies

$$A(t) \lesssim \|c(t)\|_{\ell^2(2^{\mathbb{Z}})}^{2n} \|c'(t)\|_{\ell^2(2^{\mathbb{Z}})}^{(\nu-3)n}.$$

This implies that  $A(t) < \infty$  for almost all  $t$ , hence that  $\sum_N \|P_N u(t)\|_{L^\infty(\mathbb{R}^d)} < \infty$ . Therefore  $\sum_N \|P_N u(t)\|_{L^\infty(\mathbb{R}^d)}$  converges in  $L^\infty(\mathbb{R}^d)$ . Since it converges to  $u$  in the distribution sense, so the limit is  $u(t)$ . Thus

$$\begin{aligned} \|u\|_{L^{\nu-1}(\mathbb{R}, L^\infty(\mathbb{R}^d))}^{\nu-1} &= \int_{\mathbb{R}} \|u(t)\|_{L^\infty(\mathbb{R}^d)}^{m/n} dt \lesssim \int_{\mathbb{R}} \|c(t)\|_{\ell^2(2^{\mathbb{Z}})}^2 \|c'(t)\|_{\ell^2(2^{\mathbb{Z}})}^{\nu-3} dt \\ &\lesssim \|c\|_{L_{\mathbb{R}}^2 \ell^2(2^{\mathbb{Z}})}^2 \|c'\|_{L_{\mathbb{R}}^\infty \ell^2(2^{\mathbb{Z}})}^{\nu-3} = \|u\|_{L^p(\mathbb{R}, \dot{B}_{2^*}^{\gamma_c - \gamma_{2,2^*}}(\mathbb{R}^d))}^2 \|u\|_{L^\infty(\mathbb{R}, \dot{B}_{2^*}^{\gamma_c}(\mathbb{R}^d))}^{\nu-3}. \end{aligned}$$

The proof is complete.  $\square$

### 2.3 Proof of Theorem 1.1

We now give the proof of Theorem 1.1 by using the standard fixed point argument in a suitable Banach space. Thanks to (1.2), we are able to choose  $p > \max(\nu - 1, 4)$  when  $d = 2$  and  $p > \max(\nu - 1, 2)$  when  $d \geq 3$  such that  $\gamma > d/2 - 1/p$  and then choose  $q \in [2, \infty)$  such that

$$\frac{2}{p} + \frac{d-1}{q} \leq \frac{d-1}{2}.$$

**Step 1.** Existence. Let us consider

$$X := \left\{ u \in L^\infty(I, H^\gamma) \cap L^p(I, H_q^{\gamma - \gamma_{p,q}}) \mid \|u\|_{L^\infty(I, H^\gamma)} + \|u\|_{L^p(I, H_q^{\gamma - \gamma_{p,q}})} \leq M \right\},$$

equipped with the distance

$$d(u, v) := \|u - v\|_{L^\infty(I, L^2)} + \|u - v\|_{L^p(I, H_q^{-\gamma_{p,q}})},$$

where  $I = [0, T]$  and  $M, T > 0$  to be chosen later. By the Duhamel formula, it suffices to prove that the functional

$$\Phi(u)(t) = e^{it\Lambda} u_0 + i\mu \int_0^t e^{i(t-s)\Lambda} |u(s)|^{\nu-1} u(s) ds \quad (2.13)$$

is a contraction on  $(X, d)$ . The Strichartz estimate (2.3) yields

$$\begin{aligned} \|\Phi(u)\|_{L^\infty(I, H^\gamma)} + \|\Phi(u)\|_{L^p(I, H_q^{\gamma - \gamma_{p,q}})} &\lesssim \|u_0\|_{H^\gamma} + \|F(u)\|_{L^1(I, H^\gamma)}, \\ \|\Phi(u) - \Phi(v)\|_{L^\infty(I, L^2)} + \|\Phi(u) - \Phi(v)\|_{L^p(I, H_q^{-\gamma_{p,q}})} &\lesssim \|F(u) - F(v)\|_{L^1(I, L^2)}, \end{aligned}$$

where  $F(u) = |u|^{\nu-1} u$  and similarly for  $F(v)$ . By our assumptions on  $\nu$ , Corollary 2.6 gives

$$\|F(u)\|_{L^1(I, H^\gamma)} \lesssim \|u\|_{L^{\nu-1}(I, L^\infty)}^{\nu-1} \|u\|_{L^\infty(I, H^\gamma)} \lesssim T^{1-\frac{\nu-1}{p}} \|u\|_{L^p(I, L^\infty)}^{\nu-1} \|u\|_{L^\infty(I, H^\gamma)}, \quad (2.14)$$

$$\begin{aligned} \|F(u) - F(v)\|_{L^1(I, L^2)} &\lesssim \left( \|u\|_{L^{\nu-1}(I, L^\infty)}^{\nu-1} + \|v\|_{L^{\nu-1}(I, L^\infty)}^{\nu-1} \right) \|u - v\|_{L^\infty(I, L^2)} \\ &\lesssim T^{1-\frac{\nu-1}{p}} \left( \|u\|_{L^p(I, L^\infty)}^{\nu-1} + \|v\|_{L^p(I, L^\infty)}^{\nu-1} \right) \|u - v\|_{L^\infty(I, L^2)}. \end{aligned} \quad (2.15)$$

The Sobolev embedding with the fact that  $\gamma - \gamma_{p,q} > d/q$  implies  $L^p(I, H_q^{\gamma - \gamma_{p,q}}) \subset L^p(I, L^\infty)$ . Thus, we get

$$\|\Phi(u)\|_{L^\infty(I, H^\gamma)} + \|\Phi(u)\|_{L^p(I, H_q^{\gamma - \gamma_{p,q}})} \lesssim \|u_0\|_{H^\gamma} + T^{1-\frac{\nu-1}{p}} \|u\|_{L^p(I, H_q^{\gamma - \gamma_{p,q}})}^{\nu-1} \|u\|_{L^\infty(I, H^\gamma)},$$

and

$$d(\Phi(u), \Phi(v)) \lesssim T^{1-\frac{\nu-1}{p}} \left( \|u\|_{L^p(I, H_q^{\gamma - \gamma_{p,q}})}^{\nu-1} + \|v\|_{L^p(I, H_q^{\gamma - \gamma_{p,q}})}^{\nu-1} \right) \|u - v\|_{L^\infty(I, L^2)}.$$



This shows that for all  $u, v \in X$ , there exists  $C > 0$  independent of  $u_0 \in H^\gamma$  and  $T$  such that

$$\begin{aligned} \|\Phi(u)\|_{L^\infty(I, H^\gamma)} + \|\Phi(u)\|_{L^p(I, H_q^{\gamma-\gamma p, q})} &\leq C\|u_0\|_{H^\gamma} + CT^{1-\frac{\nu-1}{p}} M^\nu, \\ d(\Phi(u), \Phi(v)) &\leq CT^{1-\frac{\nu-1}{p}} M^{\nu-1} d(u, v). \end{aligned}$$

Therefore, if we set  $M = 2C\|u_0\|_{H^\gamma}$  and choose  $T > 0$  small enough so that  $CT^{1-\frac{\nu-1}{p}} M^{\nu-1} \leq \frac{1}{2}$ , then  $X$  is stable by  $\Phi$  and  $\Phi$  is a contraction on  $X$ . By the fixed point theorem, there exists a unique  $u \in X$  so that  $\Phi(u) = u$ .

**Step 2.** Uniqueness. Consider  $u, v \in C(I, H^\gamma) \cap L^p(I, L^\infty)$  two solutions of (NLHW). Since the uniqueness is a local property (see Chapter 4 of [5]), it suffices to show  $u = v$  for  $T$  is small. We have from (2.15) that

$$d(u, v) \leq CT^{1-\frac{\nu-1}{p}} \left( \|u\|_{L^p(I, L^\infty)}^{\nu-1} + \|v\|_{L^p(I, L^\infty)}^{\nu-1} \right) d(u, v).$$

Since  $\|u\|_{L^p(I, L^\infty)}$  is small if  $T$  is small and similarly for  $v$ , we see that if  $T > 0$  small enough,

$$d(u, v) \leq \frac{1}{2} d(u, v) \text{ or } u = v.$$

**Step 3.** Item (i). Since the time of existence constructed in Step 1 only depends on  $H^\gamma$ -norm of the initial data. The blowup alternative follows by standard argument (see again Chapter 4 of [5]).

**Step 4.** Item (ii). Let  $u_{0,n} \rightarrow u_0$  in  $H^\gamma$  and  $C, T = T(u_0)$  be as in Step 1. Set  $M = 4C\|u_0\|_{H^\gamma}$ . It follows that  $2C\|u_{0,n}\|_{H^\gamma} \leq M$  for sufficiently large  $n$ . Thus the solution  $u_n$  constructed in Step 1 belongs to  $X$  with  $T = T(u_0)$  for  $n$  large enough. We have from Strichartz estimate (2.3) and (2.14) that

$$\|u\|_{L^a(I, H_b^{\gamma-\gamma a, b})} \lesssim \|u_0\|_{H^\gamma} + T^{1-\frac{\nu-1}{p}} \|u\|_{L^p(I, L^\infty)}^{\nu-1} \|u\|_{L^\infty(I, H^\gamma)},$$

provided  $(a, b)$  is admissible and  $b < \infty$ . This shows the boundedness of  $u_n$  in  $L^a(I, H_b^{\gamma-\gamma a, b})$ . We also have from (2.15) and the choice of  $T$  that

$$d(u_n, u) \leq C\|u_{0,n} - u_0\|_{L^2} + \frac{1}{2} d(u_n, u) \text{ or } d(u_n, u) \leq 2C\|u_{0,n} - u_0\|_{L^2}.$$

This yields that  $u_n \rightarrow u$  in  $L^\infty(I, L^2) \cap L^p(I, H_q^{\gamma-\gamma p, q})$ . Strichartz estimate (2.3) again implies that  $u_n \rightarrow u$  in  $L^a(I, H_b^{\gamma-\gamma a, b})$  for any admissible pair  $(a, b)$  with  $b < \infty$ . The convergence in  $C(I, H^{\gamma-\epsilon})$  follows from the boundedness in  $L^\infty(I, H^\gamma)$ , the convergence in  $L^\infty(I, L^2)$  and that  $\|u\|_{H^{\gamma-\epsilon}} \leq \|u\|_{H^\gamma}^{1-\frac{\epsilon}{\gamma}} \|u\|_{L^2}^{\frac{\epsilon}{\gamma}}$ .

**Step 5.** Item (iii). If  $u_0 \in H^\beta$  for some  $\beta > \gamma$  satisfying  $\lceil \beta \rceil \leq \nu$  if  $\nu > 1$  is not an odd integer, then Step 1 shows the existence of  $H^\beta$  solution defined on some maximal interval  $[0, T)$ . Since  $H^\beta$  solution is also a  $H^\gamma$  solution, thus  $T \leq T^*$ . Suppose that  $T < T^*$ . Then the unitary property of  $e^{it\Delta}$  and Lemma imply that

$$\|u(t)\|_{H^\beta} \leq \|u_0\|_{H^\beta} + C \int_0^t \|u(s)\|_{L^\infty}^{\nu-1} \|u(s)\|_{H^\beta} ds,$$

for all  $0 \leq t < T$ . The Gronwall's inequality then gives

$$\|u(t)\|_{H^\beta} \leq \|u_0\|_{H^\beta} \exp \left( C \int_0^t \|u(s)\|_{L^\infty}^{\nu-1} ds \right),$$

for all  $0 \leq t < T$ . Using the fact that  $u \in L_{\text{loc}}^{\nu-1}([0, T^*), L^\infty)$ , we see that  $\limsup \|u(t)\|_{H^\beta} < \infty$  as  $t \rightarrow T$  which is a contradiction to the blowup alternative in  $H^\beta$ .  $\square$

**Remark 2.8.** If we assume that  $\nu > 1$  is an odd integer or

$$\lceil \gamma \rceil \leq \nu - 1$$

otherwise, then the continuous dependence holds in  $C(I, H^\gamma)$ . To see this, we consider  $X$  as above equipped with the following metric

$$d(u, v) := \|u - v\|_{L^\infty(I, H^\gamma)} + \|u - v\|_{L^p(I, H_q^{\gamma - \gamma p, q})}.$$

Using Item (ii) of Corollary 2.6, we have

$$\begin{aligned} \|F(u) - F(v)\|_{L^1(I, H^\gamma)} &\lesssim (\|u\|_{L^{\nu-1}(I, L^\infty)}^{\nu-1} + \|v\|_{L^{\nu-1}(I, L^\infty)}^{\nu-1}) \|u - v\|_{L^\infty(I, H^\gamma)} \\ &\quad + (\|u\|_{L^{\nu-1}(I, L^\infty)}^{\nu-2} + \|v\|_{L^{\nu-1}(I, L^\infty)}^{\nu-2}) (\|u\|_{L^\infty(I, H^\gamma)} + \|v\|_{L^\infty(I, H^\gamma)}) \|u - v\|_{L^{\nu-1}(I, L^\infty)}. \end{aligned}$$

The Sobolev embedding then implies for all  $u, v \in X$ ,

$$d(\Phi(u), \Phi(v)) \lesssim T^{1 - \frac{\nu-1}{p}} M^{\nu-1} d(u, v).$$

Therefore, the continuity in  $C(I, H^\gamma)$  follows as in Step 4.

## 2.4 Proof of Theorem 1.2

We now turn to the proof of the local well-posedness and small data scattering in critical case by following the same argument as in [10].

**Step 1.** Existence. We only treat for  $d \geq 4$ , the ones for  $d = 2, d = 3$  are completely similar. Let us consider

$$X := \left\{ u \in L^\infty(I, H^{\gamma_c}) \cap L^2(I, B_{2^*}^{\gamma_c - \gamma_2, 2^*}) \mid \|u\|_{L^\infty(I, \dot{H}^{\gamma_c})} \leq M, \|u\|_{L^2(I, \dot{B}_{2^*}^{\gamma_c - \gamma_2, 2^*})} \leq N \right\},$$

equipped with the distance

$$d(u, v) := \|u - v\|_{L^\infty(I, L^2)} + \|u - v\|_{L^2(I, \dot{B}_{2^*}^{-\gamma_2, 2^*})},$$

where  $I = [0, T]$  and  $T, M, N > 0$  will be chosen later. One can check (see e.g. [4] or Chapter 4 of [5]) that  $(X, d)$  is a complete metric space. We will show that the functional

$$\Phi(u)(t) = e^{it\Lambda} u_0 + i\mu \int_0^t e^{i(t-s)\Lambda} |u(s)|^{\nu-1} u(s) ds =: u_{\text{hom}}(t) + u_{\text{inh}}(t), \quad (2.16)$$

is a contraction on  $(X, d)$ . The Strichartz estimate (2.2) yields

$$\|u_{\text{hom}}\|_{L^2(I, \dot{B}_{2^*}^{\gamma_c - \gamma_2, 2^*})} \lesssim \|u_0\|_{\dot{H}^{\gamma_c}}. \quad (2.17)$$

We see that  $\|u_{\text{hom}}\|_{L^2(I, \dot{B}_{2^*}^{\gamma_c - \gamma_2, 2^*})} \leq \varepsilon$  for some  $\varepsilon > 0$  small enough which will be chosen later, provided that either  $\|u_0\|_{\dot{H}^{\gamma_c}}$  is small or it is satisfied for some  $T > 0$  small enough by the dominated convergence theorem. Therefore, we can take  $T = \infty$  in the first case and  $T$  be this small time in the second. A similar estimate to (2.17) holds for  $\|u_{\text{hom}}\|_{L^\infty(I, \dot{H}^{\gamma_c})}$ . On the other hand, using again (2.2), we have

$$\|u_{\text{inh}}\|_{L^2(I, \dot{B}_{2^*}^{\gamma_c - \gamma_2, 2^*})} \lesssim \|F(u)\|_{L^1(I, \dot{H}^{\gamma_c})}.$$

A same estimate holds for  $\|u_{\text{inh}}\|_{L^\infty(I, \dot{H}^{\gamma_c})}$ . Corollary 2.6 and Lemma 2.7 give

$$\|F(u)\|_{L^1(I, \dot{H}^{\gamma_c})} \lesssim \|u\|_{L^{\nu-1}(I, L^\infty)}^{\nu-1} \|u\|_{L^\infty(I, \dot{H}^{\gamma_c})} \lesssim \|u\|_{L^2(I, \dot{B}_{2^*}^{\gamma_c - \gamma_2, 2^*})}^2 \|u\|_{L^\infty(I, \dot{H}^{\gamma_c})}^{\nu-2}. \quad (2.18)$$

Similarly, we have

$$\begin{aligned} \|F(u) - F(v)\|_{L^1(I, L^2)} &\lesssim \left( \|u\|_{L^{\nu-1}(I, L^\infty)}^{\nu-1} + \|v\|_{L^{\nu-1}(I, L^\infty)}^{\nu-1} \right) \|u - v\|_{L^\infty(I, L^2)} \\ &\lesssim \left( \|u\|_{L^2(I, \dot{B}_{2^*}^{\gamma_c - \gamma_2, 2^*})}^2 \|u\|_{L^\infty(I, \dot{H}^{\gamma_c})}^{\nu-3} + \|v\|_{L^2(I, \dot{B}_{2^*}^{\gamma_c - \gamma_2, 2^*})}^2 \|v\|_{L^\infty(I, \dot{H}^{\gamma_c})}^{\nu-3} \right) \|u - v\|_{L^\infty(I, L^2)}. \end{aligned} \quad (2.19)$$

This implies for all  $u, v \in X$ , there exists  $C > 0$  independent of  $u_0 \in H^{\gamma_c}$  such that

$$\begin{aligned} \|\Phi(u)\|_{L^2(I, \dot{B}_{2^*}^{\gamma_c - \gamma_2, 2^*})} &\leq \varepsilon + CN^2 M^{\nu-2}, \\ \|\Phi(u)\|_{L^\infty(I, \dot{H}^{\gamma_c})} &\leq C\|u_0\|_{\dot{H}^{\gamma_c}} + CN^2 M^{\nu-2}, \\ d(\Phi(u), \Phi(v)) &\leq CN^2 M^{\nu-3} d(u, v). \end{aligned}$$

Now by setting  $N = 2\varepsilon$  and  $M = 2C\|u_0\|_{\dot{H}^{\gamma_c}}$  and choosing  $\varepsilon > 0$  small enough such that  $CN^2 M^{\nu-3} \leq \min\{1/2, \varepsilon/M\}$ , we see that  $X$  is stable by  $\Phi$  and  $\Phi$  is a contraction on  $X$ . By the fixed point theorem, there exists a unique solution  $u \in X$  to (NLHW). Note that when  $\|u_0\|_{\dot{H}^{\gamma_c}}$  is small enough, we can take  $T = \infty$ .

**Step 2.** Uniqueness. The uniqueness in  $C^\infty(I, H^{\gamma_c}) \cap L^2(I, B_{2^*}^{\gamma_c - \gamma_2, 2^*})$  follows as in Step 2 of the proof of Theorem 1.1 using (2.19). Here  $\|u\|_{L^2(I, \dot{B}_{2^*}^{\gamma_c - \gamma_2, 2^*})}$  can be small as  $T$  is small.

**Step 3.** Scattering. The global existence when  $\|u_0\|_{\dot{H}^{\gamma_c}}$  is small is given in Step 1. It remains to show the scattering property. Thanks to (2.18), we see that

$$\begin{aligned} \|e^{-it_2\Lambda}u(t_2) - e^{-it_1\Lambda}u(t_1)\|_{\dot{H}^{\gamma_c}} &= \left\| i\mu \int_{t_1}^{t_2} e^{-is\Lambda} (|u|^{\nu-1}u)(s) ds \right\|_{\dot{H}^{\gamma_c}} \\ &\leq \|F(u)\|_{L^1([t_1, t_2], \dot{H}^{\gamma_c})} \lesssim \|u\|_{L^2([t_1, t_2], \dot{B}_{2^*}^{\gamma_c - \gamma_2, 2^*})}^2 \|u\|_{L^\infty([t_1, t_2], \dot{H}^{\gamma_c})}^{\nu-2} \rightarrow 0 \end{aligned} \quad (2.20)$$

as  $t_1, t_2 \rightarrow +\infty$ . We have from (2.19) that

$$\|e^{-it_2\Lambda}u(t_2) - e^{-it_1\Lambda}u(t_1)\|_{L^2} \lesssim \|u\|_{L^2([t_1, t_2], \dot{B}_{2^*}^{\gamma_c - \gamma_2, 2^*})}^2 \|u\|_{L^\infty([t_1, t_2], \dot{H}^{\gamma_c})}^{\nu-3} \|u\|_{L^\infty([t_1, t_2], L^2)}, \quad (2.21)$$

which also tends to zero as  $t_1, t_2 \rightarrow +\infty$ . This implies that the limit

$$u_0^+ := \lim_{t \rightarrow +\infty} e^{-it\Lambda}u(t)$$

exists in  $H^{\gamma_c}$ . Moreover, we have

$$u(t) - e^{it\Lambda}u_0^+ = -i\mu \int_t^{+\infty} e^{i(t-s)\Lambda} F(u(s)) ds.$$

The unitary property of  $e^{it\Lambda}$  in  $L^2$ , (2.20) and (2.21) imply that  $\|u(t) - e^{it\Lambda}u_0^+\|_{H^{\gamma_c}} \rightarrow 0$  when  $t \rightarrow +\infty$ . This completes the proof of Theorem 1.2.  $\square$

### 3 Ill-posedness

In this section, we will give the proof of Theorem 1.3. We follow closely the argument of [7] using small dispersion analysis and decoherence arguments.

#### 3.1 Small dispersion analysis

Now, let us consider for  $0 < \delta \ll 1$  the following equation

$$\begin{cases} i\partial_t \phi(t, x) + \delta \Lambda \phi(t, x) &= -\mu |\phi|^{\nu-1} \phi(t, x), & (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\ \phi(0, x) &= \phi_0(x), & x \in \mathbb{R}^d. \end{cases} \quad (3.1)$$

Note that (3.1) can be transformed back to (NLHW) by using

$$u(t, x) := \phi(t, \delta x).$$

**Lemma 3.1.** *Let  $k > d/2$  be an integer. If  $\nu$  is not an odd integer, then we assume also the additional regularity condition  $\nu \geq k + 1$ . Let  $\phi_0$  be a Schwartz function. Then there exists  $C, c > 0$  such that if  $0 < \delta \leq c$  sufficiently small, then there exists a unique solution  $\phi^{(\delta)} \in C([-T, T], H^k)$  of (3.1) with  $T = c|\log \delta|^c$  satisfying*

$$\|\phi^{(\delta)}(t) - \phi^{(0)}(t)\|_{H^k} \leq C\delta^{1/2}, \quad (3.2)$$

for all  $|t| \leq c|\log \delta|^c$ , where

$$\phi^{(0)}(t, x) := \phi_0(x) \exp(-i\mu t |\phi_0(x)|^{\nu-1})$$

is the solution of (3.1) with  $\delta = 0$ .

*Proof.* We refer to Lemma 2.1 of [7], where the small dispersion analysis is invented to prove the ill-posedness for the nonlinear Schrödinger equation. The same proof can be applied to the nonlinear half-wave equation without any difficulty. By using the energy method, we end up with the following estimate

$$\|\phi^{(\delta)}(t) - \phi^{(0)}(t)\|_{H^k} \leq C\delta \exp(C(1 + |t|)^C).$$

Thus, if  $|t| \leq c|\log \delta|^c$  for suitably small  $0 < \delta \leq c$ , then  $\exp(C(1 + |t|)^C) \leq \delta^{-1/2}$  and (3.2) follows.  $\square$

**Remark 3.2.** By the same argument as in [7], we can get the following better estimate

$$\|\phi^{(\delta)}(t) - \phi^{(0)}(t)\|_{H^{k,k}} \leq C\delta^{1/2}, \quad (3.3)$$

for all  $|t| \leq c|\log \delta|^c$ , where  $H^{k,k}$  is the weighted Sobolev space

$$\|\phi\|_{H^{k,k}} := \sum_{|\alpha|=0}^k \|\langle x \rangle^{k-|\alpha|} D^\alpha \phi\|_{L^2}.$$

Now, let  $\lambda > 0$  and set

$$u^{(\delta,\lambda)}(t, x) := \lambda^{-\frac{1}{\nu-1}} \phi^{(\delta)}(\lambda^{-1}t, \lambda^{-1}\delta x). \quad (3.4)$$

It is easy to see that  $u^{(\delta,\lambda)}$  is a solution of (NLHW).

**Lemma 3.3.** *Let  $\gamma \in \mathbb{R}$  and  $0 < \lambda \leq \delta \ll 1$ . Let  $\phi_0 \in \mathcal{S}$  be such that if  $\gamma \leq -d/2$ ,*

$$\hat{\phi}_0(\xi) = O(|\xi|^\kappa) \text{ as } \xi \rightarrow 0,$$

for some  $\kappa > -\gamma - d/2$ , where  $\hat{\cdot}$  is the Fourier transform. Then there exists  $C > 0$  such that

$$\|u^{(\delta,\lambda)}(0)\|_{H^\gamma} \leq C\lambda^{\gamma-\gamma}\delta^{\gamma-d/2}. \quad (3.5)$$

*Proof.* The proof of this lemma is essentially given in [7]. For reader's convenience, we give a sketch of the proof. We firstly have

$$[u^{(\delta,\lambda)}(0)]^\wedge(\xi) = \lambda^{-\frac{1}{\nu-1}} (\lambda\delta^{-1})^d \hat{\phi}_0(\lambda\delta^{-1}\xi).$$

Thus,

$$\begin{aligned} \|u^{(\delta,\lambda)}(0)\|_{H^\gamma}^2 &= \lambda^{-\frac{2}{\nu-1}} (\lambda\delta^{-1})^{2d} \int (1 + |\xi|^2)^\gamma |\hat{\phi}_0(\lambda\delta^{-1}\xi)|^2 d\xi \\ &= \lambda^{-\frac{2}{\nu-1}} (\lambda\delta^{-1})^d \int (1 + |\lambda^{-1}\delta\xi|^2)^\gamma |\hat{\phi}_0(\xi)|^2 d\xi \\ &\sim \lambda^{-\frac{2}{\nu-1}} (\lambda\delta^{-1})^{d-2\gamma} \left( \int_{|\xi| \geq \lambda\delta^{-1}} |\xi|^{2\gamma} |\hat{\phi}_0(\xi)|^2 d\xi + \lambda^{-\frac{8}{\nu-1}} (\lambda\delta^{-1})^d \int_{|\xi| \leq \lambda\delta^{-1}} |\hat{\phi}_0(\xi)|^2 d\xi \right) \\ &= \lambda^{-\frac{2}{\nu-1}} (\lambda\delta^{-1})^{d-2\gamma} \left( \int_{\mathbb{R}^d} |\xi|^{2\gamma} |\hat{\phi}_0(\xi)|^2 d\xi - \int_{|\xi| \leq \lambda\delta^{-1}} ((\lambda\delta^{-1})^{2\gamma} - |\xi|^{2\gamma}) |\hat{\phi}_0(\xi)|^2 d\xi \right). \end{aligned}$$

Using the fact that  $\lambda\delta^{-1} \leq 1$ , we obtain for  $\gamma > -d/2$  that

$$\|u^{(\delta,\lambda)}(0)\|_{H^\gamma} = c\lambda^{-\frac{1}{\nu-1}}(\lambda\delta^{-1})^{d/2-\gamma}(1 + O((\lambda\delta^{-1})^{\gamma+d/2})) \leq C\lambda^{\gamma_c-\gamma}\delta^{\gamma-d/2},$$

where  $c \neq 0$  provided that  $\phi_0$  is not identically zero. Moreover, for  $\gamma \leq -d/2$ , the assumption on  $\hat{\phi}_0$  also implies

$$\|u^{(\delta,\lambda)}(0)\|_{H^\gamma} \leq C\lambda^{\gamma_c-\gamma}\delta^{\gamma-d/2}.$$

Here we use the fact that

$$\int_{|\xi| \leq \lambda\delta^{-1}} ((\lambda\delta^{-1})^{2\gamma} - |\xi|^{2\gamma}) |\hat{\phi}_0(\xi)|^2 d\xi \leq C(\lambda\delta^{-1})^{d+2\gamma+2\kappa} \leq C.$$

This completes the proof of (3.5).  $\square$

### 3.2 Proof of Theorem 1.3

We are now able to prove Theorem 1.3. We only consider the case  $t \geq 0$ , the one for  $t < 0$  is similar. Let  $\epsilon \in (0, 1]$  be fixed and set

$$\lambda^{\gamma_c-\gamma}\delta^{\gamma-d/2} =: \epsilon, \quad (3.6)$$

equivalently

$$\lambda = \delta^\theta, \text{ where } \theta = \frac{d/2 - \gamma}{\gamma_c - \gamma} > 1,$$

hence  $0 < \lambda \leq \delta \ll 1$ . Note that we are considering here  $\gamma < \gamma_c$ . We now split the proof into several cases.

**The case  $0 < \gamma < \gamma_c$ .** Using (3.6), Lemma 3.3 gives

$$\|u^{(\delta,\lambda)}(0)\|_{H^\gamma} \leq C\epsilon.$$

Since the support of  $\phi^{(0)}(t, x)$  is independent of  $t$ , we see that for  $t$  large enough, depending on  $\gamma$ ,

$$\|\phi^{(0)}(t)\|_{H^\gamma} \sim t^\gamma,$$

whenever  $\gamma \geq 0$  provided either  $\nu > 1$  is an odd integer or  $\gamma \leq \nu - 1$  otherwise. Thus for  $\delta \ll 1$  and  $1 \ll t \leq c|\log \delta|^c$ , (3.2) implies

$$\|\phi^{(\delta)}(t)\|_{H^\gamma} \sim t^\gamma. \quad (3.7)$$

We next have

$$[u^{(\delta,\lambda)}(\lambda t)]^\wedge(\xi) = \lambda^{-\frac{1}{\nu-1}}(\lambda\delta^{-1})^d [\phi^{(\delta)}(t)]^\wedge(\lambda\delta^{-1}\xi).$$

This shows that

$$\begin{aligned} \|u^{(\delta,\lambda)}(\lambda t)\|_{H^\gamma}^2 &= \int (1 + |\xi|^2)^\gamma |[u^{(\delta,\lambda)}(\lambda t)]^\wedge(\xi)|^2 d\xi \\ &= \lambda^{-\frac{2}{\nu-1}}(\lambda\delta^{-1})^d \int (1 + |\lambda^{-1}\delta\xi|^2)^\gamma |[\phi^{(\delta)}(t)]^\wedge(\xi)|^2 d\xi \\ &\geq \lambda^{-\frac{2}{\nu-1}}(\lambda\delta^{-1})^{d-2\gamma} \int_{|\xi| \geq 1} |\xi|^{2\gamma} |[\phi^{(\delta)}(t)]^\wedge(\xi)|^2 d\xi \\ &\geq \lambda^{-\frac{2}{\nu-1}}(\lambda\delta^{-1})^{d-2\gamma} \left( c\|\phi^{(\delta)}(t)\|_{H^\gamma}^2 - C\|\phi^{(\delta)}(t)\|_{L^2}^2 \right). \end{aligned}$$

Thanks to (3.7), we have  $\|\phi^{(\delta)}(t)\|_{L^2} \ll \|\phi^{(\delta)}(t)\|_{H^\gamma}$  for  $t \gg 1$ . This yields that

$$\|u^{(\delta,\lambda)}(\lambda t)\|_{H^\gamma} \geq c\lambda^{-\frac{1}{\nu-1}}(\lambda\delta^{-1})^{d/2-\gamma} \|\phi^{(\delta)}(t)\|_{H^\gamma} \geq c\epsilon t^\gamma,$$

for  $1 \ll t \leq c|\log \delta|^c$ . We now choose  $t = c|\log \delta|^c$  and pick  $\delta > 0$  small enough so that

$$\epsilon t^\gamma > \epsilon^{-1}, \quad \lambda t < \epsilon.$$

Therefore, for any  $\epsilon > 0$ , there exists a solution of (NLHW) satisfying

$$\|u(0)\|_{H^\gamma} < \epsilon, \quad \|u(t)\|_{H^\gamma} > \epsilon^{-1}$$

for some  $t \in (0, \epsilon)$ . Thus for any  $t > 0$ , the solution map  $\mathcal{S} \ni u(0) \mapsto u(t)$  for the Cauchy problem (NLHW) fails to be continuous at 0 in the  $H^\gamma$ -topology.

**The case  $\gamma \leq -d/2$  and  $\gamma < \gamma_c$ .** Let  $u^{(\delta, \lambda)}$  be as in (3.4). Thanks to (3.6), Lemma 3.3 implies

$$\|u^{(\delta, \lambda)}(0)\|_{H^\gamma} \leq C\epsilon,$$

provided  $0 < \lambda \leq \delta \ll 1$  and  $\phi_0 \in \mathcal{S}$  satisfying

$$\hat{\phi}_0(\xi) = O(|\xi|^\kappa) \text{ as } \xi \rightarrow 0,$$

for some  $\kappa > -\gamma - d/2$ . We recall that

$$\phi^{(0)}(t, x) = \phi_0(x) \exp(-i\mu t |\phi_0(x)|^{\nu-1}).$$

It is clear that we can choose  $\phi_0$  so that

$$\left| \int \phi^{(0)}(1, x) dx \right| \geq c \text{ or } |[\phi^{(0)}(1)]^\wedge(0)| \geq c,$$

for some constant  $c > 0$ . Since  $\phi^{(0)}(1)$  is rapidly decreasing, the continuity implies that

$$|[\phi^{(0)}(1)]^\wedge(\xi)| \geq c,$$

for  $|\xi| \leq c$  with  $0 < c \ll 1$ . On the other hand, using (3.3) (note that  $H^{k,k}$  controls  $L^1$  when  $k > d/2$ ), we have

$$|[\phi^{(\delta)}(1)]^\wedge(\xi) - [\phi^{(0)}(1)]^\wedge(\xi)| \leq C\delta^{1/2},$$

and then

$$|[\phi^{(\delta)}(1)]^\wedge(\xi)| \geq c,$$

for  $|\xi| \leq c$  provided  $\delta$  is taken small enough. Moreover, we have

$$u^{(\delta, \lambda)}(\lambda, x) = \lambda^{-\frac{1}{\nu-1}} \phi^{(\delta)}(1, \lambda^{-1} \delta x)$$

and

$$[u^{(\delta, \lambda)}(\lambda)]^\wedge(\xi) = \lambda^{-\frac{1}{\nu-1}} (\lambda \delta^{-1})^d [\phi^{(\delta)}(1)]^\wedge(\lambda \delta^{-1} \xi).$$

This implies that

$$[u^{(\delta, \lambda)}(\lambda)]^\wedge(\xi) \geq c \lambda^{-\frac{1}{\nu-1}} (\lambda \delta^{-1})^d,$$

for  $|\xi| \leq c \lambda^{-1} \delta$ .

In the case  $\gamma < -d/2$ , we have

$$\|u^{(\delta, \lambda)}(\lambda)\|_{H^\gamma} \geq c \lambda^{-\frac{1}{\nu-1}} (\lambda \delta^{-1})^d = c \epsilon (\lambda \delta^{-1})^{\gamma+d/2}.$$

Here  $0 < \lambda \leq \delta \ll 1$ , thus  $(\lambda \delta^{-1})^{\gamma+d/2} \rightarrow +\infty$ . We can choose  $\delta$  small enough so that  $\lambda \rightarrow 0$  and  $(\lambda \delta^{-1})^{\gamma+d/2} \geq \epsilon^{-2}$  or

$$\|u^{(\delta, \lambda)}(\lambda)\|_{H^\gamma} \geq \epsilon^{-1}.$$

In the case  $\gamma = -d/2$ , we have

$$\begin{aligned} \|u^{(\delta, \lambda)}(\lambda)\|_{H^{-d/2}} &\geq c \lambda^{-\frac{1}{\nu-1}} (\lambda \delta^{-1})^d \left( \int_{|\xi| \leq c \lambda^{-1} \delta} (1 + |\xi|)^{-d} d\xi \right)^{1/2} \\ &= c \lambda^{-\frac{1}{\nu-1}} (\lambda \delta^{-1})^d (\log(c \lambda^{-1} \delta))^{1/2} \\ &= c \epsilon (\log(c \lambda^{-1} \delta))^{1/2}. \end{aligned}$$

By choosing  $\delta$  small enough so that  $\lambda \rightarrow 0$  and  $\log(c \lambda^{-1} \delta) \geq \epsilon^{-4}$ , we see that

$$\|u^{(\delta, \lambda)}(\lambda)\|_{H^{-d/2}} \geq \epsilon^{-1}.$$

Combining both cases, we see that the solution map fails to be continuous at 0 in  $H^\gamma$ -topology.

**The case  $\gamma = 0 < \gamma_c$ .** Let  $a, a' \in [1/2, 2]$ . Let  $\phi^{(a,\delta)}$  be the solution to (3.1) with initial data

$$\phi^{(a,\delta)}(0) = a\phi_0.$$

Then, Lemma 3.1 gives

$$\|\phi^{(a,\delta)}(t) - \phi^{(a,0)}(t)\|_{H^k} \leq C\delta^{1/2}, \quad (3.8)$$

for all  $|t| \leq c|\log \delta|^c$ , where

$$\phi^{(a,0)}(t, x) = a\phi_0(x) \exp(-i\mu a^{\nu-1}t|\phi_0(x)|^{\nu-1}) \quad (3.9)$$

is the solution of (3.1) with  $\delta = 0$  and the same initial data as  $\phi^{(a,\delta)}$ . Note that since  $a$  belongs to a compact set, then the constant  $C, c$  can be taken to be independent of  $a$ . We next define

$$u^{(a,\delta,\lambda)}(t, x) := \lambda^{-\frac{1}{\nu-1}} \phi^{(a,\delta)}(\lambda^{-1}t, \lambda^{-1}\delta x). \quad (3.10)$$

It is easy to see that  $u^{(a,\delta,\lambda)}$  is also a solution of (NLHW). Using (3.9), a direct computation shows that

$$\|\phi^{(a,0)}(t) - \phi^{(a',0)}(t)\|_{L^2} \geq c > 0,$$

for some time  $t$  satisfying  $|a - a'|^{-1} \leq t \leq c|\log \delta|^c$  provided that  $\delta$  is small enough so that  $c|\log \delta|^c \geq |a - a'|^{-1}$ . The triangle inequality together with (3.8) yields

$$\|\phi^{(a,\delta)}(t) - \phi^{(a',\delta)}(t)\|_{L^2} \geq c,$$

for all  $|a - a'|^{-1} \leq t \leq c|\log \delta|^c$ . Now let  $\epsilon$  be as in (3.6), i.e.

$$\lambda^{-\frac{1}{\nu-1}} (\lambda\delta^{-1})^{d/2} =: \epsilon,$$

or  $\lambda = \delta^\theta$  with  $\theta = \frac{d/2}{\gamma_c} > 1$ . Moreover, using the fact

$$[u^{(a,\delta,\lambda)}(\lambda t)]^\wedge(\xi) = \lambda^{-\frac{1}{\nu-1}} (\lambda\delta^{-1})^d [\phi^{(a,\delta)}(t)]^\wedge(\lambda\delta^{-1}\xi),$$

we have

$$\|u^{(a,\delta,\lambda)}(\lambda t) - u^{(a',\delta,\lambda)}(\lambda t)\|_{L^2} = \lambda^{-\frac{1}{\nu-1}} (\lambda\delta^{-1})^{d/2} \|\phi^{(a,\delta)}(t) - \phi^{(a',\delta)}(t)\|_{L^2} \geq c\epsilon.$$

Similarly, using that

$$[u^{(a,\delta,\lambda)}(0)]^\wedge(\xi) = a\lambda^{-\frac{1}{\nu-1}} (\lambda\delta^{-1})^d \hat{\phi}_0(\lambda\delta^{-1}\xi),$$

we have

$$\|u^{(a,\delta,\lambda)}(0)\|_{L^2}, \|u^{(a',\delta,\lambda)}(0)\|_{L^2} \leq C\epsilon,$$

and

$$\|u^{(a,\delta,\lambda)}(0) - u^{(a',\delta,\lambda)}(0)\|_{L^2} \leq C\epsilon|a - a'|.$$

Since  $|a - a'|$  can be arbitrarily small, this shows that for any  $0 < \epsilon, \sigma < 1$  and for any  $t > 0$ , there exist  $u_1, u_2$  solutions of (NLHW) with initial data  $u_1(0), u_2(0) \in \mathcal{S}$  such that

$$\|u_1(0)\|_{L^2}, \|u_2(0)\|_{L^2} \leq C\epsilon, \quad \|u_1(0) - u_2(0)\|_{L^2} \leq C\sigma, \quad \|u_1(t) - u_2(t)\|_{L^2} \geq c\epsilon.$$

This shows that the solution map fails to be uniformly continuous on  $L^2$ . This completes the proof of Theorem 1.3.

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