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# Adaptive boundary observer for parabolic PDEs subject to domain and boundary parameter uncertainties

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We are considering the problem of state observation for a class of infinite dimensional systems modeled by parabolic type PDEs. The model is subject to parametric uncertainty entering in both the domain equation and the boundary condition. An adaptive boundary observer, providing online estimates of the system state and parameters, is designed using finite- and infinite-dimensional backstepping-like transformations. The observer is exponentially convergent under an ad hoc persistent excitation condition.

## 1. Introduction

The problem of observer design for infinite dimensional systems (IDSs) is given an increasing interest. Several observer design methods have been developed including the infinite dimensional Luenberger observer for linear IDSs (e.g. Amann, 1989; Curtain & Zwart, 1995; Lasiecka & Triggiani, 2000), the boundary observer design of bilinear IDSs (e.g. Bounit & Hammouri, 1997; Smyshlyayev & Krstic, 2005; Vries, Keesman, & Zwart, 2007; Xu, Ligaius, & Gauthier, 1995), backstepping-like boundary observers for parabolic partial integro-differential systems (Smyshlyayev & Krstic, 2005), initial state recovery in finite time of linear and semilinear IDSs using forward and backward observers sequences (Fridman, 2013; Ramdani, Tucsnak, & Weiss, 2010; Tucsnak & Weiss, 2009), sampled-data (in time and space) observers of semilinear parabolic systems designed using Lyapunov functions and LMIs

(Fridman & Blighovsky, 2012). Another important problem in system control is one of estimating unknown parameters. In the case of stable IDSs, this issue can be coped with in open-loop using parameter identification, using variants of the least-squares technique, see Smyshlyayev, Orlov, and Krstic (2009) and references there in. In the case of unstable systems, online parameter estimation is generally involved as part of adaptive controllers (Guo & Guo, 2013; Smyshlyayev & Krstic, 2007a,b). Most adaptive controllers rely on full state measurements, e.g. Bentsman and Orlov (2001), Bohm, Demetriou, Reich, and Rosen (1998), Guo and Guo (2013), Smyshlyayev and Krstic (2007a) and Smyshlyayev and Krstic (2008). Output-feedback adaptive controllers have been proposed for some classes of IDSs including specific parabolic PDEs (Hong & Bentsman, 1994; Smyshlyayev & Krstic, 2007b) and wave PDEs subject to a boundary harmonic disturbance linearly parameterized along a known set of functions (Smyshlyayev & Krstic, 2006). Also, in most adaptive controllers, the asymptotic convergence of the parameter estimates to their true values is not guaranteed. A quite complete description of the literature on adaptive controllers of IDSs systems described by parabolic equations, where both sensing and actuation are performed at the boundary and the unknown parameters are allowed to be spatially varying, can be found in Smyshlyayev and Krstic (2010).

This study is focused on the problem of designing adaptive observers featuring exponential convergence of the state estimate

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and the unknown parameter vector estimate. The problem has recently been addressed in Ahmed-Ali, Giri, Krstic, Burlion, and Lamnabhi-Lagarrigue (2015a); Ahmed-Ali, Giri, Krstic, Lamnabhi-Lagarrigue, and Burlion (2015b) for a class of semilinear parabolic PDEs. In Ahmed-Ali et al. (2015a), an exponentially convergent adaptive observer has been proposed for parabolic PDEs containing a single unknown parameter in the boundary condition. The result of Ahmed-Ali et al. (2015b) is an exponentially convergent adaptive observer of a class of semilinear parabolic PDE subject to domain parameter uncertainty. But, the number of unknown parameters must be equal to the number of available sensors in the domain. That is, if a single boundary sensor is available, only one unknown parameter is allowed to be in the domain. The novelty of the present study is twofold: (i) the parameter uncertainty is allowed to be both in the domain and at the boundary condition; (ii) the domain uncertainty is captured through an unknown parameter vector of arbitrary finite dimension, while only a single boundary sensing is available. It turns out that the adaptive observer problems addressed in Ahmed-Ali et al. (2015a,b) are particular cases of the problem considered here, whenever a single boundary sensing is available. Compared to earlier works on adaptive control or parameter identification (e.g. Guo & Guo, 2013; Smyshlyaev & Krstic, 2007a,b; Smyshlyaev et al., 2009), the present study involves a much wider class of systems, see Remark 1 hereafter. Furthermore, the new adaptive observer enjoys, under an ad hoc persistent excitation condition, exponential convergence while the earlier adaptive identifiers ensure  $L_2$  convergence results. The parameter adaptive law is derived by using a finite-dimensional backstepping-like transformation, as in Ahmed-Ali et al. (2015a,b), and the observer domain varying gain is obtained by using an infinite-dimensional backstepping-like transformation, as in Smyshlyaev and Krstic (2005).

The paper is organized as follows: the observation problem statement is described in Section 2; the adaptive observer design and analysis are dealt with in Sections 3 and 4, respectively; a numerical simulation is made in Section 5; a conclusion and a reference list end the paper.

**Notation.** Throughout the paper,  $\mathbf{R}^n$  is the  $n$  dimensional real space and  $\mathbf{R}^{n \times m}$  is the set of all  $n \times m$  real matrices. The corresponding Euclidean norms are denoted  $\|\cdot\|$ .  $L^2[0, 1]$  is the Hilbert space of square integrable functions defined on the interval  $[0, 1]$  and  $\|\cdot\|$  is the associated  $L^2$ -norm.  $H^1(0, 1)$  is the Sobolev space of absolutely continuous functions  $\eta : [0, 1] \rightarrow \mathbf{R}$  with  $d\eta/d\zeta \in L^2[0, 1]$ . Given a function  $w : [0, 1] \times \mathbf{R}_+ \rightarrow \mathbf{R}$ ;  $(x, t) \rightarrow w(x, t)$ , the notation  $w[t]$  and  $w_x[t]$  refer to the functions defined on  $0 \leq x \leq 1$  by  $(w[t])(x) = w(x, t)$  and  $(w_x[t])(x) = \partial w(x, t)/\partial x$ .

## 2. Observation problem statement

The system under study is described by a parabolic PDE of the following form:

$$u_t(x, t) = u_{xx}(x, t) + \phi(x, t)^T q_1, \quad 0 < x < 1, t > 0 \quad (1a)$$

with the following boundary condition:

$$u_x(0, t) = -q_0 u(0, t), \quad t \geq 0 \quad (1b)$$

where  $\phi$  is a known function of class  $C^1([0, 1] \times [0, \infty); \mathbf{R}^n)$ ,  $q_1 \in \mathbf{R}^n$  and  $q_0 \in \mathbf{R}$  are unknown vector and scalar parameters, respectively. For convenience, the following extended parameter vector is introduced:

$$\theta = \begin{bmatrix} q_0 \\ q_1 \end{bmatrix} \in \mathbf{R}^{n+1}. \quad (1c)$$

The goal is to generate accurate online estimates,  $\hat{u}(x, t)$  and  $\hat{\theta}(t)$ , of the system distributed state  $u(x, t)$  ( $0 \leq x \leq 1$ ;  $t \geq 0$ )

and the parameter vector  $\theta$ , based only on the input and output measurements ( $U(t), y(t)$ ;  $t \geq 0$ ) with

$$U(t) = u(1, t), \quad t \geq 0 \text{ (control signal)} \quad (1d)$$

$$y(t) = u(0, t), \quad t \geq 0 \text{ (system output)}. \quad (1e)$$

To achieve this objective, it is supposed that the state variable  $u(x, t)$  ( $0 \leq x \leq 1$ ;  $t \geq 0$ ) is bounded.

**Remark 1.** (1) In Smyshlyaev et al. (2009), two special forms belonging to the class defined by (1a)–(1b) have been considered. The first corresponds to the case where  $q_0 = 0$ ,  $q_1 \in \mathbf{R}$ , and  $\phi(x, t) = u(0, t)$ . Then, the system involves a single uncertain parameter in the domain. The second special form is such that  $q_0 \in \mathbf{R}$ ,  $q_1 = 0$  and  $\phi(x, t) = 0$ , leading to a single uncertain parameter in the boundary. This second case has also been considered in Ahmed-Ali et al. (2015a) where an adaptive observer was developed.

(2) The class of systems (1a)–(1b) is also quite different from the one studied in Ahmed-Ali et al. (2015b). In the latter, it is supposed that a finite number of sensors are placed in the domain, while only one sensor is required here. It turns out that, in the case where the system in Ahmed-Ali et al. (2015b) contains a single sensor placed at the boundary then it falls in the class (1a)–(1b) with  $q_0 = 0$ ,  $q_1 \in \mathbf{R}$ , and  $\phi(x, t) = \psi(u(0, t), t)$ .

## 3. Adaptive observer design

### 3.1. Observer structure

The system model (1a)–(1e) suggests the following observer structure:

$$\hat{u}_t(x, t) = \hat{u}_{xx}(x, t) + \hat{q}_1^T(t)\phi(x, t) - K(x)(\hat{u}(0, t) - y(t)) + v(x, t) \quad (2a)$$

$$\hat{u}_x(0, t) = -\hat{q}_0(t)u(0, t) \quad (2b)$$

$$\hat{u}(1, t) = U(t) \quad (2c)$$

where  $K(x)$  is a (space-dependent) observer gain,  $\hat{q}_0 \in \mathbf{R}$ ,  $\hat{q}_1 \in \mathbf{R}^n$  are parameter estimates, and  $v(x, t)$  is an additional feedback term that will be determined latter. Let us introduce the following usual estimation errors:

$$\tilde{u}(x, t) = \hat{u}(x, t) - u(x, t) \quad \text{(state estimation error)} \quad (3a)$$

$$\tilde{\theta}(t) = \hat{\theta}(t) - \theta \stackrel{\text{def}}{=} \begin{bmatrix} \tilde{q}_0(t) \\ \tilde{q}_1(t) \end{bmatrix} \quad \text{(parameter estimation error)} \quad (3b)$$

where  $\hat{\theta}(t) = [\hat{q}_0(t) \hat{q}_1(t)]^T$  and

$$\tilde{q}_0(t) = \hat{q}_0(t) - q_0, \quad \tilde{q}_1(t) = \hat{q}_1(t) - q_1. \quad (3c)$$

Then, subtracting Eqs. (1a) to (2a), it follows that  $\tilde{u}(x, t)$  satisfies the following equation:

$$\tilde{u}_t(x, t) = \tilde{u}_{xx}(x, t) + \tilde{q}_1^T(t)\phi(x, t) - K(x)\tilde{u}(0, t) + v(x, t) \quad (4a)$$

with the following boundary conditions, obtained using (1b), (1d) and (2b)–(2c):

$$\tilde{u}_x(0, t) = -\tilde{q}_0(t)u(0, t) \quad (4b)$$

$$\tilde{u}(1, t) = 0. \quad (4c)$$

### 3.2. Finite-dimensional backstepping-like transformation and parameter adaptive law selection

Consider the following finite-dimensional backstepping-like transformation:

$$z(x, t) = \tilde{u}(x, t) - \lambda_0(x, t)\tilde{q}_0(t) - \lambda_1(x, t)\tilde{q}_1(t). \quad (5a)$$

This is equally expressed as follows:

$$z(x, t) = \tilde{u}(x, t) - \Lambda(x, t)\tilde{\theta}(t) \quad (5b)$$

with

$$\Lambda(x, t) = [\lambda_0(x, t) \ \lambda_1(x, t)] \in \mathbf{R}^{1 \times (n+1)} \quad (5c)$$

where  $\lambda_0(x, t) \in \mathbf{R}$  and  $\lambda_1(x, t) \in \mathbf{R}^{1 \times n}$  are auxiliary functions to be defined later. Using (4a), it follows from (5a) that  $z(x, t)$  satisfies the following equation:

$$\begin{aligned} z_t(x, t) &= \tilde{u}_{xx}(x, t) + \tilde{q}_1^T(t)\phi(x, t) - K(x)\tilde{u}(0, t) + v(x, t) \\ &\quad - \Lambda_t(x, t)\tilde{\theta}(t) - \Lambda(x, t)\dot{\tilde{\theta}}(t), \quad t \geq 0. \end{aligned} \quad (6)$$

This immediately suggests the following choice of the feedback expression for  $v(x, t)$ :

$$v(x, t) = \Lambda(x, t)\dot{\tilde{\theta}}(t). \quad (7)$$

Doing so, Eq. (6) leads to

$$\begin{aligned} z_t(x, t) &= \tilde{u}_{xx}(x, t) + \tilde{q}_1^T(t)\phi(x, t) - K(x)\tilde{u}(0, t) \\ &\quad - \Lambda_t(x, t)\tilde{\theta}(t), \quad t \geq 0. \end{aligned} \quad (8)$$

In view of (5b),  $\tilde{u}(x, t)$  is replaced by  $z(x, t) + \Lambda(x, t)\tilde{\theta}(t)$  on the right side of (8). Then, one gets

$$\begin{aligned} z_t(x, t) &= z_{xx}(x, t) + \Lambda_{xx}(x, t)\tilde{\theta}(t) - \Lambda_t(x, t)\tilde{\theta}(t) \\ &\quad + \phi^T(x, t)\tilde{q}_1(t) - K(x)z(0, t) - K(x)\Lambda(0, t)\tilde{\theta}(t) \end{aligned} \quad (9a)$$

which, in view of (3c) and (5c), also writes in the following less compact form:

$$\begin{aligned} z_t(x, t) &= z_{xx}(x, t) - K(x)z(0, t) + (\lambda_{0,xx}(x, t) - K(x)\lambda_0(0, t) \\ &\quad - \lambda_{0,t}(x, t))\tilde{q}_0(t) + (\lambda_{1,xx}(x, t) + \phi(x, t)^T \\ &\quad - K(x)\lambda_1(0, t) - \lambda_{1,t}(x, t))\tilde{q}_1(t). \end{aligned} \quad (9b)$$

Eq. (9b) suggests the following trajectory of the auxiliary states  $\lambda_i(x, t)$  ( $i = 0, 1$ ):

$$\lambda_{0,t}(x, t) = \lambda_{0,xx}(x, t) - K(x)\lambda_0(0, t), \quad (10a)$$

$$\lambda_{1,t}(x, t) = \lambda_{1,xx}(x, t) + \phi(x, t)^T - K(x)\lambda_1(0, t). \quad (10b)$$

This is completed by the following boundary and initial conditions which will prove to be judicious:

$$\lambda_0(1, t) = 0, \quad \lambda_0(x, 0) = 0, \quad \lambda_{0,x}(0, t) = -u(0, t) \quad (10c)$$

$$\lambda_1(1, t) = 0, \quad \lambda_1(x, 0) = 0, \quad \lambda_{1,x}(0, t) = 0. \quad (10d)$$

Doing so, Eq. (9b) simplifies as follows:

$$z_t(x, t) = z_{xx}(x, t) - K(x)z(0, t); \quad t \geq 0. \quad (11a)$$

In view of (10c)–(10d) and (4b)–(4c), one gets from (5b) the following boundary conditions:

$$z(1, t) = \tilde{u}(1, t) - \lambda_0(1, t)\tilde{q}_0(t) - \lambda_1(1, t)\tilde{q}_1(t) = 0 \quad (11b)$$

$$z_x(0, t) = \tilde{u}_x(0, t) - \lambda_{0,x}(0, t)\tilde{q}_0(t) - \lambda_{1,x}(0, t)\tilde{q}_1(t) = 0. \quad (11c)$$

As it will be further discussed in Remark 2, Eq. (5b) suggests the following parameter adaptive law:

$$\dot{\hat{\theta}}(t) = \frac{R(t)\Lambda^T(0, t)}{1 + \Lambda(0, t)\Lambda^T(0, t)}\tilde{u}(0, t) \quad (12a)$$

$$\dot{R}(t) = R(t) - \frac{R(t)\Lambda^T(0, t)\Lambda(0, t)R(t)}{1 + \Lambda(0, t)\Lambda^T(0, t)} \quad (12b)$$

with  $R(t) \in \mathbf{R}^{(n+1) \times (n+1)}$  where the initial conditions  $\hat{\theta}(0) = \theta_0$  and  $R(0) = R_0$  are arbitrarily chosen but  $R_0 = R_0^T > 0$ . The parameter adaptive law (12a)–(12b) is a variant of the least-squares estimator, commonly referred to forgetting-factor least-squares (Ioannou & Sun, 2006). Normalization is used to keep  $R^{-1}(t)$  bounded independently of the properties of  $\Lambda(0, t)$ . Indeed, it is easy to check that (12b) is equivalent to

$$\frac{dR^{-1}}{dt}(t) = -R^{-1}(t) + \frac{\Lambda^T(0, t)\Lambda(0, t)}{1 + \Lambda(0, t)\Lambda^T(0, t)}. \quad (12c)$$

**Remark 2.** The finite-dimensional transformation (5a)–(5c) was introduced in the early nineties for adaptive observer design of ODE systems, see e.g. Besançon (2000) and references therein. Its extension to IDSS, especially those described by parabolic type PDEs, has been derived in Ahmed-Ali et al. (2015a,b).  $\square$

### 3.3. Infinite-dimensional backstepping-like transformation and space-dependent gain selection

The gain  $K(x)$  must be selected so that the error system with states  $(z, \tilde{\theta})$ , described by (11a)–(11c) and (12a)–(12b), is exponentially stable. For convenience, this system is rewritten in the following form involving estimation errors:

$$z_t(x, t) = z_{xx}(x, t) - K(x)z(0, t), \quad t \geq 0 \quad (13a)$$

$$\begin{aligned} \dot{\tilde{\theta}}(t) &= -\frac{R(t)\Lambda^T(0, t)\Lambda(0, t)}{1 + \Lambda(0, t)\Lambda^T(0, t)}\tilde{\theta}(t) \\ &\quad + \frac{R(t)\Lambda^T(0, t)}{1 + \Lambda(0, t)\Lambda^T(0, t)}z(0, t) \end{aligned} \quad (13b)$$

$$\tilde{u}(0, t) = z(0, t) + \Lambda(0, t)\tilde{\theta}(t) \quad (13c)$$

$$z_x(0, t) = 0, \quad z(1, t) = 0 \quad (13d)$$

where the auxiliary state vector  $\Lambda(x, t)$  and the parameter adaptive law gain  $R(t)$  are defined by (10a)–(10d) and (12b), respectively. Clearly, (13a) is a simple copy of (11a); (13c) is obtained using (5b); (13b) is obtained from (12a) and (13c); and (13d) is obtained from (11b)–(11c).

The forthcoming design of the gain  $K(x)$  is performed making use of the backstepping-like approach (Smyshlyaev & Krstic, 2005). The starting point is the following backstepping transformation:

$$z(x, t) = w(x, t) - \int_0^x p(x, y)w(y, t)dy \quad (14)$$

where the kernel function  $p(x, y)$  is defined on  $T = [0, 1] \times [0, 1]$ . The kernel must be selected so that the new state variable  $w(x, t)$  satisfies the following target system equations:

$$w_t(x, t) = w_{xx}(x, t) - cw(x, t), \quad (15a)$$

$$w_x(0, t) = 0, \quad w(1, t) = 0 \quad (15b)$$

where  $c \geq 0$  is a design parameter. Exponential stability of the system (15a)–(15b) is established later (see Proposition 1). Following a by now well established method, see e.g. Smyshlyaev and Krstic (2005), the expression (14) is substituted into (13a) and (13d) which yields the following set of conditions on the kernel

$p(x, y)$ , including a hyperbolic PDE and corresponding boundary conditions:

$$p_{xx}(x, y) - p_{yy}(x, y) = -cp(x, y) \quad (16a)$$

$$p(1, y) = 0 \quad (16b)$$

$$p(x, x) = \frac{c}{2}x. \quad (16c)$$

Another byproduct of the above technique is the following relation:

$$K(x) = p_y(x, 0). \quad (17)$$

Eqs. (16a)–(16c) are shown in Smyshlyaev and Krstic (2004) to have the following explicit solution:

$$p(x, y) = -c(1-x) \frac{I_1(\sqrt{c(x-y)(2-x-y)})}{\sqrt{c(x-y)(2-x-y)}} \quad (18)$$

which, due to (17), gives

$$K(x) = p_y(x, 0) = \frac{c(1-x)}{x(2-x)} I_2(\sqrt{cx(2-x)}) \quad (19)$$

where  $I_n$  denotes the modified Bessel function of order  $n$ . The observer thus designed is constituted of Eqs. (2a)–(2c), (10a)–(10d), (12a)–(12b), and (19). For convenience, the whole observer is summarized in Table 1.

**Remark 3.** (1) The infinite-dimensional transformation (14), inspired by Smyshlyaev and Krstic (2005), is used to derive the observer gain (19). The domain-dependent nature of the gain will prove to be useful in making the observer convergence rate tunable through the design parameter  $c$  in (18)–(19). In Ahmed-Ali et al. (2015b), a domain-independent gain (i.e.  $K(x) = K = \text{const}$ ) was used. Then, the adaptive convergence rate can be set by directly tuning the gain  $K$ . The larger  $K$ , the higher the convergence rate. However, the price paid in Ahmed-Ali et al. (2015b) is the implementation of several sensors along the state domain: the larger  $K$ , the larger the required number of sensors.

(2) A link can be done between the parameter adaptive law in Table 1 and earlier ones using infinite-dimensional backstepping-like transformations, e.g. Smyshlyaev and Krstic (2007a,b). For instance, the filter (20c–20f) in Table 1 are the analogous (though not quite identical) to the filter (62)–(64) in Smyshlyaev and Krstic (2007b) or (62)–(64) in Smyshlyaev and Krstic (2007a). However, the parameter adaptive law (20g–20h) is quite different from the adaptive law (41) in Smyshlyaev and Krstic (2007a) or (61) in Smyshlyaev and Krstic (2007b).

(3) The well posedness of the system (1a)–(1b) and the adaptive observer of Table 1 can be investigated following similar existing analysis. First, it follows applying Lemma 2 in Schaum, Moreno, Fridman, and Alvarez (2013) that, for any function  $\phi(x, t)$  of class  $C^1([0, 1] \times \mathbf{R}_+; \mathbf{R})$ , any external control signal  $U(t) = u(1, t)$  of class  $C^2(\mathbf{R}_+; \mathbf{R})$ , and every  $u[0] \in H^1(0, 1)$ ,  $(u[0])(1) = U(0)$ , the initial value problem (1a)–(1d), with initial condition  $u(x, 0) = (u[0])(x)$  for  $x \in [0, 1]$ , has a strong unique solution in  $H^1(0, 1)$  for all  $t \geq 0$ . Similar results immediately follow for the initial problems (20d–20g), providing the auxiliary states  $(\lambda_0(x, t), \lambda_1(x, t))$ , and (13a), (13d) providing  $z(x, t)$ . Then, the existence and uniqueness of both  $R(t)$  and  $\theta(t)$  are immediately obtained from (20h–20i), applying the usual existence theorem of ODEs. Then, well posedness of the observer (20a–20c) can similarly be stated. Specifically, for every  $u[0], \hat{u}[0] \in H^1(0, 1)$ , with  $(u[0])(1) = (\hat{u}[0])(1) = U(0)$ , the initial value problem (20a–20c), with initial condition  $u(x, 0) = (u[0])(x)$  and  $\hat{u}(x, 0) = (\hat{u}[0])(x)$  for  $x \in [0, 1]$ , has a strong unique solution in  $H^1(0, 1)$  for all  $t \geq 0$ .

**Table 1**  
Adaptive observer.

<i>State observer:</i>	
$\dot{\hat{u}}_t(x, t) = \hat{u}_{xx}(x, t) + \hat{q}_1^T(t)\phi(x, t)$	
$\quad - K(x)(\hat{u}(0, t) - y(t)) + \Lambda(x, t)\dot{\theta}(t)$	(20a)
$\hat{u}_x(0, t) = -\hat{q}_0 u(0, t), \hat{u}(1, t) = U(t)$	(20b)
$K(x) = \frac{c(1-x)}{x(2-x)} I_2(\sqrt{cx(2-x)})$	(20c)
with $y(t) = u(0, t)$ and $\Lambda(x, t) = [\lambda_0(x, t) \lambda_1(x, t)] \in \mathbf{R}^{1 \times (n+1)}$	
<i>Filters:</i>	
$\lambda_{0,t}(x, t) = \lambda_{0,xx}(x, t) - K(x)\lambda_0(0, t); \lambda_0(x, t) \in \mathbf{R}$	(20d)
$\lambda_{1,t}(x, t) = \lambda_{1,xx}(x, t) + \phi(x, t)^T - K(x)\lambda_1(0, t); \lambda_1 \in \mathbf{R}^{1 \times n}$	(20e)
$\lambda_0(1, t) = 0, \lambda_0(x, 0) = 0, \lambda_{0,x}(0, t) = -u(0, t)$	(20f)
$\lambda_{1,1}(t) = 0, \lambda_1(x, 0) = 0, \lambda_{1,x}(0, t) = 0$	(20g)
<i>Parameter estimator:</i>	
$\dot{\theta}(t) = -\frac{R(t)\Lambda^T(0,t)}{1+\Lambda(0,t)\Lambda^T(0,t)}\hat{u}(0, t)$	(20h)
$\dot{R}(t) = R(t) - \frac{R(t)\Lambda^T(0,t)\Lambda(0,t)R(t)}{1+\Lambda(0,t)\Lambda^T(0,t)}$	(20i)
$R(t) \in \mathbf{R}^{(n+1) \times (n+1)}, R(0) = R_0; R_0 = R_0^T > 0$	(20j)

#### 4. Adaptive observer analysis

In the forthcoming analysis, the following Wirtinger's inequalities are repeatedly used (Hardy, Littlewood, & Polya, 1934):

$$\int_0^1 \varphi^2(x) dx \leq \frac{4}{\pi^2} \int_0^1 \varphi_x^2(x) dx \quad (21a)$$

$$\max_{0 \leq x \leq 1} \varphi^2(x) \leq \int_0^1 \varphi_x^2(x) dx \quad (21b)$$

for all  $\varphi \in H^1(0, 1)$  is such that  $\varphi(0) = 0$  or  $\varphi(1) = 0$ .

The first analysis step is to show that the states  $w(x, t)$  and  $z(x, t)$  are exponentially vanishing. To this end, consider the following Lyapunov functional associated to system (15a)–(15b):

$$V_1(t) = \frac{1}{2} \int_0^1 w^2(x, t) dx + \frac{1}{2} \int_0^1 w_x^2(x, t) dx. \quad (22)$$

**Proposition 1.** (1) *The system (15a)–(15b) is exponentially stable with respect to the Lyapunov functional (22). Accordingly,  $\int_0^1 w^2(x, t) dx, \int_0^1 w_x^2(x, t) dx$  and  $\max_{0 \leq x \leq 1} |w(x, t)|$  are all exponentially vanishing (as  $t \rightarrow \infty$ ). Furthermore, the larger the design parameter  $c \geq 0$ , the greater the convergence rate.*

(2) *Consequently,  $\max_{0 \leq x \leq 1} |z(x, t)|$  is exponentially vanishing and its convergence rate depends on  $c$ . The larger  $c$  the speedier the convergence.*

**Proof. Part 1.** Differentiating  $V_1(t)$  yields using (15a)–(15b):

$$\begin{aligned} \dot{V}_1(t) &= \int_0^1 w(x, t) w_t(x, t) dx + \int_0^1 w_x(x, t) w_{x,t}(x, t) dx \\ &= -c \int_0^1 w^2(x, t) dx - \int_0^1 w_x^2(x, t) dx \\ &\quad - \int_0^1 w_{xx}^2(x, t) dx + c \int_0^1 w_{xx}(x, t) w(x, t) dx \end{aligned} \quad (23)$$

where the last equality is obtained integrating by parts twice. Again, integrating by parts the last integral term, one obtains from (23) that

$$\begin{aligned} \dot{V}_1(t) &= -c \int_0^1 w^2(x, t) dx - \int_0^1 w_x^2(x, t) dx \\ &\quad - \int_0^1 w_{xx}^2(x, t) dx - c \int_0^1 w_x^2(x, t) dx \\ &\leq -c \int_0^1 w^2(x, t) dx - (1+c) \int_0^1 w_x^2(x, t) dx \\ &\leq -2cV_1(t) \end{aligned} \quad (24)$$

using (22). This gives  $V_1(t) \leq V_1(0)e^{-2ct}$  which, by using (22), implies

$$\int_0^1 w^2(x, t) dx \leq 2V_1(0)e^{-2ct} \quad \text{and} \quad (25a)$$

$$\int_0^1 w_x^2(x, t) dx \leq 2V_1(0)e^{-2ct}.$$

Then, inequality (21b) yields

$$\max_{0 \leq x \leq 1} |w^2(x, t)| \leq 2V_1(0)e^{-2ct} \quad (25b)$$

which proves Part 1.

**Proof of Part 2.** From (14) one gets, using Schwarz inequality:

$$|z(x, t)| \leq |w(x, t)| + \sqrt{\int_0^x p^2(x, y) dy} \sqrt{\int_0^x w^2(y, t) dy}. \quad (26)$$

Let us determine an upper bound on the root-square term involving the kernel  $p(x, y)$ . First, it is easily checked that  $0 \leq \max_{0 \leq y \leq x} \sqrt{c(x-y)(2-x-y)} \leq \sqrt{cx(2-x)}$  and  $0 \leq \max_{0 \leq x \leq 1} \sqrt{cx(2-x)} \leq \sqrt{c}$ . Then, it follows from (18) that

$$|p(x, y)| \leq c \max_{0 \leq \zeta \leq \sqrt{c}} \frac{I_1(\zeta)}{\zeta}. \quad (27a)$$

Let us recall the following expression of the modified Bessel functions (see e.g. Smyshlyaev & Krstic, 2010, p. 307):

$$I_n(x) = \sum_{m=0}^{\infty} \frac{(x/2)^{n+2m}}{m!(m+n)!}. \quad (27b)$$

This entails

$$\frac{I_1(x)}{x} = \frac{1}{2} \sum_{m=0}^{\infty} \frac{(x/2)^{2m}}{m!(m+1)!} \leq \frac{1}{2} \sum_{m=0}^{\infty} \frac{(x/2)^{2m}}{m!} = \frac{1}{2} e^{x^2/4}$$

which together with (27a) gives

$$|p(x, y)| \leq \frac{c}{2} e^{c/4}. \quad (28)$$

Using (28) and (25a)–(25b), it follows from (26) that

$$|z(x, t)| \leq |w(x, t)| + \frac{c}{2} e^{c/4} \sqrt{\int_0^1 w^2(y, t) dy}$$

which, together with the bound established in the proof Part 1, yields

$$\max_{0 \leq x \leq 1} |z(x, t)| \leq \left(1 + \frac{c}{2} e^{c/4}\right) \sqrt{2V_1(0)} e^{-ct} \quad (29)$$

which, in view of Part 1, proves Part 2. This completes the proof of Proposition 1. ■

The result of Proposition 1 is quite interesting but it is still insufficient to conclude that (the norm of)  $\tilde{u}(x, t)$  is exponentially vanishing. Actually, in view of (5b), one has to show that  $\tilde{\theta}(t)$  is also exponentially vanishing and  $\int_0^1 \|\Lambda(x, t)\|^2 dx$  is bounded. The last requirement is first investigated.

**Proposition 2.** Consider the system (1a)–(1b) and the adaptive observer of Table 1. Let the observer parameter  $c$  be selected such that  $0 \leq K_{\max} < \pi^2$  with  $K_{\max} = \max_{0 \leq x \leq 1} K(x)$ . Then,  $\int_0^1 \|\Lambda(x, t)\|^2 dx$  is bounded. □

See the proof in the Appendix.

**Remark 4.** The variation of the observer gain  $K(x)$  is investigated in Smyshlyaev and Krstic (2005). It is shown (see Fig. 1 therein) that,  $K_{\max} = \max_{0 \leq x \leq 1} K(x)$  is an increasing function of  $c$ . It is readily checked that the requirement  $0 \leq K_{\max} < \pi^2$  is met if the parameter  $c$  belongs to the (approximate) interval  $(0, 8)$ . □

Now, to show that  $\tilde{\theta}(t)$  is exponentially vanishing we need, in addition to Propositions 1 and 2, the following assumption:

**PE assumption.** The vector signal  $\Lambda(0, t)$  is persistently exciting (PE), in the sense that,

$$\exists \delta, \varepsilon_0 > 0, \quad \forall t > 0: \int_t^{t+\delta} \frac{\Lambda^T(0, s)\Lambda(0, s)}{1 + \Lambda(0, s)\Lambda^T(0, s)} ds > \varepsilon_0 I \quad (30)$$

where  $I \in \mathbf{R}^{(n+1) \times (n+1)}$  denotes the identity matrix.

Property (30) means that the family of vectors  $\Lambda^T(0, s)/\sqrt{1 + \Lambda(0, s)\Lambda^T(0, s)}$  ( $t \leq s \leq t + \delta$ ) spans the vector space  $\mathbf{R}^{n+1}$ . Now, it can be shown (see e.g. Ioannou & Sun, 2006) that, if (30) holds then the time-varying inverse gain matrix  $R^{-1}(t)$  (solution of (12c)) is positive definite and stays bounded away from 0. More specifically, there exist two positive real numbers  $(r_0, r_1)$ , such that

$$r_0 \leq R^{-1}(t) \leq r_1, \quad \text{for all } t \geq 0. \quad (31)$$

In the sequel, condition (30) is supposed to be true, so that one can make use of (31) and define the following Lyapunov function, used in the next theorem,

$$V_2(t) = \tilde{\theta}^T(t)R^{-1}(t)\tilde{\theta}(t). \quad (32)$$

**Theorem 1.** The adaptive observer of Table 1 is exponentially convergent in the sense that both  $\tilde{\theta}(t)$  and  $\int_0^1 \tilde{u}^2(x, t) dx$  are exponentially convergent to zero, for any initial conditions  $\tilde{\theta}(0)$  and  $\tilde{u}(x, 0)$  ( $0 < x < 1$ ), and the corresponding convergence rates grow with the parameter  $c$ . ■

**Proof.** From (32), one gets the following time-derivative:

$$\dot{V}_2(t) = \tilde{\theta}^T(t)\dot{R}^{-1}(t)\tilde{\theta}(t) + 2\tilde{\theta}^T(t)R^{-1}(t)\dot{\tilde{\theta}}(t). \quad (33)$$

Using (5b), (12c) and (20h), one gets from (33) that

$$\begin{aligned} \dot{V}_2(t) &= \tilde{\theta}^T(t) \left( -R^{-1} + \frac{\Lambda^T(0, t)\Lambda(0, t)}{1 + \Lambda(0, t)\Lambda^T(0, t)} \right) \tilde{\theta}(t) \\ &\quad - 2\tilde{\theta}^T(t) \left( \frac{\Lambda^T(0, t)\Lambda(0, t)}{1 + \Lambda(0, t)\Lambda^T(0, t)} \tilde{\theta}(t) \right. \\ &\quad \left. - \frac{\Lambda^T(0, t)z(0, t)}{1 + \Lambda(0, t)\Lambda^T(0, t)} \right) \\ &\leq -\tilde{\theta}^T(t)R^{-1}\tilde{\theta}(t) - \frac{\tilde{\theta}^T(t)\Lambda^T(0, t)\Lambda(0, t)\tilde{\theta}(t)}{1 + \Lambda(0, t)\Lambda^T(0, t)} \\ &\quad + \left\| \frac{\tilde{\theta}^T(t)\Lambda^T(0, t)}{1 + \Lambda(0, t)\Lambda^T(0, t)} \right\|^2 + z^2(0, t) \\ &\leq -V_2(t) + z^2(0, t) \end{aligned} \quad (34)$$

where the two last inequalities are obtained using Young's inequality and (32). By Proposition 1 (Part 2),  $z^2(0, t)$  is exponentially vanishing. Then, applying the comparison lemma (see e.g. Khalil, 2002), it follows from (34) that  $V_2(t)$  is exponentially vanishing and, in view of (31), so is  $\tilde{\theta}(t)$ . Furthermore, again using Proposition 1, the convergence rate is growing with the parameter  $c$ . Finally, from (5b), one gets the following:

$$\begin{aligned} \sqrt{\int_0^1 \tilde{u}^2(x, t) dx} &\leq \sqrt{\int_0^1 z^2(x, t) dx} \\ &\quad + \left\| \tilde{\theta}(t) \right\| \int_0^1 \|\Lambda(x, t)\|^2 dx \end{aligned}$$

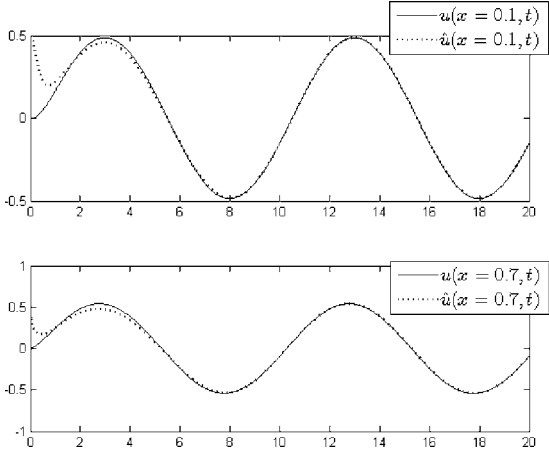


Fig. 1. State  $u(x, t)$  and its estimate  $\hat{u}(x, t)$  for  $x = 0.1$  (bottom) and  $x = 0.7$  (top).

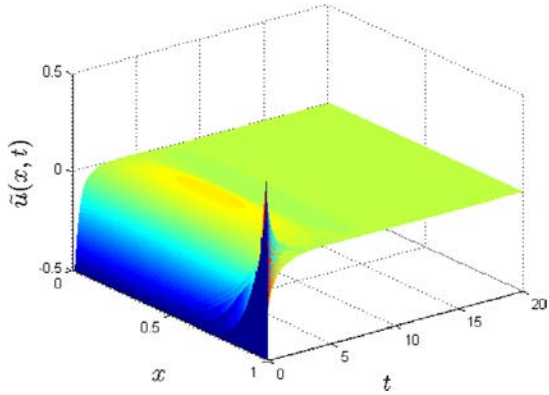


Fig. 2. State estimation error  $\tilde{u}(x, t) = \hat{u}(x, t) - u(x, t)$ .

which, together with Proposition 2, implies that  $\int_0^1 \tilde{u}^2(x, t) dx$  is exponentially vanishing, with a convergence rate growing with  $c$ . This ends the proof of Theorem 1. ■

## 5. Simulation

To illustrate the theoretical result of Theorem 1, consider the following system:

$$u_t(x, t) = u_{xx}(x, t) + q_1 e^{0.2x} u(0, t), \quad 0 < x < 1, \quad t > 0 \quad (35a)$$

$$u_x(0, t) = -q_0 u(0, t), \quad t \geq 0 \quad (35b)$$

where  $q_1 = 1.2$  and  $q_0 = 0.68$  are supposed to be unknown. Clearly, the system is of the form (1a)–(1d) with  $\phi(x, t) = e^{0.2x} u(0, t)$ . We let it be excited at the boundary by  $U(t) = u(1, t) = \sin(\omega_0 t)$  with  $\omega_0 = \pi/5$  (rd/s). Applying the adaptive observer of Table 1, with the following choice of the tuning parameters  $c = 7$  and  $R_0 = I_2$ , we get state estimates  $\hat{u}(x, t)$  ( $0 < x < 1$ ) and parameter estimates  $\hat{q}_1, \hat{q}_0$ . Fig. 1 shows the time evolution of the state estimate at two particular positions in the domain as well as the corresponding estimates. Clearly, the estimates get very close to their true variables after a transient period of 5 (s). In Fig. 2, the estimation error  $\tilde{u}(x, t) = \hat{u}(x, t) - u(x, t)$  is plotted over time for all  $0 \leq x \leq 1$ . The global view thus obtained demonstrates the good state estimation quality. Fig. 3 shows that the quality of parameter estimates gets quite accurate after a transient period of 10 (s). The above observations confirm the theoretical asymptotic performance described in Theorem 1.

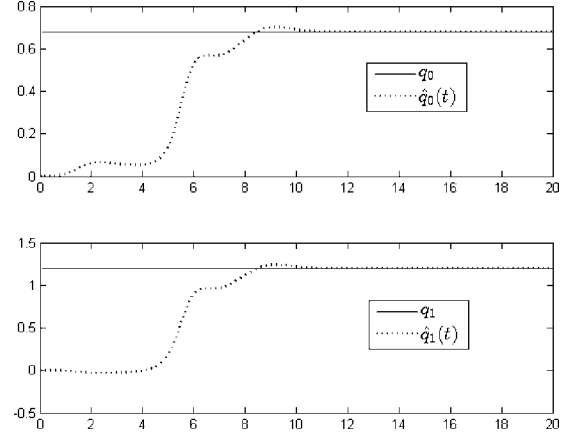


Fig. 3. Evolution in time of the parameter estimates.

## 6. Conclusion

The problem of state observation and parameter estimation is addressed for IDSs described by the model (1a)–(1b). The considered IDS in this paper is basically a parabolic PDE with parameter uncertainty entering in the domain and the boundary condition. The adaptive observer of Table 1 is designed by combining the finite- and infinite-dimensional backstepping-like transformations (14) and (5b). To our knowledge, it is the first time that an adaptive observer combines both transformations. The observer enjoys exponential convergence under the PE condition (30).

## Appendix. Proof of Proposition 2

**Part 1.** Proof that  $\int_0^1 \lambda_0^2(x, t) dx$  is bounded. Consider the following Lyapunov functional candidate associated with the system (20d) together with the boundary conditions (20f):

$$W_0(\lambda_0) = \frac{1}{2} \int_0^1 \lambda_0^2(x, t) dx. \quad (A.1)$$

Its time derivative is

$$\begin{aligned} \dot{W}_0(\lambda_0) &= \int_0^1 \lambda_0(x, t) \lambda_{0,t}(x, t) dx \\ &= - \int_0^1 \lambda_{0,x}^2(x, t) dx + u(0, t) \lambda_0(0, t) \\ &\quad - \int_0^1 K(x) \lambda_0^2(x, t) dx \\ &\quad - \int_0^1 K(x) \lambda_0(x, t) (\lambda_0(0, t) - \lambda_0(x, t)) dx \end{aligned} \quad (A.2)$$

where an integration by part has been used to get the last equality. Applying Young's inequality to the second and fourth terms on the right side of (A.2) yields

$$\begin{aligned} \dot{W}_0(\lambda_0) &\leq - \int_0^1 \lambda_{0,x}^2(x, t) dx + \frac{\varsigma}{2} \lambda_0^2(0, t) + \frac{1}{2\varsigma} u^2(0, t) \\ &\quad - \int_0^1 K(x) \lambda_0^2(x, t) dx + \frac{\vartheta}{2} \int_0^1 K(x) \lambda_0^2(x, t) dx \\ &\quad + \frac{1}{2\vartheta} \int_0^1 K(x) (\lambda_0(0, t) - \lambda_0(x, t))^2 dx \\ &\leq - \left(1 - \frac{\vartheta}{2}\right) \int_0^1 K(x) \lambda_0^2(x, t) dx \end{aligned}$$

$$\begin{aligned}
& - \left(1 - \frac{2K_{\max}}{\vartheta\pi^2} - \frac{\zeta}{2}\right) \int_0^1 \lambda_{0,x}^2(x, t) dx \\
& + \frac{1}{2\zeta} u^2(0, t)
\end{aligned} \tag{A.3}$$

whatever  $\zeta > 0$  and  $\vartheta > 0$ , where we have applied the two Wirtinger's inequalities in (21a)–(21b). Letting the free scalars be selected such that  $\vartheta = 2$  and

$$\frac{\zeta}{2} < 1 - \frac{K_{\max}}{\pi^2} \tag{A.4}$$

one gets from (A.3) and (A.1)

$$\dot{W}_0(\lambda_0) \leq -\frac{\pi^2}{2} \left(1 - \frac{K_{\max}}{\pi^2} - \frac{\zeta}{2}\right) W_0(\lambda_0) + \frac{1}{2\zeta} u^2(0, t) \tag{A.5}$$

using Wirtinger's inequality (21a). Note that the condition in (A.5) is feasible because  $\frac{K_{\max}}{\pi^2} < 1$  by assumption. Now, in view of (A.1), it follows from (A.5) that  $W_0(\lambda_0)$  is bounded, because  $u(0, t)$  is so (by assumption). Part 1 is proved.

**Part 2.** Proof that  $\int_0^1 \|\lambda_1(x, t)\|^2 dx$  is bounded. Following closely the argument in Part 1, the system (20e), subject to the boundary conditions (20g), is now analyzed by considering the Lyapunov functional candidate,

$$W_1(\lambda_1) = \frac{1}{2} \int_0^1 \lambda_1(x, t) \lambda_1^T(x, t) dx. \tag{A.6}$$

Differentiating this gives

$$\begin{aligned}
\dot{W}_1(\lambda_1) &= \int_0^1 \lambda_1(x, t) \lambda_{1,xx}^T(x, t) dx \\
&\quad - \int_0^1 K(x) \lambda_1(x, t) \lambda_1^T(0, t) dx \\
&\quad + \int_0^1 \lambda_1(x, t) \phi(x, t) dx \\
&\leq - \int_0^1 \|\lambda_{1,x}(x, t)\|^2 dx - \int_0^1 K(x) \|\lambda_1(x, t)\|^2 dx \\
&\quad - \int_0^1 K(x) \lambda_1(x, t) (\lambda_1^T(0, t) - \lambda_1^T(x, t)) dx \\
&\quad + \frac{\zeta}{2} \int_0^1 \|\lambda_1(x, t)\|^2 dx \\
&\quad + \frac{1}{2\zeta} \int_0^1 \|\phi(x, t)\|^2 dx
\end{aligned} \tag{A.7}$$

whatever the scalar  $\zeta > 0$ , where we have used an integration by parts and Young's inequality. Again applying Young's inequality to the third term on the right side of (A.7), one gets

$$\begin{aligned}
\dot{W}_1(\lambda_1) &\leq - \int_0^1 \|\lambda_{1,x}(x, t)\|^2 dx - \int_0^1 K(x) \|\lambda_1(x, t)\|^2 dx \\
&\quad + \frac{1}{2\nu} \int_0^1 K(x) \|\lambda_1(0, t) - \lambda_1(x, t)\|^2 dx \\
&\quad + \frac{\nu}{2} \int_0^1 K(x) \|\lambda_1(x, t)\|^2 dx \\
&\quad + \frac{\zeta}{2} \int_0^1 \|\lambda_1(x, t)\|^2 dx + \frac{1}{2\zeta} \int_0^1 \|\phi(x, t)\|^2 dx \\
&\leq - \int_0^1 \|\lambda_{1,x}(x, t)\|^2 dx - \int_0^1 K(x) \|\lambda_1(x, t)\|^2 dx \\
&\quad + \frac{2K_{\max}}{\nu\pi^2} \int_0^1 \|\lambda_{1,x}(x, t)\|^2 dx
\end{aligned}$$

$$\begin{aligned}
& + \frac{\nu}{2} \int_0^1 K(x) \|\lambda_1(x, t)\|^2 dx \\
& + \frac{2\zeta}{\pi^2} \int_0^1 \|\lambda_{1,x}(x, t)\|^2 dx \\
& + \frac{1}{2\zeta} \int_0^1 \|\phi(x, t)\|^2 dx
\end{aligned} \tag{A.8}$$

whatever the scalar  $\nu > 0$ , where the third and fifth terms on the right side of (A.8) are obtained using Wirtinger's inequality (21a). It turns out that

$$\begin{aligned}
\dot{W}_1(\lambda_1) &\leq - \left(1 - \frac{2K_{\max}}{\nu\pi^2} - \frac{2\zeta}{\pi^2}\right) \int_0^1 \|\lambda_{1,x}(x, t)\|^2 dx \\
&\quad - \left(1 - \frac{\nu}{2}\right) \int_0^1 K(x) \|\lambda_1(x, t)\|^2 dx \\
&\quad + \frac{1}{2\zeta} \int_0^1 \|\phi(x, t)\|^2 dx.
\end{aligned} \tag{A.9}$$

Let the free scalars,  $\zeta > 0$  and  $\nu > 0$  be selected as follows:

$$\nu = 2 \quad \text{and} \quad \frac{2\zeta}{\pi^2} < 1 - \frac{K_{\max}}{\pi^2}. \tag{A.10}$$

Then, applying Wirtinger's inequality (21a) to the first term on the right side of (A.9), one gets

$$\begin{aligned}
\dot{W}_1(\lambda_1) &\leq - \left(1 - \frac{2K_{\max}}{\nu\pi^2} - \frac{2\zeta}{\pi^2}\right) \frac{\pi^2}{4} \int_0^1 \|\lambda_1(x, t)\|^2 dx \\
&\quad + \frac{1}{2\zeta} \int_0^1 \|\phi(x, t)\|^2 dx \\
&\leq - \left(1 - \frac{2K_{\max}}{\nu\pi^2} - \frac{2\zeta}{\pi^2}\right) \frac{\pi^2}{2} W_1(\lambda_1) \\
&\quad + \frac{1}{2\zeta} \int_0^1 \|\phi(x, t)\|^2 dx
\end{aligned}$$

which, in view of (A.10), implies that  $W_1(\lambda_1)$  is bounded, because  $\phi(x, t)$  is so by assumption. Finally, note that condition (A.10) is feasible since  $\frac{K_{\max}}{\pi^2} < 1$  by assumption. Part 2 is proved completing the proof of Proposition 2. ■

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