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Some growth results for the pullback attractor of an epidemiological system

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Abstract

In this paper, we prove some exponential growth results when time goes to $-\infty$ for the pullback attractor of a non-autonomous SIR (susceptible, infected, recovered) model from epidemiology considered in Anguiano and Kloeden (2014). We prove two exponential growth results for this pullback attractor, firstly in the norm $H_0^1(\Omega)^3$, and later, under appropriate additional assumptions, in the norm $H^2(\Omega)^3$.

Keywords: SIR epidemic model with diffusion; pullback attractors; invariant sets; H^2 -exponential growth

Mathematics Subject Classifications (2010): 35B41 37B55

1 Introduction and setting of the problem

Let us consider the following problem for a temporally-forced SIR (susceptible, infected, recovered) model with diffusion

$$\left. \begin{aligned} \frac{\partial S}{\partial t} - \Delta S &= aq(t) - aS + bI - \gamma \frac{SI}{N}, \\ \frac{\partial I}{\partial t} - \Delta I &= -(a + b + c)I + \gamma \frac{SI}{N}, \\ \frac{\partial R}{\partial t} - \Delta R &= cI - aR, \end{aligned} \right\} \quad (1)$$

with the Dirichlet boundary condition

$$S(x, t) = I(x, t) = R(x, t) = 0 \text{ on } \partial\Omega \times (t_0, +\infty), \quad (2)$$

and initial condition

$$S(x, t_0) = S_0(x), \quad I(x, t_0) = I_0(x), \quad R(x, t_0) = R_0(x) \text{ for } x \in \Omega, \quad (3)$$

where $\Omega \subset \mathbb{R}^d$, $d \geq 1$, is a bounded domain with a smooth boundary $\partial\Omega$, $N(t) = S(t) + I(t) + R(t)$ and $t_0 \in \mathbb{R}$. We assume that the parameters a , b , c and γ are positive constants such that $\gamma + \frac{b}{2} + \frac{c}{2} < \lambda_1$, where $\lambda_1 > 0$ is the first eigenvalue of the negative Laplacian with zero Dirichlet boundary condition in Ω . The temporal forcing term is given by a continuous function $q : \mathbb{R} \rightarrow \mathbb{R}$ taking positive bounded values, i.e. $q(t) \in [q^-, q^+]$ for all $t \in \mathbb{R}$ where $0 < q^- \leq q^+$, such that $q' \in L_{loc}^2(\mathbb{R}; L^2(\Omega))$ and satisfies

$$\lim_{t_0 \rightarrow -\infty} e^{\lambda_1 t_0} \int_{t_0}^{t_0+1} |q'(s)|_{L^2(\Omega)}^2 ds = 0. \quad (4)$$

Epidemiology is the study of the spread of diseases with the objective of tracing factors that are responsible for or contribute to their occurrence. Mathematical models are used extensively in the study of epidemiological phenomena. Most models for the transmission of infectious (see for instance [1, 5]) descend from the classical SIR model of Kermack and McKendrick [8] established in 1927. Its classical form involves a system of autonomous ordinary differential equations for three classes, the susceptibles S , infectives I and recovered R , of a constant total population.

There is a strong biological motivation to include time-dependent terms into epidemiological models, for instance temporally varying forcing is typical of seasonal variation of a disease [7, 10].

Our model (1)-(3) is a classical and well-known model from mathematical epidemiology in the form of the SIR equations, with diffusion, in which a temporal forcing term is considered.

Several studies on this model have already been published. More precisely, in [3] we proved the existence and uniqueness of positive solutions of (1)-(3) for initial data in $L^2(\Omega)^3$. Namely, in [3, Theorem 6.2.] we establish that, if q takes positive bounded values, the process associated to (1)-(3) has a unique pullback attractor \mathcal{A} , which satisfies

$$\lim_{t_0 \rightarrow -\infty} \left(e^{\lambda_1 t_0} \sup_{(w_1, w_2, w_3) \in \mathcal{A}(t_0)} |(w_1, w_2, w_3)|_{L^2(\Omega)}^2 \right) = 0. \quad (5)$$

In Anguiano [4], we established a regularity result for the unique positive solution to problem (1)-(3), and we proved some regularity results for the pullback attractor \mathcal{A} . More precisely, we prove that, under the condition $\gamma + \frac{b}{2} + \frac{c}{2} < \lambda_1$, the set $\mathcal{A}(t_0)$, for any $t_0 \in \mathbb{R}$, is a bounded subset of $H_0^1(\Omega)^3$, and if moreover $q' \in L_{loc}^2(\mathbb{R}; L^2(\Omega))$, then $\mathcal{A}(t_0)$ is also a bounded subset of $H^2(\Omega)^3$.

However, as far as we know, there are no results in the literature concerning exponential growth of the pullback attractor of (1)-(3) as we will consider in this paper. By relying on a technique introduced by Anguiano *et al.* [2], we prove our main results (cf. Theorem 3 and Theorem 4).

The structure of the paper is as follows. In Section 2, we prove an exponential growth condition for the attractor $\mathcal{A}(t_0)$ in $H_0^1(\Omega)^3$ when $t_0 \rightarrow -\infty$. Under appropriate additional assumptions, an exponential growth condition in $H^2(\Omega)^3$ for $\mathcal{A}(t_0)$ is proved in Section 3.

2 An exponential growth condition for the pullback attractor in $H_0^1(\Omega)^3$

We denote by (\cdot, \cdot) the inner product in $L^2(\Omega)$, and by $|\cdot|_{L^2(\Omega)}$ the associated norm. By $\|\cdot\|$ we denote the norm in $H_0^1(\Omega)$, which is associated to the inner product $((\cdot, \cdot)) := (\nabla \cdot, \nabla \cdot)$. We will denote by $\langle \cdot, \cdot \rangle$ the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

In addition, X_3 denotes the space of functions $(u_1, u_2, u_3) \in L^2(\Omega)^3$ with the scalar product

$$((u_1, u_2, u_3), (v_1, v_2, v_3)) = (u_1, v_1) + (u_2, v_2) + (u_3, v_3),$$

and norm

$$|(u_1, u_2, u_3)|_{L^2(\Omega)} = |u_1|_{L^2(\Omega)} + |u_2|_{L^2(\Omega)} + |u_3|_{L^2(\Omega)},$$

for all $(u_1, u_2, u_3), (v_1, v_2, v_3) \in X_3$, while Y_3 denotes the space of functions $(u_1, u_2, u_3) \in H_0^1(\Omega)^3$ with the scalar product

$$(((u_1, u_2, u_3), (v_1, v_2, v_3))) = ((u_1, v_1)) + ((u_2, v_2)) + ((u_3, v_3)),$$

and norm

$$\|(u_1, u_2, u_3)\| = \|u_1\| + \|u_2\| + \|u_3\|,$$

for all $(u_1, u_2, u_3), (v_1, v_2, v_3) \in Y_3$. Finally, let X_3^+ be the subspace of non-negative functions in X_3 and Y_3^+ be the subspace of non-negative functions in Y_3 .

The globally defined nonnegative solutions of (1)–(3) generate a process in the Banach space X_3^+ (see [3]), i.e., a family of mappings $U_{t,t_0} : X_3^+ \rightarrow X_3^+$ with $t \geq t_0$ in \mathbb{R} satisfying

$$U_{t_0,t_0}x = x, \quad U_{t,t_0}x = U_{t,r} \circ U_{r,t_0}x,$$

for all $t_0 \leq r \leq t$ and $x \in X_3^+$. In [3, Proposition 1] we established that the 2-parameter family of mappings $U_{t,t_0} : X_3^+ \rightarrow X_3^+$, $t_0 \leq t$, given by

$$U_{t,t_0}(S_0, I_0, R_0) = (S(t), I(t), R(t)), \quad (6)$$

where $(S(t), I(t), R(t))$ is the unique positive solution of (1)–(3) with the initial value (S_0, I_0, R_0) , defines a continuous process on X_3^+ .

Recall that a pullback attractor for the process U_{t,t_0} (e.g., cf. [6]) in the space X_3^+ is a family $\mathcal{A} = \{\mathcal{A}(t), t \in \mathbb{R}\}$ of nonempty compact subsets of X_3^+ , which is invariant in the sense that

$$U_{t,t_0}\mathcal{A}(t_0) = \mathcal{A}(t), \quad \text{for all } t \geq t_0,$$

and pullback attracts bounded subsets D of X_3^+ , i.e.,

$$\text{dist}_{X_3^+}(U_{t,t_0}D, \mathcal{A}(t)) \rightarrow 0 \quad \text{as } t_0 \rightarrow -\infty,$$

where we denote by $\text{dist}_{X_3^+}(\cdot, \cdot)$ the Hausdorff semi-distance in X_3^+ .

We recall a lemma (see [9]) which is necessary for the proof of our results.

Lemma 1 *Let X, Y be Banach spaces such that X is reflexive, and the inclusion $X \subset Y$ is continuous. Assume that $\{u_n\}$ is a bounded sequence in $L^\infty(t_0, T; X)$ such that $u_n \rightharpoonup u$ weakly in $L^q(t_0, T; X)$ for some $q \in [1, +\infty)$ and $u \in C^0([t_0, T]; Y)$. Then, $u(t) \in X$ for all $t \in [t_0, T]$ and*

$$\|u(t)\|_X \leq \sup_{n \geq 1} \|u_n\|_{L^\infty(t_0, T; X)} \quad \forall t \in [t_0, T].$$

Let $A : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ be the linear operator associated with the negative Laplacian. The operator A is symmetric, coercive and continuous.

Since the space $H_0^1(\Omega)$ is included in $L^2(\Omega)$ with compact injection, as a consequence of the Hilbert-Schmidt Theorem there exists a nondecreasing sequence $0 < \lambda_1 \leq \lambda_2 \leq \dots$ of eigenvalues of A with zero Dirichlet boundary condition in Ω , with $\lim_{j \rightarrow \infty} \lambda_j = +\infty$ and there exists an orthonormal basis of Hilbert $\{w_j : j \geq 1\}$ of $L^2(\Omega)$ and orthogonal in $H_0^1(\Omega)$ with $V_n := \text{span}\{w_j : 1 \leq j \leq n\}$ densely embedded in $H_0^1(\Omega)$, such that

$$Aw_j = \lambda_j w_j \quad \text{for all } j \geq 1.$$

For each integer $n \geq 1$, we denote by $(S_n(t), I_n(t), R_n(t)) = (S_n(t; t_0, S_0), I_n(t; t_0, I_0), R_n(t; t_0, R_0))$ the Galerkin approximation of the solution $(S(t; t_0, S_0), I(t; t_0, I_0), R(t; t_0, R_0))$ of (1)–(3), which is given by

$$S_n(t) = \sum_{j=1}^n \gamma_{nj}^1(t) w_j, \quad I_n(t) = \sum_{j=1}^n \gamma_{nj}^2(t) w_j, \quad R_n(t) = \sum_{j=1}^n \gamma_{nj}^3(t) w_j,$$

and is the solution of

$$\begin{aligned} \frac{d}{dt}(S_n(t), w_j) &= \langle \Delta S_n(t), w_j \rangle + (f_1(S_n(t), I_n(t), R_n(t), t), w_j), \\ \frac{d}{dt}(I_n(t), w_j) &= \langle \Delta I_n(t), w_j \rangle + (f_2(S_n(t), I_n(t), R_n(t), t), w_j), \\ \frac{d}{dt}(R_n(t), w_j) &= \langle \Delta R_n(t), w_j \rangle + (f_3(S_n(t), I_n(t), R_n(t), t), w_j), \end{aligned}$$

with initial data

$$(S_n(t_0), w_j) = (S_0, w_j), (I_n(t_0), w_j) = (I_0, w_j), (R_n(t_0), w_j) = (R_0, w_j),$$

for all $w_j \in V_n$, where

$$\gamma_{nj}^1(t) = (S_n(t), w_j), \quad \gamma_{nj}^2(t) = (I_n(t), w_j), \quad \gamma_{nj}^3(t) = (R_n(t), w_j).$$

We denote

$$\begin{aligned} f_1(S_n(t), I_n(t), R_n(t), t) &:= aq(t) - aS_n(t) + bI_n(t) - \gamma \frac{S_n(t)I_n(t)}{N_n(t)}, \\ f_2(S_n(t), I_n(t), R_n(t), t) &:= -(a + b + c)I_n(t) + \gamma \frac{S_n(t)I_n(t)}{N_n(t)}, \\ f_3(S_n(t), I_n(t), R_n(t), t) &:= cI_n(t) - aR_n(t), \end{aligned}$$

where

$$N_n(t) = S_n(t) + I_n(t) + R_n(t).$$

On the other hand, if we denote

$$D(A) = \{v \in H_0^1(\Omega) : Av \in L^2(\Omega)\},$$

with the scalar product

$$(v, w)_{D(A)} = (Av, Aw) \quad \forall v, w \in D(A),$$

then $D(A)$ is a Hilbert space, and $D(A)$ is included in $H_0^1(\Omega)$ with continuous and dense injection. Let $D(A)^+$ be the subspace of non-negative functions in $D(A)$.

Remark 2 We note that if $\Omega \subset \mathbb{R}^d$ is a bounded C^2 domain, then we have that $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, and moreover the norm induced by $(\cdot, \cdot)_{D(A)}$ in $D(A)$ and the norm of $H^2(\Omega)$ are equivalent.

Now, in our first main result, we prove an exponential growth condition for the attractor $\mathcal{A}(t_0)$ in $H_0^1(\Omega)^3$ when $t_0 \rightarrow -\infty$.

Theorem 3 Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded C^2 domain and assume that $\gamma + \frac{b}{2} + \frac{c}{2} < \lambda_1$ where λ_1 is the first eigenvalue of the operator A on the domain Ω with Dirichlet boundary condition. Then $\mathcal{A}(t_0)$ satisfies

$$\lim_{t_0 \rightarrow -\infty} \left\{ e^{\lambda_1 t_0} \sup_{(v_1, v_2, v_3) \in \mathcal{A}(t_0)} \|(v_1, v_2, v_3)\|^2 \right\} = 0. \quad (7)$$

Proof. From the inequality (27) of [4], for any $t \geq t_0$ we have

$$\begin{aligned} & |S_n(r)|_{L^2(\Omega)}^2 + |I_n(r)|_{L^2(\Omega)}^2 + |R_n(r)|_{L^2(\Omega)}^2 + \int_{t_0}^r \left(\|S_n(s)\|^2 + \|I_n(s)\|^2 + \|R_n(s)\|^2 \right) ds \\ & \leq C_1 \left(|S_0|_{L^2(\Omega)}^2 + |I_0|_{L^2(\Omega)}^2 + |R_0|_{L^2(\Omega)}^2 + (t - t_0) \right), \end{aligned} \quad (8)$$

for all $r \in [t_0, t]$, and all $n \geq 1$, where $C_1 := \frac{\max \{1, \frac{a}{2}(q^+)^2 |\Omega|\}}{\min \{1, 2 - \lambda_1^{-1}(b + c + 2\gamma)\}}$.

From (8) and (26) in [4] we now obtain that

$$\begin{aligned} (r - t_0) \left(\|S_n(r)\|^2 + \|I_n(r)\|^2 + \|R_n(r)\|^2 \right) & \leq C_1 \left(|S_0|_{L^2(\Omega)}^2 + |I_0|_{L^2(\Omega)}^2 + |R_0|_{L^2(\Omega)}^2 + (t - t_0) \right) \\ & + (q^+)^2 |\Omega| (t - t_0)^2 (2a^2 + \frac{a}{2} k_1 C) \\ & + k_1 C \left(|S_0|_{L^2(\Omega)}^2 + |I_0|_{L^2(\Omega)}^2 + |R_0|_{L^2(\Omega)}^2 \right) (t - t_0), \end{aligned} \quad (9)$$

for any $t \geq t_0$, all $r \in [t_0, t]$, and all $n \geq 1$, where $C := (2\lambda_1 - b - c - 2\gamma)^{-1}$ and k_1 is a positive constant.

In particular, from (9) we deduce

$$\|S_n(r)\|^2 + \|I_n(r)\|^2 + \|R_n(r)\|^2 \leq C_2 \left(|S_0|_{L^2(\Omega)}^2 + |I_0|_{L^2(\Omega)}^2 + |R_0|_{L^2(\Omega)}^2 + 1 \right), \quad (10)$$

for all $r \in [t_0 + 1, t_0 + 2]$, and any $n \geq 1$, where

$$C_2 := \max \left\{ C_1 + 2k_1C, 2C_1 + 4(q^+)^2|\Omega| \left(2a^2 + \frac{a}{2}k_1C \right) \right\}.$$

Using Lemma 3 in [4], we have that $(S_n(\cdot), I_n(\cdot), R_n(\cdot)) = (S_n(\cdot; t_0, S_0), I_n(\cdot; t_0, I_0), R_n(\cdot; t_0, R_0))$ converges weakly to the unique solution to (1)-(3) $(S(\cdot), I(\cdot), R(\cdot)) = (S(\cdot; t_0, S_0), I(\cdot; t_0, I_0), R(\cdot; t_0, R_0))$ in $L^2(t_0, t; (Y^+)^3)$, for all $t > t_0$. Thus, from (10) and Lemma 1, we in particular obtain

$$\|S(t_0+1)\|^2 + \|I(t_0+1)\|^2 + \|R(t_0+1)\|^2 \leq C_2 \left(|S_0|_{L^2(\Omega)}^2 + |I_0|_{L^2(\Omega)}^2 + |R_0|_{L^2(\Omega)}^2 + 1 \right).$$

Multiplying this inequality by $e^{\lambda_1(t_0+1)}$ and using (6), there exists a constant $C_3 > 0$ such that

$$e^{\lambda_1(t_0+1)} \|U_{t_0+1, t_0}(S_0, I_0, R_0)\|^2 \leq C_3 e^{\lambda_1} \left(e^{\lambda_1 t_0} |(S_0, I_0, R_0)|_{L^2(\Omega)}^2 + e^{\lambda_1 t_0} \right), \quad (11)$$

for all $t_0 \in \mathbb{R}$, and all $(S_0, I_0, R_0) \in X_3^+$.

As $\mathcal{A}(t_0 + 1) = U_{t_0+1, t_0}\mathcal{A}(t_0)$, it follows from (11) that

$$e^{\lambda_1(t_0+1)} \|(v_1, v_2, v_3)\|^2 \leq C_3 e^{\lambda_1} \left(e^{\lambda_1 t_0} \sup_{(w_1, w_2, w_3) \in \mathcal{A}(t_0)} |(w_1, w_2, w_3)|_{L^2(\Omega)}^2 + e^{\lambda_1 t_0} \right),$$

for all $(v_1, v_2, v_3) \in \mathcal{A}(t_0 + 1)$, and any $t_0 \in \mathbb{R}$.

Finally, this inequality implies

$$e^{\lambda_1 t_0} \|(v_1, v_2, v_3)\|^2 \leq C_3 e^{\lambda_1} \left(e^{\lambda_1(t_0-1)} \sup_{(w_1, w_2, w_3) \in \mathcal{A}(t_0-1)} |(w_1, w_2, w_3)|_{L^2(\Omega)}^2 + e^{\lambda_1(t_0-1)} \right), \quad (12)$$

for all $(v_1, v_2, v_3) \in \mathcal{A}(t_0)$, and any $t_0 \in \mathbb{R}$. Taking into account (5), from (12) we obtain (7). ■

3 An exponential growth condition for the pullback attractor in $H^2(\Omega)^3$

The aim of this section is to continue with the analysis of the model in the sense of proving that the attractor $\mathcal{A}(t_0)$ satisfies also an exponential growth condition on the space $H^2(\Omega)^3$ provided some additional assumptions are fulfilled. Our second main result is the following.

Theorem 4 *In addition to the assumptions in Theorem 3, assume moreover that $q' \in L_{loc}^2(\mathbb{R}; L^2(\Omega))$, and satisfies (4). Then $\mathcal{A}(t_0)$ satisfies that*

$$\lim_{t_0 \rightarrow -\infty} \left(e^{\lambda_1 t_0} \sup_{(v_1, v_2, v_3) \in \mathcal{A}(t_0)} \|(v_1, v_2, v_3)\|_{H^2(\Omega)}^2 \right) = 0. \quad (13)$$

Proof. From inequality (35) in [4], taking $t = t_0 + 3$ and $\varepsilon = 2$, we have

$$\begin{aligned} |S'_n(r)|_{L^2(\Omega)}^2 + |I'_n(r)|_{L^2(\Omega)}^2 + |R'_n(r)|_{L^2(\Omega)}^2 &\leq (4k_3 + 1) \int_{t_0+1}^{t_0+3} \left(|S'_n(\theta)|_{L^2(\Omega)}^2 + |I'_n(\theta)|_{L^2(\Omega)}^2 + |R'_n(\theta)|_{L^2(\Omega)}^2 \right) d\theta \\ &\quad + a \int_{t_0+1}^{t_0+3} |q'(\theta)|_{L^2(\Omega)}^2 d\theta, \end{aligned} \quad (14)$$

for all $r \in [t_0 + 2, t_0 + 3]$, and any $n \geq 1$, where k_3 is a positive constant.

Analogously, and if we take $s = t_0 + 1$ and $r = t = t_0 + 3$ in inequality (25) of [4], we, in particular, have

$$\begin{aligned} \int_{t_0+1}^{t_0+3} \left(|S'_n(\theta)|_{L^2(\Omega)}^2 + |I'_n(\theta)|_{L^2(\Omega)}^2 + |R'_n(\theta)|_{L^2(\Omega)}^2 \right) d\theta &\leq \|S_n(t_0 + 1)\|^2 + \|I_n(t_0 + 1)\|^2 + \|R_n(t_0 + 1)\|^2 \\ &+ 3(q^+)^2 |\Omega| (2a^2 + \frac{a}{2} k_1 C) \\ &+ k_1 C \left(|S_0|_{L^2(\Omega)}^2 + |I_0|_{L^2(\Omega)}^2 + |R_0|_{L^2(\Omega)}^2 \right), \end{aligned} \quad (15)$$

for all $n \geq 1$, where k_1 is a positive constant and $C := (2\lambda_1 - b - c - 2\gamma)^{-1}$.

From (14) and (15), we obtain

$$\begin{aligned} |S'_n(r)|_{L^2(\Omega)}^2 + |I'_n(r)|_{L^2(\Omega)}^2 + |R'_n(r)|_{L^2(\Omega)}^2 &\leq (4k_3 + 1) (\|S_n(t_0 + 1)\|^2 + \|I_n(t_0 + 1)\|^2 + \|R_n(t_0 + 1)\|^2) \\ &+ (4k_3 + 1) 3(q^+)^2 |\Omega| (2a^2 + \frac{a}{2} k_1 C) + a \int_{t_0+1}^{t_0+3} |q'(\theta)|_{L^2(\Omega)}^2 d\theta \\ &+ (4k_3 + 1) k_1 C \left(|S_0|_{L^2(\Omega)}^2 + |I_0|_{L^2(\Omega)}^2 + |R_0|_{L^2(\Omega)}^2 \right), \end{aligned}$$

for all $r \in [t_0 + 2, t_0 + 3]$, and any $n \geq 1$.

Owing to this inequality and (10), there exists a constant $\tilde{C}_1 > 0$ such that

$$|S'_n(r)|_{L^2(\Omega)}^2 + |I'_n(r)|_{L^2(\Omega)}^2 + |R'_n(r)|_{L^2(\Omega)}^2 \leq \tilde{C}_1 \left(|S_0|_{L^2(\Omega)}^2 + |I_0|_{L^2(\Omega)}^2 + |R_0|_{L^2(\Omega)}^2 + \int_{t_0}^{t_0+3} |q'(\theta)|_{L^2(\Omega)}^2 d\theta + 1 \right),$$

for all $r \in [t_0 + 2, t_0 + 3]$, and any $n \geq 1$.

From inequality (36) of [4], and thanks to (16), we have

$$\begin{aligned} &|\Delta S_n(r)|_{L^2(\Omega)}^2 + |\Delta I_n(r)|_{L^2(\Omega)}^2 + |\Delta R_n(r)|_{L^2(\Omega)}^2 \\ &\leq 4\tilde{C}_1 \left(|S_0|_{L^2(\Omega)}^2 + |I_0|_{L^2(\Omega)}^2 + |R_0|_{L^2(\Omega)}^2 + \int_{t_0}^{t_0+3} |q'(\theta)|_{L^2(\Omega)}^2 d\theta + 1 \right) \\ &+ 8a^2(q^+)^2 |\Omega| + 4k_2 \left(|S_n(r)|_{L^2(\Omega)}^2 + |I_n(r)|_{L^2(\Omega)}^2 + |R_n(r)|_{L^2(\Omega)}^2 \right), \end{aligned}$$

for all $r \in [t_0 + 2, t_0 + 3]$, and any $n \geq 1$, where k_2 is a positive constant.

Therefore, by (8) we obtain that there exists a constant $\tilde{C}_2 > 0$ such that

$$\begin{aligned} &|\Delta S_n(r)|_{L^2(\Omega)}^2 + |\Delta I_n(r)|_{L^2(\Omega)}^2 + |\Delta R_n(r)|_{L^2(\Omega)}^2 \\ &\leq \tilde{C}_2 \left(|S_0|_{L^2(\Omega)}^2 + |I_0|_{L^2(\Omega)}^2 + |R_0|_{L^2(\Omega)}^2 + \int_{t_0}^{t_0+3} |q'(\theta)|_{L^2(\Omega)}^2 d\theta + 1 \right), \end{aligned} \quad (16)$$

for all $r \in [t_0 + 2, t_0 + 3]$, and any $n \geq 1$.

By Theorem 6 in [4], we have that $(S(\cdot; t_0, S_0), I(\cdot; t_0, I_0), R(\cdot; t_0, R_0)) \in C([t_0 + 2, t_0 + 3]; Y_3^+)$. On the other hand, in the proof of Theorem 4 in [4], we proved that $\{(S_n(\cdot; t_0, S_0), I_n(\cdot; t_0, I_0), R_n(\cdot; t_0, R_0))\}$ is bounded in $L^2(t_0, t; (D(A)^+)^3)$ for all $t > t_0$. Then, in particular, we have that $(S_n(\cdot), I_n(\cdot), R_n(\cdot)) = (S_n(\cdot; t_0, S_0), I_n(\cdot; t_0, I_0), R_n(\cdot; t_0, R_0))$ converges weakly to the unique solution, $(S(\cdot), I(\cdot), R(\cdot)) = (S(\cdot; t_0, S_0), I(\cdot; t_0, I_0), R(\cdot; t_0, R_0))$, to (1)-(3) in $L^2(t_0 + 2, t_0 + 3; (D(A)^+)^3)$.

Then, by Lemma 1, inequality (16) and the equivalence of the norms $|\Delta v|_{L^2(\Omega)}$ and $\|v\|_{H^2(\Omega)}$, we have that there exists a constant $\tilde{C}_3 > 0$ such that

$$\begin{aligned} &\|(S(r; t_0, S_0), I(r; t_0, I_0), R(r; t_0, R_0))\|_{H^2(\Omega)}^2 \\ &\leq \tilde{C}_3 \left(|S_0|_{L^2(\Omega)}^2 + |I_0|_{L^2(\Omega)}^2 + |R_0|_{L^2(\Omega)}^2 + \int_{t_0}^{t_0+3} |q'(\theta)|_{L^2(\Omega)}^2 d\theta + 1 \right), \end{aligned} \quad (17)$$

for all $r \in [t_0 + 2, t_0 + 3]$, any $t_0 \in \mathbb{R}$, and $(S_0, I_0, R_0) \in X_3^+$.

Thus, from (17), and using (6), we deduce that there exists a constant $\tilde{C}_4 > 0$ such that

$$\|U_{t_0+2,t_0}(S_0, I_0, R_0)\|_{H^2(\Omega)}^2 \leq \tilde{C}_4 \left(|(S_0, I_0, R_0)|_{L^2(\Omega)}^2 + \int_{t_0}^{t_0+3} |q'(\theta)|_{L^2(\Omega)}^2 d\theta + 1 \right),$$

for all $t_0 \in \mathbb{R}$, $(S_0, I_0, R_0) \in X_3^+$.

From this inequality, and the fact that $\mathcal{A}(t_0) = U_{t_0,t_0-2}\mathcal{A}(t_0-2)$, we obtain

$$\|(v_1, v_2, v_3)\|_{H^2(\Omega)}^2 \leq \tilde{C}_4 \left(\sup_{(w_1, w_2, w_3) \in \mathcal{A}(t_0-2)} |(w_1, w_2, w_3)|_{L^2(\Omega)}^2 + \int_{t_0-2}^{t_0+1} |q'(\theta)|_{L^2(\Omega)}^2 d\theta + 1 \right), \quad (18)$$

for all $(v_1, v_2, v_3) \in \mathcal{A}(t_0)$, and any $t_0 \in \mathbb{R}$.

Now, thanks to (4) and (5), we obtain (13) from (18). ■

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