Generalized Intervals: A Guided Tour

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Abstract

Generalized intervals are finite ordered sequences of time points (Allen's calculus is the special case of ordered pairs). In this paper, we show why generalized intervals are good candidates for reasoning about complex events (with more than two crucial time points): Binary relations between them can be easily encoded; the conversion and composition operations on disjunctive relations provide them with a structure of a relation algebra; although the whole calculus is not tractable in general, there exists a subclass of disjunctive relations, which is easily characterized in geometric terms, which is tractable (in Allen's cases, this subclass coincides with the ORD-Horn class); for binary temporal networks on this subclass, consistency is decidable in cubic time by testing path-consistency; moreover, a scenario can be computed in cubic time without backtrack (in quadratic time for consistent networks). Finally, the strong theory of n-intervals (generalized intervals with n time points) has a unique countable model (up to isomorphism), which implies its decidability. In a word, most of the pleasant properties of Allen's calculus hold in this generalized framework.

Introduction

In his 1983 paper (Allen 1983), Allen justified the introduction of his calculus about intervals in the following terms: "the temporal representation described above is notable in that it is both expressive and computationally feasible". Over the years, Allen calculus has indeed proven to be a reference in qualitative temporal reasoning, in such diverse applications as planning, natural language processing, or qualitative physics.

The interrogation about what the right temporal entity should be in a given context has been a persistent topic. Mixing interval-based and point-based calculus has been in practice since at least Vilain's 1982 paper (Vilain 1982), and is represented, in the domain of qualitative and quantitative reasoning, by Meiri's framework (Dechter, Meiri, & Pearl 1991). This consideration of temporal entities of many types is also apparent in the strand of work concerned with repeating events and their associated non-convex intervals.

Time points, intervals, and non-convex intervals are all subsumed by the notion of generalized interval. In that way, the time-point calculus, Allen's calculus, or the calculus of relations between non-convex intervals appear as specific instances of generalized interval calculus. This fact would not be of much interest if the good properties of time-point or interval calculus were lost in this generalized context. The good news is that this is not the case. Let us list some of those good properties of a time calculus:

1. The calculus is based on a finite set of basic binary relations between the basic temporal entities.
2. There is a notion of composition of two basic relations, yielding a finite disjunction of possible basic relations; composition can be easily tabulated (Allen's composition table is an example).
3. The set of disjunctions of basic relations, together with suitable operations induced by composition and conversion, is a relation algebra.
4. A framework for exploiting the temporal calculus is provided by binary temporal constraint networks labeled with disjunctions of basic relations.
5. Although the basic problems of determining the consistency of a network, computing the minimal network, or finding all scenarios associated to a given network are NP-hard (Vilain, Kautz, & van Beek 1989), constraint propagation can be used and algorithms for path-consistency provide useful filtering techniques.
6. The topological properties of the time-axis are reflected in the conceptual neighborhood structure of the basic relations. Most of the properties of specific tractable sub-classes of the calculus can be related to geometric properties of the associated relations: such is the case, for example, of the maximal tractable subclass of Allen's algebra discovered by Nebel and Bürckert (Nebel & Bürckert 1995).
7. The models of the calculus can be characterized. In particular, the strong version of the calculus has completeness properties: up to isomorphism, the only countable model is provided by the corresponding entities in the field of rational numbers.

Conceived as a guided tour, this paper will provide the reader with a round-up of all these features. We

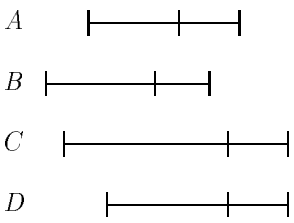

Figure 1: Four medical files

illustrate the general properties of the formalism on a particular case: the calculus of 3-intervals.

Generalized intervals: Learning the Language

Motivation

Consider an application where medical files about patients are managed. A basic event corresponds to a given patient entering the hospital (at date t_1), undergoing a medical intervention (at date t_2), then finally leaving the hospital (at date t_3). We assume that in all cases $t_1 < t_2 < t_3$. In the spirit of qualitative temporal reasoning, we are interested in dealing with qualitative information concerning a finite number of events.

For instance, assume we have the following information about four interventions (see Fig. 1):

- Patient A was admitted after patient B, and left the hospital after B did (we do not know anything about A or B's days of intervention).
- Patient B entered the hospital before C did, and was operated and released during C's pre-intervention period.
- Patient C entered the hospital before D, was operated on the same day as D, and also released on the same day as D.

We could represent the situation by using time points, and by introducing for each patient A, B, C, D three time points (a_1, a_2, a_3), $\cdots$, (d_1, d_2, d_3).

We could also use Allen's formalism, by introducing e.g. two ajoining intervals i (before intervention and j (after intervention) for each patient. Hence we would have, for the relations between A and B for example: $i_A\ m\ j_A$, $i_B\ m\ j_B$, and: $i_A\ d\ i_B$ and $j_A\ d^\smile\ j_B$, or $i_A\ f\ i_B$ and $j_A\ s^\smile\ j_B$, or $i_A\ o^\smile\ i_B$ and $j_A\ o^\smile\ j_B$, and similarly for the other pairs of events.

Notice however that using Allen's formalism suffers from two drawbacks. First, it leads to disjunctive representations which cannot be represented in a unique network. Second, the choice of i and j is somewhat arbitrary: we could for instance decide to represent each data by its total duration k, and by the initial interval i (before intervention). Then we would have $i\ s\ k$ (i starts k), rather than $i\ m\ j$.

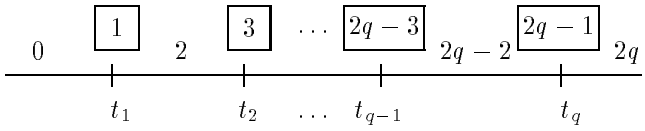

Figure 2: Zones defined by a q-interval

Our proposal i to consider the most obvious kind of objects in this case: Namely, ordered sequences of 3 time points. This idea is generalized as follows.

Basics: Objects and Relations

We consider ordered sequences of time points as basic temporal entities. If such a sequence has q points, we call it a q-interval. Hence an interval in Allen's sense is a 2-interval (t_1, t_2).

For denoting relations between a p-interval and a q-interval, we use the trick of numbering the consecutive zones defined in time by each point from 0 to $2q$:
zone 0 is $(-\infty, t_1)$; zone 1 is $\{t_1\}$; ...; zone $2q-1$ is $\{t_q\}$; zone $2q$ is $(t_q, +\infty)$.

Notice that zones with even numbers are intervals in the time line, and odd numbered zones are time points (see Fig. 2).

Using this numbering, the qualitative relation of any p-interval $(x_1, \ldots, x_q)$ with respect to a q-interval $(t_1, \ldots, t_q)$ is entirely characterized by the numbers of the zones $x_1, \ldots, x_p$ belong to. In this way, we arrive at a notation of relations as sequences of integers between 0 and $2q$. More specifically:

Definition 1 *The set of (p,q)-relations between a p-interval and a q-interval is the set of all non-decreasing sequences of integers between 0 and $2q$ such that no odd integer appears more than once.*

Example In the case of sequences of three time points, the corresponding set of relations is a subset of the set of all non-decreasing sequences of integers between 0 and 6. Using this notation, we can express the constraints about the medical files in the following way (where we use A, B, C,D to denote the four 3-intervals corresponding to the four patients):
The relation from A to B is represented by: $(2, [2,4], 6)$ (where $(2, [2,4], 6)$ stands for the disjunction of $(2,2,6)$, $(2,3,6)$, and $(2,4,6)$). The relation from B to C and C to D are represented by $(0,2,2)$ and $(0,3,5)$ respectively.

We will need later what we call the *symbol* of a (p,q)-relation: by definition, this will be an abstract sequence of p points labelled by the integers which encode the relation. The symbol of relation $(0,2,2)$ is represented in Fig. 3.

Counting the relations In this way we get a set of 63 basic relations for 3-intervals. In the general case,

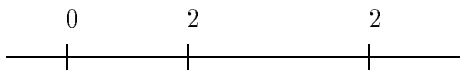

Figure 3: The symbol of relation $(0, 2, 2)$

the number of basic relations can be computed in a simple way (see (Ligozat 1990)). We also have a general characterization in terms of integral points in a convex polytope:

Consider in $\mathbf{R^q}$ the convex closure $\mathbf{L_q}$ of the $2q$ points $(\pm 1, 0, \ldots, 0)$, $(0, \pm 1, 0, \ldots, 0)$, $\cdots$, $(0, \ldots, 0, \pm 1)$. For instance, for $q = 2$, $\mathbf{L_2}$ is the square with $(1, 0)$, $(-1, 0)$, $(0, 1)$, $(0, -1)$ as vertices. Then:

Proposition 1 *The number of (p, q)-relations is the number of points with integral coordinates in p $\mathbf{L_q}$.*

The lattice of (p, q)-relations

Encoding the set of basic relations as p-tuples of integers gives it an order structure: for two relations $r = (r_1, \ldots, r_p)$ and $s = (s_1, \ldots, s_p)$, $r \leq s$ if and only if $r_i \leq s_i$, for all $1 \leq i \leq p$. This order structure reflects the order structure of the time line in a natural way. It is easily verified that the resulting ordered structure is a distributive lattice: every pair of basic relations has a greatest lower bound and a least upper bound, which, incidently, can be computed componentwise.

In the case of Allen's calculus, the resulting lattice structure has been known and used by many authors, including Nökel (Nökel 1988) and Freksa (Freksa 1992).

The lattice of $(3, 3)$-relations is represented in Fig. 4.

Following the practice in Allen's case, we define a *relation* as any subset of basic relations. In the general case, the lattice structure yields natural notions of neighborhoods which we discuss below. In particular, it gives a general definition of *convex relations*:

Definition 2 *A* convex relation *is an interval in the lattice of basic relations.*

Hence a non-empty convex relation is the set of all t such that $r \leq t \leq s$, for some $r \leq s$. We also denote this relation by $[r, s]$, using the usual interval notation applied to the lattice.

Translating into Other Languages

As the preceding section shows, generalized intervals provide an "unprejudiced" language for representing complex temporal sequences. For example, representing relations between two 3-intervals does not imply choosing any decomposition in terms of intervals. However, the language allows an easy translation from its own language to other possible languages.

For instance, consider again the medical example, and its expression in terms of $(3, 3)$-relations. It is a

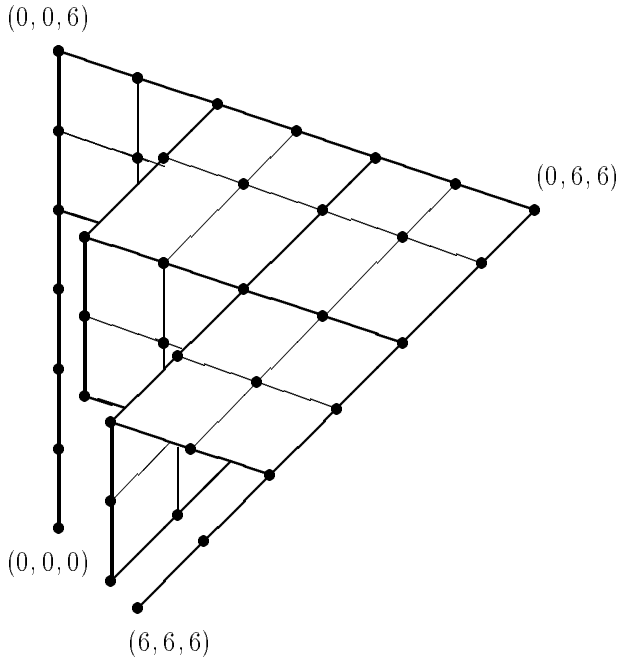

Figure 4: The lattice of $(3, 3)$-relations

general fact that, for each (p, q)-relation, we can "forget" about some of the time points in the reference q-interval. In the case of 3-intervals, we can forget about the middle points. This amounts to keeping only the first and third coordinates, and interpreting zones 2, 3, and 4 as a unique zone numbered 2, zone 5 as 3, and zone 6 as 4. Hence, $(2, [2, 4], 6)$ projects to $(2, 4)$, i.e. $o^\smile$, $(0, 2, 2)$ projects to $(0, 2)$, i.e. o, and $(0, 3, 5)$ projects to $(0, 3)$, i.e. $f^\smile$.

Projections are a general way of representing information about complex entities in terms of simpler ones. The price to be paid in general is loss of information.

Non convex intervals

Qualitative reasoning about non convex intervals as first class objects was introduced by Ladkin (Ladkin 1986).

In particular, Ladkin gives a general taxonomy of the set of all relations between two non convex intervals in terms of "interesting" relations, such as *disjoint-from* (two intervals have no point in common), *bars* (the union of the two sets of points is convex), *always-meets* (each convex component of J is met by a component of I).

The language of generalized intervals gives a nattural way of expressing Ladkin's relations. A non convex interval with m connected components can be represented by a $2m$-interval. Consider the set of $(2n, 2m)$-relations. Then, for instance:

- *disjoint-from* is the set of $\pi = (\pi(1), \ldots, \pi(2n))$ such that

1. $\pi(1) = \pi(2)$, $\pi(3) = \pi(4)$,...,$\pi(2n-1) = \pi(2n)$,
2. $\pi(i)$ is a multiple of 4, for all $0 \leq i \leq 2n$.

- *bars* is the set of $\pi = (\pi(1), \ldots, \pi(2n))$ such that
  1. $\pi(1) < 4$ and $\pi(2n) > 4m - 4$;
  2. no $\pi(i)$ is congruent to 0 (mod 4) , except for possibly $\pi(1) = 0$, $\pi(2n) = 4m$.

Generalized Intervals: a Language for Reasoning

Operations

As promised, there is a finite set of basic binary relations between the basic temporal entities. We also have two basic operations on this set, conversion and composition.

Conversion *Conversion* corresponds to switching roles in a binary relation: suppose a p-interval x is in relation $(r_1, \ldots, r_q)$ with respect to a q-interval y. What is the relation of y with respect to x? By definition, we will call it the transpose of $(r_1, \ldots, r_q)$. Obviously, it is a (q,p)-relation, which is encoded by a sequence of q-integrers between 0 and $2p$. Which one is it? The answer is given as follows.

Consider the symbol of r. The symbol of r defines $2p+1$ zones which can be numbered from 0 to $2p$. Consider also the sequence $1, 3, \ldots, 2q-1$ of the first q odd integers (let us call it the standard q-interval).

Proposition 2 *Let $r = (r_1, \ldots, r_q)$ be a (p,q)-relation. The converse $r^\smile$ of r is the (q,p)-relation of the standard q-interval with respect to the symbol of r.*

Example Consider again the medical example. The standard 3-interval in the sequence $(1,3,5)$ in N. Consider the relation of B with respect to C. Its symbol has $(0,2,2)$ as a sequence of labels. Since 1 is between 0 and 2, it is in zone number 2 of the symbol. Since 3 and 5 are both greater than 2, they both belong to zone number 6. Hence the converse of $(0,2,2)$ is $(2,6,6)$.

Composition *Composition* corresponds to composing information.

We use the following notation: for any pair of integers (m,n) with $m \leq n$, let $[[m,n]]$ (the *reduced* zone) be the largest integer interval with even endpoints contained in $[m,n]$.

Proposition 3 *Let $r = (r_1, \ldots, r_p)$ be a (p,k)-relation, and $s = (s_1, \ldots, s_k)$ be a (k,q)-relation. The composition $r \circ s$ of r with s is the set of (p,q)-relations whose i-th coordinates is in $[[s_l, s_{l+1}]]$, where $[s_l, s_{l+1}]$ is zone number r_i in the symbol of s, for $1 \leq i \leq p$.*

Example In the medical example (refer to Fig. 5), we know that B is $(0,2,2)$ to C and that C is $(0,3,5)$ to D. To get the relations of B with respect to D, we have to compute the composition of $(0,2,2)$ with $(0,3,5)$. By the above rule, this composition contains all relations such that:

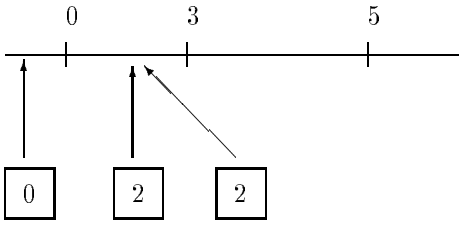

Figure 5: Composing $(0,2,2)$ with $(0,3,5)$

- Their first projection in zone number 0 of the symbol $(0,3,5)$: this zone contains only zero;
- their second projection is in the (reduced) zone number 2 of the symbol $(0,3,5)$, that is $[[0,3]]$;
- their third projection in the same range as the second one.

Since $[[0,3]]$ is $[0,2]$, we get $(0,[0,2],[0,2])$ as a result. This is the set with relations $(0,0,0)$, $(0,0,1)$, $(0,0,2)$, $(0,1,2)$, $(0,2,2)$.

Coming back to the general case, we notice that the composition of two basic relations is a convex relation. Composition can be extended in an obvious way to relations: the composition of two relations is by definition the union of the pairwise basic relations they contain. Then the preceding result is a particular case of a stronger result (Ligozat 1991):

Proposition 4 *The composition of two convex relations is itself a convex relation. More specifically, let $[r,s]$ and $[t,u]$ be two convex relations. Then the composition $[r,s] \circ [t,u]$ is the convex relation $[inf(r \circ t), sup(s \circ u)]$.*

This fact has an important consequence for reasoning about disjunctive knowledge expressed in the subclass of convex relations: independently of the number of basic relations contained in each disjunct, the composition of two convex relations can be obtained by computing only two compositions. If a table of composition is available for all basic relations (the equivalent of Allen's table), then two lookups are enough.

The algebraic structure of (p,q)-relations

Since relations (subsets of basic relations) are subsets of a given set, they inherit the structure of a Boolean algebra. The existence of conversion and composition gives them the ingredients necessary to be a relation algebra in the sense of Jónsson and Tarski (Jónsson & Tarski 1952). In particular, for any given n, the set of (n,n)-relations is such an algebra.

Binary constraint networks

A convenient way of expressing knowledge about temporal situations uses the framework of temporal con-

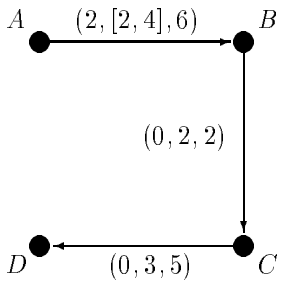

Figure 6: A network describing the medical example

straint networks. A temporal constraint network is a finite directed graph with labeled arcs. Each node X_i represents a p-interval. The arc from X_i (a p-interval) to X_j (a q-interval) is labeled with a (disjunctive) (p,q)-relation $\alpha_{i,j}$.

Example The information in the medical example can be represented by a 4-node network with nodes A, B, C, and D. The arcs (A,B), (B,C) and (C,D) are labeled by $(2,[2,4],6)$, $(0,2,2)$, and $(0,3,5)$, respectively. The corresponding network is represented in Fig. 6.

We assume that $\alpha_{j,i}$ is the converse of $\alpha_{i,j}$, and that $\alpha_{i,i}$ is equality, for all i.

A network is *path-consistent* if each label $\alpha_{i,j}$ contains at least an atomic relation, and $(\alpha_{i,j} \circ \alpha_{j,k}) \supseteq \alpha_{i,k}$ for all i, j, k, $1 \leq i,j,k \leq m$.

An *instantiation* of a variable X_i is a pair of real numbers (a_i, b_i), with $a_i < b_i$. If instantiations are given for X_i and X_j, there is exactly one atomic relation r such that the interval (a_i, b_i) is in relation r with respect to (a_j, b_j). We say that the two instantiations of X_i, X_j instantiate r.

A network is *globally consistent* if instantiations can be found for each X_i such that for each pair (i,j) an element in $\alpha_{i,j}$ is instantiated on the edge (i,j).

Matters of Tractability

The geometry of (p,q)-relations: dimension and closures

Generalized intervals in the time line can be interpreted as points in Euclidean space. For instance, a 3-interval $x = (x_1, x_2, x_3)$ is an element of the cone $\mathbf{C_3}$ in 3D-space defined as the intersection of the half-spaces $x_1 < x_2$ and $x_2 < x_3$.

If a 3-interval $a = (a_1, a_2, a_3)$ is fixed, then the condition for $x = (x_1, x_2, x_3)$ to be in one of the $(3,3)$-relations with respect to x defines a region in this cone. Those regions are connected, pairwise disjoint, and their union is the (open) cone itself.

Example x is in the $(3,3)$-relation $(0,3,4)$ with respect to a if and only if: $x_1 < a_1$, $x_2 = a_2$, and $a_2 < x_3 < a_3$.

This defines a region of dimension 2 contained in the plane $x_2 = a_2$. This region is not closed. Its topological closure in $\mathbf{C_3}$ further contains the open segment defined by: $x_1 = a_1$, $x_2 = a_2$, $a_2 < x_3 < a_3$, the open half-line $x_2 = a_2$, $x_3 = a_3$, and $x_1 < a_1$, and the point $x_1 = a_1$, $x_2 = a_2$, $x_3 = a_3$. In fact, each one of those components is associated to a relation: The segment to $(1,3,4)$, the half-line to $(0,3,5)$, and the point to $(1,3,5)$ (equality). This is a general fact.

1. The set of (p,q)-relations corresponds to a partition into connected regions of the cone $\mathbf{C_p}$ defined in $\mathbf{R^p}$ by $x_1 < x_2 \ldots x_{p-1} < x_p$.
2. The dimension of the region in $\mathbf{C_p}$ associated to $\pi = (\pi_1, \ldots, \pi_p)$ is p - #(odd integers in (π_i)).
3. The topological closure in $\mathbf{C_p}$ of the region associated to π is a union of regions associated to a subset of (p,q)-relations.
4. The region associated to $\pi' = (\pi'_i)$ belongs to the closure of the region associated to π if and only if, for all i, $1 \leq i \leq p$, either $\pi'_i = \pi_i$ or π_i is even and $|\pi'_i - \pi_i| = 1$.

Example Consider again the relation $(0,3,4)$. By the above result, the topological closure of this relation contains $(0,3,4)$ itself, as well as $(1,3,4)$, $(0,3,5)$, and $(1,3,5)$, as desired.

Definition 3 *The* dimension *of a relation is the dimension of its associated region.*

The topological closure $C(\alpha)$ *of α is the set of relations whose associated regions constitute the topological closure of the region associated to α.*

The subclass of pre-convex relations

The topological closure of a convex relation is itself convex. In fact, the closure of each atom is convex, and we have, for any convex relation $[\alpha, \beta]$, $C([\alpha,\beta]) = [inf(C(\alpha)), sup(C(\beta))]$.

Definition 4 *A relation is a* pre-convex *relation if its topological closure is convex.*

An equivalent definition uses the notion of convex closure. Recall that $I(\alpha)$ denotes the convex closure of α. Then a relation α is pre-convex if and only if $I(\alpha)$ is contained in $C(\alpha)$.

In Allen's case, pre-convex relations coincide with ORD-Horn relations (Ligozat). By Nebel and Bürckert's results, they constitute a tractable subclass in Allen's algebra (which, moreover, has a maximality property). This is no coincidence: pre-convex relations in general are tractable. Explaining why this is true is the purpose of the next subsections.

Projecting convex and pre-convex relations A crucial fact about convex regions is the fact that they are entirely determined by their projections. For pre-convex relations, this is no longer the case. However, it is *almost* true, as explained below.

Consider first convex relations. The region associated to it in the cone $\mathbf{C_p}$ is connected. Its projections on each component are connected subsets of the time line (intervals). Let α be a convex relation, R_α its associated region. Project R_α on each coordinate, obtaining $pr_i(R_\alpha)$, then "pullback" by considering the product of all projections $pr_1(R_\alpha) \times \ldots \times pr_p(R_\alpha)$, and intersect it with $\mathbf{C_p}$: The resulting set is exactly R_α.

Now if α is a pre-convex relation which is not convex, we cannot hope for the same fact to hold in general. However, it does hold for the convex closure $I(\alpha)$ of α. Because α is pre-convex, $I(\alpha)$ only differs from α by a union of atoms which belong to the topological closure of α. Hence, playing the project-then-pullback game, and intersecting with $\mathbf{C_p}$, results in a relation which is (in general) bigger than the relation we started with, but not substantially: it only adds smaller-dimensional atoms to the relation we started with:

Proposition 5 *If α is a pre-convex relation, then $R_\alpha \setminus ((pr_1(R_\alpha) \times \cdots \times pr_p(R_\alpha)) \cap \mathbf{C_p})$ is a union of regions corresponding to atoms of dimensions strictly less than $dim(\alpha)$.*

This is a crucial fact for proving tractability.

Pre-convex relations are tractable

We are now in position to describe why tractability holds (Ligozat 1996). Consider a path-consistent network whose labels are pre-convex relations. If the network has three nodes, path-consistency implies that, if we are given instantiations for two of the nodes, then the third node can be instantiated in a way such that all constraints are satisfied. In fact, a slightly more precise fact holds.

Lemma 1 *Consider a path-consistent, pre-convex network with three nodes X_1, X_2, $X3$. Choose instantiations for X_1 and X_2 such that the resulting atomic dimension is of maximal dimension in the label on (X_1, X_2). Then the third node X_3 can be instantiated in such a way that the resulting atoms on (X_3, X_1) and (X_3, X_2) are of maximal dimensions.*

The general proof is by induction on the number of nodes: it extends the preceding lemma to a network with an arbitrary number of nodes. If the network has m nodes, we show that each instantiation of n nodes corresponding to atoms of maximal dimension in their edges can be extended to $n + 1$ nodes, with the same property.

The key intuition is the following: Suppose n nodes have been instantiated. The constraints on the instantiation for the new node to be instantiated are in terms of regions associated to pre-convex relations. Because of path-consistency, those regions are pairwise intersecting. Consider first the convex closures of the constraints. Because of the properties of projection, finding a common point in the corresponding regions amounts to do it for each projection.

That this can be done is merely the fact that, for a finite family of intervals in the time line, pairwise intersection implies having a non empty global intersection[1]. If this non-empty intersection is an interval, we can choose an instantiation avoiding any finite set of values. If it is a point, we have no choice other than taking it. A technical point is to check that this is unharmful: any instantiation meeting the constraints would have to identify the corresponding projections.

Now come back to the initial, pre-convex constraints. From the regions we considered before, some lower dimensional parts have to be avoided in order to meet the stronger constraints. But those parts are defined by extra identifications on some projections, which we can avoid if the projection considered is an interval. If it is not, we can show that the identification it implies is already implied by all atoms in the corresponding pre-convex relation.

Hence we are able to instantiate the new node in a "generic" way. This proves the result. Moreover, it also gives a strategy for finding scenarios for pre-convex networks.

Finding scenarios for pre-convex networks

By definition, a scenario for a constraint network is a consistent subnetwork with atomic labels.

A direct method The preceding method of proof yields the following stategy for constructing a scenario for a path-consistent, pre-convex network.

1. Choose an arc, and consider the network where the corresponding label is replaced by an atom of maximal dimension in it.
2. Apply path-consistency to the reduced network.
3. If the result is not an atomic network, choose an arc which has a non atomic label, and repeat the procedure.

The key point is that the procedure cannot lead to an empty constraint, because each intermediate network is path-consistent and pre-convex, hence consistent.

This strategy can be implemented in $O(m^3)$ time (m is the number of nodes), by using the fact that the calculations only result in shrinking the original network. Hence a given arc can be changed at most $K - 1$ times (removing an atom from the label), if K is the number of atomic (p, q)-relations.

Computing a scenario in quadratic time Checking path-consistency takes cubic time. However, suppose a pre-convex network is given, which is known to be consistent. Then a scenario can be determined in *quadratic* time, as follows.

1. Replace the network by its convex closure (each label is replaced by its convex closure).

[1] This is an elementary case of Helly's theorem (Chvátal 1983).

2. Consider the p time point networks obtained by projecting the convex relations.
3. For each of the point network, apply van Beek's algorithm (van Beek 1990) to get a scenario, i.e. compute the strongly connected components, and apply topological sort.
4. The scenario equivalent to the set of point scenarios is a scenario for the original network.

This method[2] works because of the following remark: If van Beek's algorithm leads to identification of two endpoints, then these endpoints have to coincide in all models of the (looser) convex network. Hence they have to coincide in all models of the initial, tighter, pre-convex network: This means that van Beek's construction leads to a scenario where the resulting atoms are of maximal dimension. As a consequence, it is a scenario for the pre-convex netwwork itself.

Models of the Generalized Interval Calculus

In this section, we consider (p,p)-relations. What are the possible models of the calculus of (p,p)-relations?

Weak and strong representations

A convenient way of looking at models is in terms of (weak) representations (Ligozat 1990). Intuitively, a weak representation is just a set U of elements, which stand for p-intervals, together with the assignment to each atomic (p,p)-relation of a set of pairs (u,v) of elements in U (i.e., a binary relation in U).

To be interpreted as a model, these binary relations should satisfy the axioms corresponding to the algebraic properties of (p,p)-relations. In particular:

- Each pair (u,v) of elements of U belongs to one and only one of the binary relations.
- If (u,w) belongs to the binary relation interpreting a basic relation α, and (w,v) to the binary relation interpreting a basic relation β, then (u,v) belongs to one of the binary relations associated to the elements of $(\alpha \circ \beta)$.

Definition 5 *A* weak representation *of the algebra A_p of (p,p)-relations is a map Φ of A_p into the algebra of subsets of $U \times U$, where U is a non empty set, such that:*

1. *Φ is an homomorphism of Boolean algebras.*
2. *$\Phi(\alpha \circ \beta) contains \Phi(\alpha) \circ \Phi(\beta)$.*
3. *$\Phi(identity) = identity$.*
4. *$\Phi(\alpha^\smile)$ is the transpose of $\Phi(\alpha)$.*

A stronger notion of a model specifies that the axioms embodied in the composition table should be interpreted as necessary and sufficient conditions: namely, if a pair (u,v) belongs to the relation interpreting γ, and if γ can be obtained by composing α and β, then there should exist w in U such that (u,w) is in the interpretation of α, and (w,v) in that of β.

This stronger notion corresponds to the standard notion of a representation in algebra:

A weak representation is a *representation* if it is one-to-one and condition (2) is replaced by:

5. $\Phi(\alpha \circ \beta) = \Phi(\alpha) \circ \Phi(\beta)$.

Example The medical example described by Figure 1 is a weak representation of A_3 with 4 elements A, B, C, and D. The map Φ associates to the $(3,3)$-relation $(2,4,6)$ the pairs (A,B) and (D,A), etc.

Models of the $(1,1)$-calculus

Consider the case of A_1. A weak representation is a non empty set U (interpreted as a set of time points), together with a map Φ. Now the three atoms in A_1 are identity, $<$ and its converse $>$. The definitions imply that a weak representation is determined by a binary relation R on U, such that:

- $U \times U$ is the disjoint union of R, its transpose R^t, and $\Delta = \{(u,u) | u \in U$.
- $R = \Phi(< \circ <)$ contains $R \circ R$.

In other words, the definition boils down to the fact that R is a total order on U. Hence a weak representation is just a total order R on a set U.

A representation of A_1 will have to meet stronger requirements. If fact, it also has to satisfy:

- R is contained in $R \circ R$.
- Δ is contained in $R \circ R^t$ and in $R^t \circ R$.

The first condition expresses the fact that the total order R is *dense*. The second conditions mean that the order is unbounded on the left and on the right.

By a theorem of Cantor, there is only one such order which is countable, up to isomorphism: namely, the total order of rational numbers.

Allen's case

In Allen's case, a weak representation is a set of elements, interpreted as intervals, together with an assignment of one of the 13 atomic relations to each pair of intervals, in such a way that the structural constraints are met.

The problem of determining all representations of Allen's algebra was considered by Ladkin (Ladkin 1987) (under a different name). Ladkin proved, by using quantifier elimination, that the exists only one countable representation of Allen's algebra A_2, namely, the set of intervals in the rationals.

In the next subsection, we indicate why Ladkin's result is a special case of a general result which is true of p-intervals in general.

[2] This is essentially the method used in (Gerevini & Cristani 1997).

The general case

The following result generalizes Ladkin's rsesult to p-intervals, for any $p \geq 1$:

Theorem 1 *Up to isomorphism, the set of p-intervals in the rational numbers constitutes the only countable representation of the algebra of (p,p)-relations.*

The proof (Ligozat 1990) uses a general construction which, given a weak representation (U, Φ), associates to it a weak representation of A_1 (a total order), called its *endpoint structure*.

Example Consider the medical example described by Figure 1. Its associated endpoint structure has 10 elements, corresponding to $a_1, a_2, a_3, \ldots, d_1, d_2, d_3$ (only 10, because $c_2 = d_2$, and $c_3 = d_3$)), with $b_1 < c_1 < \ldots < c_3$.

Conversely, given a total order, the set of p-intervals in the total order, together with the (p,q)-relations induced on them, is a weak representation of A_p.

Example In the medical example described by Figure 1, the weak representation obtained from the total order on 10 endpoints includes the initial 3-intervals A, B, C, and D, but it also includes other 3-intervals, such as (a_1, b_1, c_1), which were not elements in the original structure. The weak representation considered is not closed in the following sense:

Definition 6 *A weak representation of A_p is* closed *if it is isomorphic to the p-intervals on its associated endpoint structure.*

Two weak representations are said to be *equivalent* if their associated endpoint structures are isomorphic. This defines an equivalence relation.

Now the following facts imply the desired result:

- Representations are closed.
- The endpoint structure of a representation is a representation.

To prove the result, consider a countable representation of A_p. Its associated endpoint structure is a countable representation of A_1 (considering the endpoint structure introduces at most p endpoints for each p-interval). Hence, by Cantor's theorem, it is isomorphic to the rational numbers. Since the original representation is closed, it is isomorphic to p-intervals in the rationals.

This expresses the completeness of the theory embodied by the relation algebra: given the strong interpretation of the axioms, there is essentially no other countable model other than the set of p-intervals in the field of rational numbers.

Conclusions

The language of generalized intervals enjoys most of the nice properties of Allen's calculus; it offers a natural language for dealing with complex temporal events; it fits into a well-known algebraic framework; a substantial subset of disjunctive relations, which has a simple geometric characterization, is tractable and can be processed by using constraint propagation techniques. Finally, the model theory associated to this language is well understood (and decidable).

These property make it a good candidate for reasoning about time in many contexts, such as reasoning about processes with interrupts, diagnosis, and reasoning about time in natural language processing.

References

Allen, J. F. 1983. Maintaining knowledge about temporal intervals. *Communications of the ACM* 26(11):832–843.

Chvátal, V. 1983. *Linear Programming.* W. H. Freeman and Company, New York.

Dechter, R.; Meiri, I.; and Pearl, J. 1991. Temporal constraint networks. *Artificial Intelligence* 49.

Freksa, C. 1992. Temporal reasoning based on semi-intervals. *Artificial Intelligence* 54:129–227.

Gerevini, A., and Cristani, M. 1997. On Finding a Solution in Temporal Constraint Satisfaction Problems. In *Proceedings of IJCAI-97*, 1460–1465.

Jónsson, B., and Tarski, A. 1952. Boolean algebras with operators, part II. *American J. of Mathematics* 74:127–162.

Ladkin, P. B. 1986. Time representation: A taxonomy of interval relations. In *Proc. of AAAI-86*, 360–366.

Ladkin, P. 1987. Models of axioms for time intervals. In *Proc. of AAAI-87.*

Ligozat, G. In Anger, F. D., and G; eds., *Proc. of the ECAI-94 Workshop on Spatial and Temporal Reasoning.*

Ligozat, G. 1990. Weak Representations of Interval Algebras. In *Proceedings of AAAI-90*, 715–720.

Ligozat, G. 1991. On generalized interval calculi. In *Proceedings of AAAI-91*, 234–240.

Ligozat, G. 1996. A New Proof of Tractability for ORD-Horn Relations. In *Proc. of AAAI-96*, 395–401.

Nebel, B., and Bürckert, H.-J. 1995. Reasoning about temporal relations: A maximal tractable subclass of Allen's interval algebra. *Journal of the ACM* 42(1):43–66.

Nökel, K. 1988. Convex relations between time intervals. Technical Report SR-88-17, Universität Kaiserslautern.

van Beek, P. 1990. Reasoning about qualitative temporal information. In *Proceedings of AAAI-90*, 728–734.

Vilain, M.; Kautz, H. A.; and van Beek, P. G. 1989. Constraint propagation algorithms for temporal reasoning: A revised report. In Weld, D., and de Kleer, J., eds., *Readings in Qualitative Reasoning about Physical Systems.* Morgan Kaufmann.

Vilain, M. B. 1982. A system for reasoning about time. In *Proceedings of AAAI-82*, 197–201.