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MULTIPLICATIVE ERGODICITY OF LAPLACE TRANSFORMS FOR ADDITIVE FUNCTIONAL OF MARKOV CHAINS WITH APPLICATION TO AGE-DEPENDENT BRANCHING PROCESS.

LOÏC HERVÉ, SANA LOUHICHI, AND FRANÇOISE PÈNE

Abstract. We study the exponential growth of bifurcating processes with ancestral dependence. We suppose here that the lifetimes of the cells are dependent random variables, that the numbers of new cells are random and dependent. Lifetimes and new cells's numbers are also assumed to be dependent. We illustrate our results by examples, including some Markov models. Our approach is related to the behaviour of the Laplace transform of nonnegative additive functional of Markov chains and require weak moment assumption (no exponential moment is needed).

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1. Introduction

Mathematical models for the growth of populations have been widely studied and applied in many fields especially animal, cell biology (mitosis), plant and forestry sciences. Branching process is a mathematical model often used to model reproduction (see for instance [2], [12] and [18]). It is described as follows. A single ancestor object (that may be a particle, a neutron, a cosmic ray, a cell and so on) is born at time 0. It lives for a random time. At the moment of death, the object produces a random number of progenies. Each of the first generation progeny behaves, independently of each other and of the ancestor: the objects do not interfere with one another. Many authors were interested by generalizing this classical model by trying to describe the interference of objects between them (see, for instance, [23] and the references therein).

In the spirit of the branching processes and for the sake of generalization, we consider in this paper a model of reproduction with a random number of children and with random life duration. Our approach, to develop this mathematical model, is to associate to each object v a parameter x_v, this parameter may depend on its energy, its position or on other non-observed factors and it can be seen as the characteristics of the given object. When v runs over the set of all objects, we assume that the set of parameters $(x_v)_v$ is a realisation of a random process $(X_v)_v$. We do not specify either the type of interactions between the objects nor the dependence conditions on the sequence $(X_v)_v$. As will be described later in

this paper, our results are first announced in the case where $(X_v)_v$ is a stationary process on some paths of objects: the parameters processes have the same law-behavior for all objects belonging to the same generation. The classical independent identically distributed (i.i.d. for short) case is a particular case of stationarity. Next we will use, as was done by many authors, Markov processes as mathematical models for this parameter process. We discuss, in particular, the linear autoregressive process.

We are interested in this paper by the mitosis model and from now an object will be a cell. This mitosis model starts with one single initial cell. After a random time, this initial cell is divided into a random number of cells and the process continue. For technical reasons we will suppose that the random number of children is always larger than 2. We summarize our model, used throughout the paper, as follows,

- to each cell v, is associated a parameter $x_v \in \mathbb{X}$ (with $(\mathbb{X}, \mathcal{X})$ a measurable space) which determines its lifetime $\xi(x_v)$ and the number of new cells $\kappa(x_v)$ in which the cell splits at the end of its lifetime (where ξ and κ are two measurable functions with values in $[0, +\infty)$ and in $\mathbb{Z}_+$ respectively);
- for each line $(v_n)_{n\geq 0}$ of cells, the parameters along this line are given by copies (non necessarily independent) of a process $(X_n)_{n\geq 0}$ with values in $\mathbb{X}$;
- $\kappa(x) \geq 2$ for any x, i.e. each cell gives birth to more than two children.

For every $t > 0$, let us consider the Belleman-Harris age-dependent branching process $(N_t)_{t\geq 0}$, that is N_t is the number of cells alive at time t (see [2] and [12] for more about). The exponential growth behavior of N_t as t tends to infinity was studied in the book of Harris [12] in the case where $(X_n)_n$ is a sequence of i.i.d. rv's, that is when the lifetimes are modeled by a sequence of i.i.d. random variables independent of the random numbers of the news cells which are also assumed to be i.i.d. The growth rate ν_0 (also called the Malthusian parameter) was defined, in this context, as the positive root of the equation,

$$\mathbb{E}[\kappa(X_1)]\,\mathbb{E}\left[e^{-\nu_0\xi(X_1)}\right] = 1, \tag{1}$$

as soon as the distribution of $\xi(X_1)$ is not lattice (cf. [12, Theorem 17.1]). Louhichi and Ycart [23] extend some results of Harris to the case where the lifetimes are a sequence of dependent random variables and when each cell is divided, after a random lifetime, into two cells: $(X_n)_n$ is a stationary process and $\kappa(x) = 2$ for any x. Under those assumptions the Malthusian parameter ν_1 is expressed in terms of the Laplace transform of the random variable S_n

$$S_n := \sum_{k=0}^{n} \xi(X_k) \tag{2}$$

which models the birth date of the $(n+1)$-th individual of a same line. More precisely,

$$\nu_1 = \inf\left\{\gamma > 0,\ \sum_{n\geq 0} 2^n \mathbb{E}\left[e^{-\gamma S_n}\right] < \infty\right\}. \tag{3}$$

In this paper we are interested by the general case where the lifetimes and the numbers of new cells are dependent random variables. In this case, the growth rate of N_t is given by:

$$\nu := \inf\left\{\gamma > 0,\ \sum_{n\geq 0} g_n(\gamma) < \infty\right\}, \tag{4}$$

where $g_n(\gamma)$ is expressed in terms of a perturbed Laplace transform of S_n,

$$g_n(\gamma) := \mathbb{E}\left[\left(\prod_{j=0}^{n-1}\kappa(X_j)\right)(\kappa(X_n)-1)e^{-\gamma S_n}\right].$$

One task of the paper, is to give exact evaluations of $\mathbb{E}(N_t)$ and $\mathbb{E}(N_t N_{t+\tau})$, for any $t,\tau \geq 0$, under general and minimal conditions on the parameter process (X_n) as described above. Those calculations are the main ingredients to get the convergence almost surely of $e^{-\nu t}N_t$ to a non-negative random variable W. A second task of the paper is to discuss the conditions yielding the previous results and to give some Markovian models for which the growth parameter ν is finite.

This paper is organized as follows. In section 2, we give with proofs an exact evaluation of $\mathbb{E}(N_t)$ (see Proposition 2.1). An immediate consequence of this calculation, when ν is supposed finite, is the convergence in mean of $e^{-\nu t}\mathbb{E}(N_t)$ to some constant C_ν given by

$$C_\nu := \lim_{\gamma\to 0}\frac{\gamma}{\gamma+\nu}\sum_{n\geq 0} g_n(\gamma+\nu),$$

see Corollary 2.2 for a precise statement. As noticed in Subsection 2.2, from a multiplicative ergodicity behavior of $g_n(\gamma)$, see Definition 2.3, one can deduce that ν and C_ν are both finite. For this, we study this multiplicative ergodicity property in the context of additive functional of Markov chains with Markov kernel P and initial law μ. In Subsection 2.2, we state our spectral assumptions, after noting that

$$\forall n \geq 1, \quad g_n(\gamma) = \mu\left(\kappa\, e^{-\gamma\xi}\, P_\gamma^{n-1}\left(Ph_{\kappa,\gamma}\right)\right),$$

where P_γ is a Laplace-type kernel associated with P, ξ and κ (see Formula (11) for more details about the notations). Those spectral assumptions are on P_γ supposed to continuously act on some suitable Banach space $\mathcal{B}$ and on its spectral radius $r(\gamma)$ (see Hypotheses 2.5, 2.7 and 2.7*). Those assumptions are needed in order to obtain a spectral multiplicative ergodicity property for the iterated operator P_γ^n. This spectral property is the main tool to prove the multiplicative ergodicity behavior of $g_n(\gamma)$ and then to deduce that ν is finite and is given by

$$\nu = \inf\{\gamma > 0,\ r(\gamma) < 1\},$$

(see Theorem 2.8). The existence and the finiteness of C_ν are discussed in Thereom 2.9. Thereom 2.9 needs reinforcing assumptions by considering a longer chain of Banach spaces. We discuss in Subsection 2.3 two Markovian examples satisfying Theorems's 2.8 and 2.9 assumptions, see Theorem 2.11 and Theorem 2.12 for respectively a toy model involving some Knudsen gas (see [3] for other results on Knudsen gases) and for linear autoregressive processes, both under weak integrability assumptions on the observable ξ (the lifetime).

Our approach for Markov models is based on the method of perturbation of operators. This method, introduced by Nagaev [25, 26] and by Le Page and Guivarc'h [21, 10] to prove a wide class of limit theorems (central limit theorem, local limit theorem, large and moderate deviations principles), has known an impressive development in the past decades (e.g. see [4, 14] and the references therein). The price to pay for our weak moment assumptions is that, in general, the classical perturbation method does not apply in our context to the family of Laplace operators and that we have to consider several Banach spaces instead of a single one (see Remark 2.6 for details). This is allowed by the Keller and Liverani perturbation theorem

[17, 1](e.g. see [16] and the references therein). The fact that we work with several spaces (due to our weak moment assumptions) complicates our study compared to the classical approach.

In Section 3, we study the behavior of $\mathbb{E}(N_t N_{t+\tau})$, for any $t > 0$, $\tau \geq 0$, in the very general setting of dependence (cf. Proposition 3.2). Proposition 3.2 is obtained under a suitable behavior of the sequence $(\kappa(X_i))_{1\leq i\leq n}$, the birthtimes S_n and $S_m^{(k)}$ of two cells belonging respectively to the n-th and m-th generation and having the last common ancestor in the k-th generation. More precisely, Proposition 3.2 gives the following formula,

$$\mathbb{E}(N_t N_{t+\tau}) = \mathbb{E}(N_t) + \mathbb{E}(N_{t+\tau}) - 1 + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{\min(n,m)-1} \mathbb{E}\left(A_{n,m,k}\mathbb{I}_{S_{n-1}\leq t}\mathbb{I}_{S_{m-1}^{(k)}\leq t+\tau}\right),$$

for suitable integer sequence $A_{n,m,k}$ expressed only by the numbers of children as is defined in Subsection 3.1. We study in Corollary 3.3, the mean quadratic and the almost sure convergence of $e^{-\nu t}N_t$ as t tends to infinity. The purpose of Subsection 3.2 is to discuss Corollary's 3.3 assumptions, yielding the almost sure convergence of $e^{-\nu t}N_t$ as t tends to infinity, in the particular case where lifetimes and new cells numbers are independent and new cells numbers are modelled by a sequence of iid random variables. A main step to check in this particular case is to establish (see Lemma 3.7),

$$\left| e^{-\nu t}\sum_{n\geq 0}\kappa_1^n(\kappa_1 - 1)\mathbb{P}(\sum_{i=1}^{n+1}\xi(X_i) \leq t|X_0 = x) - C_0(x)\right| \leq \frac{\Psi_0(x)}{2\pi}e^{-\delta t}. \tag{5}$$

The bound (5) is the main tool to obtain (see Proposition 3.8 for more details),

$$\mathbb{E}(N_t) = e^{\nu t}\mathbb{E}(\kappa(X_0))\mathbb{E}(e^{-\nu\xi(X_0)}\tilde{C}_0(X_0))[1 + O(e^{-\epsilon_1 t})],\ as\ t \to \infty,\ \text{for some } \epsilon_1 > 0.$$

Under additional assumptions, the bound (5) allows also to obtain (see Proposition 3.9),

$$\mathbb{E}(N_t N_{t+\tau}) = e^{\nu(2t+\tau)}\kappa_2\sum_{k=0}^{\infty}\kappa_1^k\mathbb{E}(\tilde{C}_0^2(X_k)e^{-2\nu S_k})[1 + ae^{-\epsilon_1 t}],\ as\ t \to \infty,$$

where $\kappa_2 = \mathbb{E}(\kappa(X_1)(\kappa(X_1) - 1))$ and where a, ϵ_1 are positive constants independent of t and τ. Lemma 3.7 gives sufficient conditions ensuring (5). Theorem 3.6 studies the convergence in mean quadratic and almost surely of $e^{-\nu t}N_t$ to a random variable W and gives the expressions of the first and of the second moment of this limit W. In Subsection 3.3 we discuss mainly the conditions of Lemma 3.7 (and then sufficient conditions for the bound (5)).

The rest of the sections are devoted to the proofs of Theorems 2.8 and 2.9 dedicated to the multiplicative ergodicity property for additive Markov chains. In Section 4, we discuss the tools and the assumptions needed to those two main theorems. We study in particular the monotonicity, the positivity and the derivative of the spectral radius $r(\gamma)$ of the Laplace kernel P_γ. In Section 5, we apply our general method to a toy model of Knudsen gas in the particular case where $\kappa \equiv 2$. The proofs concerning the less elementary case of linear autoregressive models are given in Section 6. Theorems 2.8 and 2.9 are completely proved in Section 7. Section 8 gives the proof of Proposition 4.5 needed to check Hypothesis 2.5 which is one of the main assumptions of Theorems 2.8 and 2.9. Finally, Section 9 gives an example of a kernel P_γ for which its spectral radius is greater than one and hence the growth rate ν does not exist.

2. Behavior of the first moment, multiplicative ergodicity, examples

2.1. **First moment of N_t.** The following proposition evaluates, for any $t>0$, the expectation of N_t when it exists in terms of the lifetimes $(\xi(X_i))_{i\geq 0}$ and of the numbers of new cells $(\kappa(X_i))_{i\geq 0}$.

Proposition 2.1. *Let $t>0$ be fixed. If $\sum_{n\geq 0}\mathbb{E}\left[\left(\prod_{j=0}^{n}\kappa(X_j)\right)\mathbf{1}_{\{S_n\leq t\}}\right]<\infty$, then $\mathbb{E}(N_t)<\infty$ and*

$$\mathbb{E}(N_t)=1+\sum_{n\geq 0}\mathbb{E}\left[\left(\prod_{j=0}^{n-1}\kappa(X_j)\right)(\kappa(X_n)-1)\,\mathbf{1}_{\{S_n\leq t\}}\right]$$

(with the usual convention $\prod_{j=0}^{-1}\kappa(X_j)=1$).

Proof. For every $n\geq 0$, we write $\Sigma_n(t)$ for the number of cells of generation n alive at time t. Let us write D_0 and T_0 for respectively the number of children and the lifetime of the initial cell. For every $k\in\{1,\cdots,D_0\}$, we write $x_{0,k}$ for the parameter (resp. $D_{0,k}$ and $T_{0,k}$ for the number of children and the lifetime) of the k-th child of the initial cell. More generally, we write $x_{0,k_1,\cdots,k_n}$ for the parameter (resp. $D_{0,k_1,\cdots,k_n}$ and $T_{0,k_1,\cdots,k_n}$ for the number of children and the lifetime) of the cell of the n-th generation which is the k_n-th child of the k_{n-1}-th child of the ... of the k_1-th child of the initial cell. Observe that $\mathbb{E}[\Sigma_0(t)]=\mathbb{P}(\xi(X_0)>t)$ and that, for every $n\geq 1$,

$$\begin{aligned}
\mathbb{E}[\Sigma_n(t)] &= \mathbb{E}\left[\sum_{k_1=1}^{D_0}\sum_{k_2=1}^{D_{0,k_1}}\cdots\sum_{k_n=1}^{D_{0,k_1,\cdots,k_{n-1}}}\mathbf{1}_{\{T_0+T_{0,k_1}+\cdots+T_{0,k_1,\cdots,k_{n-1}}\leq t<T_0+T_{0,k_1}+\cdots+T_{0,k_1,\cdots,k_n}\}}\right]\\
&= \mathbb{E}\left[D_0\mathbb{E}\left[\sum_{k_2=1}^{D_{0,1}}\cdots\sum_{k_n=1}^{D_{0,1,k_2,\cdots,k_{n-1}}}\mathbf{1}_{\{T_0+T_{0,1}+\cdots+T_{0,1,k_2,\cdots,k_{n-1}}\leq t<T_0+T_{0,1}+\cdots+T_{0,1,k_2,\cdots,k_n}\}}\,\middle|\,X_0\right]\right]\\
&= \mathbb{E}\left[D_0D_{0,1}\mathbb{E}\left[\sum_{k_3=1}^{D_{0,1,1}}\cdots\sum_{k_n=1}^{D_{0,1,1,k_3,\cdots,k_{n-1}}}\mathbf{1}_{\{T_0+T_{0,1}+\cdots+T_{0,1,1,k_3,\cdots,k_{n-1}}\leq t<T_0+T_{0,1}+\cdots+T_{0,1,1,k_3,\cdots,k_n}\}}\,\middle|\,X_0,X_{0,1}\right]\right]
\end{aligned}$$

Hence, for every $n\geq 1$,

$$\begin{aligned}
\mathbb{E}[\Sigma_n(t)] &= \mathbb{E}\left[D_0D_{0,1}\cdots D_{0,1^{n-1}}\mathbf{1}_{\{T_0+T_{0,1}+\cdots+T_{0,1^{n-1}}\leq t<T_0+T_{0,1}+\cdots+T_{0,1^n}\}}\right]\\
&= \mathbb{E}\left[\left(\prod_{j=0}^{n-1}\kappa(X_j)\right)\left(\mathbf{1}_{\{S_{n-1}\leq t\}}-\mathbf{1}_{\{S_n\leq t\}}\right)\right].
\end{aligned}$$

Since $\sum_{n\geq 0}\mathbb{E}\left[\left(\prod_{j=0}^{n}\kappa(X_j)\right)\mathbf{1}_{\{S_n\leq t\}}\right]<\infty$ and $N_t=\mathbf{1}_{T_0>t}+\sum_{n=1}^{\infty}\Sigma_n(t)$ a. s., we obtain

$$\begin{aligned}
\mathbb{E}[N_t] &= \mathbb{P}(\xi(X_0)>t)+\mathbb{E}\left[\kappa(X_0)\mathbf{1}_{\xi(X_0)\leq t}\right]+\sum_{n\geq 1}\mathbb{E}\left[\left(\prod_{j=0}^{n-1}\kappa(X_j)\right)(\kappa(X_n)-1)\mathbf{1}_{\{S_n\leq t\}}\right]\\
&= 1+\mathbb{E}\left[(\kappa(X_0)-1)\mathbf{1}_{\xi(X_0)\leq t}\right]+\sum_{n\geq 1}\mathbb{E}\left[\left(\prod_{j=0}^{n-1}\kappa(X_j)\right)(\kappa(X_n)-1)\mathbf{1}_{\{S_n\leq t\}}\right].
\end{aligned}$$

□

It follows from Proposition 2.1 that $\mathbb{E}[N_t] < \infty$ if ν defined by (4) is finite (using the fact that $\mathbf{1}_{\{S_n \le t\}} \le e^{\gamma t} e^{-\gamma S_n}$). Now using Proposition 2.1 and arguing exactly as for the proof of Theorem 2.1 in [23], we obtain the following exponential behavior in mean of $\mathbb{E}(N_t)$ in a very general setting of dependence with the use of the function G given by

$$G(\gamma) := \sum_{n \ge 0} g_n(\gamma), \quad \text{recall that} \quad g_n(\gamma) = \mathbb{E}\left[\left(\prod_{j=0}^{n-1} \kappa(X_j)\right)(\kappa(X_n) - 1)e^{-\gamma S_n}\right]. \tag{6}$$

Corollary 2.2. *Assume that $\nu < \infty$ and that the following limit exists*

$$C_\nu := \lim_{\gamma \to 0} \frac{\gamma}{\gamma + \nu} G(\nu + \gamma)\,, \tag{7}$$

then,

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t e^{-\nu s} \mathbb{E}(N_s) ds = C_\nu. \tag{8}$$

2.2. **Multiplicative ergodicity, application to Markov chains.** We adapt the notion of "multiplicative ergodicity", as introduced in [19] and [20], to our context.

Definition 2.3. *Let $\gamma_1 > 0$. We say that $(S_n, \kappa(X_n))_n$ is* **multiplicatively ergodic** *on $J = [0, \gamma_1)$ if there exist two continuous maps A and ρ from J to $(0, +\infty)$ such that, for every compact subset K of $(0, \gamma_1)$, there exist $M_K > 0$ and $\theta_K \in (0, 1)$ such that, for every $n \ge 1$,*

$$\forall \gamma \in K, \quad |g_n(\gamma) - A(\gamma)(\rho(\gamma))^n| \le M_K(\rho(\gamma)\theta_K)^n. \tag{9}$$

When $\kappa(\cdot)$ is constant, we will simply say that $(S_n)_n$ is multiplicatively ergodic on J.

Remark 2.4. *Assume that $(S_n, \kappa(X_n))_n$ is multiplicatively ergodic on $J = [0, \gamma_1)$. Then*

- *For every $\gamma \in J$ we have: $G(\gamma) = \sum_{n \ge 0} g_n(\gamma) < \infty \iff \rho(\gamma) < 1$.*
- *For every compact subset K of J, we obtain from the definition of ν in (4) that*

$$\forall \gamma \in K \cap (\nu, +\infty), \quad \left| G(\gamma) - \frac{A(\gamma)}{1 - \rho(\gamma)} \right| \le \frac{M_K}{1 - \rho(\gamma)\theta_K}.$$

- *$\nu < \gamma_1$ means that*

$$\nu = \inf\{\gamma \in J \ : \ \rho(\gamma) < 1\} < \gamma_1. \tag{10}$$

- *If moreover ρ is differentiable at ν with $\rho(\nu) = 1$ and $\rho'(\nu) \ne 0$, then (7) follows with $C_\nu = -\frac{A(\nu)}{\nu \rho'(\nu)}$. Actually, to obtain (10), we can relax the continuity assumptions on A and ρ on $J = [0, \gamma_1)$. For (7), we just need the continuity of A and the differentiability of ρ at ν (with $\rho'(\nu) \ne 0$).*

We investigate now the geometric ergodicity property in the following context of additive functional of Markov chains. We assume throughout this section that $X = (X_n)_n$ is a Markov chain on $(\mathbb{X}, \mathcal{X})$ with Markov kernel $P(x, dy)$, invariant probability π, and initial probability μ (i.e. μ is the distribution of X_0). Assume moreover that, for every $n \ge 1$, the random

variable $\prod_{j=0}^{n}\kappa(X_j)$ is integrable. We set $h_{\kappa,\gamma} := \big(\kappa-1\big)\,e^{-\gamma\xi}$. Let $\gamma\in(0,+\infty)$. For $n\geq 1$,

$$\begin{aligned} g_n(\gamma) &= \mathbb{E}\left[\left(\prod_{j=0}^{n-1}\kappa(X_j)e^{-\gamma\xi(X_j)}\right)h_{\kappa,\gamma}(X_n)\right] \\ &= \mathbb{E}\left[\left(\prod_{j=0}^{n-1}\kappa(X_j)e^{-\gamma\xi(X_j)}\right)(Ph_{\kappa,\gamma})(X_{n-1})\right], \end{aligned}$$

with $(Ph)(x) := \int_{\mathbb{X}} h(y)\,P(x,dy)$. If $n\geq 2$, we continue and obtain

$$g_n(\gamma) = \mathbb{E}\left[\left(\prod_{j=0}^{n-2}\kappa(X_j)e^{-\gamma\xi(X_j)}\right)(P_\gamma(Ph_{\kappa,\gamma}))(X_{n-2})\right],$$

with $P_\gamma h := P(h\kappa e^{-\gamma\xi})$. An easy induction gives

$$\forall n\geq 1,\quad g_n(\gamma) = \mu\left(\kappa\, e^{-\gamma\xi}\,P_\gamma^{n-1}\,(Ph_{\kappa,\gamma})\right). \tag{11}$$

We also define $P_\infty h := P(h\kappa\mathbf{1}_{\{\xi=0\}})$.

Before stating our spectral assumptions on P_γ, we need to introduce some standard notations. When $(\mathcal{B},\|\cdot\|_{\mathcal{B}})$ and $(\mathcal{B}_1,\|\cdot\|_{\mathcal{B}_1})$ are two Banach spaces, the space of continuous $\mathbb{C}$-linear operators from $\mathcal{B}$ to $\mathcal{B}_1$ will be written $\mathcal{L}(\mathcal{B},\mathcal{B}_1)$. We simply write $\mathcal{L}(\mathcal{B})$ for $\mathcal{L}(\mathcal{B},\mathcal{B})$, and the topological dual space of $\mathcal{B}$ is denoted by $(\mathcal{B}^*,\|\cdot\|_{\mathcal{B}^*})$. When, for some $\gamma\in[0,+\infty]$, the kernel P_γ continuously acts on $\mathcal{B}$, the spectral radius of $P_{\gamma|\mathcal{B}}$ and its essential spectral radius are denoted by $r(\gamma)$ and $r_{ess}(P_\gamma)$ respectively, that is:

$$r(\gamma) := r(P_{\gamma|\mathcal{B}}) = \lim_n \|(P_{\gamma|\mathcal{B}})^n\|_{\mathcal{B}}^{1/n} \quad\text{and}\quad r_{ess}(P_\gamma) := \lim_n \inf_{F\in\mathcal{L}(\mathcal{B})\text{ compact}} \|(P_\gamma)^n - F\|_{\mathcal{B}}^{1/n}.$$

Hypothesis 2.5. *Let $\mathcal{B}$ be a Banach space composed of functions on $\mathbb{X}$ (or of classes of such functions modulo the $\pi-$almost sure equality) such that $\mathcal{B}\subset\mathbb{L}^1(\pi)$. Let J be a subinterval of $[0,+\infty]$. We assume that, for every $\gamma\in J$, P_γ continuously acts on $\mathcal{B}$ and that*

(i) $r(\gamma) := r(P_{\gamma|\mathcal{B}})>0$, and P_γ is quasi-compact on $\mathcal{B}$ (i.e. $r_{ess}\big(P_{\gamma|\mathcal{B}}\big)<r(\gamma)$)
(ii) $r(\gamma)$ is the only eigenvalue of modulus $r(\gamma)$ for P_γ, and $r(\gamma)$ is a first order pole of P_γ with moreover $\dim\mathrm{Ker}(P_\gamma - r(\gamma)I) = \dim\mathrm{Ker}(P_\gamma - r(\gamma)I)^2 = 1$.

Theorem 2.8 will ensure that, under Hypothesis 2.5, for every $\gamma\in J$ there exists a rank-one non-negative projector $\Pi_\gamma\in\mathcal{L}(\mathcal{B})$ (i.e. the eigenprojector associated with the eigenvalue $r(\gamma)$), and some constants $\theta_\gamma\in(0,1)$ and $M_\gamma\in(0,+\infty)$ such that

$$\forall\gamma\in J,\quad \forall f\in\mathcal{B},\quad \big\|P_\gamma^n f - r(\gamma)^n\Pi_\gamma f\big\|_{\mathcal{B}} \leq M_\gamma\,\big(\theta_\gamma\, r(\gamma)\big)^n\|f\|_{\mathcal{B}}. \tag{12}$$

If moreover $\mu(\kappa e^{-\gamma\xi}\cdot)\in\mathcal{B}^*$, $P(h_{\kappa,\gamma})\in\mathcal{B}$, $B(\gamma) := \mu(\kappa\, e^{-\gamma\xi}\Pi_\gamma(Ph_{\kappa,\gamma}))$ is positive, then we deduce (9) with $A = A/r$ and $\rho = r$ from (11) in the specific case $K=\{\gamma\}$.

Remark 2.6. *To establish the multiplicative ergodicity of Definition 2.3, further regularity properties are needed. Due to (12), a natural way is to apply the perturbation theory of linear operators. Unfortunately, the classical operator perturbation method* [25, 26, 10, 11]. *does not apply to our context. Indeed, because we do not assume any exponential moment condition on ξ (contrarily to the papers mentioned in Introduction), the map $\gamma\mapsto P_\gamma$ is (in general)* **not continuous** *from $(0,+\infty)$ to $\mathcal{L}(\mathcal{B})$. For instance, for linear autoregressive models (Theorem*

2.13), we will work with Banach spaces $\mathcal{B}_a = \mathcal{C}_{V^a}$ linked to some weighted-supremum Banach spaces. For the Knudsen gas (Theorem 2.11), we will work with $\mathcal{B}_a = \mathbb{L}^a(\pi)$. In these two cases, the map $\gamma \mapsto P_\gamma$ is not continuous in general from $(0,+\infty)$ to $\mathcal{L}(\mathcal{B}_a)$, but only from $(0,+\infty)$ to $\mathcal{L}(\mathcal{B}_a,\mathcal{B}_b)$ for $a<b$ for the linear autoregressive models (and for $b<a$ for the Knudsen gas). This is the reason why we use the Keller-Liverani perturbation theorem [17]*. The price to pay is to consider "a chain of Banach spaces" instead of a single one.*

We introduce two sets of assumptions (Hypotheses 2.7 and 2.7*): both of them will be relevant for our examples in Sections 5 and 6 (Knudsen gas and linear autoregressive model). Below the notation $\mathcal{B}_0 \hookrightarrow \mathcal{B}_1$ means that $\mathcal{B}_0$ is continuously injected in $\mathcal{B}_1$.

Hypothesis 2.7. *Let $\mathcal{B}_0$ and $\mathcal{B}_1$ be two Banach spaces, let J be a subinterval of $[0,+\infty]$. We will say that $(P_\gamma, J, \mathcal{B}_0, \mathcal{B}_1)$ satisfies Hypothesis 2.7 if*

- $\mathcal{B}_0 \hookrightarrow \mathcal{B}_1$,
- *for every $\gamma \in J$, $P_\gamma \in \mathcal{L}(\mathcal{B}_0) \cap \mathcal{L}(\mathcal{B}_1)$,*
- *the map $\gamma \mapsto P_\gamma$ is continuous from J to $\mathcal{L}(\mathcal{B}_0,\mathcal{B}_1)$,*
- *there exist $c_0>0$, $\delta_0>0$, $M>0$ such that*

$$\forall \gamma \in J, \quad r_{ess}\big(P_{\gamma|\mathcal{B}_0}\big) \le \delta_0 \tag{13a}$$

$$\forall \gamma \in J,\ \forall n \ge 1,\ \forall f \in \mathcal{B}_0, \quad \|P_\gamma^n f\|_{\mathcal{B}_0} \le c_0\big(\delta_0^n \|f\|_{\mathcal{B}_0} + M^n \|f\|_{\mathcal{B}_1}\big) \tag{13b}$$

Hypothesis 2.7*. *$(P_\gamma, J, \mathcal{B}_0, \mathcal{B}_1)$ satisfies all the conditions of Hypothesis 2.7, except for (13a) and (13b) which are replaced by the following ones:*

$$\forall \gamma \in J, \quad r_{ess}\big((P_\gamma^*)_{|\mathcal{B}_1^*}\big) \le \delta_0 \tag{14a}$$

$$\forall \gamma \in J,\ \forall n \ge 1,\ \forall f^* \in \mathcal{B}_1^*, \quad \|(P_\gamma^*)^n f^*\|_{\mathcal{B}_1^*} \le c_0(\delta_0^n \|f^*\|_{\mathcal{B}_1^*} + M^n \|f^*\|_{\mathcal{B}_0^*}) \tag{14b}$$

Hypothesis 2.7* can be seen as a dual version of Hypothesis 2.7, but it is worth noticing that the conditions (14a)-(14b) cannot be deduced from (13a)-(13b) (and conversely). Under Hypothesis 2.7 or 2.7* we define the following set:

$$J_0 := \{\gamma \in J : r(\gamma) > \delta_0\}. \tag{15}$$

Theorem 2.8. *Let $\mathcal{B}_0 \hookrightarrow \mathcal{B}_3 \hookrightarrow \mathbb{L}^1(\pi)$ be two Banach spaces such that $\mathbf{1}_{\mathbb{X}} \in \mathcal{B}_0$, let J be a subinterval of $[0,+\infty]$. Assume that $(P_\gamma, J, \mathcal{B}_0, \mathcal{B}_3)$ satisfies Hypothesis 2.7 or 2.7* and that*

- *Hypothesis 2.5 holds on J_0 and $\mathcal{B} := \mathcal{B}_0$ under Hypothesis 2.7*
- *Hypothesis 2.5 holds on J_0 and $\mathcal{B} := \mathcal{B}_3$ under Hypothesis 2.7*.*

Then

$$\forall \gamma_0 \in J, \quad \limsup_{\gamma \to \gamma_0} r(\gamma) \le \max(\delta_0, r(\gamma_0))\,, \tag{16}$$

and the function $\gamma \mapsto r(\gamma) := r(P_{\gamma|\mathcal{B}})$ is continuous on J_0. Moreover there exists a map $\gamma \mapsto \Pi_\gamma$ from J_0 to $\mathcal{L}(\mathcal{B})$ which is continuous from J_0 to $\mathcal{L}(\mathcal{B}_0,\mathcal{B}_3)$ such that, for every compact subset K of J_0, there exist $\theta_K \in (0,1)$ and $M_K \in (0,+\infty)$ such that

$$\forall \gamma \in K,\ \forall f \in \mathcal{B}, \quad \big\|P_\gamma^n f - r(\gamma)^n \Pi_\gamma f\big\|_{\mathcal{B}} \le M_K \left(\theta_K\, r(\gamma)\right)^n \|f\|_{\mathcal{B}}. \tag{17}$$

Consequently, under the previous assumptions, the following assertions hold:

(i) If the maps $\gamma \mapsto Ph_{\kappa,\gamma}$ and $\gamma \mapsto \mu(\kappa e^{-\gamma\xi}\cdot)$ are continuous from J_0 to $\mathcal{B}_0$ and to $\mathcal{B}_3^*$ respectively, and if

$$\forall \gamma \in J_0, \quad B(\gamma) := \mu\left(\kappa\, e^{-\gamma\xi}\Pi_\gamma(Ph_{\kappa,\gamma})\right) > 0, \tag{18}$$

then $(S_n, \kappa(X_n))_n$ is multiplicatively ergodic on J_0 with $A(\gamma) := \frac{B(\gamma)}{r(\gamma)}$ and $\rho(\gamma) = r(\gamma)$.

(ii) If moreover $\inf_{\gamma\in J_0} r(\gamma) < 1 < \sup_{\gamma\in J_0} r(\gamma)$, then ν is finite and

$$\nu = \inf\{\gamma > 0 \,:\, r(\gamma) < 1\}. \tag{19}$$

Formula (17) can be interpreted as a **spectral multiplicative ergodicity property**. To prove the existence of C_ν, we reinforce our assumptions by considering a longer chain of Banach spaces.

Theorem 2.9. *Assume $\pi(\xi > 0) > 0$. Let $\mathcal{B}_0 \hookrightarrow \mathcal{B}_1 \hookrightarrow \mathcal{B}_2 \hookrightarrow \mathcal{B}_3 \hookrightarrow \mathbb{L}^1(\pi)$ be Banach spaces containing $\mathbf{1}_{\mathbb{X}}$ and let J be a subinterval of $[0, +\infty]$. Assume that one of the two following conditions holds*

(a) Either: for $i = 0, 1, 2$, $(P_\gamma, J, \mathcal{B}_i, \mathcal{B}_{i+1})$ satisfies Hypothesis 2.7, and Hypothesis 2.5 holds with $(J_0, \mathcal{B}_i)$; in this case we set $\mathcal{B} := \mathcal{B}_0$.
(b) Or: for $i = 0, 1, 2$, $(P_\gamma, J, \mathcal{B}_i, \mathcal{B}_{i+1})$ satisfies Hypothesis 2.7, and Hypothesis 2.5 holds with $(J_0, \mathcal{B}_{i+1})$; in this case we set $\mathcal{B} := \mathcal{B}_3$.*

Assume moreover that the map $\gamma \mapsto P_\gamma$ is continuous from J to $\mathcal{L}(\mathcal{B}_i, \mathcal{B}_{i+1})$ for $i \in \{0, 2\}$ and C^1 from J to $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ with derivative $P'_\gamma f = P_\gamma(-\xi f)$ ($f \mapsto \xi f$ being in $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$). Then (17) holds with C^1-smooth maps $\gamma \mapsto r(\gamma) := r(P_{\gamma|\mathcal{B}})$ and $\gamma \mapsto \Pi_\gamma$ from J_0 into $\mathbb{R}$ and into $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_3)$ respectively. Consequently, under the previous assumptions, the assertions (i)-(ii) in Theorem 2.8 can be specified and completed as follows:

(i') If the additional assumptions in Assertion (i) of Theorem 2.8 hold with the present spaces $\mathcal{B}_0$ and $\mathcal{B}_3$, then the functions $A(\cdot)$ and $\rho(\cdot) := r(\cdot)$ are C^1-smooth on J_0.
(ii') If moreover $\inf_{\gamma\in J_0} r(\gamma) < 1 < \sup_{\gamma\in J_0} r(\gamma)$ and if $r'(\nu) \neq 0$, then the constant C_ν of (7) is well defined and finite, and Property (8) holds true.

Note that the two above results require, not only to check the spectral property (13a) (or (14a)) and the Doeblin-Fortet inequalities (13b) (or (14b)), but also to prove (18) and moreover $r'(\nu) > 0$ in Theorem 2.9. This will be discussed in section 4.

2.3. **Markovian examples.** Here we present two examples which will be derived from the general results of the previous section (see Sections 5 and 6 for the proofs). The first one is a toy model on which our general spectral assumptions are easily checked. In this model, at each step, either we follow a Markov chain $Z = (Z_n)_n$ (with probability $(1 - \alpha)$) or we generate an independent random variable with distribution the invariant probability measure of Z (with probability α). See [3] for more about this model.

Example 2.10 (Knudsen gas)**.** *Let $\mathbb{X} := \mathbb{R}^d$, let π be some Borel probability measure on $\mathbb{X}$, and let U a Markov operator with stationary probability π. We fix $\alpha \in (1/2, 1)$. Let $X = (X_n)_n$ be a Markov chain with transition kernel $P := \alpha\pi + (1 - \alpha)\, U$.*

Theorem 2.11. *Assume that $X = (X_n)_n$ is a Knudsen gas with initial distribution admitting a density with respect to π, which belongs to $\mathcal{L}^p(\pi)$ for some $p > 1$. Moreover assume that $\kappa \equiv 2$ and that $\pi(\xi > 0) = 1$. Then ν defined by (4) is finite. If, moreover, $\pi(\xi^\tau) < \infty$ for some $\tau \in (1, p/(p-1))$, then (7) is well defined and Property (8) holds with $C_\nu \in (0, +\infty)$.*

Example 2.12 (Linear autoregressive model)**.** *$\mathbb{X} := \mathbb{R}$ and $X_n = \alpha X_{n-1} + \vartheta_n$ for $n \geq 1$, where X_0 is a real-valued random variable, $\alpha \in (-1, 1)$, and $(\vartheta_n)_{n\geq 1}$ is a sequence of i.i.d. real-valued random variables independent of X_0. Let $r_0 > 0$. We assume that ϑ_1 has a continuous Lebesgue probability density function $p > 0$ on $\mathbb{X}$ satisfying the following condition: for all $x_0 \in \mathbb{R}$, there exist a neighbourhood V_{x_0} of x_0 and a non-negative function $q_{x_0}(\cdot)$ such that $y \mapsto (1 + |y|)^{r_0}\, q_{x_0}(y)$ is Lebesgue-integrable and such that :*

$$\forall y \in \mathbb{R},\ \forall v \in V_{x_0},\ p(y + v) \leq q_{x_0}(y). \tag{20}$$

Note that ϑ_1 admits a moment of order r_0.

Recall that $\xi : \mathbb{R} \to [0, +\infty)$ is said to be coercive if $\lim_{|x|\to+\infty} \xi(x) = +\infty$, i.e. if, for every β, the set $[\xi \leq \beta]$ is bounded.

Theorem 2.13 (Linear autoregressive model)**.** *Assume that $X = (X_n)_n$ is a linear autoregressive model satisfying the above assumption and that the distribution of X_0 is either the stationary probability measure π or δ_x for some $x \in \mathbb{R}$. Let N_0 be a positive integer. Assume that κ is bounded, that ξ is coercive, that the Lebesgue measure of the set $[\xi = 0]$ is zero, and that $\sup_{x\in\mathbb{R}} \frac{\xi(x)}{(1+|x|)^{r_0}} < \infty$.*

Then, ν given by (4) is well defined (and is independent of the choice of the distribution of X_0 as above). If moreover there exists $\tau > 0$ such that $\sup_{x\in\mathbb{R}} \frac{\xi(x)^{1+\tau}}{(1+|x|)^{r_0}} < \infty$, then the constant C_ν given by (7) is well defined in $(0, +\infty)$ and Property (8) holds.

Recall that $\int_{\mathbb{R}} |x|^{r_0}\, d\pi(x) < \infty$ under the assumptions of Theorem 2.13 (see [5, 6]). Hence $\sup_{x\in\mathbb{R}} \frac{\xi(x)^{1+\tau}}{(1+|x|)^{r_0}} < \infty$ implies that $\int_{\mathbb{R}} |\xi|^{1+\tau}\, d\pi < \infty$.

3. Behavior of the second moment and almost sure convergence

3.1. Second moment and applications.

We make some stationarity assumption.

Hypothesis 3.1. *For each $k \in \mathbb{N}$, there exists a process $X^{(k)} = (X_n^{(k)})_{n\geq 0}$ such that*

$$\begin{cases} (X_n^{(k)})_{0\leq n\leq k} = (X_n)_{0\leq n\leq k} & \text{a.s.} \\ (X_n^{(k)})_{n\geq 0} = (X_n)_{n\geq 0} & \text{in law,} \end{cases} \tag{21}$$

and such that, for every couple of lines of cells $((v_n)_n, (w_n)_n)$ coinciding up to the k-th generation but not at the $(k+1)$-th generation, the corresponding sequence of parameters have the same distribution as $(X, X^{(k)})$.

Now define, for any integers $n \geq 1, m \geq 1$ and $\min(n,m)-1 \geq k \geq 0$ the random variables $A_{n,m,k}$ as follows:

$$A_{n,m,k} = \left(\prod_{i=0}^{n-2} \kappa(X_i)\right) \left(\prod_{j=\min(k+1,n-1)}^{m-2} \kappa(X_j^{(k)})\right) \left(\prod_{j\in\{k\}\setminus\{n-1,m-1\}} (\kappa(X_j)-1)\right) (\kappa(X_{n-1})-1)\left(\kappa(X_{m-1}^{(k)})-1\right),$$

with the usual convention $\prod_{i=k+1}^{\ell} \cdots = 1$ if $\ell \leq k$. Define also $S_n^{(k)} := \sum_{j=0}^{n} \xi(X_j^{(k)})$. The main result of this section is the following proposition.

Proposition 3.2. *Assume that Hypothesis 3.1 holds. Let $t > 0$ and $\tau \geq 0$ be fixed. If $\mathbb{E}(N_{t+\tau}) < \infty$, then*

$$\mathbb{E}[N_t N_{t+\tau}] = \mathbb{E}[N_t] + \mathbb{E}[N_{t+\tau}] - 1 + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{\min(n,m)-1} \mathbb{E}\left[A_{n,m,k}\mathbf{1}_{\{S_{n-1}\leq t, S_{m-1}^{(k)}\leq t+\tau\}}\right]. \quad (22)$$

Proof. We have, using the notations of the proof of Proposition 2.1, $N_t = \mathbf{1}_{\{T_0>t\}} + \sum_{n=1}^{\infty} \Sigma_n(t) \quad a.s.$, with

$$\Sigma_n(t) = \sum_{k_1=1}^{D_0}\sum_{k_2=1}^{D_{0,k_1}} \cdots \sum_{k_n=1}^{D_{0,k_1,\cdots,k_{n-1}}} \mathbf{1}_{\{T_0+T_{0,k_1}+\cdots+T_{0,k_1,\cdots,k_{n-1}}\leq t\}} - \mathbf{1}_{\{T_0+T_{0,k_1}+\cdots+T_{0,k_1,\cdots,k_n}\leq t\}}.$$

Now due to $\mathbb{E}(N_t) < \infty$ (see the proof of Proposition 2.1),

$$\sum_{k_1=1}^{D_0}\sum_{k_2=1}^{D_{0,k_1}} \cdots \sum_{k_n=1}^{D_{0,k_1,\cdots,k_{n-1}}} \mathbf{1}_{\{T_0+T_{0,k_1}+\cdots+T_{0,k_1,\cdots,k_{n-1}}\leq t\}} < \infty \quad a.s..$$

Therefore $N_t = 1 + \sum_{n=1}^{\infty} \tilde{\Sigma}_n(t) \quad a.s.$, with

$$\tilde{\Sigma}_n(t) = \sum_{k_1=1}^{D_0}\sum_{k_2=1}^{D_{0,k_1}} \cdots \sum_{k_{n-1}=1}^{D_{0,k_1,\cdots,k_{n-2}}} (D_{0,k_1,\cdots,k_{n-1}}-1)\mathbf{1}_{\{T_0+T_{0,k_1}+\cdots+T_{0,k_1,\cdots,k_{n-1}}\leq t\}}.$$

Consequently,

$$\begin{aligned} N_t N_{t+\tau} &= 1 + \sum_{n=1}^{\infty}\tilde{\Sigma}_n(t) + \sum_{n=1}^{\infty}\tilde{\Sigma}_n(t+\tau) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\tilde{\Sigma}_n(t)\tilde{\Sigma}_m(t+\tau) \\ &= N_t + N_{t+\tau} - 1 + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\tilde{\Sigma}_n(t)\tilde{\Sigma}_m(t+\tau). \end{aligned}$$

For any positive integers n, m and $t > 0, \tau \geq 0$, we have,

$$\tilde{\Sigma}_n(t)\tilde{\Sigma}_m(t+\tau) = \sum_{k=0}^{\min(n-1,m-1)} \sum_{(\boldsymbol{\ell},\tilde{\boldsymbol{\ell}})\in E_{n,m,k}} (D_{\boldsymbol{\ell}}-1)(D_{\tilde{\boldsymbol{\ell}}}-1)\mathbf{1}_{\{\mathcal{S}_{n-1}(\boldsymbol{\ell})\leq t, \mathcal{S}_{m-1}(\tilde{\boldsymbol{\ell}})\leq t+\tau\}}$$

where $E_{n,m,k}$ is the set of $(\ell,\tilde{\ell})$, with $\boldsymbol{\ell} = (0,\ell_1,...,\ell_{n-1}) \in (\mathbf{N}^*)^n$ and $\tilde{\boldsymbol{\ell}} = (0,\tilde{\ell}_1,...,\tilde{\ell}_{m-1}) \in (\mathbf{N}^*)^m$ having the same coordinates up to time k, i.e. such that $\min\{j = 0,...,\min(n,m)$:

$\ell_j \neq \tilde{\ell}_j\} = k+1$, with the notation $\mathcal{S}_{n-1}(\boldsymbol{\ell}) := T_0 + T_{0,\ell_1} + \cdots + T_{0,\ell_1,\cdots,\ell_{n-1}}$. We conclude by proceding exactly as in the proof of Proposition 2.1. □

As it was done in [12] in the case of independence (cf. Lemma 19.1 and Theorem 21.1 there), Proposition 3.2 is the main ingredient for the proofs of the quadratic mean and of the almost sure convergence of $e^{-\nu t}N_t$, for the growth rate ν already defined in (4).

Corollary 3.3. *Assume Hypothesis 3.1, that $\nu < \infty$, that $\limsup_{t\to\infty} e^{-\nu t}\mathbb{E}[N_t] < \infty$ and that there exists $K > 0$ such that*

$$\lim_{t\to\infty} \sup_{\tau\geq 0} \left| e^{-\nu(2t+\tau)} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sum_{k=0}^{\min(n,m)-1} \mathbb{E}\left[A_{n,m,k}\mathbf{1}_{\{S_{n-1}\leq t, S^{(k)}_{m-1}\leq t+\tau\}}\right] - K \right| = 0. \tag{23}$$

Then there exists a square integrable random variable W such that $(e^{-\nu t}N_t)_{t\geq 0}$ converges in quadratic mean to W as t tends to infinity.

If moreover the convergence in (23) is exponentially fast and if $W > 0$, then $(e^{-\nu t}N_t)_{t\geq 0}$ converges almost surely to W as t tends to infinity.

Proof of Corollary 3.3. Due to Corollary 3.3 and Proposition 3.2,

$$\lim_{t\to\infty} \mathbb{E}\left[\left(e^{-\nu t}N_t - e^{-\nu(t+\tau)}N_{t+\tau}\right)^2\right] = 0,$$

for any $\tau \geq 0$, uniformly in τ. The Cauchy criterion ensures then the convergence in quadratic mean of $e^{-\nu t}N_t$ as t tends to infinity to a random variable W with finite second moment.

For the last point, we deduce from Proposition 3.2 that $\int_0^\infty \mathbb{E}\left[\left(e^{-\nu t}N_t - W\right)^2\right] dt < \infty$. This yields (arguing as for the proof of Theorem 21.1 in [12]) the almost sure convergence, as t tends to infinity, of $e^{-\nu t}N_t$ to W. □

We will apply these results in the two following sections under some additional independence assumptions.

3.2. **Some extensions of Harris' results.** For further results, we will make the following stronger assumption.

Hypothesis 3.4. *$(X_n)_n$ is a stationary Markov process, $(\kappa(X_n))_n$ is a sequence of i.i.d. square integrable random variables of expectation κ_1, which is independent of $(\xi(X_n))_n$. Moreover, for all $k \in \mathbb{N}$, $(X_n^{(k)})_{n\geq k+1}$ and $(X_n)_{n\geq k+1}$ are independent given X_k and ν (as defined in (4)) satisfies,*

$$\forall x \in \mathbb{X}, \quad \nu = \inf\left\{\gamma > 0,\ \sum_{n\geq 0} \kappa_1^n \mathbb{E}\left[e^{-\gamma S_{n+1}} | X_0 = x\right] < \infty\right\} < \infty\,. \tag{24}$$

We set $\kappa_2 := \mathbb{E}[\kappa(X_1)(\kappa(X_1)-1)]$.

Define $f_{x,0}(t) = (\kappa_1 - 1)e^{-\nu t} \sum_{n\geq 0} \kappa_1^n \mathbb{P}\left(S_{n+1} - S_0 \leq t | X_0 = x\right)$. We will make the following assumption involving the Laplace transform $\tilde{f}_{x,0}$ of $f_{x,0}$:

$$\forall \gamma > 0, \quad \tilde{f}_{x,0}(\gamma) = \int_0^\infty e^{-\gamma t} f_{x,0}(t)dt = \frac{\kappa_1 - 1}{\gamma + \nu} \sum_{n\geq 0} \kappa_1^n \mathbb{E}\left[e^{-(\gamma+\nu)(S_{n+1}-S_0)} | X_0 = x\right]. \quad (25)$$

Hypothesis 3.5. *Suppose that there exist two positive reals $\delta < \nu$ and ϵ such that, for any x, the Laplace transform $\tilde{f}_{x,0}$, extended on the complex plane, satisfies the following conditions:*

(1) $\tilde{f}_{x,0}$ *is analytic in* $\{z = u + iy, |u| < \delta + \epsilon, y \in \mathbb{R}\} \setminus \{0\}$,
(2) $\tilde{f}_{x,0}$ *has a simple pole at* 0, *with residue* $\tilde{C}_0(x)$,
(3) $\int_{-\infty}^{+\infty} |\tilde{f}_{x,0}(\delta + iy)|dy < \infty$,
(4) $\lim_{y\to\pm\infty} \tilde{f}_{x,0}(u + iy) = 0$, *uniformly in* $u \in [-\delta, \delta]$,
(5) $\Psi_0(x) := \int_{-\infty}^{+\infty} |\tilde{f}_{x,0}(-\delta + iy)|dy < \infty$.

We generalize the approach of [23] to obtain the following result extending [12, Theorem 19.1] to the case where the lifetimes are dependent.

Theorem 3.6. *Assume that Hypotheses 3.4 and 3.5 are satisfied and that*

$$\mathbb{E}\left[e^{-\nu\xi(X_0)}\tilde{C}_0(X_0)\right] + \mathbb{E}\left[e^{-(\nu-\delta)\xi(X_0)}\Psi_0(X_0)\right] + \sum_{k=0}^{\infty} \kappa_1^k \mathbb{E}\left[\tilde{C}_0^2(X_k)e^{-2\nu S_k}\right] < \infty,$$

$$\text{and} \quad \sum_{k=0}^{\infty} \kappa_1^k \mathbb{E}\left[\tilde{C}_0(X_k)\Psi_0(X_k)e^{-(2\nu-\delta)S_k} + \Psi_0^2(X_k)e^{-2(\nu-\delta)S_k}\right] < \infty,$$

$$\sum_{k=0}^{\infty} \kappa_1^k \mathbb{E}\left[(\tilde{C}_0(X_k) + \Psi_0(X_k) + k)e^{-(\nu+\delta)S_k}\right] < \infty, \quad (26)$$

then there exists a square integrable random variable W such that $(e^{-\nu t}N_t)_{t\geq 0}$ converges in quadratic mean to W as t tends to infinity, with

$$\mathbb{E}[W] = \kappa_1 \mathbb{E}\left[e^{-\nu\xi(X_0)}\tilde{C}_0(X_0)\right]$$

$$Var(W) = \kappa_2 \sum_{k=0}^{\infty} \kappa_1^k \mathbb{E}\left[\tilde{C}_0^2(X_k)e^{-2\nu S_k}\right] - \kappa_1^2 \left(\mathbb{E}\left[e^{-\nu\xi(X_0)}\tilde{C}_0(X_0)\right]\right)^2.$$

If, moreover, $W > 0$ almost surely then $e^{-\nu t}N_t$ converges almost surely to W.

The rest of this subsection is devoted to the proof of Theorem 3.6. Define, for $r \leq m$, the partial sums $S_{r,m} := \sum_{i=r}^m \xi(X_i)$, so that $S_n = S_{0,n}$ and $S_{r,m}^{(k)} := \sum_{i=r}^m \xi(X_i^{(k)})$. The following lemma is very useful in order to prove the exponential speed of convergence in (23) and then to obtain the almost sure convergence of $e^{-\nu t}N_t$. Its proof is an immediate consequence of Lemma 2.2 in [23] and we omit it.

Lemma 3.7. *Assume Hypothesis 3.5. Then, for any $t > 0$,*

$$\left| e^{-\nu t} \sum_{n\geq 0} \kappa_1^n (\kappa_1 - 1)\mathbb{P}(S_{1,n+1} \leq t | X_0 = x) - \tilde{C}_0(x) \right| \leq \frac{\Psi_0(x)}{2\pi} e^{-\delta t}. \quad (27)$$

Lemma 3.7 together with Proposition 2.1 allows to give an exact asymptotic behavior of $\mathbb{E}(N_t)$ as t tends to infinity, as shows the following proposition.

Proposition 3.8. *Suppose that $(X_n)_{n\geq 0}$ is a stationary sequence of random variables, that $(\kappa(X_i))_{i\geq 0}$ is a sequence of i.i.d. random variables with finite first moment. Let $\kappa_1 = \mathbb{E}(\kappa(X_1))$. Suppose moreover that $(\kappa(X_i))_{i\geq 0}$ is independent of $(\xi(X_i))_{i\geq 0}$, that ν satisfies (24) and that Hypothesis 3.5 is satisfied for some positive δ strictly less than ν. If $\mathbb{E}(e^{-\nu\xi(X_0)}\tilde{C}_0(X_0)) < \infty$ and if $\mathbb{E}(e^{-(\nu-\delta)\xi(X_0)}\Psi_0(X_0)) < \infty$, then $\mathbb{E}(N_t) < \infty$ and there exists $\epsilon_1 > 0$ such that*

$$\mathbb{E}(N_t) = e^{\nu t}\kappa_1\mathbb{E}(e^{-\nu\xi(X_0)}\tilde{C}_0(X_0))[1 + O(e^{-\epsilon_1 t})], \text{ as } t \to \infty\,.$$

Proof. Recall that $f_{x,0}(u) = e^{-\nu u}\sum_{n\geq 1}\kappa_1^{n-1}(\kappa_1 - 1)\mathbb{P}(S_{1,n} \leq u|X_0 = x)$. Moreover

$$\begin{aligned}
&\sum_{n\geq 1}\kappa_1^n(\kappa_1 - 1)\mathbb{P}(S_n \leq t) = \mathbb{E}\left[\sum_{n\geq 1}\kappa_1^n(\kappa_1 - 1)\mathbb{P}(S_{1,n} \leq t - \xi(X_0)|X_0)\mathbf{1}_{\{\xi(X_0)\leq t\}}\right]\\
&= e^{\nu t}\kappa_1\mathbb{E}\left[e^{-\nu\xi(X_0)}f_{X_0,0}(t - \xi(X_0))\mathbf{1}_{\{\xi(X_0)\leq t\}}\right]\\
&= e^{\nu t}\kappa_1\mathbb{E}\left[e^{-\nu\xi(X_0)}\left(f_{X_0,0}(t - \xi(X_0)) - \tilde{C}_0(X_0)\right)\mathbf{1}_{\{\xi(X_0)\leq t\}}\right] + e^{\nu t}\kappa_1\mathbb{E}\left[e^{-\nu\xi(X_0)}\tilde{C}_0(X_0)\mathbf{1}_{\{\xi(X_0)\leq t\}}\right].
\end{aligned}$$

Now Lemma 3.7 gives,

$$\mathbb{E}\left[e^{-\nu\xi(X_0)}\left(f_{X_0,0}(t - \xi(X_0)) - \tilde{C}_0(X_0)\right)\mathbf{1}_{\{\xi(X_0)\leq t\}}\right] \leq \frac{e^{-\delta t}}{2\pi}\mathbb{E}\left[\Psi_0(X_0)e^{-(\nu-\delta)\xi(X_0)}\right].$$

Consequently $\sum_{n\geq 1}\kappa_1^n(\kappa_1 - 1)\mathbb{P}(S_n \leq t) < \infty$ by the requirements of this proposition and

$$\sum_{n\geq 1}\kappa_1^n(\kappa_1 - 1)\mathbb{P}(S_n \leq t) = e^{\nu t}\kappa_1\left[\mathbb{E}\left(e^{-\nu\xi(X_0)}\tilde{C}_0(X_0)\mathbf{1}_{\{\xi(X_0)\leq t\}}\right) + e^{-\delta t}D(t)\right] \tag{28}$$

with $0 < D(t) \leq (2\pi)^{-1}\mathbb{E}\left(\Psi_0(X_0)e^{-(\nu-\delta)\xi(X_0)}\right)$. This allows to deduce thanks to Proposition 2.1 that $\mathbb{E}(N_t) < \infty$ and

$$\mathbb{E}(N_t) = e^{\nu t}\kappa_1\mathbb{E}[e^{-\nu\xi(X_0)}\tilde{C}_0(X_0)\mathbf{1}_{\{\xi(X_0)\leq t\}}]\left[1 + e^{-\nu t}A(t) + e^{-\delta t}D(t)\right],$$

with $0 < A(t) < \sup_{t>0} A(t) < \infty, 0 < D(t) < \sup_{t>0} D(t) < \infty$. □

The following proposition gives an exact asymptotic behavior of $\mathbb{E}(N_tN_{t+\tau})$ under further assumptions. Theorem 3.6 follows directly from the following proposition combined with Corollary 3.3.

Proposition 3.9. *Assume that Hypotheses 3.4 and 3.5 are satisfied. Suppose also that $\sum_{k=0}^{\infty}\kappa_1^k\mathbb{E}(\tilde{C}_0^2(X_k)e^{-2\nu S_k}) < \infty$. If Condition (26) is satisfied, then, for any $t > 0$, $\tau \geq 0$, $\mathbb{E}(N_tN_{t+\tau}) < \infty$ and*

$$\mathbb{E}(N_tN_{t+\tau}) = e^{\nu(2t+\tau)}\kappa_2\sum_{k=0}^{\infty}\kappa_1^k\mathbb{E}\left(\tilde{C}_0^2(X_k)e^{-2\nu S_k}\right)[1 + ae^{-\epsilon_1 t}], \text{ as } t \to \infty,$$

where a and ϵ_1 are positive constants independent of t and τ.

Proof. The task is to apply Proposition 3.2. For $0 \leq k \leq \min(n,m)-1$, we have

$$\mathbb{E}\left[A_{n,m,k}\mathbf{1}_{\{S_{n-1}\leq t,\ S^{(k)}_{m-1}\leq t+\tau\}}\right] = \mathbb{E}\left[A_{n,m,k}\right]\mathbb{P}\left(S_{n-1}\leq t,\ S^{(k)}_{m-1}\leq t+\tau\right).$$

Observe that

$$\mathbb{E}\left[A_{n,m,k}\right] = \begin{cases} \mathbb{E}[(\kappa(X_0)-1)^2]\kappa_1^{n-1} & \text{if } k=n-1=m-1 \\ \kappa_2(\kappa_1-1)\kappa_1^{\max(n-2,m-2)} & \text{if } (k=m-1 \textit{ ou } k=n-1),\ n\neq m \\ \kappa_2(\kappa_1-1)^2\kappa_1^{n-2+m-2-k} & \text{if } (k\neq m-1 \textit{ et } k\neq n-1). \end{cases}$$

Now we have for $k\neq m-1$,

$$\begin{aligned} &\mathbb{P}\left(S_{n-1}\leq t,\ S^{(k)}_{m-1}\leq t+\tau\right) = \\ &\mathbb{E}\left[\mathbb{P}\left(S_{k+1,n-1}\leq t-S_k,\ \ S^{(k)}_{k+1,m-1}\leq t+\tau-S_k|S_k,X_k\right)\mathbf{1}_{\{S_k\leq t\}}\right] \\ &= \mathbb{E}\left[\mathbb{P}\left(S_{k+1,n-1}\leq t-S_k|S_k,X_k\right)\mathbb{P}\left(S^{(k)}_{k+1,m-1}\leq t+\tau-S_k|S_k,X_k\right)\mathbf{1}_{\{S_k\leq t\}}\right]. \end{aligned}$$

When $k=m-1$ (and then necessarily $m\leq n$)

$$\mathbb{P}\left(S_{n-1}\leq t,\ S_{m-1}\leq t+\tau\right) = \mathbb{P}\left(S_{n-1}\leq t\right).$$

Consequently,

$$\begin{aligned} &\sum_{n=1}^{\infty}\sum_{m=1}^{n}\mathbb{E}\left[A_{n,m,m-1}\mathbf{1}_{\{S_{n-1}\leq t,\ S_{m-1}+S^{(m-1)}_{m,m-1}\leq t+\tau\}}\right] \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{n-1}\kappa_1^{n-2}(\kappa_1-1)\kappa_2\mathbb{P}\left(S_{n-1}\leq t\right) + \sum_{n=1}^{\infty}\kappa_1^{n-1}\mathbb{E}[(\kappa(X_0)-1)^2]\mathbb{P}\left(S_{n-1}\leq t\right) \\ &= (\kappa_1-1)\kappa_2\sum_{n=1}^{\infty}(n-1)\kappa_1^{n-2}\mathbb{P}\left(S_{n-1}\leq t\right) + \mathbb{E}[(\kappa(X_0)-1)^2]\sum_{n=1}^{\infty}\kappa_1^{n-1}\mathbb{P}\left(S_{n-1}\leq t\right), \end{aligned}$$

and for any $0<\delta<\nu$

$$\begin{aligned} &e^{-\nu(2t+\tau)}\sum_{n=1}^{\infty}\sum_{m=1}^{n}\mathbb{E}\left[A_{n,m,m-1}\mathbf{1}_{\{S_{n-1}\leq t,\ S_{m-1}+S^{(m-1)}_{m,m-1}\leq t+\tau\}}\right] \\ &\leq e^{-t(\nu-\delta)}(\kappa_1-1)\kappa_2\sum_{n=1}^{\infty}(n-1)\kappa_1^{n-2}\mathbb{E}[e^{-(\nu+\delta)S_{n-1}}] \\ &+e^{-t(\nu-\delta)}\mathbb{E}[(\kappa(X_0)-1)^2]\sum_{n=1}^{\infty}\kappa_1^{n-1}\mathbb{E}[e^{-(\nu+\delta)S_{n-1}}]. \end{aligned} \tag{29}$$

We also have,

$$\begin{aligned} &\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\sum_{k=0}^{\min(n,m)-2}\mathbb{E}\left[A_{n,m,k}\mathbf{1}_{\{S_{n-1}\leq t,\, S_k+S^{(k)}_{k+1,m-1}\leq t+\tau\}}\right] \\ &= \kappa_2\sum_{k=0}^{\infty}\kappa_1^{k}\mathbb{E}\Bigg[\sum_{n\geq k+2}\kappa_1^{n-2-k}(\kappa_1-1)\mathbb{P}\left(S_{k+1,n-1}\leq t-S_k|S_k,X_k\right) \\ &\times\sum_{m\geq k+2}\kappa_1^{m-2-k}(\kappa_1-1)\mathbb{P}\left(S^{(k)}_{k+1,m-1}\leq t+\tau-S_k|S_k,X_k\right)\mathbf{1}_{\{S_k\leq t\}}\Bigg] \end{aligned} \tag{30}$$

Now the bound (27) gives,

$$\left| e^{-\nu(t-\sum_{i=0}^{k}\xi(x_i))} \sum_{n\geq k+2} \kappa_1^{n-2-k}(\kappa_1-1)\mathbb{P}\left(S_{k+1,n-1}\leq t-\sum_{i=0}^{k}\xi(x_i)|X_k=x_k\right)-\tilde{C}_0(x_k)\right|$$
$$\leq \Psi_0(x_k)e^{-\delta(t-\sum_{i=0}^{k}\xi(x_i))}$$
$$\left| e^{-\nu(t+\tau-\sum_{i=0}^{k}\xi(x_i))} \sum_{m\geq k+2} \kappa_1^{m-2-k}(\kappa_1-1)\mathbb{P}\left(S^{(k)}_{k+1,m-1}\leq t+\tau-\sum_{i=0}^{k}\xi(x_i)|X_k=x_k\right)-\tilde{C}_0(x_k)\right|$$
$$\leq \Psi_0(x_k)e^{-\delta(t+\tau-\sum_{i=0}^{k}\xi(x_i))}.$$

The two last bounds together with (30) give then

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\sum_{k=0}^{\min(n,m)-2}\mathbb{E}\left[A_{n,m,k}\mathbf{1}_{\{S_{n-1}\leq t,\ S_k+S^{(k)}_{k+1,m-1}\leq t+\tau\}}\right]$$
$$= e^{\nu(2t+\tau)}\kappa_2\left[\sum_{k=0}^{\infty}\kappa_1^k\mathbb{E}(\tilde{C}_0^2(X_k)e^{-2\nu S_k}\mathbf{1}_{S_k\leq t})+e^{-\delta t}a\right], \tag{31}$$

where $a\leq 2\sum_{k=0}^{\infty}\kappa_1^k\mathbb{E}(\tilde{C}_0(X_k)\Psi_0(X_k)e^{-(2\nu-\delta)S_k})+e^{-\delta t}\sum_{k=0}^{\infty}\kappa_1^k\mathbb{E}(\Psi_0^2(X_k)e^{-2(\nu-\delta)S_k})$.
Finally,

$$\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}\mathbb{E}\left(A_{n,m,n-1}\mathbf{1}_{\{S_{n-1}\leq t,\ S_{n-1}+S^{(k)}_{n,m-1}\leq t+\tau\}}\right) \tag{32}$$
$$=\kappa_2(\kappa_1-1)\sum_{n=1}^{\infty}\mathbb{E}\left[\sum_{m=n+1}^{\infty}\kappa_1^{m-2}\mathbb{P}\left(S^{(n-1)}_{n,m-1}\leq t+\tau-S_{n-1}|S_{n-1},X_{n-1}\right)\mathbf{1}_{\{S_{n-1}\leq t\}}\right]$$
$$=\kappa_2(\kappa_1-1)\mathbb{E}\left[\sum_{n=1}^{\infty}\kappa_1^{n-1}\mathbf{1}_{\{S_{n-1}\leq t\}}\sum_{m=n+1}^{\infty}\kappa_1^{m-n-1}\mathbb{P}\left(S^{(n-1)}_{n,m-1}\leq t+\tau-S_{n-1}|S_{n-1},X_{n-1}\right)\right]$$

Now the bound (27) gives

$$\left|(\kappa_1-1)e^{-\nu(t+\tau-\sum_{i=0}^{n-1}\xi(x_i))}\sum_{m\geq n+1}\kappa_1^{m-n-1}\right.$$
$$\left.\mathbb{P}\left(S^{(n-1)}_{n,m-1}\leq t+\tau-\sum_{i=0}^{n-1}\xi(x_i)|X_{n-1}=x_{n-1}\right)-\tilde{C}_0(x_{n-1})\right|$$
$$\leq \Psi_0(x_{n-1})e^{-\delta(t+\tau-\sum_{i=0}^{n-1}\xi(x_i))}. \tag{33}$$

We have also,

$$e^{-\nu t}\sum_{n\geq 1}\kappa_1^{n-1}\mathbb{E}\left(\tilde{C}_0(X_{n-1})e^{-\nu S_{n-1}}\mathbf{1}_{\{S_{n-1}\leq t\}}\right)\leq e^{-(\nu-\delta)t}\sum_{n\geq 1}\kappa_1^{n-1}\mathbb{E}\left[\tilde{C}_0(X_{n-1})e^{-(\nu+\delta)S_{n-1}}\right], \tag{34}$$

and

$$\begin{aligned}
&e^{-\nu t}\mathbb{E}\left[\sum_{n\geq 1}\kappa_1^{n-1}\mathbf{1}_{\{S_{n-1}\leq t\}}\Psi_0(X_{n-1})e^{-\nu S_{n-1}}e^{-\delta(t+\tau-S_{n-1})}\right]\\
&\leq e^{-(\nu+\delta)t}\sum_{n\geq 1}\kappa_1^{n-1}\mathbb{E}\left(\mathbf{1}_{\{S_{n-1}\leq t\}}\Psi_0(X_{n-1})e^{-(\nu-\delta)S_{n-1}}\right)\\
&\leq e^{-(\nu-\delta)t}\sum_{n\geq 1}\kappa_1^{n-1}\mathbb{E}\left[\Psi_0(X_{n-1})e^{-(\nu+\delta)S_{n-1}}\right] \qquad (35)
\end{aligned}$$

We get collecting Inequalities (32), (33), (34) and (35)

$$\begin{aligned}
&e^{-\nu(2t+\tau)}\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}\mathbb{E}\left(A_{n,m,n-1}\mathbf{1}_{\{S_{n-1}\leq t,\ S_{n-1}+S^{(k)}_{n,m-1}\leq t+\tau\}}\right)\\
&\leq e^{-(\nu-\delta)t}\kappa_2\sum_{n\geq 1}\kappa_1^{n-1}\mathbb{E}\left((\tilde{C}_0(X_{n-1})+\Psi_0(X_{n-1}))e^{-(\nu+\delta)S_{n-1}}\right). \qquad (36)
\end{aligned}$$

We then obtain combining (29), (31) and (36),

$$\begin{aligned}
&e^{-\nu(2t+\tau)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{\min(n,m)-1}\mathbb{E}\left[A_{n,m,k}\mathbf{1}_{\{S_{n-1}\leq t,\ S_k+S^{(k)}_{k+1,m-1}\leq t+\tau\}}\right]\\
&=\kappa_2\sum_{k=0}^{\infty}\kappa_1^k\mathbb{E}[\tilde{C}_0^2(X_k)e^{-2\nu S_k}\mathbf{1}_{\{S_k\leq t\}}]+e^{-(\nu-\delta)t}A+e^{-\delta t}\kappa_2 a, \qquad (37)
\end{aligned}$$

where a is as defined in (31) and

$$\begin{aligned}
A\leq\ &\kappa_2\sum_{n\geq 1}\kappa_1^{n-1}\mathbb{E}\left[(\tilde{C}_0(X_{n-1})+\Psi_0(X_{n-1}))e^{-(\nu+\delta)S_{n-1}}\right]\\
&+(\kappa_1-1)\kappa_2\sum_{n=1}^{\infty}(n-1)\kappa_1^{n-2}\mathbb{E}(e^{-(\nu+\delta)S_{n-1}})+\mathbb{E}[(\kappa(X_0)-1)^2]\sum_{n=1}^{\infty}\kappa_1^{n-1}\mathbb{E}[e^{-(\nu+\delta)S_{n-1}}].
\end{aligned}$$

Consequently, we obtain collecting (37) together with (22) and Proposition 3.8, that there exists $\epsilon_1>0$, such that, for any $t,\tau\geq 0$,

$$\mathbb{E}[N_tN_{t+\tau}]=e^{\nu(2t+\tau)}\kappa_2\sum_{k=0}^{\infty}\kappa_1^k\mathbb{E}[\tilde{C}_0^2(X_k)e^{-2\nu S_k}](1+c_{t,\tau}e^{-\epsilon_1 t}),\ \text{as } t\to\infty,$$

where $\sup_{t,\tau}c_{t,\tau}<\infty$. □

Proof of Theorem 3.6. The bound (37) proves that the convergence in (23) is exponentially fast and satisfied with

$$K=\kappa_2\sum_{k=0}^{\infty}\kappa_1^k\mathbb{E}[\tilde{C}_0^2(X_k)e^{-2\nu S_k}].$$

The proof of Theorem 3.6 is complete by using Corollary 3.3, Proposition 3.9, Proposition 3.8, and the fact that,

$$\mathbb{E}[W] = \lim_{t\to\infty} \mathbb{E}[e^{-\nu t} N_t)] = \kappa_1 \mathbb{E}[e^{-\nu\xi(X_0)} \tilde{C}_0(X_0)], \text{ by Proposition 3.8,}$$

$$\mathbb{E}[W^2] = \lim_{t\to\infty} \mathbb{E}[e^{-2\nu t} N_t^2] = \kappa_2 \sum_{k=0}^{\infty} \kappa_1^k \mathbb{E}[\tilde{C}_0^2(X_k) e^{-2\nu S_k}] \text{ by Proposition 3.9.}$$

□

3.3. **About Hypothesis 3.5.** The purpose of this section is to discuss Hypothesis 3.5 yielding to the key bound (27). We assume Hypothesis 3.4 along this paragraph.

3.3.1. *The i.i.d. case.* If moreover the lifetimes are i.i.d. then the growth rate ν as defined in (24) is also that defined in (1), that is $\kappa_1 \mathbb{E}\left[e^{-\nu\xi(X_0)}\right] = 1$. The following lemma gives additional assumptions ensuring (27) in this i.i.d. case. Although this case is classical, its study is important to make comparison with previous results (obtained with other methods of proofs). Also proofs for more general results will be obtained in the spirit of this classical case, see the paragraph below.

Lemma 3.10 (i.i.d. case)**.** *Assume Hypothesis 3.4, that $(\xi(X_n))_n$ is a sequence of i.i.d. random variables admitting a density $g \in \mathcal{L}^p$ for some $p > 1$. If, for some $M > 0$, $\nu > \delta > 0$, $\inf_{y\in\mathbb{R}} \left|1 - \kappa_1 \mathbb{E}\left[e^{-(-\delta+\nu+iy)\xi(X_0)}\right]\right| > 0$ and $\inf_{|y|\geq M} \inf_{u\in[-\delta,\delta]} |1-\kappa_1 \mathbb{E}\left[e^{-(u+\nu+iy)\xi(X_0)}\right]| > 0$, then Hypothesis 3.5 is satisfied for any $x \in \mathbb{X}$, with*

$$\tilde{C}_0(x) = \frac{\kappa_1 - 1}{\kappa_1^2 \nu} \left(\mathbb{E}\left[\xi(X_0) e^{-\nu\xi(X_0)}\right]\right)^{-1},$$

$$\Psi_0(x) \leq \textit{Const.} \left(\inf_{y\in\mathbb{R}} \left|1 - \kappa_1 \mathbb{E}\left[e^{-(-\delta+\nu+iy)\xi(X_0)}\right]\right|\right)^{-1} \left[\int_{-\infty}^{+\infty} g^p(u) du\right]^{1/(p-1)}.$$

Remark 1. *Note that if $\xi(X_0)$ is exponentially distributed with parameter λ, then for any $\delta > 0$ sufficiently small,*

$$\nu = \lambda(\kappa_1 - 1)$$

$$\inf_{y\in\mathbb{R}} \left|1 - \kappa_1 \mathbb{E}\left[e^{-(-\delta+\nu+iy)\xi(X_0)}\right]\right| > 0$$

$$\inf_{|y|\geq M} \inf_{u\in[-\delta,\delta]} \left|1 - \kappa_1 \mathbb{E}\left[e^{-(u+\nu+iy)\xi(X_0)}\right]\right| \geq M\lambda\kappa_1(\lambda+\delta)^{-1}((\delta+\lambda+\nu)^2 + M^2)^{-1/2} > 0$$

Remark 2. *Due to Lemma 3.10, Propositions 3.8 and 3.9 are respectively Theorem 17.2 and Lemma 18.1 in* [12] *(m, $h''(1)$ and n_1 there being respectively κ_1, κ_2 and the constant function $\tilde{C}_0$).*

Proof of Lemma 3.10. The task is to check the conditions on the Laplace transform $\tilde{f}_{x,0}$ (as calculated in (25)). In this i.i.d. case, we have, for $\gamma = s + iy$ and $s > 0$,

$$\tilde{f}_{x,0}(\gamma) = \frac{\kappa_1 - 1}{\gamma + \nu} \frac{\mathbb{E}[e^{-(\gamma+\nu)\xi(X_0)}]}{1 - \kappa_1 \mathbb{E}[e^{-(\gamma+\nu)\xi(X_0)}]}.$$

The right hand side of the last equality is analytic in a sufficiently narrow strip $\{z = u + iy, |u| \leq \delta + \epsilon, y \in \mathbb{R}\} \setminus \{0\}$ for $\delta + \epsilon < \nu$. It has a simple pole at 0 with residue $\tilde{C}_0(x)$, since

$$\lim_{z \to 0} z \tilde{f}_{x,0}(z) = \frac{\kappa_1 - 1}{\kappa_1^2 \nu} \left(\mathbb{E}\left[\xi(X_0) e^{-\nu \xi(X_0)} \right] \right)^{-1} = \tilde{C}_0(x).$$

We have, since $|1 - \kappa_1 \mathbb{E}(e^{-(\delta+\nu+iy)\xi(X_0)})|$ is bounded below by a strictly positive constant (this follows from $\sup_{y \in \mathbb{R}} |\mathbb{E}[e^{-(\delta+\nu+iy)\xi(X_0)}]| < \mathbb{E}[e^{-\nu\xi(X_0)}] = 1/\kappa_1$), and letting $p' = p/(p-1)$,

$$\int_{-\infty}^{+\infty} |\tilde{f}_{x,0}(\delta + iy)| dy \leq \text{Cst} \int_{-\infty}^{+\infty} \frac{|\mathbb{E}[e^{-(\delta+iy+\nu)\xi(X_0)}]|}{\sqrt{(\nu+\delta)^2 + y^2}} dy$$
$$\leq \text{Cst} \left(\int_{-\infty}^{+\infty} |\mathbb{E}[e^{-(\delta+iy+\nu)\xi(X_0)}]|^{p'} dy \right)^{1/p'} \left(\int_{-\infty}^{+\infty} ((\nu+\delta)^2 + y^2)^{-p/2} dy \right)^{1/p}.$$

Now we have, arguing as for the proof of Lemma 3 in Harris (1963) page 163,

$$\int_{-\infty}^{+\infty} |\mathbb{E}[e^{-(\delta+iy+\nu)\xi(X_0)}]|^{p'} dy \leq \tilde{C}_p \left[\int_{-\infty}^{+\infty} e^{-p(\delta+\nu)t} g^p(t) dt \right]^{1/(p-1)}.$$

Consequently $\int_{-\infty}^{+\infty} |\tilde{f}_{x,0}(\delta + iy)| dy < \infty$ since the density g is supposed to be in $\mathcal{L}^p$. Using the same arguments, we prove that $\int_{-\infty}^{+\infty} |\tilde{f}_{x,0}(-\delta + iy)| dy < \infty$, in fact,

$$\int_{-\infty}^{+\infty} |\tilde{f}_{x,0}(-\delta + iy)| dy$$
$$\leq \kappa_1 \tilde{C}_p \left(\inf_{y \in \mathbb{R}} \left| 1 - \kappa_1 \mathbb{E}[e^{-(-\delta+\nu+iy)\xi(X_0)}] \right| \right)^{-1} \left[\int_{-\infty}^{+\infty} g^p(t) dt \right]^{1/p} \left(\int_{-\infty}^{+\infty} ((\nu-\delta)^2 + y^2)^{-p/2} dy \right)^{1/p},$$

which is finite by the requirements of the lemma. Now, we have for any $u \in [-\delta, \delta]$ and for any $|y| > M > 0$,

$$|\tilde{f}_{x,0}(u + iy)| \leq (\kappa_1 - 1)((\nu + u)^2 + y^2)^{-1/2} \left| 1 - \kappa_1 \mathbb{E}[e^{-(u+\nu+iy)\xi(X_0)}] \right|^{-1}$$
$$\leq \frac{\kappa_1}{|y|} \left(\inf_{|y| \geq M} \inf_{u \in [-\delta,\delta]} \left| 1 - \kappa_1 \mathbb{E}[e^{-(u+\nu+iy)\xi(X_0)}] \right| \right)^{-1}$$

this proves that, for all $M > 0$, $\sup_{|y|>M} \sup_{u \in [-\delta,\delta]} \left| \tilde{f}_{x,0}(u + iy) \right| < \infty$ and then $\lim_{y \to \pm\infty} \tilde{f}_{x,0}(u + iy) = 0$, uniformly in $u \in [-\delta, \delta]$. Hypothesis (3.5) is then satisfied. □

3.3.2. *Multiplicative ergodic case.* Hypothesis 3.5 can also be satisfied by Markov Chains having multiplicative ergodic sums (cf. Definition 3.3 and Section 3 in [23]), the multiplicative ergodic property supposes that, for any $\gamma > 0$, any $x \in \mathbb{X}$ and any $n \in \mathbb{N}$,

$$\mathbb{E}\left[e^{-\gamma S_{1,n+1}(\xi,X)} | X_0 = x \right] = \alpha(\gamma, x) L^{n+1}(\gamma) + r_{n+1}(\gamma, x) \tag{38}$$

for suitable non-negative functions α, L and $(r_n)_n$. We suppose here that,

(a) For all $x \in \mathbb{X}$, the functions $\alpha(\cdot, x)$, L and $r_n(\cdot, x)$ can be extended to analytic functions in $\{z = u + iy, |u| \leq \delta + \epsilon < \nu, y \in \mathbb{R}\}$
(b) L is positive and non-increasing on $\mathbb{R}_+^*$. The equation $\kappa_1 L(z) = 1$, has a unique positive solution in $\mathbb{C}$, denoted by ν.
(c) The mapping L is holomorphic at ν and $L'(\nu) < 0$.

(d) The series $\sum_{n>0}\kappa_1^n r_n(\gamma,x)$ converges uniformly in γ in a neighborhood of ν uniformly in x.
(e) There exists $p'>1$ such that $\int_{-\infty}^{\infty}|\alpha(\pm\delta+\nu+iy,x)L(\pm\delta+\nu+iy)|^{p'}dy<\infty$ and that $\int_{-\infty}^{\infty}|\sum_{n\geq 0}\kappa_1^n r_n(\pm\delta+\nu+iy,x)|^{p'}dy<\infty$.
(f) $\inf_{y\in\mathbb{R}}|1-\kappa_1 L(\pm\delta+iy+\nu)|>0$, $\infty>\sup_{|y|>M}\sup_{u\in[-\delta,\delta]}|\alpha(u+iy+\nu,x)L(u+iy+\nu)|>0$, $\infty>\sup_{|y|>M}\sup_{u\in[-\delta,\delta]}\sum_{n\geq 0}\kappa_1^n|r_n(u+iy+\nu,x)|>0$ and $\inf_{|y|>M}\inf_{u\in[-\delta,\delta]}|1-\kappa_1 L(u+iy+\nu)|>0$, for some $M>0$.

Remark 3. *Under Hypotheses (3.4) and (3.5), the multiplicative ergodic property stated in (38) ensures (9), in fact, in this case*

$$g_n(\gamma)=(\kappa_1-1)\kappa_1^n L^n(\gamma)\mathbb{E}(\alpha(\gamma,X_0))+(\kappa_1-1)\kappa_1^n\mathbb{E}(r_n(\gamma,X_0)),\ \textit{for any}\ \gamma>0.$$

Lemma 3.11. *Hypothesis 3.5 is satisfied under Conditions (a)-$\cdots$-(f) with*

$$\tilde{C}_0(x)=-\frac{(\kappa_1-1)}{\kappa_1^2\nu L'(\nu)}\alpha(\nu,x)$$
$$\Psi_0(x)\leq\kappa_1\left(\inf_{y\in\mathbb{R}}|1-\kappa_1 L(-\delta+iy+\nu,x)|\right)^{-1}\times$$
$$\left(\int_{-\infty}^{\infty}|\alpha(-\delta+\nu+iy,x)L(-\delta+\nu+iy)|^{p'}dy\right)^{1/p'}\left(\int_{-\infty}^{+\infty}((\nu-\delta)^2+y^2)^{-p/2}dy\right)^{1/p}$$
$$+\kappa_1\left(\int_{-\infty}^{\infty}|\sum_{n\geq 0}\kappa_1^n r_{n+1}(-\delta+\nu+iy,x)|^{p'}dy\right)^{1/p'}\left(\int_{-\infty}^{+\infty}((\nu-\delta)^2+y^2)^{-p/2}dy\right)^{1/p}$$

for any $x\in\mathbb{X}$.

Proof of Lemma 3.11. We have, in this case,

$$\tilde{f}_{x,0}(\gamma)=\frac{\kappa_1-1}{\gamma+\nu}\sum_{n\geq 0}\kappa_1^n\mathbb{E}[e^{-(\gamma+\nu)S_{1,n+1}(\xi,X)}|X_0=x]$$
$$=\frac{\kappa_1-1}{\gamma+\nu}\frac{\alpha(\gamma+\nu,x)L(\gamma+\nu)}{1-\kappa_1 L(\gamma+\nu)}+\frac{\kappa_1-1}{\gamma+\nu}\sum_{n\geq 0}\kappa_1^n r_{n+1}(\gamma+\nu,x),$$

which can be extended thanks to Conditions (a) and (d) to an analytic function in $\{z=u+iy,|u|\leq\delta+\epsilon<\nu,y\in\mathbb{R}\}\setminus\{0\}$. Conditions (a), (b), (c) and (d) allow to deduce that

$$\lim_{z\to 0}z\tilde{f}_{x,0}(z)=-\frac{(\kappa_1-1)}{\kappa_1^2\nu L'(\nu)}\alpha(\nu,x)=:\tilde{C}_0(x).$$

We have arguing as for the proof of Lemma 3.10, for any $|y|>M$ and any $u\in[-\delta,\delta]$,

$$|\tilde{f}_{x,0}(u+iy)|\leq\frac{\kappa_1}{|y|}\frac{\sup_{|y|>M}\sup_{u\in[-\delta,\delta]}|\alpha(u+iy+\nu,x)L(u+iy+\nu)|}{\inf_{|y|>M}\inf_{u\in[-\delta,\delta]}|1-\kappa_1 L(u+iy+\nu)|}$$
$$+\frac{\kappa_1}{|y|}\sup_{|y|>M}\sup_{u\in[-\delta,\delta]}\sum_{n\geq 0}\kappa_1^n|r_{n+1}(u+iy+\nu,x)|,$$

which proves, thanks to (f), that $\lim_{y\to\pm\infty}\sup_{u\in[-\delta,\delta]}|\tilde{f}_{x,0}(u+iy)|=0$. Now,

$$\int|\tilde{f}_{x,0}(\pm\delta+iy)|dy\leq\kappa_1\left(\inf_{y\in\mathbb{R}}|1-\kappa_1L(\pm\delta+iy+\nu)|\right)^{-1}\times$$
$$\left(\int_{-\infty}^{\infty}|\alpha(\pm\delta+\nu+iy,x)L(\pm\delta+\nu+iy)|^{p'}dy\right)^{1/p'}\left(\int_{-\infty}^{+\infty}((\nu\pm\delta)^2+y^2)^{-p/2}dy\right)^{1/p}$$
$$+\kappa_1\left(\int_{-\infty}^{\infty}|\sum_{n\geq0}\kappa_1^nr_{n+1}(\pm\delta+\nu+iy,x)|^{p'}dy\right)^{1/p'}\left(\int_{-\infty}^{+\infty}((\nu\pm\delta)^2+y^2)^{-p/2}dy\right)^{1/p},$$

which is finite thanks to Conditions (e) and (f). Hypothesis (3.5) is then satisfied. □

3.3.3. *Bifurcating Markov chains.* Hypothesis 3.4 supposes also that the two sequences $(X_n^{(k)})_{n\geq k+1}$ and $(X_n)_{n\geq k+1}$ are independent given X_k for any $k\in\mathbb{N}$. This condition can be satisfied by bifurcating Markov chains as defined in Section 3 of [23].

4. Tools for Markov models

In this section, we discuss assumptions of Theorems 2.8 and 2.9. We assume that $X=(X_n)_n$ is a Markov chain on $(\mathbb{X},\mathcal{X})$ with Markov kernel $P(x,dy)$, invariant probability π, and initial probability μ (i.e. μ is the distribution of X_0).

4.1. **Notations, definitions and hypotheses on Banach spaces.** Recall that the Laplace-type kernels $(P_\gamma)_\gamma$ are given by

$$\forall\gamma\in[0,+\infty),\ P_\gamma f=P(\kappa e^{-\gamma\xi}f)\ \text{ and }\ P_\infty f=P(\kappa\mathbf{1}_{\{\xi=0\}}f). \tag{39}$$

For any normed complex vector spaces $(\mathcal{B}_0,\|\cdot\|_{\mathcal{B}_0})$ and $(\mathcal{B}_1,\|\cdot\|_{\mathcal{B}_1})$, we endow the space $\mathcal{L}(\mathcal{B}_0,\mathcal{B}_1)$ with the operator norm $\|\cdot\|_{\mathcal{B}_0,\mathcal{B}_1}$ given by

$$\forall Q\in\mathcal{L}(\mathcal{B}_0,\mathcal{B}_1),\quad \|Q\|_{\mathcal{B}_0,\mathcal{B}_1}=\sup_{f\in\mathcal{B}_0,\ \|f\|_{\mathcal{B}_0}=1}\|Qf\|_{\mathcal{B}_1}.$$

We simply write $(\mathcal{L}(\mathcal{B}),\|\cdot\|_{\mathcal{B}})$ for $(\mathcal{L}(\mathcal{B},\mathcal{B}),\|\cdot\|_{\mathcal{B},\mathcal{B}})$. For any $Q\in\mathcal{L}(\mathcal{B})$, we denote by Q^* its adjoint operator. We write $\sigma(Q)=\sigma(Q_{|\mathcal{B}})$ for the spectrum of Q:

$$\sigma(Q):=\{\lambda\in\mathbb{C}\ :\ (Q-\lambda I)\text{ is non invertible}\},$$

where I denotes the identity operator on $\mathcal{B}$. Recall that Q and Q^* have the same norm in $\mathcal{L}(\mathcal{B})$ and $\mathcal{L}(\mathcal{B}^*)$ respectively, as well as the same spectrum. The spectral radius of $Q_{|\mathcal{B}}$ (resp. its essential spectral radius) is denoted by $r(Q)=r(Q_{|\mathcal{B}})$ (resp. $r_{ess}(Q)=r_{ess}(Q_{|\mathcal{B}})$). Recall that

$$r_{ess}(Q):=\sup\{|\lambda|\ :\ \lambda\in\mathbb{C}\text{ and }(Q-\lambda I)\text{ is non Fredholm}\}.$$

To check the assumptions of Theorems 2.8 and 2.9, we work with Banach spaces contained in $\mathbb{L}^1(\pi)$ (as stated in Hypothesis 2.5) and we use some assumptions involving the notion of positivity and non-negativity on such a space (or on its dual space), as defined below.

Definition 4.1. *Let $\mathcal{B}$ be a Banach space composed of functions $f : \mathbb{X} \to \mathbb{C}$ (or of classes of such functions modulo π). If $f \in \mathcal{B}$ is a class of functions, we say that it is* **non-negative** *(resp.* **positive***) if one of its representant is so. We say that it is* **non-null** *if the null function is not one of its representant. An element $\psi \in \mathcal{B}^*$ is said to be* **non-negative** *if for every non-negative $f \in \mathcal{B}$, we have $\psi(f) \geq 0$. An element $\psi \in \mathcal{B}^*$ is said to be* **positive** *if for every non-negative non-null $f \in \mathcal{B}$, we have $\psi(f) > 0$.*

Hypothesis 4.2. *For every $\phi \in \mathcal{B}$, $\phi \geq 0$, $\phi \neq 0$, there exists $\psi \in \mathcal{B}^*$, $\psi \geq 0$, such that $\psi(\phi) > 0$. For every $\psi \in \mathcal{B}^*$, $\psi \geq 0$, $\psi \neq 0$, there exists $\phi \in \mathcal{B}$, $\phi \geq 0$ such that $\psi(\phi) > 0$.*

Hypothesis 4.3. *Let $J \subset [0, +\infty]$ such that: $\forall \gamma \in J$, $P_\gamma \in \mathcal{L}(\mathcal{B})$. For every $\gamma \in J$ such that $r(\gamma) > 0$, the following properties hold: if $\phi \in \mathcal{B}$ is non-null and non-negative, then $P_\gamma \phi > 0$ (modulo π) and every non-null non-negative $\psi \in \mathcal{B}^* \cap \mathrm{Ker}(P_\gamma^* - r(\gamma)I)$ is positive.*

Note that Hypothesis 4.2 is quite general. Let us recall the definition of a Banach lattice.

Definition 4.4. *A complex Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ of functions $f : \mathbb{X} \to \mathbb{C}$ (or of classes of such functions modulo π) is said to be a* **complex Banach lattice** *if it is stable by $|\cdot|$, by real part and if*

$$\forall f, g \in \mathcal{B}, \quad f(\mathbb{X}) \cup g(\mathbb{X}) \subset \mathbb{R} \quad \Rightarrow \quad \min(f, g), \max(f, g) \in \mathcal{B},$$
$$\forall f, g \in \mathcal{B}, \quad |f| \leq |g| \quad \Rightarrow \quad \| \, |f| \, \|_{\mathcal{B}} = \|f\|_{\mathcal{B}} \leq \|g\|_{\mathcal{B}} = \| \, |g| \, \|_{\mathcal{B}}.$$

Such a space satisfies Hypothesis 4.2. Classical instances of Banach lattices of functions are the spaces $(\mathbb{L}^p(\pi), \|\cdot\|_p)$ and $(\mathcal{B}_V, \|\cdot\|_V)$ (see (44) and (52)), as well as the space $(\mathcal{L}^\infty(\mathbb{X}), \|\cdot\|_\infty)$ composed of all the bounded measurable $\mathbb{C}$-valued functions on $\mathbb{X}$, and equipped with its usual norm $\|f\|_\infty := \sup_{x \in \mathbb{X}} |f(x)|$.

4.2. About Hypothesis 2.5.

Proposition 4.5. *Let J be a subinterval of $[0, +\infty]$ and let $\mathcal{B}$ be a Banach lattice of functions on $\mathbb{X}$ (or of classes of such functions modulo π). We assume that Hypothesis 4.3 holds, that $\mathcal{B} \subset \mathbb{L}^1(\pi)$, and that for every $\gamma \in J$*

(i) $P_\gamma \in \mathcal{L}(\mathcal{B})$, $r(\gamma) := r(P_{\gamma|\mathcal{B}}) > 0$, and P_γ is quasi-compact on $\mathcal{B}$,
(ii) for every $f, g \in \mathcal{B}$ with $f > 0$, $P_\gamma f = r(\gamma) f$ and $P_\gamma g = r(\gamma) g$, we have $g \in \mathbb{C} \cdot f$,

Assume moreover that the Markov kernel P satisfies the following condition: 1 is the only complex number λ of modulus 1 such that $P(h/|h|) = \lambda h/|h|$ in $\mathbb{L}^1(\pi)$ for some $h \in \mathcal{B}$, $|h| > 0$ (modulo π). Then Hypothesis 2.5 is fulfilled with J and $\mathcal{B}$.

Proposition 4.5 is proved in Section 8.

Proposition 4.6. *Assume that Hypothesis 2.5 holds for some $\mathcal{B}$ and J. Let $\gamma \in J$. Then*

$$\Pi_\gamma = \lim_n r(\gamma)^{-n} P_\gamma^n \quad \text{in } \mathcal{L}(\mathcal{B}), \tag{40}$$

and there exist some nonzero elements $\widehat{\pi}_\gamma \in \mathcal{B}^* \cap \mathrm{Ker}(P_\gamma^* - r(\gamma)I)$ and $\widehat{\phi}_\gamma \in \mathcal{B} \cap \mathrm{Ker}(P_\gamma - r(\gamma)I)$ such that $\widehat{\pi}_\gamma(\widehat{\phi}_\gamma) = 1$ and

$$\forall f \in \mathcal{B}, \quad \Pi_\gamma f = \widehat{\pi}_\gamma(f) \, \widehat{\phi}_\gamma \qquad \text{and} \qquad \forall f^* \in \mathcal{B}^*, \quad \Pi_\gamma^* f^* = f^*(\widehat{\phi}_\gamma) \, \widehat{\pi}_\gamma. \tag{41}$$

If $\mathcal{B}$ satisfies Hypothesis 4.2, then $\widehat{\phi}_\gamma$ and $\widehat{\pi}_\gamma$ are non-negative in $\mathcal{B}$ and $\mathcal{B}^*$ respectively. Under the additional Hypothesis 4.3, every non-null non-negative $\phi \in \mathcal{B} \cap \mathrm{Ker}(P_\gamma - r(\gamma)I)$ is positive $\pi-$a.s. and, for every non-null and non-negative $f \in \mathcal{B}$, we have $\Pi_\gamma f > 0$ $\pi-$a.s..

Proof. Properties (40) and the existence of $\widehat{\phi}_\gamma$ and $\widehat{\pi}_\gamma$ in (41) follow from Hypothesis 2.5. Now assume that Hypothesis 4.2 holds. Then (40) and the first assertion in Hypothesis 4.2, applied with $\phi = \widehat{\phi}_\gamma$ and the associated $\psi_\gamma \in \mathcal{B}^*$, $\psi_\gamma \geq 0$, imply that, for every $g \in \mathcal{B}$, $g \geq 0$, we have $0 \leq \lim_n r(\gamma)^{-n}\psi_\gamma(P_\gamma^n g) = \widehat{\pi}_\gamma(g)\,\psi_\gamma(\widehat{\phi}_\gamma)$, hence $\widehat{\pi}_\gamma \geq 0$ since $\psi_\gamma(\widehat{\phi}_\gamma) > 0$. Next the second assertion in Hypothesis 4.2, applied with $\psi = \widehat{\pi}_\gamma$ and the associated $\phi_\gamma \in \mathcal{B}$, $\phi_\gamma \geq 0$, gives $0 \leq \lim_n r(\gamma)^{-n} P_\gamma^n \phi_\gamma = \widehat{\pi}_\gamma(\phi_\gamma)\widehat{\phi}_\gamma$, hence $\widehat{\phi}_\gamma \geq 0$ since $\widehat{\pi}_\gamma(\phi_\gamma) > 0$. Finally, under Hypotheses 4.2 and 4.3, if $\gamma \in J$ and if $f \in \mathcal{B}$, $f \neq 0$, $f \geq 0$, then $\widehat{\pi}_\gamma(f) > 0$. Thus $\Pi_\gamma f = \widehat{\pi}_\gamma(f)\,\widehat{\phi}_\gamma$ is positive modulo π since so is $\widehat{\phi}_\gamma > 0$ from Hypothesis 4.3. □

Corollary 4.7. *Assume that, for some subinterval $J \subset [0, +\infty)$, Hypothesis 2.5 holds on two Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$, both containing $1_{\mathbb{X}}$ and satisfying Hypotheses 4.2 and 4.3. Then*

$$\forall \gamma \in J, \quad r(P_{\gamma|\mathcal{B}_1}) = r(P_{\gamma|\mathcal{B}_2}) = \lim_{n\to+\infty} (\pi(P_\gamma^n \mathbf{1}_{\mathbb{X}}))^{1/n}.$$

If moreover $\mathcal{B}_1 \hookrightarrow \mathcal{B}_2$ and if, for $i = 1, 2$, $\Pi_{\gamma,i}$ denotes the rank-one eigen-projector associated with $P_{\gamma|\mathcal{B}_i}$ in (12), then the restriction of $\Pi_{\gamma,2}$ to $\mathcal{B}_1$ equals to $\Pi_{\gamma,1}$.

Proof. For $i = 1, 2$, Proposition 4.6 applied to $P_{\gamma|\mathcal{B}_i}$ (with the notations $\widehat{\phi}_{\gamma,i}$ and $\widehat{\pi}_{\gamma,i}$) gives

$$\pi(P_\gamma^n \mathbf{1}_{\mathbb{X}}) = (r(P_{\gamma|\mathcal{B}_i}))^n \widehat{\pi}_{\gamma,i}(\mathbf{1}_{\mathbb{X}})\pi(\widehat{\phi}_{\gamma,i}) + \mathrm{o}\big(r(P_{\gamma|\mathcal{B}_i})^n\big)$$

with $\pi(\widehat{\phi}_{\gamma,i})\,\widehat{\pi}_{\gamma,i}(\mathbf{1}_{\mathbb{X}}) > 0$ from Hypothesis 4.3. Hence the first assertion holds. Now let $f \in \mathcal{B}_1$. Then $\Pi_{\gamma,1} f = \lim_n r(P_{\gamma|\mathcal{B}_2})^{-n}(P_{\gamma|\mathcal{B}_1})^n f$ in $\mathcal{B}_1$ from Proposition 4.6 applied to $P_{\gamma|\mathcal{B}_1}$ and from the previous fact. It follows from $\mathcal{B}_1 \hookrightarrow \mathcal{B}_2$ that this convergence holds in $\mathcal{B}_2$ too. Now Proposition 4.6 applied to $P_{\gamma|\mathcal{B}_2}$ gives $\Pi_{\gamma,1} f = \Pi_{\gamma,2} f$. □

4.3. **About Condition** (18).

Proposition 4.8. *Assume that $(P_\gamma)_\gamma$ and μ satisfy the assumptions of Theorem 2.8, excepted (18). Then the real number $B(\gamma)$ given in (18) is well-defined for every $\gamma \in J_0$, and*

$$\forall \gamma \in J_0, \quad B(\gamma) = \widehat{\pi}_\gamma(Ph_{\kappa,\gamma})\,\mu\left(\kappa\, e^{-\gamma\xi}\widehat{\phi}_\gamma\right), \tag{42}$$

where $\widehat{\phi}_\gamma$ and $\widehat{\pi}_\gamma$ are given in (41). Assume moreover that the space $\mathcal{B}$ involved in the assumptions of Theorem 2.8 satisfies Hypotheses 4.2 and 4.3 on J_0 and that one of the following assumptions holds true

(i) μ is absolutely continuous with respect to π,
(ii) the first part in Hypothesis 4.3 is reinforced as follows: for every $\gamma \in J$, if $\phi \in \mathcal{B}$ is non-null and non-negative, then $P_\gamma \phi > 0$ everywhere on $\mathbb{X}$.

Then (18) holds.

Proof. Let $\gamma \in J_0$ (thus $r(\gamma) > 0$). First, by assumption $Ph_{\kappa,\gamma} \in \mathcal{B}_0$. Thus $\Pi_\gamma(Ph_{\kappa,\gamma}) \in \mathcal{B}_3$ since $\Pi_\gamma \in \mathcal{L}(\mathcal{B}_0, \mathcal{B}_3)$ (see Theorem 2.8), and $B(\gamma)$ is then well-defined from the assumptions on μ in Theorem 2.8. Formula (42) follows from (41). Now, under the additional assumptions

in (i), we know from Proposition 4.6 that $\widehat{\phi}_\gamma \geq 0$ and $\widehat{\pi}_\gamma \geq 0$. Moreover $Ph_{\kappa,\gamma} \geq 0$ and $Ph_{\kappa,\gamma} \neq 0$ in $\mathbb{L}^1(\pi)$ since $\pi(Ph_{\kappa,\gamma}) = \pi(h_{\kappa,\gamma}) > 0$. Since $\mathcal{B}_0 \hookrightarrow \mathcal{B}_3 \hookrightarrow \mathbb{L}^1(\pi)$ by hypothesis, it follows that $Ph_{\kappa,\gamma} \neq 0$ in $\mathcal{B}_1$ and in $\mathcal{B}_3$. Thus $\widehat{\pi}_\gamma(Ph_{\kappa,\gamma}) > 0$ from Hypothesis 4.3. Also note that $\widehat{\phi}_\gamma > 0$ (modulo π) from Proposition 4.6. Then (i) follows from the previous remarks and from (42). Assertion (ii) is obvious since, in this case, $\widehat{\phi}_\gamma > 0$ everywhere. □

4.4. About the monotonicity of the spectral radius.

Proposition 4.9. *If $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a complex Banach lattice of functions $f : \mathbb{X} \to \mathbb{C}$ (or of classes of functions modulo π), and if $P_\gamma \in \mathcal{L}(\mathcal{B})$ for every $\gamma \in [0, +\infty)$, then the map $\gamma \mapsto r(\gamma)$ is non-increasing on $[0, +\infty)$.*

Proof. For any $0 \leq \gamma < \gamma' \leq \infty$ and for any $f, g \in \mathcal{B}$ such that $|f| \leq |g|$, we have $e^{-\gamma'\xi}|f| \leq e^{-\gamma\xi}|g|$ and so $P_{\gamma'}|f| \leq P_\gamma|g|$, which implies by induction that $P^n_{\gamma'}|f| \leq P^n_\gamma|f|$ for every integer $n \geq 1$. We conclude that $\|P^n_{\gamma'}\|_{\mathcal{B}} \leq \|P^n_\gamma\|_{\mathcal{B}}$ since $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a Banach lattice. This implies that $r(\gamma') \leq r(\gamma)$ and so the desired statement. □

Proposition 4.10. *Assume that Hypothesis 2.5 is fulfilled for some J and for some $\mathcal{B}$ satisfying Hypotheses 4.2 and 4.3. Then $\gamma \to r(\gamma)$ is non-increasing on J.*

Proof. We use the notations of Proposition 4.6. Let $\gamma_1, \gamma_2 \in J$ such that $\gamma_2 < \gamma_1$. Then $\widehat{\pi}_{\gamma_1}(P^n_{\gamma_1}\widehat{\phi}_{\gamma_1}) \leq \widehat{\pi}_{\gamma_1}(P^n_{\gamma_2}\widehat{\phi}_{\gamma_1})$ since $\widehat{\pi}_{\gamma_1}$ is non-negative. Moreover $\widehat{\pi}_{\gamma_1}(P^n_{\gamma_1}\widehat{\phi}_{\gamma_1}) = (r(\gamma_1))^n$ and

$$\widehat{\pi}_{\gamma_1}(P^n_{\gamma_2}\widehat{\phi}_{\gamma_1}) = (r(\gamma_2))^n \widehat{\pi}_{\gamma_2}(\widehat{\phi}_{\gamma_1})\widehat{\pi}_{\gamma_1}(\widehat{\phi}_{\gamma_2}) + o\left((r(\gamma_2))^n\right).$$

Since $\widehat{\pi}_{\gamma_1}$ and $\widehat{\pi}_{\gamma_2}$ are positive and since ϕ_{γ_1} and ϕ_{γ_2} are non-null non-negative, Hypothesis 4.3 gives $\widehat{\pi}_{\gamma_2}(\widehat{\phi}_{\gamma_1})\widehat{\pi}_{\gamma_1}(\widehat{\phi}_{\gamma_2}) > 0$. Thus $(r(\gamma_1))^n \leq O\left((r(\gamma_2))^n\right)$, so $r(\gamma_1) \leq r(\gamma_2)$. □

Remark 4.11. *The non-increasingness of r on J implies that the set J_0 in (15) is an interval. Let us also indicate that it can happen that $r(\cdot)$ is constant on $[0, +\infty)$ (see Appendix 9).*

The next result is relevant to check the condition $r'(\nu) \neq 0$ in Theorem 2.9.

Proposition 4.12. *Assume that the assumptions of Theorem 2.9 hold.*

(i) Let $\gamma \in J_0$ be such that $\pi\big(\Pi_\gamma(\xi\,\Pi_\gamma \mathbf{1}_{\mathbb{X}})\big) > 0$. Then $r'(\gamma) < 0$.

(ii) Assume moreover that the space $\mathcal{B}_2$ involved in the assumptions of Theorem 2.9 satisfy Hypotheses 4.2 and 4.3 on J_0, and that $\pi(\{\xi = 0\}) < 1$. Then $r'(\cdot) < 0$ on J_0, thus $\gamma \mapsto r(\gamma)$ is strictly decreasing on J_0.

Proof. Assertion (i) follows from Proposition 7.5. Let us derive (ii) from (i). Let $\gamma \in J_0$. First note that $\xi\Pi_\gamma \mathbf{1}_{\mathbb{X}} \in \mathcal{B}_2$. Indeed it follows from the assumptions of Theorem 2.9 that $\Pi_\gamma \mathbf{1}_{\mathbb{X}} \in \mathcal{B}_1$ (use also Corollary 4.7) and that the map $f \mapsto \xi f$ is in $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$. Hence $\xi\Pi_\gamma \mathbf{1}_{\mathbb{X}} \in \mathcal{B}_2$. Moreover, under the assumptions in (ii), we know from the last assertion of Proposition 4.6 (applied on $\mathcal{B}_2$) that $\Pi_\gamma \mathbf{1}_{\mathbb{X}} > 0$ modulo π, thus $\xi\,\Pi_\gamma \mathbf{1}_{\mathbb{X}} \neq 0$ in $\mathbb{L}^1(\pi)$ (since $\pi(\{\xi = 0\}) < 1$), and so $\xi\,\Pi_\gamma \mathbf{1}_{\mathbb{X}} \neq 0$ in $\mathcal{B}_2$ from $\mathcal{B}_2 \hookrightarrow \mathbb{L}^1(\pi)$. Then it follows again from the last assertion of Proposition 4.6 (applied on $\mathcal{B}_2$) that $\Pi_\gamma(\xi\,\Pi_\gamma 1_{\mathbb{X}}) > 0$ modulo π, thus $\pi\big(\Pi_\gamma(\xi\,\Pi_\gamma \mathbf{1}_{\mathbb{X}})\big) > 0$. Hence $r'(\gamma) < 0$ from (i). □

A consequence of the monotonicity of $\gamma \mapsto r(\gamma)$ the following characterisation of $\nu < \infty$.

Proposition 4.13. *Assume that the assumptions of Theorem 2.8 hold (with J and δ_0 given in Hypothesis 2.7 or 2.7*); in particular for every $\gamma \in J$ the Laplace kernel P_γ is assumed to continuously act on the Banach space $\mathcal{B}$ chosen in the assumptions of Theorem 2.8.*

(i) If $\delta_0 < 1$, then for every $\gamma \in J$ we have: $G(\gamma) < \infty \Leftrightarrow r(\gamma) < 1$.

(ii) If r is non-increasing, if $J = (a, +\infty]$ for some $a \geq 0$ and if $\delta_0 < 1$, then $\nu < \infty \Leftrightarrow r(\infty) = r(P_{\infty|\mathcal{B}}) < 1$.

Proof. First, if $\gamma \in J_0$, then $G(\gamma) < \infty \Leftrightarrow r(\gamma) < 1$ due to Theorem 2.8 and to Remark 2.4. Second, if $\gamma \in J \setminus J_0$, then $r(\gamma) \leq \delta_0 < 1$, so that, for some fixed $\delta \in (\delta_0, 1)$, we can deduce from the definition of $g_n(\gamma)$ and $r(\gamma)$, and from assumptions of Theorem 2.8, that there exists $\tilde{C}_\delta > 0$ such that $G(\gamma) \leq \sum_{n=0}^{+\infty} \tilde{C}_\delta \delta^n < \infty$. Hence (i) is fulfilled. Now, under the assumptions of (ii), it follows from Proposition 4.9 and from (i) that: $\nu < \infty \Leftrightarrow \limsup_{\gamma \to +\infty} r(\gamma) < 1$. Moreover, due to Theorem 2.8, we know that $\limsup_{\gamma \to +\infty} r(\gamma) \leq \max(\delta_0, r(\infty))$, and even that $\lim_{\gamma \to +\infty} r(\gamma) = r(\infty)$ if $r(\infty) > \delta_0$. Considering the cases $r(\infty) \leq \delta_0$ and $r(\infty) > \delta_0$ then gives the desired equivalence in (ii). □

4.5. **Positivity of the spectral radius.** Recall that we have set $J_0 := \{\gamma \in J : r(\gamma) > \delta_0\}$ under the assumptions of Theorem 2.8. Another consequence of the monotonicity of $\gamma \mapsto r(\gamma)$ is the following lemma.

Lemma 4.14. *Let $\gamma_1, \gamma_2, \gamma_3$ be such that $0 \leq \gamma_1 < \gamma_2 < \gamma_3$. Assume that the assumptions of Theorem 2.8 hold with $J = (\gamma_1, \gamma_2)$ and that r is non-increasing on J. Moreover suppose that, for every $\gamma \in (\gamma_1, \gamma_3)$, P_γ continuously acts on $\mathcal{B}$ and that the map $f \mapsto \pi(\kappa e^{-\gamma\xi} f)$ is in $\mathcal{B}^*$, where $\mathcal{B}$ is the space given in Theorem 2.8. Finally suppose that*

$$\Delta_0 := \limsup_n \left(\pi\big(\kappa\, P_0^n \mathbf{1}_{\mathbb{X}}\big) \right)^{\frac{1}{n}} < \infty. \tag{43}$$

If $J_0 \neq \{0\}$, then we have $r(\gamma) > 0$ for every $\gamma \in (\gamma_1, \gamma_3)$.

Proof. Let $\gamma_0 \in J_0$, $\gamma_0 \neq 0$. Then $r(\gamma) \geq r(\gamma_0) > 0$ for every $\gamma \in (\gamma_1, \gamma_0]$. Next let $\gamma \in (\gamma_0, \gamma_3)$ and set $p := \gamma/\gamma_0 > 1$. Since $\widehat{\pi}_{\gamma_0}$ is positive and $\widehat{\phi}_{\gamma_0}$ is non-negative and non-null (modulo π), we have

$$0 < r(\gamma_0) = r\left(\frac{\gamma}{p}\right) = \lim_{n \to +\infty} \left(\pi\big(\kappa\, e^{-\frac{\gamma}{p}\xi}\, P_{\frac{\gamma}{p}}^n \mathbf{1}_{\mathbb{X}}\big) \right)^{\frac{1}{n}} = \lim_{n \to +\infty} \left(\mathbb{E}_\pi\left[\left(\prod_{j=0}^n \kappa(X_j) \right) e^{-\frac{\gamma}{p} S_n} \right] \right)^{\frac{1}{n}}$$

due to (17) since $\mathbf{1}_{\mathbb{X}} \in \mathcal{B}$ and $f \mapsto \pi(\kappa e^{-\frac{\gamma}{p}\xi} f)$ is in $\mathcal{B}^*$, and due to (11) (in which we replace $h_{\kappa,\gamma}$ by $\kappa e^{-\gamma\xi}$). Let $q = p/(p-1)$. Writing $\kappa(X_j) = \kappa(X_j)^{1/q} \kappa(X_j)^{1/p}$, it follows from the Hölder inequality that

$$\begin{aligned} r(\gamma_0) &\leq \limsup_{n \to +\infty} \left(\mathbb{E}_\pi\left[\prod_{j=0}^n \kappa(X_j) \right] \right)^{\frac{1}{nq}} \times \limsup_{n \to +\infty} \left(\mathbb{E}_\pi\left[\left(\prod_{j=0}^n \kappa(X_j) \right) e^{-\gamma S_n} \right] \right)^{\frac{1}{np}} \\ &\leq \limsup_{n \to +\infty} \left(\pi\big(\kappa\, P_0^n \mathbf{1}_{\mathbb{X}}\big) \right)^{\frac{1}{nq}} \times \limsup_{n \to +\infty} \left(\pi\big(\kappa\, e^{-\gamma\xi}\, P_\gamma^n \mathbf{1}_{\mathbb{X}}\big) \right)^{\frac{1}{np}}. \end{aligned}$$

The above first limit superior equals to $\Delta_0^{1/q}$ by hypothesis, and the second limit superior is less than $(r(\gamma))^{\frac{1}{p}}$ from the definition of $r(\gamma)$ and from the fact that $\mathbf{1}_{\mathbb{X}} \in \mathcal{B}$ and $f \mapsto \pi(\kappa e^{-\gamma\xi} f)$ is in $\mathcal{B}^*$. Thus $0 < r(\gamma_0) \leq \Delta_0^{1/q} (r(\gamma))^{\frac{1}{p}}$. □

Remark 4.15. *Condition (43) holds if $0 \in J$ (in particular $P_0 \in \mathcal{L}(\mathcal{B})$) and if the map $f \mapsto \pi(\kappa f)$ is in $\mathcal{B}^*$ since $\Delta_0 \leq r(0)$ from the definition of the spectral radius $r(0)$ of P_0. Moreover note that (43) holds too if κ is bounded by some constant $d > 0$ since $P_0 \leq d\,P$.*

5. Knudsen gas: Proof of Theorem 2.11

In this section, we apply our general results for the Knudsen gas. Here the Laplace-type kernels P_γ act on the usual Lebesgue space $(\mathbb{L}^a(\pi), \|\cdot\|_a)$ for some suitable $a \in [1, +\infty)$, where

$$\|f\|_a := \left(\int_{\mathbb{X}} |f(x)|^a \, d\pi(x) \right)^{\frac{1}{a}}. \tag{44}$$

Theorem 2.11 directly follows from the next more precise theorem.

Theorem 5.1 (Knudsen gas)**.** *Assume that $X = (X_n)_n$ is a Knudsen gas as described in Example 2.10, and that its initial distribution μ on $\mathbb{X}$ is absolutely continuous with respect to π, with density in $\mathbb{L}^p(\pi)$. Assume moreover that $\kappa \equiv 2$ and that the function $\xi : \mathbb{X} \to [0, +\infty)$ in (2) is measurable. Then $(S_n)_n$ is multiplicatively ergodic on the interval $J_0 = \{\gamma > 0 : r(\gamma) > 2(1-\alpha)\}$, where $r(\gamma)$ denotes le spectral radius of P_γ on $\mathbb{L}^b$ with $b := \frac{p}{p-1}$. If moreover $\alpha > 1/2$ and if*

$$2\alpha \sum_{n \geq 0} (2(1-\alpha))^n \mathbb{P}_\pi \left(\sum_{k=0}^{n} \xi(Z_k) = 0 \right) < 1, \tag{45}$$

where $(Z_n)_n$ is a Markov process with transition U, then ν defined in (4) is finite. Finally, if $\pi(\xi^\tau) < \infty$ for some $\tau > 1$ and if $p > \frac{\tau}{\tau-1}$, then the constant C_ν in (7) is well defined and finite, and consequently the conclusion (8) of Corollary 2.2 holds true.

Theorem 5.1 straightforwardly extends to the case $\kappa(\cdot) \equiv m$, where $m \geq 2$ is any integer. To prove Theorem 5.1, we apply Theorems 2.8 and 2.9. First we prove the following.

Lemma 5.2. *Let $1 \leq b < a$.*

(i) For every $\gamma \geq 0$, $r_{ess}(P_{\gamma|\mathbb{L}^a(\pi)}) \leq 2(1-\alpha)$.
(ii) The function $\gamma \to P_\gamma$ is continuous from $(0, +\infty]$ to $\mathcal{L}(\mathbb{L}^a(\pi), \mathbb{L}^b(\pi))$.
(iii) For any $\gamma \in [0, +\infty]$ and any $f \in \mathbb{L}^a(\pi)$, $\|P_\gamma f\|_a \leq 2((1-\alpha)\|f\|_a + \alpha\|f\|_1)$.
(iv) For any $\gamma > 0$, for any non-null non-negative $f \in \mathbb{L}^a(\pi)$ and every non-null non-negative $g \in \mathbb{L}^{a'}(\pi)$ with $a' = \frac{a}{a-1}$, we have $\pi(gP_\gamma f) > 0$ and $P_\gamma f > 0$.
(v) If $r(\gamma) > 2(1-\alpha)$, for every $f, g \in \mathbb{L}^a(\pi)$ with $f > 0$, $P_\gamma f = r(\gamma) f$ and $P_\gamma g = r(\gamma) g$, then we have $g \in \mathbb{C} \cdot f$.
(vi) 1 is the only complex number λ of modulus 1 such that $P(h/|h|) = \lambda h/|h|$ in $\mathbb{L}^1(\pi)$ for some $h \in \mathcal{B}$, $|h| > 0$ (modulo π).

Proof.
(i) Observe that $P_\gamma = 2(\alpha\pi(e^{-\gamma\xi}\cdot) + (1-\alpha)U_\gamma)$ with $U_\gamma := U(e^{-\gamma\xi}\cdot)$. Since the sum of

a Fredholm operator with a compact operator is Fredholm, we directly obtain $r_{ess}(P_\gamma) = 2(1-\alpha)r_{ess}(U_\gamma) \leq 2(1-\alpha)$.

(ii) For every $0 \leq \gamma, \gamma' < \infty$ and every $f \in \mathcal{B}$ such that $\|f\|_a = 1$, we have

$$\begin{aligned} \|P_\gamma f - P_{\gamma'} f\|_b &= 2\|P((e^{-\gamma\xi} - e^{-\gamma'\xi})f)\|_b \\ &\leq 2\|(e^{-\gamma\xi} - e^{-\gamma'\xi})f\|_b \leq 2\|e^{-\gamma\xi} - e^{-\gamma'\xi}\|_c, \end{aligned}$$

where c is such that $\frac{1}{a} + \frac{1}{c} = \frac{1}{b}$. Hence $\|P_\gamma - P_{\gamma'}\|_{\mathbb{L}^a(\pi),\mathbb{L}^b(\pi)} \leq 2\|e^{-\gamma\xi} - e^{-\gamma'\xi}\|_c$, which converges to 0 as γ' goes to γ, by the dominated convergence theorem. In the same way, we prove that $\|P_\gamma - P_\infty\|_{\mathbb{L}^a(\pi),\mathbb{L}^b(\pi)} \leq 2\|e^{-\gamma\xi}\|_c$ and hence the continuity of $\gamma \mapsto P_\gamma$ at infinity.

(iii) For every $\gamma \in [0, +\infty]$ and every $f \in \mathbb{L}^a(\pi)$, $\|P_\gamma f\|_a \leq 2\|Pf\|_a \leq 2((1-\alpha)\|f\|_a + \alpha\|f\|_1)$ since $\|Uf\|_a \leq \|f\|_a$. This gives the Doeblin-Fortet inequality.

(iv) For any non-null non-negative $f \in \mathbb{L}^a(\pi)$, we have $P_\gamma f \geq 2\alpha\pi(e^{-\gamma\xi} f)1_{\mathbb{X}} > 0$. The other assertion of *(iv)* is then obvious.

(v) Let $f, g \in \mathbb{L}^a(\pi)$ such that $f > 0$, $P_\gamma f = r(\gamma)f$ and $P_\gamma g = r(\gamma)g$ in $\mathbb{L}^a(\pi)$. Set $\beta := \frac{\pi(e^{-\gamma\xi}g)}{\pi(e^{-\gamma\xi}f)}$ and $h := g - \beta f$. Then $\pi(e^{-\gamma\xi}h) = 0$ and $P_\gamma h = r(\gamma)\, h$, which gives $r(\gamma)\, h = 2(1-\alpha)\, U(e^{-\gamma\xi}h)$, so that $r(\gamma)\,|h| \leq 2(1-\alpha)\, U(|h|)$. Since $\pi\, U = \pi$, we obtain: $r(\gamma)\,\pi(|h|) \leq 2(1-\alpha)\,\pi(|h|)$. Finally we conclude that $\pi(|h|) = 0$ because $r(\gamma) > 2(1-\alpha)$ and so $g = \beta f$ in $\mathbb{L}^a(\pi)$.

(vi) Let $k \in \mathbb{L}^1(\pi)$ and $\lambda \in \mathbb{C}$ be such that $|\lambda| = 1$, $|k| \equiv \mathbf{1}_{\mathbb{X}}$ and $P(k) = \lambda k$. Then $\lambda k = \alpha\pi(k) + (1-\alpha)U(k)$. Taking the modulus, we obtain $1 \leq \alpha|\pi(k)| + (1-\alpha)U(\mathbf{1}_{\mathbb{X}}) \leq 1$. By convexity we conclude that $|\pi(k)| = 1$ and that k is constant modulo π, so that $\lambda = 1$. □

Proof of the multiplicative ergodicity. We apply Theorem 2.8. Let $b := \frac{p}{p-1}$ and $a > b$. From Assertion *(i)*-(iii) of Lemma 5.2, P_γ satisfies Hypothesis 2.7 with $J = [0, +\infty)$, $\mathcal{B}_0 = \mathbb{L}^a(\pi)$ and $\mathcal{B}_1 = \mathbb{L}^b(\pi)$ (to obtain (13b), iterate Inequality *(iii)* of Lemma 5.2 and use $\|\cdot\|_1 \leq \|\cdot\|_b$). Next, if $\gamma \in J_0$ (i.e. $r(\gamma) > 2(1-\alpha)$), then P_γ is quasi-compact from Assertion *(i)* of Lemma 5.2. Note that Assertion *(iv)* of Lemma 5.2 implies that Hypothesis 4.3 holds on the space $\mathbb{L}^a(\pi)$. Then P_γ satisfies the hypotheses of Proposition 4.5 on $\mathcal{B}_0 = \mathbb{L}^a(\pi)$ from Assertions *(v)* and *(vi)* of Lemma 5.2. Thus Hypothesis 2.5 holds on J_0 with $\mathcal{B}_0 = \mathbb{L}^a(\pi)$. Consequently $(S_n)_n$ is multiplicatively ergodic on J_0 with respect to $\mathbb{P}_\mu$, if we prove that $\mu_\gamma : f \mapsto \mu(e^{-\gamma\xi}f)$ is in $(\mathbb{L}^b(\pi))^*$, that the function $\gamma \mapsto \mu_\gamma$ is continuous from J_0 to $(\mathbb{L}^b(\pi))^*$, and finally that Condition (18) holds. Since μ is absolutely continuous with respect to π with density g_μ in $\mathbb{L}^p(\pi)$, we have

$$\forall f \in \mathbb{L}^b(\pi), \quad \mu_\gamma(f) = \int_{\mathbb{R}^d} f(y)\, e^{-\gamma\xi(y)}\, g_\mu(y)\, d\pi(y)$$

thus $\mu_\gamma \in (\mathbb{L}^b(\pi))^*$ since $e^{-\gamma\xi}\, g_\mu \in \mathbb{L}^p(\pi)$. Moreover the norm in $(\mathbb{L}^b(\pi))^*$ of $(\mu_\gamma - \mu_{\gamma'})$ equals to $\|(e^{-\gamma\xi} - e^{-\gamma'\xi})\, g_\mu\|_p$, which converges to 0 as $\gamma' \to \gamma$ from Lebesgue's theorem. Finally Condition (18) holds from Assertion *(i)* of Proposition 4.8. □

In view of (4), we study the spectral radius $r(\gamma)$ of P_γ. First observe that the non-increasingness of $r(\cdot)$ follows from Proposition 4.9 since $\mathbb{L}^a(\pi)$ is a Banach lattice. Consequently the set $J_0 := \{\gamma > 0 : r(\gamma) > 2(1-\alpha)\}$ is an interval with $\min J_0 = 0$ since $r(0) = 2$. Next set $h_\gamma := e^{-\gamma\xi}$ for $\gamma \geq 0$ and $h_\infty := \mathbf{1}_{\{\xi=0\}}$. Recall that $P_\gamma f = 2[\alpha\pi(f\, h_\gamma) + (1-\alpha)U(f\, h_\gamma)]$. We set $\tilde{U}_\gamma(\cdot) := h_\gamma\, U(\cdot)$.

Lemma 5.3. *Let $\gamma \in [0,\infty]$ and $a \in (1,+\infty)$. Let λ be an eigenvalue of $(P_\gamma)_{|\mathbb{L}^a(\pi)}$ such that $\lambda > 2(1-\alpha)\rho(\tilde{U}_\gamma)$. Then*

$$\lambda = 2\alpha \sum_{n\geq 0} \frac{2^n(1-\alpha)^n}{\lambda^n} \pi(\tilde{U}_\gamma^n(h_\gamma)). \tag{46}$$

In particular if $r(\gamma) > 2(1-\alpha)$, then $\lambda = r(\gamma)$ satisfies (46).

Proof. Let $\gamma \in [0,\infty]$. Let $\lambda \in \mathbb{C}$ and $f \in \mathbb{L}^a(\pi)$, $f \neq 0$, be such that $P_\gamma f = \lambda f$ in $\mathbb{L}^a(\pi)$, i.e. $\lambda f = 2[\alpha\pi(f\,h_\gamma) + (1-\alpha)U(f\,h_\gamma)]$ that can be rewritten

$$\lambda f\,h_\gamma = 2(\alpha\pi(f\,h_\gamma)\,h_\gamma + (1-\alpha)\tilde{U}_\gamma(f\,h_\gamma)).$$

Observe that $\pi(f\,h_\gamma) \neq 0$. Indeed $\pi(f\,h_\gamma) = 0$ would imply $\lambda f\,h_\gamma = 2(1-\alpha)\tilde{U}_\gamma(f\,h_\gamma)$, which contradicts the fact that $\lambda/(2-2\alpha)$ is not in the spectrum of $\tilde{U}_\gamma$. Now setting $g := f\,h_\gamma/\pi(f\,h_\gamma)$, we have

$$\lambda g = 2(\alpha\,h_\gamma + (1-\alpha)\tilde{U}_\gamma(g)) \tag{47}$$

and so

$$\left[id - \frac{2(1-\alpha)}{\lambda}\tilde{U}_\gamma\right](g) = \frac{2\alpha}{\lambda}\,h_\gamma.$$

Hence

$$g = \frac{2\alpha}{\lambda}\left[id - \frac{2(1-\alpha)}{\lambda}\tilde{U}_\gamma\right]^{-1}(h_\gamma) = \frac{2\alpha}{\lambda}\sum_{n\geq 0}\frac{2^n(1-\alpha)^n}{\lambda^n}\tilde{U}_\gamma^n h_\gamma$$

and so

$$\lambda = \lambda\pi(g) = 2\alpha\sum_{n\geq 0}\frac{2^n(1-\alpha)^n}{\lambda^n}\pi\left(\tilde{U}_\gamma^n\,h_\gamma\right).$$

□

Let $(Z_n)_n$ be a Markov process with transition U. Note that $\pi(\tilde{U}_\gamma^n(h_\gamma)) = \mathbb{E}_\pi[e^{-\gamma\sum_{k=0}^n Z_k}]$ if $\gamma \in [0,\infty)$ and $\pi(\tilde{U}_\infty^n(h_\infty)) = P_\pi[\sum_{k=0}^n Z_k = 0]$. Hence (46) can be rewritten

$$\lambda = 2\alpha\sum_{n\geq 0}\frac{2^n(1-\alpha)^n}{\lambda^n}\mathbb{E}_\pi[e^{-\gamma\sum_{k=0}^n Z_k}]. \tag{48}$$

Assume $\alpha > 1/2$ and $\pi(\xi > 0) = 1$ (so that $r(\infty) = 0$). Let $c := \sup J_0$. Due to (16), we know that $\lim_{\gamma\to c^-} r(\gamma) \leq 2(1-\alpha)$ and that r is continuous on J_0. So ν is well defined and is such that $2\alpha\sum_{n\geq 0} 2^n(1-\alpha)^n\mathbb{E}_\pi[e^{-\gamma\sum_{k=0}^n Z_k}] = 1$ (note that this quantity is 2 when $\gamma = 0$ and that its limit at infinity is smaller than 1).

Proof of the existence of the constant C_ν in (7). We apply Theorem 2.9. Assume that $\alpha > 1/2$, that (45) holds, and that $\pi(\xi^\tau) < \infty$ for some $\tau > 1$. Let $p > \frac{\tau}{\tau-1}$ and set $a_3 := \frac{p}{p-1}$ (ie. $1/p + 1/a_3 = 1$). Note that $a_3 < \tau$. Let a_2 be such that $a_3 < a_2 < \tau$. Since $\lim_{a\to+\infty}\frac{\tau a}{\tau+a} = \tau$, we can chose $a_1 > a_2$ such that $a_2 < \frac{\tau a_1}{\tau+a_1}$. Next let $a_0 > a_1$. From Lemma 5.2 we deduce that the assumptions of Theorem 2.9 hold with the spaces $\mathcal{B}_i = \mathbb{L}^{a_i}(\pi)$ for $i = 0,1,2,3$, so that we can apply Theorem 2.9: we conclude that r is C^1 on $[0,\theta_1)$. The fact that $r' < 0$ can easily be proved using Proposition 4.12. In this particular case, we can also use the fact that r is given by an implicit formula $F(r(\gamma),\gamma) = 0$ (see (46)), with F C^1 with non-null derivatives at $(r(\gamma),\gamma)$. Moreover C_ν in (7) is well-defined, provided that

$\mu_\gamma : f \mapsto \mu(e^{-\gamma\xi} f)$ is in $(\mathbb{L}^{a_3}(\pi))^*$ and that the function $\gamma \mapsto \mu_\gamma$ is continuous from J_0 to $(\mathbb{L}^{a_3}(\pi))^*$. These conditions hold since μ is absolutely continuous with respect to π with density in $\mathbb{L}^p(\pi)$ (see the proof of the multiplicative ergodicity). Moreover (18) has been proved together with the multiplicative ergodicity. □

6. Linear autoregressive model: proof of Theorem 2.13

Let $(X_n)_{n\in\mathbb{N}}$ be a linear autoregressive model as described in Example 2.12. Then $(X_n)_{n\in\mathbb{N}}$ is a Markov chain with transition kernel

$$P(x, A) = \int_{\mathbb{R}} \mathbf{1}_A(\alpha x + y)p(y)\, dy = \int_{\mathbb{R}} \mathbf{1}_A(y)p(y - \alpha x)\, dy. \tag{49}$$

Recall that the density p is assumed to satisfy the domination condition (20) ensuring that p has a moment of order r_0, that is

$$\int |x|^{r_0} p(x)dx < \infty. \tag{50}$$

In fact the domination condition (20) means that p satisfies (50) under a (local) uniform domination way.

Set $V(x) := (1 + |x|)^{r_0}$, $x \in \mathbb{R}$. Recall that, under Assumption (50), P satisfies the following drift condition (see [24])

$$\forall \delta > |\alpha|^{r_0},\ \exists L \equiv L(\delta) > 0, \quad PV \le \delta\, V + L\, \mathbf{1}_{\mathbb{X}}. \tag{51}$$

Moreover it is well-known that $(X_n)_{n\in\mathbb{N}}$ is V-geometrically ergodic, see [24]. Let $(\mathcal{B}_V, \|\cdot\|_V)$ be the weighted-supremum Banach space

$$\mathcal{B}_V := \{\ f : \mathbb{X} \to \mathbb{C},\ \text{measurable}\ : \|f\|_V := \sup_{x\in\mathbb{X}} |f(x)|V(x)^{-1} < \infty\ \}. \tag{52}$$

Let $(\mathcal{C}_V, \|\cdot\|_V)$ denote the following subspace of $\mathcal{B}_V$:

$$\mathcal{C}_V := \left\{\ f \in \mathcal{B}_V :\ f \text{ is continuous and } \lim_{|x|\to\infty} \frac{f(x)}{V(x)} \text{ exists in } \mathbb{C}\right\},$$

where the symbol $\lim_{|x|\to\infty}$ means that the limits when $x \to \pm\infty$ exist and are equal. Note that $V \in \mathcal{C}_V$ and that $\mathcal{C}_V$ is a closed subspace of $(\mathcal{B}_V, \|\cdot\|_V)$. For every $f \in \mathcal{C}_V$ we define

$$\ell_V(f) := \lim_{|x|\to\infty} \frac{f(x)}{V(x)}.$$

Let $\mathcal{C}_{0,V} := \{f \in \mathcal{C}_V : \ell_V(f) = 0\}$. Finally we denote by $(\mathcal{C}_b, \|\cdot\|_\infty)$ the space of bounded continuous complex-valued functions on $\mathbb{R}$ endowed with the supremum norm $\|\cdot\|_\infty$. We will see below that, for every $\gamma \in (0, +\infty]$, P_γ continuously acts on $\mathcal{C}_V$ (see Lemma 6.2). For $\gamma \in (0, +\infty]$, we denote by $r(\gamma)$ the spectral radius of P_γ on $\mathcal{C}_V$, that is:

$$r(\gamma) \equiv r(P_\gamma) := \lim_n \|P_\gamma^n\|_V^{1/n} = \lim_n \|P_\gamma^n V\|_V^{1/n}$$

where $\|\cdot\|_V$ also denotes the operator norm on $\mathcal{C}_V$. We will also prove that $\lim_{\gamma\to 0_+} r(\gamma) \ge 2$.

Recall that $\xi : \mathbb{X} \to [0, +\infty)$ is a measurable function and that $S_n = \sum_{k=0}^n \xi(X_k)$. Let $\kappa : \mathbb{X} \to \{2, ...\}$ be a measurable function. Theorem 6.1 below directly follows from the next more precise theorem when applied with $\mu = \delta_x$ or $\mu = \pi$.

Theorem 6.1. *Assume that the previous assumptions hold. Assume that the distribution μ of X_0 belongs to $\mathcal{C}_V^*$, namely satisfying $\mu(V) < \infty$. Assume moreover that ξ is coercive, that κ is bounded, that p is continuous, and that $\sup_{\mathbb{R}} \xi/V < \infty$. Then*

a) $(S_n, \kappa(X_n))_n$ is multiplicatively ergodic on $(0, +\infty)$ with $\rho = r$ on $[0, +\infty)$.
b) If moreover the Lebesgue measure of the set $[\xi = 0]$ is zero, then $\lim_{\gamma\to+\infty} r(\gamma) = 0$. Hence ν is finite.
c) Moreover, if there exists $\tau > 0$ such that $\sup_{\mathbb{R}} \xi^{1+\tau}/V < \infty$, then $\gamma \mapsto r(\gamma)$ admits a negative derivative on $[0, +\infty)$. Hence (7) holds also with $C_\nu \in (0, +\infty)$.

The next subsections are devoted to the proof of Theorem 6.1.

6.1. **Study of Hypothesis 2.7*.** In this subsection we prove that $(P_\gamma, J, \mathcal{B}_0, \mathcal{B}_1)$ satisfies Hypothesis 2.7* with $J = (0, +\infty]$, $\mathcal{B}_0 = \mathcal{C}_b$, and $\mathcal{B}_1 = \mathcal{C}_V$.

Lemma 6.2. *Assume that Assumption (20) holds (thus (50)), that p is continuous and that ξ is coercive. Then, for every $\gamma \in (0, +\infty]$, P_γ continuously acts on both $\mathcal{C}_b$ and $\mathcal{C}_V$. For every $\gamma \in [0, +\infty]$, P_γ is compact from $\mathcal{C}_b$ into $\mathcal{C}_V$. For every $\gamma \in (0, +\infty]$, we have $P_\gamma(\mathcal{B}_V) \subset \mathcal{C}_{0,V}$.*

Proof. Let $\gamma \in [0, +\infty]$. From (51) it easily follows that $P_\gamma V \le PV \le (\delta + L)V$, so that P_γ continuously acts on $\mathcal{B}_V$. Let $f \in \mathcal{B}_V$. Then

$$\forall x \in \mathbb{R}, \quad (P_\gamma f)(x) = \int_{\mathbb{R}} \psi_\gamma(x, y)\, dy \tag{53}$$

$$\text{with} \quad \begin{cases} \psi_\gamma(x, y) := \kappa(y) e^{-\gamma\xi(y)}\, f(y)\, p(y - \alpha x) & \text{if} \quad 0 \le \gamma < \infty \\ \psi_\gamma(x, y) := \kappa(y) 1_{\{\xi=0\}}(y)\, f(y)\, p(y - \alpha x) & \text{if} \quad \gamma = +\infty. \end{cases}$$

Let $A > 0$. We deduce from Assumption (20) and from a usual compactness argument ($[-A, A]$ is compact) that there exists a non-negative function $q \equiv q_A$ such that $y \mapsto V(y)\, q(y)$ is Lebesgue-integrable and

$$\forall v \in [-A, A],\ \forall y \in \mathbb{R}, \quad p(y + v) \le q(y).$$

Thus we have for every $x \in [-A, A]$ and for every $y \in \mathbb{R}$

$$|\psi_\gamma(x, y)| \ \le\ \|\kappa\|_\infty \|f\|_V\, (1 + |y|)^{r_0}\, p(y - \alpha x) \le \|\kappa\|_\infty \|f\|_V V(y)\, q(y). \tag{54}$$

Since $x \mapsto \psi_\gamma(x, y)$ is continuous for every $y \in \mathbb{R}$ from the continuity of p, we deduce from Lebesgue's theorem that the function $P_\gamma f$ is continuous on $\mathbb{R}$. We have proved that, if $\gamma \in [0, +\infty]$ and if $f \in \mathcal{B}_V$, then $P_\gamma f$ is continuous on $\mathbb{R}$. Thus P_γ continuously acts on $\mathcal{C}_b$.

Now, if $\gamma \in (0, +\infty]$, then

$$\forall x \in \mathbb{R}, \quad \frac{(P_\gamma f)(x)}{V(x)} = \int_{\mathbb{R}} \chi_\gamma(x, y)\, dy \qquad \text{with} \ \ \chi_\gamma(x, y) := \theta_\gamma(\alpha x + y)\, \frac{f(\alpha x + y)}{V(x)}\, p(y)$$

$$\text{where} \quad \theta_\gamma(\alpha x + y) := \begin{cases} \kappa(y) e^{-\gamma\xi(\alpha x + y)} & \text{if} \quad 0 < \gamma < \infty \\ \kappa(y) \mathbf{1}_{\{\xi=0\}}(\alpha x + y) & \text{if} \quad \gamma = +\infty. \end{cases}$$

For every $(x, y) \in \mathbb{R}^2$, we obtain that

$$|\chi_\gamma(x, y)| \ \le\ \theta_\gamma(\alpha x + y)\, \|f\|_V \left(\frac{1 + |x| + |y|}{1 + |x|}\right)^{r_0} p(y) \ \le\ \|\kappa\|_\infty \|f\|_V \big(1 + |y|\big)^{r_0} p(y).$$

Moreover $\lim_{|x|\to+\infty}\theta_\gamma(\alpha x+y)=0$ since ξ is coercive. It follows again from Lebesgue's theorem that

$$\lim_{|x|\to+\infty}\frac{(P_\gamma f)(x)}{V(x)}=0,$$

thus $P_\gamma f\in\mathcal{C}_{0,V}$. We have proved that, if $\gamma\in(0,+\infty]$, then $P_\gamma(\mathcal{B}_V)\subset\mathcal{C}_{0,V}$, thus the last assertion of Lemma 6.2 holds and P_γ continuously acts on $\mathcal{C}_V$. Finally, to prove the compactness property, let $\gamma\in[0,+\infty]$ and consider P_γ as written in (53). Since p is continuous, the image by P_γ of the unit ball $\{f\in\mathcal{C}_b:\|f\|_\infty\le 1\}$ is equicontinuous from Scheffé's lemma. Then P_γ is compact from $\mathcal{C}_b$ into $\mathcal{C}_V$ from Ascoli's theorem and from $\lim_{|x|\to\infty}V(x)=+\infty$. □

With the usual convention $V^0:=1$, we have the identification $\mathcal{C}_{V^0}=\mathcal{C}_b$. Then the continuity of $\gamma\mapsto P_\gamma$ from $(0,+\infty]$ into $\mathcal{L}(\mathcal{C}_b,\mathcal{C}_V)$ follows from the following.

Lemma 6.3. *Let $0\le a<a+b\le 1$. Assume that $\xi\le cV$ for some positive constant c. Then the following operator-norm inequality holds for every $(\gamma,\gamma')\in[0,+\infty)^2$*

$$\|P_\gamma-P_{\gamma'}\|_{\mathcal{C}_{V^a},\mathcal{C}_{V^{a+b}}}\ :=\sup_{f\in\mathcal{C}_{V^a},\|f\|_{V^a}\le 1}\|P_\gamma f-P_{\gamma'}f\|_{V^{a+b}}\ \le\|\kappa\|_\infty(c|\gamma-\gamma'|)^b\|P\|_{V^{a+b}}\,.$$

Proof. Let $(\gamma,\gamma')\in[0,+\infty)^2$. For all $(u,v)\in[0,+\infty)^2$, we have $|e^{-u}-e^{-v}|\le|e^{-u}-e^{-v}|^b\le|u-v|^b$ from Taylor's inequality. Thus we obtain for any $f\in\mathcal{C}_{V^a}$

$$\begin{aligned}\big|(P_\gamma f)(x)-(P_{\gamma'}f)(x)\big| &\le \|\kappa\|_\infty\|f\|_{V^a}\int_{\mathbb{R}}\big|e^{-\gamma\xi(y)}-e^{-\gamma'\xi(y)}\big|(V(y))^a p(y-\alpha x)\,dy\\ &\le \|\kappa\|_\infty\|f\|_{V^a}(c\,|\gamma-\gamma'|)^b\int_{\mathbb{R}}(V(y))^{a+b}\,p(y-\alpha x)\,dy\\ &\le \|\kappa\|_\infty\|f\|_{V^a}(c\,|\gamma-\gamma'|)^b PV^{a+b}(x),\end{aligned}$$

from which we deduce the desired inequality. □

Lemma 6.4. *Assume that Assumption (20) holds (thus (50)) and that ξ is coercive. Then*

$$\|P_\gamma-P_\infty\|_{\mathcal{C}_b,\mathcal{C}_V}:=\sup_{f\in\mathcal{C}_b,\|f\|_\infty\le 1}\|P_\gamma f-P_\infty f\|_V\ \longrightarrow 0\quad\textit{when }\gamma\to+\infty.$$

Proof. Let $\varepsilon>0$. Let $f\in\mathcal{C}_b$ be such that $\|f\|_\infty\le 1$. From $|P_\gamma f|\le P\mathbf{1}_{\mathbb{X}}=\mathbf{1}_{\mathbb{X}}$ it follows that there exists $A\equiv A(\varepsilon)$ such that :

$$|x|>A\ \Rightarrow\ \forall\gamma\in(0,+\infty),\ \frac{|(P_\gamma f)(x)|}{V(x)}\le\varepsilon. \tag{55}$$

Moreover, for any $\beta>0$ and $x\in\mathbb{R}$ such that $|x|\le A$, we obtain that

$$\begin{aligned}\big|(P_\gamma f-P_\infty f)(x)\big| &\le \|\kappa\|_\infty e^{-\gamma\beta}\int_{[\xi>\beta]}p(y-\alpha x)\,dy+\|\kappa\|_\infty\int_{[0<\xi\le\beta]}p(y-\alpha x)\,dy\\ &\le \|\kappa\|_\infty\left(e^{-\gamma\beta}+\int_{[0<\xi\le\beta]}q(y)\,dy\right)\end{aligned}$$

where $q\equiv q_A$ is the function given in (54). Since q is Lebesgue-integrable on $\mathbb{R}$, we have $\int_{[0<\xi\le\beta]}q(y)\,dy\to 0$ when $\beta\to 0$, so that there exists $\beta_0\equiv\beta_0(\varepsilon)>0$ such that

$$\|\kappa\|_\infty\int_{[0<\xi\le\beta_0]}q(y)\,dy\le\frac{\varepsilon}{2}.$$

Finally let $\gamma_0 \equiv \gamma_0(\varepsilon) > 0$ be such that : $\forall \gamma > \gamma_0,\ \|\kappa\|_\infty e^{-\gamma\beta_0} \le \varepsilon/2$. Then

$$|x| \le A \ \Rightarrow\ \forall \gamma \in (\gamma_0, \infty),\ \frac{|(P_\gamma f - P_\infty f)(x)|}{V(x)} \le |(P_\gamma f)(x) - (P_\infty f)(x)| \le \varepsilon. \tag{56}$$

Inequalities (55) and (56) provides the desired statement. □

To study the conditions (14a) and (14b) of Hypothesis 2.7* with $J = (0, +\infty]$, $\mathcal{B}_0 = \mathcal{C}_b$, and $\mathcal{B}_1 = \mathcal{C}_V$, we use the duality arguments of [15, prop. 5.4]. The topological dual spaces of $\mathcal{C}_V$ and $\mathcal{C}_b$ are denoted by $(\mathcal{C}_V^*, \|\cdot\|_V)$ and $(\mathcal{C}_b^*, \|\cdot\|_\infty)$ respectively (for the sake of simplicity we use the same notation for the dual norms). For any $\gamma > 0$, we denote by P_γ^* the adjoint operator of P_γ on $\mathcal{C}_V$. Note that each P_γ^* is a contraction with respect to the dual norm $\|\cdot\|_\infty$ because so is P_γ on $\mathcal{C}_b$.

In the sequel, $\delta > |\alpha|^{r_0}$ is fixed, as well as the associated constant $L \equiv L(\delta)$ in (51).

Lemma 6.5. *Assume that (50) holds, that κ is bounded, and that ξ is coercive. Then, for every $\gamma > 0$ and for every $\beta > 0$, there exists a positive constant L_β such that*

$$P_\gamma V \le \|\kappa\|_\infty (e^{-\gamma\beta}\,\delta\, V + L_\beta\, \mathbf{1}_{\mathbb{X}}). \tag{57}$$

Moreover

$$P_\infty V \le \|\kappa\|_\infty \left(\sup_{[\xi=0]} V\right) \mathbf{1}_{\mathbb{X}}. \tag{58}$$

Proof. We have for every $\gamma > 0$ and for every $\beta > 0$

$$\begin{aligned} P_\gamma V &= P(\kappa e^{-\gamma\xi} V) = P\big(\kappa e^{-\gamma\xi}\mathbf{1}_{[\xi>\beta]} V\big) + P\big(\kappa e^{-\gamma\xi}\mathbf{1}_{[\xi\le\beta]} V\big) \\ &\le \|\kappa\|_\infty \left(e^{-\gamma\beta}\big(\delta\, V + L\,\mathbf{1}_{\mathbb{X}}\big) + \int_{[\xi\le\beta]} V(y) P(\cdot, dy)\right) \qquad \text{(from (51))} \\ &\le \|\kappa\|_\infty \left(e^{-\gamma\beta}\,\delta\, V + \big(L + \sup_{[\xi\le\beta]} V\big)\mathbf{1}_{\mathbb{X}}\right) \end{aligned}$$

from which we deduce the first desired statement. For P_∞, we have

$$P_\infty V = P(\kappa \mathbf{1}_{\{\xi=0\}} V) \le \left(\sup_{[\xi=0]} V\right) P(\kappa) \le \left(\sup_{[\xi=0]} V\right) \|\kappa\|_\infty \mathbf{1}_{\mathbb{X}}.$$

□

Corollary 6.6. *Assume that Assumption (50) holds true, that κ is bounded, and that ξ is coercive. Then, for every $\gamma_1 > 0$ and for every $\varepsilon > 0$, there exists a constant $D > 0$ such that*

$$\forall \gamma \in [\gamma_1, +\infty],\ \forall f^* \in \mathcal{C}_V^*, \quad \|P_\gamma^* f^*\|_V \le \varepsilon\, \|f^*\|_V + D\, \|f^*\|_\infty. \tag{59}$$

Moreover, for every $\gamma \in (0, +\infty]$, the essential spectral radius $r_{ess}(P_\gamma)$ is zero.

Proof. Choose $\beta = \beta(\gamma_1, \varepsilon) > 0$ such that $\|\kappa\|_\infty e^{-\gamma_1\beta}\,\delta < \varepsilon$. Then we deduce from Lemma 6.5 that $P_{\gamma_1} V \le \varepsilon\, V + D\,\mathbf{1}_{\mathbb{X}}$, where $D \equiv D(L, \gamma_1, \varepsilon)$ is a positive constant. Now let $\gamma \in [\gamma_1, +\infty]$. Since $P_\gamma V \le P_{\gamma_1} V$, we also have $P_\gamma V \le \varepsilon\, V + D\,\mathbf{1}_{\mathbb{X}}$. This inequality easily rewrites as (59) (see the proof in [9, p. 190]). Finally, since P_γ^* is compact from $\mathcal{C}_V^*$ into $\mathcal{C}_b^*$ (Lemma 6.2), we deduce from [13] that $r_{ess}(P_\gamma) \le \varepsilon$. We obtain $r_{ess}(P_\gamma) = 0$ because ε is arbitrary. □

Remark 6.7. *Let $\gamma_1 > 0$, $\varepsilon > 0$ and $0 \le a \le a+b \le 1$. Observe that Corollary 6.6 holds also if we replace V by V^{a+b} (since ϑ_1 admits a moment of order $r_0(a+b)$). Moreover notice that (59) with V^{a+b} instead of V directly gives that there exists a constant $D_{\varepsilon,a+b} > 0$ such that*

$$\forall \gamma \in [\gamma_1, +\infty],\ \forall f^* \in \mathcal{C}^*_{V^{a+b}}, \quad \|P^*_\gamma f^*\|_{V^{a+b}} \le \varepsilon \, \|f^*\|_{V^{a+b}} + D_{\varepsilon,a+b} \, \|f^*\|_{V^a} \tag{60}$$

since $\|f^*\|_\infty \le \|f^*\|_{V^a}$.

6.2. A preliminary useful statement on $r(\gamma)$.

Proposition 6.8. *Assume that Assumption (50) holds true, that κ is bounded, that ξ is coercive and finally that the function ξ/V is bounded on $\mathbb{R}$. Then $\lim_{\gamma \to 0_+} r(\gamma) \ge 2$.*

Proof. We need the following lemma concerning the special case $\kappa \equiv 2$. To avoid confusion we write below $\widetilde{P}_\gamma$ and $\widetilde{r}(\gamma)$ in place of P_γ and $r(\gamma)$ when $\kappa \equiv 2$.

Lemma 6.9. *Assume that (50) holds, that $\kappa \equiv 2$, and that ξ is coercive. Then $\widetilde{P}_0$ continuously acts on $\mathcal{C}_V$. Moreover the function $\gamma \mapsto \widetilde{r}(\gamma)$ is continuous at $\gamma = 0$, with $\widetilde{r}(0) = 2$.*

From $P^n_\gamma \ge \widetilde{P}^n_\gamma$ (since $\kappa \ge 2$), we deduce that $r(\gamma) = r(P_\gamma) \ge r(\widetilde{P}_\gamma) = \widetilde{r}(\gamma)$. It follows from Proposition 4.9 that $\lim_{\gamma \to 0_+} r(\gamma) \ge \lim_{\gamma \to 0_+} r(\gamma) = 2$. □

Proof of Lemma 6.9. Note that $\widetilde{P}_0 = 2P$. If $f \in \mathcal{C}_V$, then

$$\forall x \in \mathbb{R}, \quad \frac{(Pf)(x)}{V(x)} = \int_{\mathbb{R}} \chi_0(x,y)\, dy \qquad \text{with} \quad \chi_0(x,y) := \frac{f(\alpha x + y)}{V(x)}\, p(y).$$

We know that $|\chi_0(x,y)| \le \|f\|_V \big(1 + |y|\big)^{r_0} p(y)$, and writing $\chi_0(x,y) := \frac{f(\alpha x+y)}{V(\alpha x+y)} \frac{V(\alpha x+y)}{V(x)}\, p(y)$ proves that $\lim_{|x| \to +\infty} \chi_0(x,y) = \ell_V(f)\, |\alpha|^{r_0}\, p(y)$. It follows from Lebesgue's theorem that $\lim_{|x| \to +\infty} \frac{(Pf)(x)}{V(x)} = \ell_V(f)\, |\alpha|^{r_0}$. Thus $P(\mathcal{C}_V) \subset \mathcal{C}_V$ and P continuously acts on $\mathcal{C}_V$.

Iterating Inequality (51) proves that P is power-bounded on $\mathcal{C}_V$ (i.e. $\sup_{n \ge 1} \|P^n V\|_V < \infty$), thus $r(P) = 1$ since P is Markov. Moreover (51) rewrites as the following (dual) Doeblin-Fortet inequality (see the proof in [9, p. 190]):

$$\forall f^* \in \mathcal{C}^*_V, \quad \|P^* f^*\|_V \le \delta \, \|f^*\|_V + L \, \|f^*\|_\infty. \tag{61}$$

Since P is compact from $\mathcal{C}_b$ into $\mathcal{C}_V$ (apply the same argument as in Lemma 6.2), so is P^* from $\mathcal{C}^*_V$ into $\mathcal{C}^*_b$. Then we deduce from [13] that, under Assumption (50), P is a quasi-compact operator on $\mathcal{C}_V$ and its essential spectral radius $r_{ess}(P)$ satisfies the following bound (see also [27, Sect. 8]): $r_{ess}(P) \le \delta$. It follows that

$$\widetilde{r}(0) = r(\widetilde{P}_0) = 2 \quad \text{and} \quad r_{ess}(\widetilde{P}_0) \le 2\delta. \tag{62}$$

Observe that (51) and Inequality $\widetilde{P}_\gamma V \le 2PV$give $\widetilde{P}_\gamma V \le 2\delta\, V + 2L\, \mathbf{1}_{\mathbb{X}}$, which rewrites as the following Doeblin-Fortet inequality:

$$\forall \gamma \in [0, +\infty),\ \forall f^* \in \mathcal{C}^*_V, \quad \|\widetilde{P}^*_\gamma f^*\|_V \le 2\delta \, \|f^*\|_V + 2L \, \|f^*\|_\infty. \tag{63}$$

Using (62), (63), Corollary 6.6 and Lemma 6.3 (with $a = 0$ et $b = 1$), it follows from Theorem 2.8 (applied with $\delta_0 = 2\delta$) that $\gamma \mapsto \widetilde{r}(\gamma)$ is continuous at $\gamma = 0$ since $\widetilde{r}(0) = 2 > 2\delta$. □

6.3. **Study of Hypothesis 2.5.** In this subsection we prove that Hypothesis 2.5 holds with respect to $(J_1, \mathcal{B}_1)$ with $\mathcal{B}_1 = \mathcal{C}_V$ and $J_1 := (0, \theta_1)$, where

$$\theta_1 := \sup\{\gamma > 0 \,:\, r(\gamma) > 0\}. \tag{64}$$

We know from Proposition 6.8 that $\theta_1 > 0$. Since $\mathcal{C}_V$ is a Banach lattice, we use Proposition 4.5 to prove Hypothesis 2.5.

Observe that, for any $\gamma \in (0, \theta_1)$, P_γ is quasi-compact on $\mathcal{C}_V$ from Corollary 6.6 since $r(\gamma) > 0$. The other conditions of Proposition 4.5 follow from Remark 6.11 and Lemmas 6.12-6.13 below. First we state the following.

Lemma 6.10. *For any non-null $e^* \in \mathcal{C}_V^*$, $e^* \geq 0$, there exists a nonnegative measure $\mu \equiv \mu_{e^*}$ on $(\mathbb{R}, \mathcal{X})$ such that*

$$\forall f \in \mathcal{C}_V, \quad e^*(f) = \mu\left(\frac{f}{V} - \ell_V(f)\,\mathbf{1}_{\mathbb{R}}\right) + e^*(V)\,\ell_V(f). \tag{65}$$

Remark 6.11. *Due to Lemma 6.10, Hypothesis 4.3 is fulfilled with J_1 and $\mathcal{B} = \mathcal{C}_V$. Indeed, let $\gamma \in J_1$ and let $\phi \in \mathcal{C}_V$ be non-null and non-negative. Then, we have $P_\gamma \phi > 0$ everywhere from the definition of P and the strict positivity of the function $p(\cdot)$. Now prove that, if $\psi \in \mathcal{B}^* \cap \mathrm{Ker}(P_\gamma^* - r(\gamma)I)$ is non-null and non-negative, then $\psi(P_\gamma \phi) > 0$, so ψ is positive. Let $\gamma > 0$. First observe that $\psi \neq c\,\ell_V$ for every $c \in \mathbb{C}$ because $r(\gamma) > 0$ and $P_\gamma^*(\ell_V) = 0$ from Lemma 6.2. Second note that $\mu = 0$ in (65) implies that $e^* = e^*(V)\,\ell_V$. Thus the nonnegative measure $\mu \equiv \mu_\psi$ associated with ψ in (65) is non-null. Since $\ell_V(P_\gamma \phi) = 0$ from Lemma 6.2, we deduce from (65) (applied with $e^* = \psi$) and from $P_\gamma \phi > 0$ that $\psi(P_\gamma \phi) = \mu(P_\gamma \phi / V) > 0$.*

Proof of Lemma 6.10. Let $(\mathcal{C}, \|\cdot\|)$ denote the following space

$$\mathcal{C} := \left\{ g : \mathbb{R} \to \mathbb{C} \ \text{ continuous } \ : \|g\| := \sup_{x \in \mathbb{R}} |g(x)| < \infty \ \text{ and } \ \lim_{|x| \to \infty} g(x) \text{ exists in } \mathbb{C} \right\}.$$

For every $g \in \mathcal{C}$, we set: $\ell(g) := \lim_{|x| \to \infty} g(x)$. We denote by $\mathcal{C}^*$ the topological dual space of $\mathcal{C}$. Let $e^* \in \mathcal{C}_V^*$, $e^* \geq 0$, and let $\widetilde{e}^* \in \mathcal{C}^*$ be defined by:

$$\forall g \in \mathcal{C}, \quad \widetilde{e}^*(g) := e^*(gV).$$

Next let $\widetilde{e}_0^*$ be the restriction of $\widetilde{e}^*$ to $\mathcal{C}_0 := \{g \in \mathcal{C} : \ell(g) = 0\}$. From the Riesz representation theorem, there exists a unique positive measure μ on $(\mathbb{R}, \mathcal{X})$ such that

$$\forall g \in \mathcal{C}_0, \quad \widetilde{e}_0^*(g) = \mu(g) := \int_{\mathbb{R}} g \, d\mu.$$

Then, writing $g = (g - \ell(g)\,\mathbf{1}_{\mathbb{R}}) + \ell(g)\,\mathbf{1}_{\mathbb{R}}$ for any $g \in \mathcal{C}$, we obtain that

$$\widetilde{e}^*(g) = \mu\big(g - \ell(g)\,\mathbf{1}_{\mathbb{R}}\big) + \widetilde{e}^*(\mathbf{1}_{\mathbb{R}})\,\ell(g).$$

We conclude by observing that, for any $f \in \mathcal{C}_V$, we have $e^*(f) = \widetilde{e}^*(f/V)$. □

Lemma 6.12. *If $f, g \in \mathcal{C}_V$ are such that $P_\gamma f = r(\gamma) f$ and $P_\gamma g = r(\gamma) g$ with $f > 0$, then $g \in \mathbb{C} \cdot f$.*

Proof. Let $f, g \in \mathrm{Ker}(P_\gamma - r(\gamma)I)$ with $f > 0$. Let $\beta \in \mathbb{C}$ be such that $h := g - \beta f$ vanishes at 0. Since $h \in \mathrm{Ker}(P_\gamma - r(\gamma)I)$ we deduce from Proposition 8.1 that $P_\gamma |h| = r(\gamma)|h|$. Then $|h|(0) = 0$, the positivity of $p(\cdot)$ and finally the continuity of $|h|$ show that $h = 0$. □

Lemma 6.13. *Let $h \in \mathcal{C}_V$ with $|h| > 0$ and $\lambda \in \mathbb{C}$ be such that $|\lambda| = 1$ and $P\frac{h}{|h|} = \lambda\frac{h}{|h|}$ in $\mathbb{L}^1(\pi)$. Then $\lambda = 1$.*

Proof. Observe that $\frac{h}{|h|}$ is in $\mathcal{C}_b$ so in $\mathcal{B}_V$. But it is known from [24] that $(X_n)_n$ is V-geometrically ergodic, so λ=1. □

6.4. **Proof of Theorem 6.1.** To prove Assertion a) of Theorem 6.1, we apply Theorem 2.8. Let γ_1 be such that $0 < \gamma_1 < \theta_1$, with θ_1 given in (64). Let $\varepsilon \in (0,1)$. Then, from the results of the previous subsections, the assumptions of Theorem 2.8 hold with $J = (\gamma_1, \theta_1)$, $\mathcal{B}_0 = \mathcal{C}_b$, $\mathcal{B}_1 = \mathcal{C}_V$, $\delta_0 = \varepsilon$, thus with $J_0 := \{\gamma \in J : r(\gamma) > \varepsilon\}$. A first consequence is that $\theta_1 = +\infty$ from Lemma 4.14 and Remark 4.15. Moreover, for every $\gamma \in J_0$, the map $\mu_\gamma : f \mapsto \mu(\kappa e^{-\gamma\xi} f)$ is in $(\mathcal{C}_V)^*$ from $\mu(V) < \infty$, and we have for every $\gamma, \gamma' \in J_0$ and for every $f \in \mathcal{C}_V$

$$\big|\mu_\gamma(f) - \mu_{\gamma'}(f)\big| \leq \|\kappa\|_\infty \|f\|_V \mu\left(\big|e^{-\gamma\xi} - e^{-\gamma\xi}\big|V\right)$$

so that the norm of $(\mu_\gamma - \mu_{\gamma'})$ in $(\mathcal{C}_V)^*$ is less than $\|\kappa\|_\infty\, \mu(|e^{-\gamma\xi} - e^{-\gamma\xi}|V)$ which converges to 0 as $\gamma' \to \gamma$ from Lebesgue's theorem. Finally note that the additional condition in Assertion (ii) of Proposition 4.8 clearly holds from the form of P (see (49)) and from the positivity of the density p. Thus (18) holds. Since γ_1 and ε are arbitrarily small, we deduce from Theorem 2.8 that $(S_n, \kappa(X_n))_n$ is multiplicatively ergodic on $(0, +\infty)$ with $\rho_Y(\gamma) = r(\gamma) > 0$ on $(0, +\infty)$. We have proved Assertion a) of Theorem 6.1.

For Assertion b), observe that $\mathrm{Leb}(\xi = 0) = 0$ implies that $P_\infty = 0$, in particular the spectral radius $r(\infty)$ of P_∞ is zero. Then Lemma 6.4 and Theorem 2.8 give $\lim_{\gamma\to+\infty} r(\gamma) = r(\infty) = 0$. Consequently ν is finite and satisfies (19), and so (4), with respect to $\mathbb{P}_\mu$, provided that μ is a probability distribution belonging to $\mathcal{C}_V^*$.

Finally, to prove Part (3) of Theorem 6.1, we assume now that $\xi \in \mathcal{B}_{V^{\frac{1}{1+\tau}}}$ for some $\tau > 0$ and that $[\xi = 0]$ has Lebesgue measure 0. Consider any

$$0 < a_0 < a_1 < a_1 + \frac{1}{1+\tau} < a_2 < a_3 = 1.$$

Let us prove that the additional assumptions of Theorem 2.9 hold true with $\mathcal{B}_i := \mathcal{C}_{V^{a_i}}$ for $i \in \{0,1,2,3\}$. Let $i \in \{0,1,2\}$. The fact that $(P_\gamma)_\gamma$ satisfies the conditions of Hypothesis 2.7* on $(J, \mathcal{B}_i, \mathcal{B}_{i+1})$ comes from Lemma 6.3 and Remark 6.7. The fact that Hypothesis 2.5 is satisfied on $\mathcal{B}_{i+1}$ follows from Proposition 4.5: apply the results of Subsection 6.3 with $V^{a_{i+1}}$ in place of V. Observe that

$$\|\xi f\|_{\mathcal{B}_2} = \sup \frac{\|\xi f\|}{V^{a_2}} \leq \sup \frac{\|\xi\|}{V^{\frac{1}{1+\tau}}} \sup \frac{\|f\|}{V^{a_1}} \leq \|f\|_{\mathcal{B}_1} \sup \frac{\|\xi\|}{V^{\frac{1}{1+\tau}}}.$$

Hence we have proved that $f \mapsto \xi f$ is in $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$. The fact that $\gamma \mapsto P_\gamma$ is C^1 from $(0, +\infty)$ to $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ and that $P'_\gamma := P_\gamma(-\xi f)$ comes from the proof of [16, Lemma 10.4]. Finally, due to Proposition 4.12, we have $r'(\nu) < 0$. We conclude by Theorem 2.9.

7. Proof of Theorems 2.8 and 2.9

Let us state the Keller-Liverani perturbation theorem.

Theorem 7.1 (Keller-Liverani Perturbation Theorem [17, 22, 8]). *Let $(\mathcal{X}_0, \|\cdot\|_{\mathcal{X}_0})$ be a Banach space and $(\mathcal{X}_1, \|\cdot\|_{\mathcal{X}_1})$ be a normed space such that $\mathcal{X}_0 \hookrightarrow \mathcal{X}_1$. Let $J \subset [-\infty, +\infty]$ be an interval and let $(Q(t))_{t\in J}$ be a family of operators. We assume that*

- *For every $t \in J$, $Q(t) \in \mathcal{L}(\mathcal{X}_0) \cap \mathcal{L}(\mathcal{X}_1)$,*
- *$t \mapsto Q(t)$ is a continuous map from J in $\mathcal{L}(\mathcal{X}_0, \mathcal{X}_1)$,*
- *There exist $\delta_0 > 0$, $c_0, M_0 > 0$ such that for every $t \in J$*

$$\forall f \in \mathcal{X}_0,\ \forall n \in \mathbb{Z}_+, \quad \|(Q(t))^n f\|_{\mathcal{X}_0} \le c_0\big(\delta_0^n \|f\|_{\mathcal{X}_0} + M_0^n \|f\|_{\mathcal{X}_1}\big).$$

Let $t_0 \in J$. Then, for every $\varepsilon > 0$ and every $\delta > \delta_0$, there exists $I_0 \subset J$ containing t_0 such that

$$\sup_{t\in I_0,\, z\in\mathcal{D}(\delta,\varepsilon)} \|(zI - Q(t))^{-1}\|_{\mathcal{X}_0} < \infty,$$

with $\mathcal{D}(\delta, \varepsilon) := \{z \in \mathbb{C},\ d(z, \sigma(Q(t_0)_{|\mathcal{X}_0})) > \varepsilon,\ |z| > \delta\}$.

Furthermore the map $t \mapsto (zI - Q(t))^{-1}$ from J to $\mathcal{L}(\mathcal{X}_0, \mathcal{X}_1)$ is continuous at t_0 in a uniform way with respect to $z \in \mathcal{D}(\delta, \varepsilon)$, i.e.

$$\lim_{t\to t_0,\, t\in J} \sup\big\{\|(zI - Q(t))^{-1} - (zI - Q(t_0))^{-1}\|_{\mathcal{X}_0,\mathcal{X}_1} \ :\ z \in \mathcal{D}(\delta, \varepsilon)\big\} = 0.$$

In particular, $\limsup_{t\to t_0} r((Q(t))_{|\mathcal{X}_0}) \le \max(\delta_0, r((Q(t_0))_{|\mathcal{X}_0}))$. Finally the map $t \mapsto r((Q(t))_{|\mathcal{X}_0})$ is continuous on $\{t \in J : r((Q(t))_{|\mathcal{X}_0}) > \delta_0 \ge r_{ess}((Q(t))_{|\mathcal{X}_0})\}$.

We use the notations of Section 4.1 and identify $(X_n)_n$ with the canonical Markov chain. From now on, to simplify notations, we write $R_z(\gamma) := (zI - P_\gamma)^{-1}$ for the resolvent when it is well defined. Recall that $J_0 := \{t \in J : r(\gamma) > \delta_0\}$.

7.1. **Proof of Theorem 2.8 under Hypothesis 2.7.** Here we assume that $(P_\gamma)_\gamma$ satisfies Hypothesis 2.7 with $(J, \mathcal{B}_0, \mathcal{B}_1)$ and that Hypothesis 2.5 is fulfilled on J_0 with $\mathcal{B} := \mathcal{B}_0$. The property (16) and the continuity on J_0 of the function $\gamma \mapsto r(\gamma) := r((P_\gamma)_{|\mathcal{B}_0})$ follow from Theorem 7.1. Moreover we deduce from Hypothesis 2.5 on J_0 with $\mathcal{B} := \mathcal{B}_0$ that (12) holds. It remains to prove that, if K is a compact subset of J_0, then the constants θ_γ and M_γ and in (12) are uniformly bounded by some $\theta_K \in (0, 1)$ and $M_K \in (0, +\infty)$, and that $\gamma \mapsto \Pi_\gamma$ is continuous from J_0 to $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_1)$. To that effect we use below the spectral definition of Π_γ.

Let $\chi : J_0 \to (0, +\infty)$ be defined by $\chi(\gamma) := \max\big(\delta_0, \lambda(\gamma)\big)$, where we have set $\lambda(\gamma) := \max\{|\lambda| : \lambda \in \sigma(P_{\gamma|\mathcal{B}_0}) \setminus \{r(\gamma)\}\}$. Due to Theorem 7.1, χ is continuous on J_0. Let K be a compact subset of J_0. We set $\theta := \max_K \frac{\chi}{r}$. Since $\chi(\gamma) < r(\gamma)$ for every $\gamma \in K$ and since $r(\cdot)$ and $\chi(\cdot)$ are continuous, we conclude that $\theta \in (0, 1)$. Next we consider any $\eta > 0$ such that $\theta + 2\eta < 1$. Next let us construct the map $\gamma \mapsto \Pi_\gamma$ from K to $\mathcal{L}(\mathcal{B}_0)$. Let $\gamma_0 \in K$. Since r is continuous on K, there exists $\varepsilon > 0$ such that, for every $\gamma \in K$ such that $|\gamma - \gamma_0| \le \varepsilon$, we have $|r(\gamma) - r(\gamma_0)| < \eta r(\gamma_0)$. Let us write $K(\gamma_0)$ for the set of $\gamma \in K$ such that $|\gamma - \gamma_0| \le \varepsilon$. Observe that, for any $\gamma \in K(\gamma_0)$,

$$\chi(\gamma) \le \theta r(\gamma) < \theta(1 + \eta) r(\gamma_0) < (\theta + \eta) r(\gamma_0) < (1 - \eta) r(\gamma_0)$$

and so the eigenprojector Π_γ on $\mathrm{Ker}(P_\gamma - r(\gamma)I)$ can be defined by

$$\Pi_\gamma = \frac{1}{2i\pi} \oint_{\Gamma_1(\gamma_0)} R_z(\gamma)\, dz, \tag{66}$$

where $\Gamma_1(\gamma_0)$ is the oriented circle centered on $r(\gamma_0)$ with radius $\eta\, r(\gamma_0)$. Due to Theorem 7.1, $\gamma \mapsto \Pi_\gamma$ is well defined from $K(\gamma_0)$ to $\mathcal{L}(\mathcal{B}_0)$ and is continuous from $K(\gamma_0)$ to $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_1)$.

Now, for every $\gamma \in K$, we define the oriented circle $\Gamma_0(\gamma) := \big\{z \in \mathbb{C} : |z| = (\theta+\eta)\, r(\gamma)\big\}$. By definition of θ, for every $\gamma \in K$, we have $\chi(\gamma) \leq \theta\, r(\gamma)$ and so $\chi(\gamma) < (\theta+\eta)\, r(\gamma) < r(\gamma)$. Hence, by definition of $\chi(\gamma)$, $R_z(\gamma)$ is well-defined in $\mathcal{L}(\mathcal{B}_0)$ for every $\gamma \in K$ and $z \in \Gamma_0(\gamma)$. From spectral theory, it comes that

$$N_\gamma^n := P_\gamma^n - r(\gamma)^n \Pi_\gamma = \frac{1}{2i\pi} \oint_{\Gamma_0(\gamma)} z^n\, R_z(\gamma)\, dz \tag{67}$$

and so

$$\|P_\gamma^n - r(\gamma)^n \Pi_\gamma\|_{\mathcal{B}_0} \leq M_\gamma \big((\theta+\eta)\, r(\gamma)\big)^{n+1} \quad \text{with} \quad M_\gamma := \sup_{|z|=(\theta+\eta)\, r(\gamma)} \|R_z(\gamma)\|_{\mathcal{B}_0}. \tag{68}$$

We have to prove that

$$M_K := \sup_{\gamma \in K} M_\gamma < \infty. \tag{69}$$

Let $\gamma_0 \in K$. Since $\gamma \mapsto r(\gamma)$ is continuous at γ_0, there exists $\alpha \equiv \alpha(\gamma_0) > 0$ such that, for every $\gamma \in K$ such that $|\gamma - \gamma_0| < \alpha$, we have

$$\frac{\theta + \frac{\eta}{2}}{\theta+\eta}\, r(\gamma_0) < r(\gamma) < \frac{\theta + \frac{3\eta}{2}}{\theta+\eta}\, r(\gamma_0).$$

Set $\delta := \frac{\eta}{2}\, r(\gamma_0)$. If $|\gamma - \gamma_0| < \alpha$ and if $|z| = (\theta+\eta)\, r(\gamma)$, we obtain since $\delta_0 \leq \chi(\gamma_0) \leq \theta\, r(\gamma_0)$ and $\theta + 2\eta < 1$:

$$\delta_0 + \delta \leq \chi(\gamma_0) + \delta \leq \big(\theta + \frac{\eta}{2}\big)\, r(\gamma_0) < |z| < \big(\theta + \frac{3\eta}{2}\big)\, r(\gamma_0) < r(\gamma_0) - \delta.$$

From the previous inequalities, let us just keep in mind that $\chi(\gamma_0) + \delta < |z| < r(\gamma_0) - \delta$. Then, by definition of $\chi(\gamma_0)$, we conclude that every complex number z such that $|z| = (\theta+\eta)\, r(\gamma)$ satisfies

$$|z| > \delta_0 + \delta \ \text{ and } \ d\big(z, \sigma(Q(\gamma_0))\big) > \delta.$$

Hence, up to a change of α, due to Theorem 7.1, we obtain that

$$\sup_{\gamma > 0\,:\, |\gamma - \gamma_0| < \alpha} M_\gamma = \sup\big\{\|R_z(\gamma)\|_{\mathcal{B}_0} \ : \ |\gamma - \gamma_0| < \alpha,\ |z| = (\theta+\eta)\, r(\gamma)\big\} < \infty.$$

By a standard compacity argument, we have proved (69). Consequently, with $\theta_K := \theta + \eta$, we deduce from (68) that

$$\|P_\gamma^n - r(\gamma)^n \Pi_\gamma\|_{\mathcal{B}_0} \leq M_K \big(\theta_K\, r(\gamma)\big)^n$$

from which we derive (17). The proof of Theorem 2.8 under Hypothesis 2.7 is achieved.

7.2. **Proof of Theorem 2.9 under Hypothesis 2.7.** First we prove the following lemma.

Lemma 7.2. *For every $\gamma \in J_0$ and for $i = 1, 2$, the spectral radius of $P_{\gamma|\mathcal{B}_i}$ is equal to $r(\gamma) := r\big(P_{\gamma|\mathcal{B}_0}\big)$.*

Proof. For $i = 0, 1, 2$ set $r_i(\gamma) := r((P_\gamma)_{|\mathcal{B}_i})$. Due to Theorem 2.8 applied to $(P_\gamma, J, \mathcal{B}_i, \mathcal{B}_{i+1})$, there exists $c_i > 0$ such that $\pi(P_\gamma^n \mathbf{1}_{\mathbb{X}}) \sim c_i\, r_i(\gamma)^n$ as n goes to infinity. This proves the equality of the spectral radius. □

Proof of Theorem 2.9 under Hypothesis 2.7. We define χ_i as χ in the proof of Theorem 2.8 for each $\mathcal{B}_i$ $(i = 0, 1, 2)$. We define now $\chi := \max(\chi_0, \chi_1, \chi_2)$. Let us prove the differentiability of r and Π on J_0. Let $\gamma_0 \in J_0$. Let $\eta > 0$ be such that $r(\gamma_0) > \chi(\gamma_0) + 2\eta$ and let $\varepsilon > 0$ be such that for every $\gamma \in J_0$ satisfying $|\gamma - \gamma_0| < \varepsilon$, we have $r(\gamma) > r(\gamma_0) - \eta > \chi(\gamma_0) + \eta > \chi(\gamma)$. We set $I_0 := J_0 \cap (\gamma_0 - \varepsilon, \gamma_0 - \varepsilon)$ and

$$\mathcal{D}_0 := \{z \in \mathbb{C} \,:\, \chi(\gamma_0) + \eta < |z| < r(\gamma_0) - \eta\} \cup \{z \in \mathbb{C} : |z - r(\gamma_0)| = \eta\}. \tag{70}$$

Due to the hypotheses of Theorem 2.9 and to an easy adaptation of [16, Lemma A.2] (see Remark 7.3), we obtain that, for every $z \in \mathcal{D}_0$, the map $\gamma \mapsto R_z(\gamma)$ is C^1 from I_0 to $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_3)$ with $R'_z(\gamma) = R_z(\gamma) P'_\gamma R_z(\gamma)$ and

$$\lim_{h \to 0} \sup_{z \in \mathcal{D}_0} \frac{\|R_z(\gamma_0 + h) - R_z(\gamma_0) - hR'_z(\gamma_0)\|_{\mathcal{B}_0, \mathcal{B}_3}}{|h|} = 0. \tag{71}$$

Moreover, for every $\gamma \in I_0$, we deduce from spectral theory that

$$\Pi_\gamma = \frac{1}{2i\pi} \oint_{\Gamma_1} R_z(\gamma)\, dz \quad \text{and} \quad N_\gamma = \frac{1}{2i\pi} \oint_{\Gamma_0} z R_z(\gamma)\, dz,$$

where Γ_1 is the oriented circle centered at $r(\gamma_0)$ with radius η and Γ_0 is the oriented circle centered at 0 with some radius ϑ_0 satisfying $\chi(\gamma_0) + \eta < \vartheta_0 < r(\gamma_0) - \eta$. Thus $\gamma \mapsto \Pi_\gamma$ and $\gamma \mapsto N_\gamma$ are C^1-smooth from J_0 to $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_3)$. Since $1_{\mathbb{X}} \in \mathcal{B}_0$ by hypothesis this implies the continuous differentiability of $\gamma \mapsto N_\gamma \mathbf{1}_{\mathbb{X}}$ and of $\gamma \mapsto \Pi_\gamma \mathbf{1}_{\mathbb{X}}$ from J_0 to $\mathcal{B}_3$. Since $r(\gamma) = \frac{(P_\gamma - N_\gamma)(\mathbf{1}_{\mathbb{X}})}{\Pi_\gamma(\mathbf{1}_{\mathbb{X}})}$ and $\gamma \mapsto P_\gamma \mathbf{1}_{\mathbb{X}}$ is C^1 from I_0 to $\mathcal{B}_3$ by hypothesis, we obtain the continuous differentiability of r on I_0. □

Remark 7.3 (Proof of the differentiability of $\gamma \mapsto R_z(\gamma)$)**.** *We adapt the arguments of* [16, Lemma A.2]*, writing*

$$R_z(\gamma) = R_z(\gamma_0) \;+\; R_z(\gamma_0)\,[P_\gamma - P_{\gamma_0}]\,R_z(\gamma_0) \;+\; \vartheta_z(\gamma),$$

$$\textit{with} \qquad \vartheta_z(\gamma) := R_z(\gamma_0)\,[P_\gamma - P_{\gamma_0}]\,R_z(\gamma_0)\,[P_\gamma - P_{\gamma_0}]\,R_z(\gamma).$$

Then

$$\frac{\|\vartheta_z(\gamma)\|_{\mathcal{B}_0, \mathcal{B}_3}}{|\gamma - \gamma_0|} \le \|R_z(\gamma_0)\|_{\mathcal{B}_2} \left\| \frac{P_\gamma - P_{\gamma_0}}{\gamma - \gamma_0} \right\|_{\mathcal{B}_1, \mathcal{B}_2} \|R_z(\gamma_0)\|_{\mathcal{B}_1} \|P_\gamma - P_{\gamma_0}\|_{\mathcal{B}_0, \mathcal{B}_1} \|R_z(\gamma)\|_{\mathcal{B}_0}. \tag{72}$$

From the hypotheses of Theorem 2.9 and from the resolvent bounds derived from Theorem 7.1, the last term goes to 0, uniformly in $z \in \mathcal{D}$, when γ goes to γ_0. Similarly we have:

$$\begin{aligned} &\left\| R_z(\gamma_0)(P_\gamma - P_{\gamma_0}) R_z(\gamma_0) - (\gamma - \gamma_0) R_z(\gamma_0) P'_{\gamma_0} R_z(\gamma_0) \right\|_{\mathcal{B}_0, \mathcal{B}_3} \\ &\qquad\qquad \le \; M \|P_\gamma - P_{\gamma_0} - (\gamma - \gamma_0) P'_{\gamma_0}\|_{\mathcal{B}_1, \mathcal{B}_2} = o(\gamma - \gamma_0) \end{aligned}$$

when again the finite positive constant M is derived from the resolvent bounds of Theorem 7.1. This shows that $R'_z(\gamma_0) = R_z(\gamma_0) P'_{\gamma_0} R_z(\gamma_0)$ in $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_3)$. To prove that $\gamma \mapsto R'_z(\gamma)$ is continuous from J_0 to $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_3)$ in a uniform way with respect to $z \in \mathcal{D}$, observe that $\gamma \mapsto R_z(\gamma)$ is $\mathcal{C}^0$ from J_0 to $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_1)$ (use Theorem 7.1), that $\gamma \mapsto P'_\gamma$ is $\mathcal{C}^0$ (uniformly in $z \in \mathcal{D}$) from J_0 to $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ by hypothesis, and finally that $\gamma \mapsto R_z(\gamma)$ is $\mathcal{C}^0$ (uniformly in $z \in \mathcal{D}$) from J_0 to $\mathcal{L}(\mathcal{B}_2, \mathcal{B}_3)$ (again use Theorem 7.1). Observe that (72) gives the differentiability at γ_0 of the map $\gamma \mapsto R_z(\gamma)$ considered from J to $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_2)$. The additional space $\mathcal{B}_3$ is only required to obtain the continuous differentiability.

7.3. **Proof of Theorems 2.8 and 2.9 under Hypothesis 2.7*.** Here the Keller-Liverani perturbation theorem must be applied to the dual family $(P_\gamma^*)_\gamma$. Actually the hypotheses of Theorem 2.8 are:

- $\mathcal{B}_1^* \hookrightarrow \mathcal{B}_0^*$,
- For every $\gamma \in J$, $P_\gamma^* \in \mathcal{L}(\mathcal{B}_0^*) \cap \mathcal{L}(\mathcal{B}_1^*)$,
- $\gamma \mapsto P_\gamma^*$ is a continuous map from J in $\mathcal{L}(\mathcal{B}_1^*, \mathcal{B}_0^*)$,
- There exist $\delta_0, c_0, M_0 > 0$ such that, for all $\gamma \in J$, $r_{ess}\big((P_\gamma)^*_{|\mathcal{B}_1^*}\big) \leq \delta_0$ and
$$\forall n \geq 1,\ \forall f^* \in \mathcal{B}_1^*, \quad \|(P_\gamma^*)^n f^*\|_{\mathcal{B}_1^*} \leq c_0(\delta_0^n \|f^*\|_{\mathcal{B}_1^*} + M^n \|f^*\|_{\mathcal{B}_0^*}).$$
- Hypothesis 2.5 holds on $(J_0, \mathcal{B}_1)$.

Proof of Theorem 2.8 under Hypothesis 2.7.* Under these assumptions it follows from Theorem 7.1 applied to $(P_\gamma^*)_{\gamma\in J}$ with respect to $(\mathcal{B}_1^*, \mathcal{B}_0^*)$ that, for every $\varepsilon > 0$ and every $\delta > \delta_0$, the map $t \mapsto (zI - P_\gamma^*)^{-1}$ is well defined from J_0 to $\mathcal{L}(\mathcal{B}_1^*)$, provided that $z \in \mathcal{D}(\delta, \varepsilon)$ with
$$\mathcal{D}(\delta,\varepsilon) := \{z \in \mathbb{C},\ d(z, \sigma\big((P_{\gamma_0}^*)_{|\mathcal{B}_2^*}\big)) > \varepsilon,\ |z| > \delta\} = \{z \in \mathbb{C},\ d(z, \sigma\big((P_{\gamma_0})_{|\mathcal{B}_2}\big)) > \varepsilon,\ |z| > \delta\}.$$
In addition, the map $t \mapsto (zI - P_\gamma^*)^{-1}$, considered from J_0 to $\mathcal{L}(\mathcal{B}_1^*, \mathcal{B}_0^*)$, is continuous at every $\gamma_0 \in J_0$ in a uniform way with respect to $z \in \mathcal{D}(\delta, \varepsilon)$. By duality this implies that $t \mapsto (zI - P_\gamma)^{-1}$ is well defined from J_0 to $\mathcal{L}(\mathcal{B}_1)$. Moreover, when this map is considered from J_0 to $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_1)$, it is continuous at γ_0 in a uniform way with respect to $z \in \mathcal{D}(\delta, \varepsilon)$. Finally Hypothesis 2.5 on $(J_0, \mathcal{B}_1)$ enables us to identify the spectral elements associated with $r(\gamma) := r\big((P_\gamma)_{|\mathcal{B}_1}\big)$. Consequently one can prove as in Subsection 7.1 that there exists a map $\gamma \mapsto \Pi_\gamma$ from J_0 to $\mathcal{L}(\mathcal{B}_1)$, which is continuous from J_0 to $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_1)$, such that (17) holds with $\mathcal{B} := \mathcal{B}_1$. □

Proof of Theorem 2.9 under Hypothesis 2.7.* When Theorem 2.9 is stated with Hypothesis 2.7*, then Theorem 2.8 applies on $(\mathcal{B}_0, \mathcal{B}_1)$, $(\mathcal{B}_1, \mathcal{B}_2)$ and $(\mathcal{B}_2, \mathcal{B}_3)$ (with Hypothesis 2.7* in each case). Thus, for every $\gamma \in J_0$, the spectral radius $r_i(\gamma) := r((P_\gamma)_{|\mathcal{B}_i})$ are equal for $i = 1, 2, 3$ (See the proof of Lemma 7.2). Observe that, from our hypotheses, Hypothesis 2.5 holds on $(J_0, \mathcal{B}_i)$ for $i = 1, 2, 3$. Since P_γ^* on $\mathcal{B}_i^*$ inherits the spectral properties of P_γ on $\mathcal{B}_i$, we can prove as above that, for every $\gamma_0 \in J_0$ and for every $\varepsilon > 0$ and $\delta > \delta_0$, the map $\gamma \mapsto (zI - P_\gamma^*)^{-1}$ is well defined from some subinterval I_0 of J_0 containing γ_0 into $\mathcal{L}(\mathcal{B}_3^*)$, provided that $z \in \mathcal{D}_0$ where the set $\mathcal{D}_0$ is defined in (70). In addition, by applying Remark 7.3 with the adjoint operators $(P_\gamma^*)_\gamma$ and the spaces $\mathcal{B}_3^* \hookrightarrow \mathcal{B}_2^* \hookrightarrow \mathcal{B}_1^* \hookrightarrow \mathcal{B}_0^*$, we can prove that the map $\gamma \mapsto (zI - P_\gamma^*)^{-1}$, considered from J_0 to $\mathcal{L}(\mathcal{B}_3^*, \mathcal{B}_0^*)$, is $\mathcal{C}^1$ in a uniform way with respect to $z \in \mathcal{D}_0$. By duality, this gives (71). We conclude the differentiability of $\gamma \mapsto \Pi_\gamma^*$ from J_0 to $\mathcal{L}(\mathcal{B}_3^*, \mathcal{B}_1^*)$ and so the differentiability of $\gamma \mapsto \Pi_\gamma$ from J_0 to $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_3)$. □

7.4. **Complements on the derivative of $r(\cdot)$.** Let us first prove the following.

Lemma 7.4. *Let $J_1 = (a, b) \subset [0, +\infty)$ and let $\mathcal{B}_1 \hookrightarrow \mathcal{B}_2$ be two Banach spaces such that $f \mapsto \xi f \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$. Assume that, for every $\gamma \in J_1$, $P_\gamma \in \mathcal{L}(\mathcal{B}_1) \cap \mathcal{L}(\mathcal{B}_2)$ and that there exist elements $\phi_\gamma \in \mathcal{B}_1$ and $\pi_\gamma \in \mathcal{B}_2^*$ such that $P_\gamma \phi_\gamma = r(\gamma)\phi_\gamma$ and $P_\gamma^* \pi_\gamma = r(\gamma)\pi_\gamma$. Moreover assume that $\gamma \mapsto P_\gamma$ and $\gamma \mapsto r(\gamma)$ are differentiable from J_1 to $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ and to $\mathbb{C}$ respectively, with respective derivatives at $\gamma \in J_1$ given by $P_\gamma' : f \mapsto P_\gamma(-\xi f)$ and $r'(\gamma)$. Finally assume that $\gamma \mapsto \phi_\gamma$ is continuous from J_1 to $\mathcal{B}_1$ and differentiable from J_1 to $\mathcal{B}_2$ with derivative $\gamma \mapsto \phi_\gamma'$.*

Then we have for every $\gamma \in J_1$: $r'(\gamma)\pi_\gamma(\phi_\gamma) = -r(\gamma)\pi_\gamma\big(\xi\phi_\gamma\big)$. In particular, if $r(\gamma) > 0$, $\pi_\gamma(\phi_\gamma) \geq 0$ and $\pi_\gamma\big(\xi\phi_\gamma\big) > 0$, then $r'(\gamma) < 0$.

Proof. Let $\gamma, \gamma_0 \in J_1$. We have $P_\gamma\phi_\gamma = r(\gamma)\phi_\gamma$ in $\mathcal{B}_2$. From $P_\gamma\phi_\gamma - P_{\gamma_0}\phi_{\gamma_0} = P_{\gamma_0}(\phi_\gamma - \phi_{\gamma_0}) + (P_\gamma - P_{\gamma_0})(\phi_\gamma)$, we obtain that

$$P_{\gamma_0}(\phi'_{\gamma_0}) + P_{\gamma_0}(-\xi\phi_{\gamma_0}) = r(\gamma_0)\phi'_{\gamma_0} + r'(\gamma_0)\phi_{\gamma_0} \quad \text{in } \mathcal{B}_2.$$

We conclude by composing by π_{γ_0} and using the fact that $\pi_{\gamma_0} \circ P_{\gamma_0} = r(\gamma_0)\pi_{\gamma_0}$. □

Proposition 7.5. *Assume that the assumptions of Theorem 2.9 hold. For every $\gamma \in J_0$, set $\phi_\gamma := \Pi_\gamma \mathbf{1}_{\mathbb{X}}$ and $\pi_\gamma := \Pi_\gamma^*\pi$. Then the assumptions of Lemma 7.4 hold with $J_1 = J_0$ and with respect to the spaces $\mathcal{B}_1 \hookrightarrow \mathcal{B}_2$ (resp. $\mathcal{B}_1 \hookrightarrow \mathcal{B}_3$) when the assumptions of Theorem 2.9 hold with Hypothesis 2.7 (resp. with Hypothesis 2.7*). Consequently, for every $\gamma \in J_0$,*

$$\pi(\Pi_\gamma(\xi\,\Pi_\gamma 1_{\mathbb{X}})) > 0 \implies r'(\gamma) < 0.$$

Proof of Proposition 7.5 under Hypothesis 2.7. We have $\pi_\gamma \in \mathcal{B}_2^*$ since $\pi \in \mathcal{B}_2^*$ and Π_γ^* is well defined in $\mathcal{L}(\mathcal{B}_2^*)$. Moreover $\phi_\gamma \in \mathcal{B}_1$ since $\mathbf{1}_{\mathbb{X}} \in \mathcal{B}_1$ and $\Pi_\gamma \in \mathcal{L}(\mathcal{B}_1)$, and $\gamma \mapsto \phi_\gamma$ is continuous from J to $\mathcal{B}_1$ by Theorem 2.8. Finally $\gamma \mapsto \phi_\gamma$ is differentiable from J to $\mathcal{B}_2$ (see the end of Remark 7.3). We have proved that the assumptions of Lemma 7.4 hold as stated under Hypothesis 2.7. Finally, since $r(\gamma) > 0$ when $\gamma \in J_0$, the desired implication in Proposition 7.5 follows from the conclusion of Lemma 7.4 because $\pi_\gamma(\phi_\gamma) = \pi(\phi_\gamma)$ and $\pi_\gamma(\xi\,\phi_\gamma) = \pi(\Pi_\gamma(\xi\,\Pi_\gamma 1_{\mathbb{X}}))$. □

Proof of Proposition 7.5 under Hypothesis 2.7.* Note that $\pi_\gamma := \Pi_\gamma^*\pi \in \mathcal{B}_3^*$ since $\pi \in \mathcal{B}_3^*$ and Π_γ^* is well defined in $\mathcal{L}(\mathcal{B}_3^*)$. The function $\gamma \mapsto P_\gamma$ is differentiable from J_0 to $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$, thus from J_0 to $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_3)$. We have $\phi_\gamma := \Pi_\gamma \mathbf{1}_{\mathbb{X}} \in \mathcal{B}_1$ since $1_{\mathbb{X}} \in \mathcal{B}_1$ and Π_γ is well defined in $\mathcal{L}(\mathcal{B}_1)$. Moreover $\gamma \mapsto \phi_\gamma$ is continuous from J to $\mathcal{B}_1$ since Π_γ is well defined in $\mathcal{L}(\mathcal{B}_1)$, continuous from J_0 to $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_1)$, and $\mathbf{1}_{\mathbb{X}} \in \mathcal{B}_0$. Finally $\gamma \mapsto \phi_\gamma$ is differentiable from J to $\mathcal{B}_3$ since Π_γ is well defined in $\mathcal{L}(\mathcal{B}_3)$ and differentiable from J_0 to $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_3)$ and $\mathbf{1}_{\mathbb{X}} \in \mathcal{B}_1$. □

8. Proof of Proposition 4.5

Proposition 4.5 directly follows from the following statement.

Proposition 8.1. *Let $\mathcal{B}$ be a non null complex Banach lattice of functions $f : \mathbb{X} \to \mathbb{C}$ (or of classes of such functions modulo π). Let Q be a (nonnull) nonnegative quasicompact operator on $\mathcal{B}$ such that $r(Q) \neq 0$ and such that for every nonnull nonnegative $f \in \mathcal{B}$ and for every nonnull nonnegative $\psi \in \mathcal{B}^* \cap \mathrm{Ker}(Q^* - r(Q)I)$, we have $Qf > 0$ and $\psi(Qf) > 0$. Then*

(a) $r(Q)$ is a first order pole of Q, and there exists a positive $\phi \in \mathcal{B}$ and a positive $\psi \in \mathcal{B}^*$ such that

$$\psi(\phi) = 1, \qquad Q\phi = r(Q)\phi \qquad \text{and} \qquad Q^*\psi = r(Q)\psi. \tag{73}$$

(b) Let $\lambda \in \mathbb{C}$ and $h \in \mathcal{B}$ such that $|\lambda| = r(Q)$ and $Qh = \lambda h$. Then $Q|h| = r(Q)|h|$ in $\mathcal{B}$.

(c) If moreover Q is of the form $Q = P(\kappa e^{-\gamma\xi}\cdot)$ where P is the operator associated with a Markov kernel, if $1_{\mathbb{X}} \in \mathcal{B} \hookrightarrow \mathbb{L}^1(\pi)$, if $\mathrm{Ker}(Q - r(Q)I) = \mathbb{C} \cdot \phi$ and if 1 is the only complex number λ of modulus 1 such that $P(h/|h|) = \lambda h/|h|$ in $\mathbb{L}^1(\pi)$ for some $h \in \mathcal{B}$ with $|h| > 0$, then $r(Q)$ is the only eigenvalue of modulus $r(Q)$ of Q.

Proof. The fact that $r(Q)$ is a finite pole of Q is classical for a nonnegative quasi-compact operator Q on a Banach lattice. Let us just remember the main arguments. From quasi-compactness we know that there exists a finite pole $\lambda \in \sigma(Q)$ such that $|\lambda| = r(Q)$. Thus, setting $\lambda_n := \lambda(1+1/n)$ for any $n \geq 1$, we deduce from $\lambda \in \sigma(Q)$ that

$$\lim_n \|(\lambda_n I - Q)^{-1}\|_{\mathcal{B}} = +\infty.$$

Since $\mathcal{B}$ is a Banach lattice, we deduce from the Banach-Steinhaus theorem that there exists a nonnegative and nonnull element $f \in \mathcal{B}$ such that

$$\lim_n \|(\lambda_n I - Q)^{-1} f\|_{\mathcal{B}} = +\infty.$$

Next define $r_n := r(Q)(1+1/n)$ and observe that

$$\big|(\lambda_n I - Q)^{-1} f\big| = \big|\sum_{k\geq 0} \lambda_n^{-(k+1)}\, Q^k f\big| \leq \sum_{k\geq 0} r_n^{-(k+1)}\, Q^k f.$$

Since $\mathcal{B}$ is a Banach lattice, the last inequality is true in norm, that is

$$\big\|(\lambda_n I - Q)^{-1} f\big\| \leq \big\|\sum_{k\geq 0} r_n^{-(k+1)}\, Q^k f\big\|$$

from which we deduce that $\lim_n \|(r_n I - Q)^{-1}\|_{\mathcal{B}} = +\infty$, thus $r(Q) \in \sigma(Q)$. Finally $r(Q)$ is a finite pole of Q from quasi-compactness.

Let q denote the order of the pole $r(Q)$, namely $r(Q)$ is a pole of order q of the resolvent function $z \mapsto (zI - Q)^{-1}$. Then there exists $\rho > 0$ such that $(zI - Q)^{-1}$ admits the following Laurent series provided that $|z - r(Q)| < \rho$ and $z \neq r(Q)$:

$$(zI - Q)^{-1} = \sum_{k=-q}^{+\infty} (z - r(Q))^k A_k,$$

where A_k are bounded linear operators on $\mathcal{B}$. By quasi-compactness, A_{-1} is a projection onto the finite subspace $\mathrm{Ker}(Q - r(Q)I)^q$. Moreover we know that

$$A_{-q} = \big(Q - r(Q)I\big)^{q-1} \circ A_{-1} = A_{-1} \circ \big(Q - r(Q)I\big)^{q-1}. \tag{74}$$

and that, setting $r_n := r(Q)(1+1/n)$,

$$\begin{aligned} A_{-q} &= \lim_{n\to+\infty} \big(r_n - r(Q)\big)^q \big(r_n I - Q\big)^{-1} \\ &= \lim_{n\to+\infty} \big(r_n - r(Q)\big)^q \sum_{k\geq 0} r_n^{-(k+1)}\, Q^k. \end{aligned} \tag{75}$$

Since Q is a nonnull nonnegative operator on $\mathcal{B}$, so is A_{-q}. Since $A_{-q} \neq 0$, we take a nonnegative $h_0 \in \mathcal{B}$ such that $\phi := A_{-q} h_0 \neq 0$ in $\mathcal{B}$. We have $(Q - r(Q)I)A_{-q} = 0$, so $r(Q)\phi = Q\phi$. Similarly there exists a nonnegative $\psi_0 \in \mathcal{B}^*$ such that $\psi_1 := A_{-q}^* \psi_0$ is a nonzero and nonnegative element of $\mathrm{Ker}(Q^* - r(Q)I)$, where A_{-q}^* is the adjoint operator of A_{-q}. Note that $\psi_1(\phi) = \psi_1(Q\phi)/r(Q) > 0$ from our hypotheses, so that ϕ and $\psi := \psi_1/\psi_1(\phi)$ satisfy (73). To conclude the proof of Assertion (a), let us prove by reductio ad absurdum that $q = 1$. Assume that $q \geq 2$. Then $A_{-q}^2 = 0$ from (74) and $A_{-1}(\mathcal{B}) = \mathrm{Ker}(Q - r(Q)I)^q$, so that $\psi_1(\phi) = (A_{-q}^* \psi_0)(A_{-q} h_0) = \psi_0(A_{-q}^2 h_0) = 0$. This contradicts the above fact.

To prove (b), recall that, from our hypotheses, we have $\psi(g) = \psi(Qg)/r(Q) > 0$ for every nonnull nonnegative $g \in \mathcal{B}$. Let $\lambda \in \mathbb{C}$ and $h \in \mathcal{B}$ such that $|\lambda| = r(Q)$ and $Qh = \lambda h$. The

positivity of Q gives $|\lambda h| = r(Q)|h| = |Qh| \leq Q|h|$, thus $g_0 := Q|h| - r(Q)|h| \geq 0$. From $\psi(g_0) = 0$, it follows that $g_0 = 0$, that is: $Q|h| = r(Q)|h|$ in $\mathcal{B}$.

Now let us prove Assertion (c) of Proposition 8.1. Recall that the above nonnull nonnegative function $\phi \in \mathcal{B}$ is such that $Q\phi = r(Q)\phi$. From our hypotheses we deduce that $\phi > 0$ (modulo π). Let $\lambda \in \mathbb{C}$ and $h \in \mathcal{B}$ be such that $|\lambda| = r(Q)$, $h \neq 0$ and $Qh = \lambda h$. Due to the previous point and to our assumptions, we obtain that $Q|h| = r(Q)\,|h|$ and $|h| = \beta\phi$ for some $\beta > 0$. In particular $h \neq 0$ $\pi-$ a.s.. One may assume that $\beta = 1$ for the sake of simplicity. Let $A = \{x \in \mathbb{X} : |h(x)| = \phi(x) > 0\}$ and

$$B = \{x \in \mathbb{X} : (P_\gamma \phi)(x) = r(Q)\phi(x)\}, \quad C = \{x \in \mathbb{X} : (P_\gamma h)(x) = \lambda h(x)\}.$$

Let $A^c = \mathbb{X} \setminus A$. It follows from $\pi(A^c) = 0$ and from the invariance of π that

$$\pi\left(P\big(1_{A^c}(\phi + |h|)\,\kappa e^{-\gamma\xi}\big)\right) = \pi\left(1_{A^c}\big(\phi + |h|\big)\,\kappa e^{-\gamma\xi}\right) = 0.$$

Let $D := \{x \in \mathbb{X} : \big(P(1_{A^c}\,(\phi + |h|)\,\kappa e^{-\gamma\xi})\big)(x) = 0\}$. Then we have $\pi(D) = 1$. Now define $E = A \cap B \cap C \cap D$. Then $\pi(E) = 1$, and we obtain that

$$\forall x \in E, \quad |h(x)| = \phi(x) > 0 \tag{76a}$$

$$\forall x \in E, \quad \lambda\, h(x) = \big(P(1_A h\,\kappa e^{-\gamma\xi})\big)(x) = \int_A h(y)\,\kappa(y) e^{-\gamma\xi(y)}\, P(x, dy) \tag{76b}$$

$$\forall x \in E, \quad r(Q)\,\phi(x) = \big(P(1_A \phi\,\kappa e^{-\gamma\xi})\big)(x) = \int_A \phi(y)\,\kappa(y) e^{-\gamma\xi(y)}\, P(x, dy). \tag{76c}$$

Let $x \in E$ and define the probability measure: $\eta_x(dy) := (r(Q)\,\phi(x))^{-1}\phi(y)\,\kappa(y)e^{-\gamma\xi(y)}\, P(x, dy)$. We have

$$\int_A \frac{r(Q)\,\phi(x)\,h(y)}{\lambda\,\phi(y)\,h(x)}\, \eta_x(dy) = 1.$$

Since $|h(x)| = \phi(x)$ and $|h| = \phi$ on A, the previous integrand is of modulus one. Then a standard convexity argument ensures that the following equality holds for $P(x, \cdot)-$almost every $y \in \mathbb{X}$:

$$r(Q)\,\phi(x)\,h(y) = \lambda\,\phi(y)\,h(x).$$

This implies that $r(Q)P\frac{h}{|h|} = \lambda\frac{h}{|h|}$ everywhere on E, thus $r(Q)P\frac{h}{|h|} = \lambda\frac{h}{|h|}$ in $\mathbb{L}^1(\pi)$. So $\lambda = r(Q)$ from the hypothesis of Assertion (c) of Proposition 8.1. □

9. A counter-example

Assume that $(\mathbb{X}, d)$ is a metric space equipped with its Borel σ-algebra. Let $\mathcal{L}^\infty$ denote the set of bounded functions $f : \mathbb{X} \to \mathbb{C}$, endowed with the supremum norm.

Proposition 9.1. *Assume that P is a Markov kernel satisfying the following condition : there exists $S \in (0, +\infty)$ such that, for every $x \in \mathbb{X}$, the support of $P(x, dy)$ is contained in the ball $B(x, S)$ centered at x with radius S. Assume that $\kappa \equiv 2$ and that $\xi(y) \to 0$ when $d(y, x_0) \to +\infty$, where x_0 is some fixed point in $\mathbb{X}$. Then, for every $\gamma \in [0, +\infty)$, the kernel $P_\gamma := 2P(e^{-\gamma\xi}\cdot)$ continuously acts on $\mathcal{L}^\infty$ and its spectral radius $r(\gamma) = r((P_\gamma)_{|\mathcal{L}^\infty})$ satisfies the following*

$$\forall \gamma \in [0, +\infty), \quad r(\gamma) = 2.$$

Proof. We clearly have $r(\gamma) \leq 2$ since $P_\gamma \leq P$ and P is Markov. For any $\beta > 0$, we obtain with $f = \mathbf{1}_{[\xi \leq \beta]}$

$$\forall x \in \mathbb{X}, \quad (P_\gamma f)(x) = 2 \int_{[\xi \leq \beta]} e^{-\gamma \xi(y)} \, P(x, dy) \geq 2e^{-\gamma\beta} \, P\big(x, [\xi \leq \beta]\big).$$

The set $[\xi \leq \beta]$ contains $\mathbb{X} \setminus B(x_0, R)$ for some $R > 0$ since $\xi(y) \to 0$ when $d(y, x_0) \to +\infty$. Thus, for $d(x, x_0)$ sufficiently large ($d(x, x_0) > R + S$), we have $P\big(x, [\xi \leq \beta]\big) = 1$, so that $\|P_\gamma\|_{\mathcal{L}^\infty} \geq \|P_\gamma f\|_{\mathcal{L}^\infty} \geq 2e^{-\gamma\beta}$. This gives $\|P_\gamma\|_{\mathcal{L}^\infty} = 2$ when $\beta \to 0$. Similarly we obtain with $f = \mathbf{1}_{[\xi \leq \beta]}$, that, $\forall x \in \mathbb{X} \setminus B(x_0, R + 2S)$,

$$\begin{aligned} (P_\gamma^2 f)(x) &= 4 \int e^{-\gamma(\xi(y) + \xi(z))} \, \mathbf{1}_{[\xi \leq \beta]}(z) \, P(y, dz) \, P(x, dy) \\ &\geq 4e^{-\gamma\beta} \int_{\mathbb{X} \setminus B(x_0, R+S)} e^{-\gamma \xi(y)} \, P\big(y, [\xi \leq \beta]\big) P(x, dy) \geq 4e^{-2\gamma\beta} \end{aligned}$$

and so $\forall \beta > 0, \quad \|P_\gamma^2\|_{\mathcal{L}^\infty} \geq \|P_\gamma^2 f\|_{\mathcal{L}^\infty} \geq 4e^{-2\gamma\beta}$. Again this provides $\|P_\gamma^2\|_{\mathcal{L}^\infty} = 4$ since β can be taken arbitrarily small. Similarly we can prove that $\|P_\gamma^n\|_{\mathcal{L}^\infty} = 2^n$ for every $n \geq 1$, thus $r(\gamma) = 2$. □

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