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Strong law of large numbers for the capacity of the Wiener sausage in dimension four

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Abstract

We prove a strong law of large numbers for the capacity of a Wiener sausage in dimension four: the capacity divided by its mean converges almost surely to one. We also obtain upper and lower bounds on the mean, and conjecture that the lower bound is sharp.

Keywords and phrases. Capacity, Green kernel, Law of large numbers.
MSC 2010 subject classifications. Primary 60F05, 60G50.

1 Introduction

We denote by $(\beta_s, s \geq 0)$ a Brownian motion on \mathbb{R}^4 , and for $r > 0$ and $0 \leq s \leq t \leq \infty$, let

$$W_r(s, t) = \{z \in \mathbb{R}^4 : |z - \beta_u| \leq r \text{ for some } s \leq u \leq t\}, \quad (1.1)$$

be the Wiener sausage of radius r in the time interval $[s, t]$. Let \mathbb{P}_x and \mathbb{E}_x be the law and expectation with respect to the Brownian motion started at site x , and let G denote Green's function and H_A denote the hitting time of A . The Newtonian capacity of a compact set $A \subset \mathbb{R}^4$ may be defined as

$$\text{Cap}(A) = \lim_{|x| \rightarrow \infty} \frac{\mathbb{P}_x[H_A < +\infty]}{G(x)}. \quad (1.2)$$

Our main result is the following:

Theorem 1.1. *In dimension four, almost surely,*

$$\lim_{t \rightarrow \infty} \frac{\text{Cap}(W_1(0, t))}{\mathbb{E}[\text{Cap}(W_1(0, t))]} = 1. \quad (1.3)$$

Moreover, there exists a constant $C > 0$, such that for all $t \geq 2$,

$$(1 + o(1)) \pi^2 \cdot \frac{t}{\log t} \leq \mathbb{E}[\text{Cap}(W_1(0, t))] \leq C \cdot \frac{t}{\log t}. \quad (1.4)$$

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We conjecture that the lower bound in (1.4) is sharp, based on results of Albeverio and Zhou, see Remark 3.2 below.

From (1.2), the capacity of Wiener sausage is equivalent to the probability that two Wiener sausages intersect. Estimating such a probability has a long tradition: pioneering works were produced by Dvoretzky, Erdős and Kakutani [4] and Aizenman [1]; Aizenman's results have been subsequently improved by Albeverio and Zhou [2], Peres [14], Pemantle, Peres and Shapiro [13] and Khoshnevisan [7] (and references therein). In the discrete setting, the literature is even larger and older, and analogous results are presented in Lawler's comprehensive book [8].

We note that the problem of obtaining a law of large numbers for the capacity of the Wiener sausage has been raised recently by van den Berg, Bolthausen and den Hollander in connection with torsional rigidity [15] – a new geometrical characteristic of the Wiener sausage on a torus.

The proof of (1.3) presents some similarities with the proof in the discrete case, which is given in our companion paper [3], but also some substantial differences. One remarkable feature is that in spite of not having an estimate for $\mathbb{E}[\text{Cap}(W_1(0, t))]$, we obtain an almost sure asymptotics only based on recent large deviation estimates of Erhard and Poisat [5]. In particular the upper bound in (1.4) directly follows from their results. The lower bound in (1.4) on the other hand is obtained via the Paley-Zygmund inequality and standard estimates on the volumes of one, or the intersection of two Wiener sausages.

It may seem odd that the fluctuations result we obtain in our analysis of the discrete model [3] are not directly transposable in the continuous setting. However, it has been noticed some thirty years ago by Le Gall [9] that *it does not seem easy to deduce Wiener sausage estimates from random walks estimates*, and vice-versa. Let us explain one reason for that. The capacity of a set A can be represented as the integral of the *equilibrium measure* of set A , very much as in the discrete formula for the capacity of the range $\mathcal{R}[0, n]$ of a random walk (with obvious notation)

$$\text{Cap}(\mathcal{R}[0, n]) = \sum_{x \in \mathcal{R}[0, n]} \mathbb{P}_x[H_{\mathcal{R}[0, n]}^+ = \infty].$$

Whereas Lawler [8] has established deep non-intersection results for two random walks in dimension four, the corresponding results for the equilibrium measure of $W_1(0, t)$ are still missing.

It is interesting to note that the decomposition formula that we use for the capacity of the union of two sets is of a different nature to the one presented in [3] for the discrete setting. In Section 2.2 we prove the following decomposition formula for the capacity of the union of two sets: for any two compact sets A and B and for any r larger than the inradius of A and B

$$\text{Cap}(A \cup B) = \text{Cap}(A) + \text{Cap}(B) - \chi_r(A, B) - \varepsilon_r(A, B), \quad (1.5)$$

where we use the notation \mathcal{S}_r for the boundary of the ball of radius r , and

$$\chi_r(A, B) = 2\pi^2 r^2 \cdot \frac{1}{|\mathcal{S}_r|} \int_{\mathcal{S}_r} (\mathbb{P}_z[H_A < H_B < \infty] + \mathbb{P}_z[H_B < H_A < \infty]) dz, \quad (1.6)$$

and

$$\varepsilon_r(A, B) = 2\pi^2 r^2 \cdot \frac{1}{|\mathcal{S}_r|} \int_{\mathcal{S}_r} \mathbb{P}_z[H_A = H_B < \infty] dz. \quad (1.7)$$

In particular that $\varepsilon_r(A, B) \leq \text{Cap}(A \cap B)$. The right hand side of (1.5) only involves intersection (or equivalently hitting) probabilities. This makes some notable difference in the way we bound the cross term χ (see Section 2.2 below).

The paper is organised as follows. Section 2 contains preliminary results: in Section 2.1 we gather some well-known facts about Brownian motion and in Section 2.2 we prove (1.5); in Section 2.3 we recall some known results about capacity and some large deviations estimates for the capacity of a Wiener sausage, due to Erhard and Poisat. We then give a corollary concerning intersection probabilities of Wiener sausages. The proof of Theorem 1.1 is given in Section 3: we first prove (1.4) in Section 3.1. Then Section 3.2 deals with the second moment of the cross term χ appearing in the decomposition of the capacity mentioned above, and eventually we conclude the proof in Section 3.3.

2 Preliminaries

2.1 Notation and basic estimates

We write $f \lesssim g$, for two functions f and g , when there exists a constant $C > 0$, such that $f(x) \leq C \cdot g(x)$, for all x and $f \asymp g$ when $f \lesssim g$ and $g \lesssim f$. We denote by \mathbb{P}_z the law of a Brownian motion starting from z , and simply write \mathbb{P} when z is the origin. Likewise $\mathbb{P}_{z,z'}$ will denote the law of two independent Brownian motions starting respectively from z and z' , and similarly for $\mathbb{P}_{z,z',z''}$. For any $x \in \mathbb{R}^4$ and $r > 0$, we denote by $\mathcal{S}(x, r)$ and $\mathcal{B}(x, r)$ respectively the sphere and the ball of radius r centered at x , that we abbreviate in \mathcal{S}_r and \mathcal{B}_r when x is the origin. We write $|A|$ for the Lebesgue measure of a Borel set A . We recall, see Theorem 3.33 in [12], that

$$G(x) = \frac{1}{2\pi^2} \cdot \frac{1}{|x|^2}, \quad (2.1)$$

Recall also, see Corollary 3.19 in [12], that for any $z \in \mathbb{R}^4$, with $|z| > r$,

$$\mathbb{P}_z[H_{\mathcal{B}_r} < \infty] = \frac{r^2}{|z|^2}. \quad (2.2)$$

We will furthermore need the following well-known estimate (see Remark 2.22 in [12]): there exist positive constants c and C , such that for any $t > 0$ and $r > 0$

$$\mathbb{P}\left[\sup_{s \leq t} |\beta_s| > r\right] \leq C \cdot \exp(-cr^2/t). \quad (2.3)$$

2.2 A decomposition formula for the capacity of two sets

For $A \subset \mathbb{R}^4$ define

$$\text{rad}(A) := \sup\{|z| : z \in A\},$$

the inradius of A . If A is a compact subset of \mathbb{R}^4 , then for any $r \geq \text{rad}(A)$, one has:

$$\begin{aligned} \text{Cap}(A) &= \lim_{|x| \rightarrow \infty} \frac{\mathbb{P}_x[H_A < \infty]}{G(x)} = \lim_{|x| \rightarrow \infty} \frac{\mathbb{P}_x[H_{\mathcal{S}_r} < \infty]}{G(x)} \cdot \int_{\mathcal{S}_r} \mathbb{P}_z[H_A < \infty] d\rho_x(z) \\ &= 2\pi^2 r^2 \cdot \frac{1}{|\mathcal{S}_r|} \int_{\mathcal{S}_r} \mathbb{P}_z[H_A < \infty] dz, \end{aligned} \quad (2.4)$$

where ρ_x is the law of the Brownian motion starting from x at time $H_{\mathcal{S}_r}$, conditioned on the event that this hitting time is finite. The second equality above follows from the Markov property, and the last equality just expresses the fact that the harmonic measure from infinity of a ball, which by Theorem 3.46 in [12] is also the weak limit of ρ_x as x goes to infinity, is the uniform measure on the boundary of the ball, by rotational invariance of the law of Brownian motion.

The decomposition (1.5) follows.

2.3 Intersection of Wiener sausages

Our aim in this section is to obtain some bounds on the probability of intersection of two Wiener sausages, or equivalently on the hitting probability of a Wiener sausage. Our first tool for this is the following basic result (see Corollary 8.12 and Theorem 8.27 in [12]):

Lemma 2.1. *Let A be a compact set in \mathbb{R}^4 . Then for any $x \in \mathbb{R}^4 \setminus A$,*

$$\mathbb{P}_x[H_A < \infty] \leq \frac{1}{2\pi^2 d(x, A)^2} \cdot \text{Cap}(A),$$

where $d(x, A) := \inf\{|x - y|^2 : y \in A\}$.

The other estimate we need was proved by Erhard and Poisat, see Equation (5.55) in [5]:

Lemma 2.2 (Erhard–Poisat [5]). *There exist positive constants t_0 , c and R_0 , such that for all $t \geq t_0$ and all $R \geq R_0$,*

$$\mathbb{P}\left[\text{Cap}(W_1(0, t)) \geq R \cdot \frac{t}{\log t}\right] \leq t^{-cR}.$$

Our main estimate on the intersection event is the following. We consider two independent Brownian motions $(\beta_t, t \geq 0)$ and $(\tilde{\beta}_t, t \geq 0)$ starting respectively from 0 and z , and denote their corresponding Wiener sausages by W and \tilde{W} . We want to estimate the probability that $W_{1/2}(0, t)$ intersects $\tilde{W}_{1/2}(0, \infty)$ (or equivalently that $\tilde{\beta}$ ever hit $W_1(0, t)$) when $|z|$ is of order \sqrt{t} up to logarithmic factors.

Proposition 2.3. *For any $\alpha > 0$, there exist positive constants C and t_0 , such that for all $t > t_0$ and $z \in \mathbb{R}^4$, with $t/(\log t)^\alpha \leq |z|^2 \leq t \cdot (\log t)^\alpha$,*

$$\mathbb{P}_{0,z}\left[W_{1/2}(0, t) \cap \tilde{W}_{1/2}(0, \infty) \neq \emptyset\right] \leq C \cdot \left(1 \wedge \frac{t}{|z|^2}\right) \cdot \frac{(\log \log t)^2}{\log t}. \quad (2.5)$$

We divide the proof of Proposition 2.3 into two lemmas. The first one deals with $|z|$ large.

Lemma 2.4. *For any $\alpha > 0$, there exist positive constants C and t_0 , such that for all $t > t_0$ and all $z \in \mathbb{R}^4$ nonzero, with $|z| \leq \sqrt{t} \cdot (\log t)^\alpha$,*

$$\mathbb{P}_{0,z}\left[W_{1/2}(0, t) \cap \tilde{W}_{1/2}(0, \infty) \neq \emptyset\right] \leq C \cdot \frac{t}{|z|^2} \cdot \frac{\log \log t}{\log t}. \quad (2.6)$$

The second lemma improves on Lemma 2.4 in the region $|z|$ small.

Lemma 2.5. *For any $\alpha > 0$, there exist positive constants C and t_0 , such that for all $t > t_0$ and all $z \in \mathbb{R}^4$, with $t \cdot (\log t)^{-\alpha} \leq |z|^2 \leq t$,*

$$\mathbb{P}_{0,z}\left[W_{1/2}(0, t) \cap \tilde{W}_{1/2}(0, \infty) \neq \emptyset\right] \leq C \cdot \frac{(\log \log t)^2}{\log t}. \quad (2.7)$$

Proof of Lemma 2.4. Let $r := \sqrt{t/\log t}$. Assume that $|z| > r$, otherwise there is nothing to prove. Using (2.2), we see that estimating (2.6) amounts to bounding the term

$$\mathbb{P}_{0,z}\left[W_{1/2}(0, t) \cap \tilde{W}_{1/2}(0, \infty) \neq \emptyset, W_1(0, t) \cap \mathcal{B}(z, r) = \emptyset\right].$$

Using Lemma 2.2, it suffices to bound the term

$$\mathbb{P}_{0,z} \left[W_{1/2}(0,t) \cap \widetilde{W}_{1/2}(0,\infty) \neq \emptyset, d(z, W_1(0,t)) \geq r, \text{Cap}(W_1(0,t)) \leq C \frac{t}{\log t} \right],$$

with C some appropriate constant. Now, by first conditioning on $W_1(0,t)$, and then applying Lemma 2.1, we deduce that the latter is upper bounded, up to a constant factor, by

$$\mathbb{E} \left[\frac{\mathbf{1}\{d(z, W_1(0,t)) \geq r\}}{d(z, W_1(0,t))^2} \right] \cdot \frac{t}{\log t}.$$

Furthermore, on the event $\{d(z, W_1(0,t)) \geq r\}$, for t sufficiently large we have

$$\frac{1}{2}d(z, \beta[0,t]) \leq d(z, \beta[0,t]) - 1 \leq d(z, W_1(0,t)) \leq d(z, \beta[0,t]),$$

with $\beta[0,t]$ the trace of β on the time interval $[0,t]$. Now by using again (2.2) and the bound $|z| \leq \sqrt{t}(\log t)^\alpha$, we get

$$\begin{aligned} \mathbb{E} \left[\frac{\mathbf{1}\{d(z, \beta[0,t]) \geq r\}}{d(z, \beta[0,t])^2} \right] &= 2 \int_{1/|z|}^{1/r} u \cdot \mathbb{P}[d(z, \beta[0,t]) \leq 1/u] du \\ &\lesssim \frac{\log(|z|/r)}{|z|^2} \lesssim (\alpha + 1) \frac{\log \log t}{|z|^2}, \end{aligned}$$

which concludes the proof. \square

Proof of Lemma 2.5. Set $t_1 = 0$, $t_2 = |z|^2$ and for $k \geq 3$, denote $t_k = 2t_{k-1}$. Let K be the smallest integer such that $2^{K-1} \geq (\log t)^\alpha$. In particular $t \leq 2^{K-1}|z|^2 = t_{K+1}$ by hypothesis. Then,

$$\mathbb{P}_{0,z} \left[W_{1/2}(0,t) \cap \widetilde{W}_{1/2}(0,\infty) \neq \emptyset \right] \leq \sum_{k=1}^K \mathbb{P}_{0,z} \left[W_{1/2}(t_k, t_{k+1}) \cap \widetilde{W}_{1/2}(0,\infty) \neq \emptyset \right].$$

We now bound each term of the sum on the right hand side. The first one (corresponding to $k = 1$) is bounded using directly Lemma 2.4: for some positive constant C ,

$$\mathbb{P}_{0,z} \left[W_{1/2}(0, |z|^2) \cap \widetilde{W}_{1/2}(0,\infty) \neq \emptyset \right] \leq C \cdot \frac{\log \log t}{\log t}.$$

Now for the other terms, we first observe that for some constant $\kappa > 0$,

$$\mathbb{E} \left[\frac{1}{|\beta_{t_k} - z|^2} \right] \lesssim \frac{1}{t_k^2} \cdot \int \frac{1}{|z-x|^2} e^{-\kappa|x|^2/t_k} dx \lesssim \frac{1}{t_k}. \quad (2.8)$$

Then, we obtain, for some positive constant C ,

$$\begin{aligned} \mathbb{P}_{0,z} \left[W_{1/2}(t_k, t_{k+1}) \cap \widetilde{W}_{1/2}(0,\infty) \neq \emptyset \right] &\leq \mathbb{E} \left[\mathbb{P}_{0,z-\beta_{t_k}} \left[W_{1/2}(0, t_{k+1} - t_k) \cap \widetilde{W}_{1/2}(0,\infty) \neq \emptyset \right] \right] \\ &\leq C \mathbb{E} \left[\frac{1}{|\beta_{t_k} - z|^2} \right] \cdot \frac{t_k \cdot \log \log t}{\log t} \leq C \cdot \frac{\log \log t}{\log t}, \end{aligned}$$

using again Lemma 2.4 and (2.3) at the second line and (2.8) for the third inequality. We conclude the proof recalling that K is of order $\log \log t$. \square

We now prove an estimate on the intersection of a Wiener sausage with two Brownian motions. So we consider this time three independent Brownian motions β , γ and $\tilde{\gamma}$, whose joint law is denoted by $\mathbb{P}_{0,z,z'}$ when they start respectively from 0, z and z' .

Lemma 2.6. *For any $\alpha > 0$ and $c \in (0, 1)$, there exist positive constants C and t_0 , such that for all $t > t_0$ and all $z, z' \in \mathbb{R}^4$, with $\sqrt{t} \cdot (\log t)^{-\alpha} \leq |z|, |z'| \leq \sqrt{t} \cdot (\log t)^\alpha$,*

$$\mathbb{P}_{0,z,z'} [W_1(0, t) \cap \gamma(0, \infty) \neq \emptyset, W_1(0, t) \cap \tilde{\gamma}(0, \infty) \neq \emptyset] \leq C \frac{(\log \log t)^4}{(\log t)^2} (1 \wedge \frac{t}{|z'|^2}) (1 \wedge \frac{t}{|z|^2}). \quad (2.9)$$

Proof. Define the stopping times

$$\sigma := \inf\{s : W_1(0, s) \cap \gamma(0, \infty) \neq \emptyset\}, \quad \text{and} \quad \tilde{\sigma} := \inf\{s : W_1(0, s) \cap \tilde{\gamma}(0, \infty) \neq \emptyset\}.$$

Note that

$$\mathbb{P}_{0,z,z'} [W_1(0, t) \cap \gamma(0, \infty) \neq \emptyset, W_1(0, t) \cap \tilde{\gamma}(0, \infty) \neq \emptyset] = \mathbb{P}_{0,z,z'} [\sigma < \tilde{\sigma} \leq t] + \mathbb{P}_{0,z,z'} [\tilde{\sigma} < \sigma \leq t].$$

By symmetry, we only need to deal with $\mathbb{P}_{0,z,z'} [\sigma < \tilde{\sigma} \leq t]$. Now conditionally on γ , σ is a stopping time for β . In particular, conditionally on σ and β_σ , $W_1(\sigma, t)$ is equal in law to $\beta_\sigma + W'_1(0, t - \sigma)$, with W' a Wiener sausage, independent of everything else. Therefore

$$\begin{aligned} \mathbb{P}_{0,z,z'} [\sigma < \tilde{\sigma} \leq t] &\leq \mathbb{E}_{0,z} [\mathbf{1}\{\sigma \leq t\} \mathbb{P}_{0,z,z'} [\sigma < \tilde{\sigma} \leq t \mid \sigma, \gamma, \beta_\sigma]] \\ &\leq \mathbb{E}_{0,z} [\mathbf{1}\{\sigma \leq t\} \mathbb{P}_{0,z'-\beta_\sigma} [W'_1(0, t - \sigma) \cap \tilde{\gamma}(0, \infty) \neq \emptyset \mid \sigma]] \\ &\leq \mathbb{E}_{0,z} [\mathbf{1}\{\sigma \leq t\} \mathbb{P}_{0,z'-\beta_\sigma} [W'_1(0, t) \cap \tilde{\gamma}(0, \infty) \neq \emptyset]]. \end{aligned}$$

To simplify notation, write $D = |z' - \beta_\sigma|$. Note that one can assume $D > \sqrt{t} \cdot (\log t)^{-3\alpha-1}$, since by using (2.2) and the hypothesis on $|z'|$ we have

$$\mathbb{P}[\sigma \leq t, D \leq \sqrt{t} \cdot (\log t)^{-3\alpha-1}] \leq \frac{t}{|z'|^2 \cdot (\log t)^{6\alpha+2}} \leq (\log t)^{-4\alpha-2},$$

and the right hand side in (2.9) is always larger than $(\log t)^{-4\alpha-2}$ by the hypothesis on z and z' . Then by applying Proposition 2.3 we get

$$\begin{aligned} \mathbb{E}_{0,z} \left[\mathbf{1} \left\{ \sigma \leq t, D > \frac{\sqrt{t}}{(\log t)^{3\alpha+1}} \right\} \mathbb{P}_{0,z'-\beta_\sigma} [W'_1(0, t) \cap \tilde{\gamma}(0, \infty) \neq \emptyset] \right] \\ \lesssim \mathbb{E}_{0,z} \left[\mathbf{1}\{\sigma \leq t\} (1 \wedge \frac{t}{D^2}) \right] \cdot \frac{(\log \log t)^2}{\log t} \\ \lesssim \mathbb{P}_{0,z} [W_1(0, t) \cap \tilde{\gamma}(0, \infty) \neq \emptyset] \cdot (1 \wedge \frac{t}{|z'|^2}) \cdot \frac{(\log \log t)^2}{\log t} + \mathbb{P}_{0,z} [\sigma \leq t, D \leq |z'|/4] \cdot \frac{(\log \log t)^2}{\log t} \\ \lesssim \frac{(\log \log t)^4}{(\log t)^2} \cdot (1 \wedge \frac{t}{|z|^2}) \cdot (1 \wedge \frac{t}{|z'|^2}) + \mathbb{P}_{0,z} [\sigma \leq t, D \leq |z'|/4] \cdot \frac{(\log \log t)^2}{\log t}. \end{aligned}$$

Now define

$$\tau_{z,z'} := \begin{cases} \inf\{s : \beta_s \in \mathcal{B}(z', |z'|/4)\} & \text{if } |z - z'| > |z'|/2 \\ \inf\{s : \beta_s \in \mathcal{B}(z', 3|z'|/4)\} & \text{if } |z - z'| \leq |z'|/2. \end{cases}$$

Note that by construction $|z - \beta_{\tau_{z,z'}}| \geq \max(|z - z'|, |z'|)/4$, and that on the event $\{D \leq |z'|/4\}$, one has $\sigma \geq \tau_{z,z'}$. Therefore by conditioning first on $\tau_{z,z'}$ and the position of β at this time, and then by using Proposition 2.3, we obtain for some constant $\kappa > 0$,

$$\begin{aligned} \mathbb{P}_{0,z} [\sigma \leq t, D \leq |z'|/4] &\leq \mathbb{P}_{0,z} [\tau_{z,z'} \leq \sigma \leq t] \\ &\lesssim (1 \wedge \frac{t}{|z - z'|^2}) \cdot \frac{(\log \log t)^2}{\log t} \cdot \mathbb{P}[\tau_{z,z'} \leq t] \end{aligned}$$

$$\begin{aligned}
&\lesssim (1 \wedge \frac{t}{|z - z'|^2}) \cdot \frac{(\log \log t)^2}{\log t} \cdot e^{-\kappa \cdot |z'|^2/t} \\
&\lesssim (1 \wedge \frac{t}{|z|^2})(1 \wedge \frac{t}{|z'|^2}) \cdot \frac{(\log \log t)^2}{\log t},
\end{aligned}$$

using (2.3) at the third line and considering two cases for the last inequality: $|z'| \geq |z|/2$, in which case we bound the exponential term by the product and $|z'| < |z|/2$, in which case using the triangle inequality gives $|z - z'| \geq |z|/2$. The proof of the lemma now follows. \square

3 Proof of Theorem 1.1

3.1 A first moment estimate

In this section we prove (1.4) that we state as a proposition:

Proposition 3.1. *There exist positive constants t_0 and C , such that for all $t \geq t_0$,*

$$(1 + o(1)) \pi^2 \cdot \frac{t}{\log t} \leq \mathbb{E}[\text{Cap}(W_1(0, t))] \leq C \cdot \frac{t}{\log t}.$$

Proof. The upper bound follows from Lemma 2.2, so we concentrate on the lower bound now. We first give a rough bound on the second moment of $\text{Cap}(W_1(0, t))$. For this we use that if $A \subset B$, then $\text{Cap}(A) \leq \text{Cap}(B)$, so for any compact set A , $\text{Cap}(A) \leq 2\pi^2 \text{rad}(A)^2$, as the capacity of a ball of radius r is equal to $2\pi^2 r^2$. It follows that

$$\mathbb{E}[\text{Cap}(W_1(0, t))^2] \lesssim \mathbb{E}[\text{rad}(W_1(0, t))^4] \lesssim \mathbb{E}\left[\sup_{s \leq t} |\beta_s|^4\right] \lesssim t^2. \quad (3.1)$$

Now we fix $r = \sqrt{t} \cdot \log t$, and using (2.3) and Cauchy-Schwarz we arrive at

$$\mathbb{E}[\text{Cap}(W_1(0, t))] = \mathbb{E}[\text{Cap}(W_1(0, t)) \mathbf{1}\{\text{rad}(W_1(0, t)) \leq r\}] + \mathcal{O}(1).$$

Then using (2.4) and (2.3) again, we obtain

$$\mathbb{E}[\text{Cap}(W_1(0, t))] = 2\pi^2 r^2 \cdot \frac{1}{|\mathcal{S}_r|} \int_{\mathcal{S}_r} \mathbb{P}_{0,z}[H_{W_1(0,t)} < \infty] dz + \mathcal{O}(1). \quad (3.2)$$

So it just amounts now to finding a lower bound for the hitting probabilities of $W_1(0, t)$ when starting at distance r . For this we define

$$Z_t = |W_{1/2}(0, t) \cap \widetilde{W}_{1/2}(0, \infty)|,$$

with \widetilde{W} a Wiener sausage independent of W . Then for any $z \neq 0$,

$$\mathbb{P}_{0,z}[H_{W_1(0,t)} < \infty] = \mathbb{P}_{0,z}[Z_t \neq 0] \geq \frac{\mathbb{E}_{0,z}[Z_t]^2}{\mathbb{E}_{0,z}[Z_t^2]}, \quad (3.3)$$

where we used Paley-Zygmund's inequality. We next have

$$\mathbb{E}_{0,z}[Z_t] = \int \mathbb{P}[x \in W_{1/2}(0, t)] \cdot \mathbb{P}_z[x \in \widetilde{W}_{1/2}(0, \infty)] dx$$

$$\begin{aligned}
&= \int \mathbb{P}[H_{\mathcal{B}(x,1/2)} \leq t] \cdot \mathbb{P}_z[H_{\mathcal{B}(x,1/2)} < \infty] dx \\
&= \int \mathbb{P}[H_{\mathcal{B}(x,1/2)} \leq t] \cdot (1 \wedge \frac{1}{4|z-x|^2}) dx.
\end{aligned}$$

Assume now that $|z| = r = \sqrt{t} \cdot \log t$, and let $r' = \sqrt{t} \cdot \log \log t$. Using (2.2) and (2.3), leads to

$$\begin{aligned}
\mathbb{E}_{0,z}[Z_t] &= \int_{\mathcal{B}(0,r')} \mathbb{P}[H_{\mathcal{B}(x,1/2)} \leq t] \cdot (1 \wedge \frac{1}{4|z-x|^2}) dx + \mathcal{O}(\exp(-c_1(\log \log t)^2)) \\
&= \frac{1 + \mathcal{O}(\frac{\log \log t}{\log t})}{4|z|^2} \int_{\mathcal{B}(0,r')} \mathbb{P}[H_{\mathcal{B}(x,1/2)} \leq t] dx + \mathcal{O}(\exp(-c_1(\log \log t)^2)) \\
&= \frac{1 + \mathcal{O}(\frac{\log \log t}{\log t})}{4|z|^2} \mathbb{E}[|W_{1/2}(0,t)|] + \mathcal{O}(\exp(-c_1(\log \log t)^2)) \\
&\sim \frac{\pi^2}{8} \cdot \frac{1}{(\log t)^2},
\end{aligned} \tag{3.4}$$

where c_1 is a positive constant and for the last line we use the classical result of Kesten, Spitzer, and Whitman on the volume of the Wiener sausage, see e.g. [10] or [11] and references therein, which implies:

$$\lim_{t \rightarrow \infty} \frac{1}{t} \cdot \mathbb{E}[|W_{1/2}(0,t)|] = \text{Cap}(\mathcal{B}_{1/2}) = \frac{\pi^2}{2}. \tag{3.5}$$

It remains to bound the second moment of Z_t . To this end we will need Gettoor's result [6] on the volume of the intersection of two Wiener sausages, see also [10] or [11], which implies

$$\lim_{t \rightarrow \infty} \frac{1}{\log t} \cdot \mathbb{E}[|W_{1/2}(0,t) \cap \widetilde{W}_{1/2}(0,\infty)|] = \frac{\text{Cap}(\mathcal{B}_{1/2})^2}{4\pi^2} = \frac{\pi^2}{16}. \tag{3.6}$$

To simplify notation write $\tau_x = H_{\mathcal{B}(x,1/2)}$. Note that for any $x \neq x'$, one has a.s. $\tau_x \neq \tau_{x'}$, since, even when $|x - x'| < 1$, for the event $\{\tau_x = \tau_{x'}\}$ to hold, the Brownian motion has to hit a two-dimensional sphere which is a polar set, by Kakutani's theorem (see [12, Theorem 8.20]). Observe also that the function $\mathbb{P}_x[\tau_{x'} < \infty]$ is symmetric in x and x' . Therefore,

$$\begin{aligned}
\mathbb{E}_{0,z}[Z_t^2] &= \int \int \mathbb{P}[\tau_x \leq t, \tau_{x'} \leq t] \cdot \mathbb{P}_z[\tau_x < \infty, \tau_{x'} < \infty] dx dx' \\
&= 2 \int \int \mathbb{P}[\tau_x \leq t, \tau_{x'} \leq t] \cdot \mathbb{P}_z[\tau_x < \tau_{x'} < \infty] dx dx' \\
&\leq \frac{1}{2} \int \frac{dx}{|z-x|^2} \int \mathbb{P}[\tau_x \leq t, \tau_{x'} \leq t] \cdot \mathbb{P}_x[\tau_{x'} < \infty] dx' \\
&= \int \frac{dx}{|z-x|^2} \int \mathbb{P}[\tau_x < \tau_{x'} \leq t] \cdot \mathbb{P}_x[\tau_{x'} < \infty] dx' \\
&\leq \int \frac{dx}{|z-x|^2} \mathbb{E} \left[\mathbf{1}\{\tau_x \leq t\} \cdot \int \mathbb{P}_x[\tau_{x'} \leq t - \tau_x \mid \tau_x] \cdot \mathbb{P}_x[\tau_{x'} < \infty] dx' \right] \\
&= \int \mathbb{E} \left[\mathbf{1}\{\tau_x \leq t\} \cdot \mathbb{E}[Z_{t-\tau_x} \mid \tau_x] \right] \frac{dx}{|z-x|^2}.
\end{aligned} \tag{3.7}$$

Now assume that $|z| = r = \sqrt{t} \cdot \log t$. Note that (3.5) and (3.6) give respectively

$$\int \mathbb{P}[t - t/\log t \leq \tau_x \leq t] dx \leq \mathbb{E}[|W_{1/2}(0, t/\log t)|] \lesssim \frac{t}{\log t},$$

and

$$\mathbb{E}[Z_{t-\tau_x} | \tau_x] \leq \mathbb{E}[Z_t] \sim \frac{\pi^2}{16} \log t.$$

Therefore by cutting the last integral in (3.7) in two pieces, one on the set $\{|x| > r'\}$, and the other one on $\{|x| < r'\}$, we arrive, similarly as for the first moment computation, to

$$\begin{aligned} \mathbb{E}_{0,z}[Z_t^2] &\leq \frac{1+o(1)}{t \cdot (\log t)^2} \cdot \int \mathbb{E} \left[\mathbf{1}\{\tau_x \leq t - t/\log t\} \cdot \mathbb{E}[Z_{t-\tau_x} | \tau_x] \right] dx \\ &\leq \frac{\pi^2}{16} \cdot \frac{1+o(1)}{t \cdot (\log t)^2} \int \mathbb{E} \left[\mathbf{1}\{\tau_x \leq t - t/\log t\} \cdot \log(t - \tau_x) \right] dx \\ &\leq \frac{\pi^4}{32} \cdot \frac{1+o(1)}{\log t}. \end{aligned} \tag{3.8}$$

Now (3.2), (3.3), (3.4) and (3.8) finish the proof of the lower bound. \square

Remark 3.2. We believe that the lower bound is sharp in Proposition 3.1. Indeed Albeverio and Zhou [2] have shown that for any positive function $t \mapsto a(t)$ converging to infinity and such that $\log(1 + t/a(t))/\log t$ goes to 0,

$$\mathbb{P}[\widetilde{W}_{1/2}(0, \infty) \cap W_{1/2}(a(t), a(t) + t) \neq \emptyset] \sim \frac{1}{2} \log\left(1 + \frac{t}{a(t)}\right) \cdot (\log t)^{-1}.$$

Now if we could apply this result by replacing $a(t)$ with the stopping time $\tau_t := \inf\{s : |\beta_s| = \sqrt{t} \cdot \log t\}$, whose mean is equal to $t(\log t)^2$, then using this estimate in (3.2) would give a matching upper bound. A problem with this argument is that $\tau_t/t(\log t)^2$ is not concentrated, and is equal in law to τ_1 by scaling.

3.2 A second moment estimate

Here we estimate the second moment of the crossing term χ from the decomposition (1.5).

Proposition 3.3. *Let β and $\widetilde{\beta}$ be two independent Brownian motions and let W and \widetilde{W} be their corresponding Wiener sausages. Then with $r(t) = \sqrt{t} \cdot \log t$,*

$$\mathbb{E} \left[\chi_{r(t)}(W_1(0, t), \widetilde{W}_1(0, t))^2 \mathbf{1}\{\text{rad}(W_1(0, t)) \leq r(t), \text{rad}(\widetilde{W}_1(0, t)) \leq r(t)\} \right] = \mathcal{O} \left(\frac{t^2 (\log \log t)^8}{(\log t)^4} \right).$$

Proof of Proposition 3.3. First note that for any compact sets A and B and any r larger than the inradius of A and B , we can upper bound $\chi_r(A, B)^2$ by

$$\begin{aligned} \chi_r(A, B)^2 &\lesssim \frac{r^4}{|\mathcal{S}_r|^2} \int_{\mathcal{S}_r \times \mathcal{S}_r} \left(\mathbb{P}_{z,z'}[H_A < H_B < \infty, \widetilde{H}_A < \widetilde{H}_B < \infty] \right. \\ &+ \mathbb{P}_{z,z'}[H_B < H_A < \infty, \widetilde{H}_B < \widetilde{H}_A < \infty] + \mathbb{P}_{z,z'}[H_A < H_B < \infty, \widetilde{H}_B < \widetilde{H}_A < \infty] \\ &\left. + \mathbb{P}_{z,z'}[H_B < H_A < \infty, \widetilde{H}_A < \widetilde{H}_B < \infty] \right) dz dz', \end{aligned} \tag{3.9}$$

where H and \widetilde{H} refer to the hitting times of two independent Brownian motions γ and $\widetilde{\gamma}$ starting respectively from z and z' in \mathcal{S}_r . To simplify notation, denote by $A = W_1(0, t)$, $B = \widetilde{W}_1(0, t)$, and $r = r(t)$. By using (2.2), we obtain

$$\mathbb{P}_{z,z'}[H_A < H_B < \infty, \widetilde{H}_A < \widetilde{H}_B < \infty]$$

$$= \mathbb{P}_{z,z'} \left[H_A < H_B < \infty, \tilde{H}_A < \tilde{H}_B < \infty, H_{\mathcal{B}_{\sqrt{t} \cdot (\log t)^{-3}}} = \infty, \tilde{H}_{\mathcal{B}_{\sqrt{t} \cdot (\log t)^{-3}}} = \infty \right] + \mathcal{O} \left(\frac{1}{(\log t)^8} \right).$$

Now, to bound the probability on the right-hand side, we use the Markov property at times H_A and \tilde{H}_A for γ and $\tilde{\gamma}$ respectively. We then have using Lemma 2.6 twice

$$\begin{aligned} \mathbb{P}_{z,z'} [H_A < H_B < \infty, \tilde{H}_A < \tilde{H}_B < \infty] &\lesssim \mathbb{P}_{z,z'} \left[H_A < \infty, \tilde{H}_A < \infty \right] \cdot \frac{(\log \log t)^4}{(\log t)^2} + \frac{1}{(\log t)^8} \\ &\lesssim (1 \wedge \frac{t}{|z'|^2}) \cdot (1 \wedge \frac{t}{|z|^2}) \frac{(\log \log t)^8}{(\log t)^4} \\ &\lesssim \frac{(\log \log t)^8}{(\log t)^8}. \end{aligned} \quad (3.10)$$

By symmetry, we get as well

$$\mathbb{P}_{z,z'} [H_B < H_A < \infty, \tilde{H}_B < \tilde{H}_A < \infty] = \mathcal{O} \left(\frac{(\log \log t)^8}{(\log t)^8} \right). \quad (3.11)$$

Now to bound the last two terms in (3.9), we can first condition on $A = W_1(0, t)$ and $B = \tilde{W}_1(0, t)$, and then using the inequality $ab \leq a^2 + b^2$ for $a, b > 0$, together with (3.10) and (3.11), this gives

$$\begin{aligned} \mathbb{P}_{z,z'} [H_A < H_B < \infty, \tilde{H}_B < \tilde{H}_A < \infty] &\leq \mathbb{P}_{z,z} [H_A < H_B < \infty, \tilde{H}_A < \tilde{H}_B < \infty] \\ &\quad + \mathbb{P}_{z',z'} [H_B < H_A < \infty, \tilde{H}_B < \tilde{H}_A < \infty] \\ &= \mathcal{O} \left(\frac{(\log \log t)^8}{(\log t)^8} \right). \end{aligned} \quad (3.12)$$

By symmetry it also gives

$$\mathbb{P}_{z,z'} [H_{\tilde{W}_1(0,t)} < H_{W_1(0,t)} < \infty, \tilde{H}_{W_1(0,t)} < \tilde{H}_{\tilde{W}_1(0,t)} < \infty] = \mathcal{O} \left(\frac{(\log \log t)^8}{(\log t)^8} \right). \quad (3.13)$$

Then the proof follows from (3.9), (3.10), (3.11), (3.12), and (3.13). \square

3.3 Law of large numbers

The final step of the proof of Theorem 1.1 is entirely similar to the discrete case [3]. The only notable difference is that in the discrete case, to bound the error term ε , we used that the capacity of any set is bounded by its size. In the continuous setting this is no longer true, but one has the following

Lemma 3.4. *Let \tilde{W} be an independent copy of W . One has a.s. for all $t > 0$,*

$$\text{Cap}(W_1(0, t) \cap \tilde{W}_1(0, t)) \leq C \cdot |W_3(0, t) \cap \tilde{W}_3(0, t)|,$$

with $C = \text{Cap}(\mathcal{B}(0, 3))/|\mathcal{B}(0, 1)|$.

Proof. Let $(\mathcal{B}(x_i, 1))_{i \leq M}$ be a finite covering of $W_1(0, t) \cap \tilde{W}_1(0, t)$ by balls of radius one whose centers are all assumed to belong to say $\beta[0, t]$, the trace of the Brownian motion driving $W_1(0, t)$. Then by removing one by one some balls if necessary, one can obtain a sequence of disjoint balls $(\mathcal{B}(x_i, 1))_{i \leq M'}$, such that the enlarged balls $(\mathcal{B}(x_i, 3))_{i \leq M'}$ cover the set $W_1(0, t) \cap \tilde{W}_1(0, t)$, and such

that all of them intersect $W_1(0, t) \cap \widetilde{W}_1(0, t)$. But since the centers (x_i) also belong to $\beta[0, t]$, all the balls $\mathcal{B}(x_i, 1)$ belong to the enlarged intersection $W_3(0, t) \cap \widetilde{W}_3(0, t)$. So one has on one hand

$$\text{Cap}(W_1(0, t) \cap \widetilde{W}_1(0, t)) \leq M' \cdot \text{Cap}(\mathcal{B}(0, 3)),$$

since for any A and B one has $\text{Cap}(A \cup B) \leq \text{Cap}(A) + \text{Cap}(B)$, and on the other hand

$$|\cup_{i=1}^{M'} \mathcal{B}(x_i, 1)| = M' |\mathcal{B}(0, 1)| \leq |W_3(0, t) \cap \widetilde{W}_3(0, t)|,$$

since by construction the balls $\mathcal{B}(x_i, 1)$, $i \leq M'$, are disjoint. This proves the lemma. \square

One can now conclude the proof. For the convenience of the reader we recall the argument from [3]:

Proof of Theorem 1.1. Let L be such that $(\log t)^4 \leq 2^L \leq 2(\log t)^4$, so that $L \asymp \log \log t$. Let $r = \sqrt{t} \cdot \log t$, and let E be the event $\{\text{rad}(W_1(0, t)) \leq r\}$. On this event, by applying successively the decomposition formula (2.4) we obtain

$$\text{Cap}(W_1(0, t)) = \sum_{i=1}^{2^L} X_{i,L}(t) - \sum_{\ell=1}^L \sum_{i=1}^{2^{\ell-1}} \chi_{i,\ell}(t) + \varepsilon(t),$$

where $\varepsilon(t)$ is a sum of the ε_r -terms appearing in (1.7),

$$X_{i,L}(t) = \text{Cap}\left(W_1((i-1)2^{-L}t, i2^{-L}t)\right),$$

and

$$\chi_{i,\ell}(t) = \chi_r\left(W_1((2i-2)2^{-\ell}t, (2i-1)2^{-\ell}t), W_1((2i-1)2^{-\ell}t, i2^{-\ell+1}t)\right),$$

with χ_r given by (1.6). Note that χ_r is defined independently of the fact that E holds or not. Thus the $(X_{i,L})_i$ are i.i.d., and for any fixed $\ell \leq L$, the $(\chi_{i,\ell}(t))_i$ also.

Recall that by using (2.3) and Proposition 3.3, we know that

$$\mathbb{E}[\chi_{i,\ell}(t)^2] = \mathcal{O}\left(\frac{t^2 \cdot (\log \log t)^8}{2^{2\ell} (\log t)^4}\right), \quad (3.14)$$

for all $i \leq 2^{\ell-1}$. Therefore, letting

$$\chi_L(t) := \sum_{\ell=1}^L \sum_{i=1}^{2^{\ell-1}} \chi_{i,\ell}(t),$$

we get using the triangle inequality for the L^2 -norm and the Cauchy-Schwarz inequality,

$$\text{Var}(\chi_L(t)) \leq L \cdot \sum_{\ell=1}^L 2^{\ell-1} \text{Var}(\chi_{1,\ell}(t)) = \mathcal{O}\left(t^2 \cdot \frac{(\log \log t)^9}{(\log t)^4}\right).$$

Now, fix some $\varepsilon > 0$. Using Chebyshev's inequality we deduce from the previous bound that

$$\mathbb{P}\left[|\chi_L(t) - \mathbb{E}[\chi_L(t)]| \geq \varepsilon \frac{t}{\log t}\right] \lesssim \frac{(\log \log t)^9}{(\log t)^2}. \quad (3.15)$$

Next, using (3.1), and Chebyshev's inequality again gives

$$\mathbb{P} \left[\left| \sum_{i=1}^{2^L} (X_{i,L}(t) - \mathbb{E}[X_{i,L}(t)]) \right| \geq \varepsilon \frac{t}{\log t} \right] = \mathcal{O} \left(\frac{1}{(\log t)^2} \right). \quad (3.16)$$

Then to finish the proof it remains to control the mean of $\varepsilon(t)$. By combining Lemma 3.4 with (3.6), we deduce that

$$\mathbb{E} \left[\text{Cap}(W_1(0, t) \cap \widetilde{W}_1(0, t)) \right] = \mathcal{O}(\log t).$$

So by the definition of $\varepsilon(t)$ as a sum of the ε_r -terms appearing in (1.7), we obtain

$$\mathbb{E}[\varepsilon(t)] = \mathcal{O}(2^{L+1} \log t) = \mathcal{O}((\log t)^5).$$

A first consequence is that by Markov's inequality

$$\mathbb{P} \left[\varepsilon(t) \geq \varepsilon \frac{t}{\log t} \right] \lesssim \frac{(\log t)^6}{t}, \quad (3.17)$$

and another consequence is that

$$\mathbb{E}[\text{Cap}(W_1(0, t))] \sim \sum_{i=1}^{2^L} \mathbb{E}[X_{i,L}(t)] - \mathbb{E}[\chi_L(t)]. \quad (3.18)$$

Combining (3.15), (3.16), (3.17) and (3.18), we obtain

$$\mathbb{P} \left[|\text{Cap}(W_1(0, t)) - \mathbb{E}[\text{Cap}(W_1(0, t))]| \geq 4\varepsilon \frac{t}{\log t} \right] = \mathcal{O} \left(\frac{(\log \log t)^9}{(\log t)^2} \right).$$

Now consider the sequence $a_n = \exp(n^{3/4})$. Since the previous bound holds for all $\varepsilon > 0$, by using Borel-Cantelli's lemma and Proposition 3.1, we deduce that a.s.

$$\lim_{n \rightarrow \infty} \frac{\text{Cap}(W_1(0, a_n))}{\mathbb{E}[\text{Cap}(W_1(0, a_n))]} = 1.$$

Let now $t > 0$, and choose $n = n(t) > 0$, so that $a_n \leq t < a_{n+1}$. Using that the map $t \mapsto \text{Cap}(W_1(0, t))$ is a.s. nondecreasing (since for any sets $A \subset B$, one has $\text{Cap}(A) \leq \text{Cap}(B)$), we can write

$$\frac{\text{Cap}(W_1(0, a_n))}{\mathbb{E}[\text{Cap}(W_1(0, a_{n+1}))]} \leq \frac{\text{Cap}(W_1(0, t))}{\mathbb{E}[\text{Cap}(W_1(0, t))]} \leq \frac{\text{Cap}(W_1(0, a_{n+1}))}{\mathbb{E}[\text{Cap}(W_1(0, a_n))]},$$

and the proof of the strong law of large numbers follows once we observe that

$$\mathbb{E}[\text{Cap}(W_1(a_n, a_{n+1}))] \asymp \frac{a_{n+1} - a_n}{\log(a_{n+1} - a_n)} = o \left(\frac{a_n}{\log a_n} \right),$$

and use that for any sets A and B one has $\text{Cap}(A) \leq \text{Cap}(A \cup B) \leq \text{Cap}(A) + \text{Cap}(B)$. \square

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