Flag-approximability of convex bodies and volume growth of Hilbert geometries

Constantin Vernicos, Cormac Walsh

FLAG-APPROXIMABILITY OF CONVEX BODIES AND VOLUME GROWTH OF HILBERT GEOMETRIES

CONSTANTIN VERNICOS AND CORMAC WALSH

Abstract. We show that the volume entropy of a Hilbert geometry on a convex body is exactly twice the flag-approximability of the body. We then show that both of these quantities are maximized in the case of the Euclidean ball.

We also compute explicitly the asymptotic volume of a convex polytope, which allows us to prove that simplices have the least asymptotic volume, as was conjectured by the first author.

Introduction

It is possible to define various topologies on the set of convex bodies. The most commonly used are induced by a distance and have the property that the subset of convex polytopes is dense. A standard question is to know, given a convex body, how complex a polytope has to be in order to approximate it well. One may for example want to approximate to within ϵ in the Hausdorff distance, and one may use the number of vertices to measure the complexity of a polytope. Schneider and Wieacker [12] defined the approximability of a convex body to be, roughly speaking, the power in $1/\epsilon$ by which the complexity grows as ϵ tends to zero.

The present paper follows on from the first author's paper [16], in which it was shown that in dimension two and three the approximability of a convex body is equal to the exactly half the volume entropy of the Hilbert geometry on the convex body. In higher dimension the author was only able to show that the former is less than or equal to the latter. Motivation for this result was to prove the entropy upper bound conjecture, which states that the volume entropy of every convex body is no greater than $d-1$. This would follow from the equality of the approximability and half the volume entropy, using the well known result, proved by Fejes–Toth [14] in dimension two and by Bronshteyn–Ivanov [6] in the general case, that the approximability of any convex body is no greater than $(d-1)/2$.

A slight change of perspective proves to be fruitful. Instead of approximating the convex body with a polytope having the least number of vertices, we approximate with one having the least number of *maximal flags*. We introduce the *flag-approximability* of a convex body as follows. Let $N_{\mathrm{f}}(\epsilon,\Omega)$ be the least number of maximal flags of a polytope whose Hausdorff distance to Ω is less than $\epsilon>0$. Then the (upper) flag approximability of Ω is defined to be

$$a_{\mathrm{f}}(\Omega) := \limsup_{\epsilon\to 0} \frac{\log N_{\mathrm{f}}(\epsilon,\Omega)}{-\log \epsilon}.$$

This is analogous to how Schneider and Wieacker [12] defined the (vertex) approximability, where the least number of vertices was used instead of the least number of flags.

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Let $\Omega \subset \mathbb{R}^d$ be a bounded open convex set. For any $p \in \Omega$ and $R > 0$, denote by $B_\Omega(p, R)$ closed ball in the Hilbert geometry centered at p of radius R. Let $\mathrm{Vol}^{\mathrm{H}}$ denote the Holmes–Thompson volume. The (upper) volume entropy of the Hilbert geometry on Ω is defined to be

$$\mathrm{Ent}(\Omega) := \limsup_{R\to\infty} \frac{\log \mathrm{Vol}^{\mathrm{H}}{}_{\Omega}\big(B_\Omega(p, R)\big)}{R}.$$

Observe that this does not depend on the base point p. Moreover, neither does it change if one takes instead the Busemann volume.

We can also define the lower flag approximability and the lower volume entropy taking infimum limits instead of supremum ones. Although the two entropies do not generally coincide, as shown by the first author in [16], all our results and proofs hold when replacing lim sup by lim inf.

Theorem 1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open convex set. Then,*

$$\mathrm{Ent}(\Omega) = 2a_{\mathrm{f}}(\Omega).$$

We show, using a slight modification of the technique in Arya–da Fonseca–Mount [2], that the Bronshteyn–Ivanov [6] bound on the (vertex) approximability also holds for the flag approximability.

Theorem 2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open convex set. Then*

$$a_{\mathrm{f}}(\Omega) \leq (d-1)/2.$$

It remains an open question whether the flag-approximability is the same as the vertex-approximability.

From Theorems 1 and 2, we deduce the following corollary. This result was also proved recently by N. Tholozan [13] using a different method.

Corollary 3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open convex set. Then*

$$\mathrm{Ent}(\Omega) \leq d - 1.$$

For many Hilbert geometries, such as hyperbolic space, the volume of balls grows exponentially. However, for some Hilbert geometries, the volume grows only polynomially. In this case it is useful to make the following definition. Fix some notion of volume Vol. The *asymptotic volume* of the Hilbert geometry on Ω is defined to be

$$\mathrm{Asvol}(\Omega) := \limsup_{R\to\infty} \frac{\mathrm{Vol}(B(x, R))}{R^d}.$$

Note that, unlike in the case of the volume entropy, the asymptotic volume depends on the choice of volume.

The first author has shown in [15] that the asymptotic volume of a convex body is finite if and only if the body is a polytope.

In the next theorem, we again see a connection appearing between volume in Hilbert geometries and the number of flags.

We denote by $\mathrm{Flags}(\mathcal{P})$ the set of maximal flags of a polytope $\mathcal{P}$. Let Σ be a simplex of dimension d. Observe that $\mathrm{Flags}(\Sigma)$ consists of $(d + 1)!$ elements.

Theorem 4. *Let $\mathcal{P}$ be a convex polytope of dimension d, and fix some notion of volume $\mathrm{Vol}_{\mathcal{P}}$. Then,*

$$\mathrm{Asvol}(\mathcal{P}) = \frac{|\mathrm{Flags}(\mathcal{P})|}{(d+1)!}\,\mathrm{Asvol}(\Sigma).$$

An immediate consequence is that the simplex has the smallest asymptotic volume among all convex bodies. This was conjectured in [15].

Corollary 5. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open convex set. Then,*

$$\operatorname{Asvol}(\Omega) \geq \operatorname{Asvol}(\Sigma),$$

with equality if and only if Ω is a simplex.

Another corollary is the following result, proved originally by Foertsch and Karlsson [8].

Corollary 6. *If a Hilbert geometry on a bounded open convex set Ω is isometric to a finite-dimensional normed space, then Ω is a simplex.*

1. Preliminaries

A *proper* open set in $\mathbb{R}^d$ is an open set not containing a whole line. A non-empty proper open convex set will be called a *convex domain*. The closure of a bounded convex domain is usually called a *convex body*.

A convex body will be said to be in *canonical form* if it is contained within an ellipsoid $\mathcal{E}$ centered at the origin, and it contains $\frac{1}{2d}\mathcal{E}$. In other words it is inside the unit ball of some Euclidean metric, and it contains the ball of radius $1/2d$ of that same Euclidean metric.

1.1. Hilbert geometries.

A Hilbert geometry (Ω, d_Ω) is a convex domain Ω in $\mathbb{R}^d$ with the Hilbert distance d_Ω defined as follows. For any distinct points p and q in Ω, the line passing through p and q meets the boundary $\partial\Omega$ of Ω at two points a and b, labeled so that the line passes consecutively through a, p, q, and b. We define

$$d_\Omega(p,q) := \frac{1}{2}\log[a,p,q,b],$$

where $[a,p,q,b]$ is the cross ratio of (a,p,q,b), that is,

$$[a,p,q,b] := \frac{|qa|}{|pa|}\frac{|pb|}{|qb|} > 1,$$

with $|xy|$ denoting the Euclidean distance between x and y in $\mathbb{R}^d$. If either a or b is at infinity, the corresponding ratio is taken to be 1.

Note that the invariance of the cross ratio by a projective map implies the invariance of d_Ω by such a map. In particular, since any convex domain is projectively equivalent to a bounded convex domain, most of our proofs will reduce to that case without loss of generality.

1.2. The Holmes–Thompson and the Busemann volumes.

Hilbert geometries are naturally endowed with a C^0 Finsler metric F_Ω as follows. If $p \in \Omega$ and $v \in T_p\Omega = \mathbb{R}^d$ with $v \neq 0$, the straight line passing through p and directed by v meets $\partial\Omega$ at two points p_Ω^+ and p_Ω^- . Let t^+ and t^- be two positive numbers such that $p + t^+v = p_\Omega^+$ and $p - t^-v = p_\Omega^-$. These numbers correspond to the time necessary to reach the boundary starting at p with velocities v and $-v$, respectively. We define

$$F_\Omega(p,v) = \frac{1}{2}\left(\frac{1}{t^+} + \frac{1}{t^-}\right) \quad \text{and} \quad F_\Omega(p,0) = 0.$$

Should p_Ω^+ or p_Ω^- be at infinity, the corresponding ratio will be taken to be 0.

The Hilbert distance d_Ω is the distance induced by F_Ω. We shall denote by $B_\Omega(p,r)$ the metric ball of radius r centered at the point $p \in \Omega$, and by $S_\Omega(p,r)$ the corresponding metric sphere.

From the Finsler metric, we can construct two important Borel measures on Ω.

The first is called the *Busemann* volume and is denoted by ${\mathrm{Vol}^{\mathrm{B}}}_{\Omega}$. It is actually the Hausdorff measure associated to the metric space (Ω, d_Ω); see [7], example 5.5.13. It is defined as follows. For any $p \in \Omega$, let

$$\beta_\Omega(p) = \{v \in \mathbb{R}^d \mid F_\Omega(p, v) < 1\}$$

be the open unit ball in $T_p\Omega = \mathbb{R}^d$ of the norm $F_\Omega(p, \cdot)$, and let ω_d be the Euclidean volume of the open unit ball of the standard Euclidean space $\mathbb{R}^d$. Consider the (density) function $h^B_\Omega \colon \Omega \longrightarrow \mathbb{R}$ given by $h^B_\Omega(p) := \omega_d / \operatorname{Leb}\bigl[\beta_\Omega(p)\bigr]$, where Leb is the canonical Lebesgue measure of $\mathbb{R}^d$, equal to 1 on the unit "hypercube". Then for any Borel set A in Ω,

$${\mathrm{Vol}^{\mathrm{B}}}_{\Omega}(A) := \int_A h^B_\Omega(p) \,\mathrm{d}\operatorname{Leb}(p).$$

The second, called the *Holmes–Thompson* volume, will be denoted by ${\mathrm{Vol}^{\mathrm{H}}}_{\Omega}$, and is defined as follows. Let $\beta^*_\Omega(p)$ be the polar dual of $\beta_\Omega(p)$, and let $h_\Omega \colon \Omega \longrightarrow \mathbb{R}$ be the density defined by $h_\Omega(p) := \operatorname{Leb}\bigl[\beta^*_\Omega(p)\bigr]/\omega_d$. Then ${\mathrm{Vol}^{\mathrm{H}}}_{\Omega}$ is the measure associated to this density.

In what follows, we will denote by Area_Ω and Area^B_Ω, respectively, the $d-1$-dimensional measures associated to the Holmes–Thompson and Busemann measures.

Lemma 7 (Monotonicity of the Holmes–Thompson measure)**.** *Let (Ω, d_Ω) be a Hilbert geometry in $\mathbb{R}^d$. The Holmes–Thompson area measure is monotonic on the set of convex bodies in Ω, that is, for any pair of convex bodies K_1 and K_2 in Ω, such that $K_1 \subset K_2$, one has*

$$\mathrm{Area}_\Omega(\partial K_1) \leq \mathrm{Area}_\Omega(\partial K_2). \tag{1}$$

Proof. If $\partial\Omega$ is C^2 with everywhere positive Gaussian curvature, then the tangent unit spheres of the Finsler metric are quadratically convex.

According to Álvarez Paiva and Fernandes [1, Theorem 1.1 and Remark 2], there exists a Crofton formula for the Holmes–Thompson area, from which inequality (1) follows.

Such smooth convex bodies are dense in the set of all convex bodies in the Hausdorff topology. By approximation, it follows that inequality (1) is valid for any Ω. □

The next result was essentially proved in [4, Lemma 2.13].

Lemma 8 (Co-area inequalities)**.** *Let Ω be a Hilbert geometry, with base-point o, and let L be a cone with apex o. Then, for some constant $C > 1$ depending only on the dimension d,*

$$\frac{1}{C}\,\mathrm{Area}^B\bigl(S(R) \cap L\bigr) \leq \frac{d}{dR}\,\mathrm{Vol}^{\mathrm{B}}\bigl(B(R) \cap L\bigr) \leq C\,\mathrm{Area}^B\bigl(S(R) \cap L\bigr),$$

for all $R \geq 0$.

The results presented in this paper are actually mostly independent of the definition of volume chosen; what really matters is that the volume one uses satisfies the following properties: continuity with respect to the Hausdorff pointed topology, monotony with respect to inclusion, and invariance under projective transformations. As a normalisation, we furthermore ask that the volume coincides with the standard one in the case of an ellipsoid (see [16] for more details).

1.3. **Asymptotic balls.** Let Ω be a bounded open convex set. For each $R \geq 0$ and $y \in \Omega$, we call the dilation of Ω about y by a factor $1 - \exp(-2R)$ the *asymptotic ball* of radius R about y, and we denote it by

$$\mathrm{AsB}_\Omega(y, R) := y + (1 - e^{-2R})(\Omega - y).$$

Some authors dilate by a factor $\tanh R$ instead, but there is very little difference when R is large. By convention, we take $\mathrm{AsB}_\Omega(y, R)$ to be empty if $R \leq 0$.

The following lemma shows the close connection between asymptotic balls and the balls of the Hilbert geometry.

Lemma 9. *Let Ω be a bounded open convex set, containing a point y. Assume that Ω contains the Euclidean ball of radius $l > 0$ about x, and is contained in the Euclidean ball of radius $L > 0$ about x. Then for all $R > 0$ we have*

$$\mathrm{AsB}_\Omega\Big(y, R - \frac{1}{2}\log\big(1 + \frac{L}{l}\big)\Big) \subset B_\Omega(y, R) \subset \mathrm{AsB}_\Omega(y, R).$$

Proof. Let $x \in \Omega$, and let w and z be the points in the boundary of Ω that are collinear with x and y, labelled so that w, x, y, and z lie in this order. Observe that $|xy| \leq L$ and $|yz| \geq l$. Therefore,

$$1 \leq \frac{|xz|}{|yz|} = 1 + \frac{|xy|}{|yz|} \leq 1 + \frac{L}{l}.$$

We may write the ball as

$$B_\Omega(y, R) = \Big\{x \in \Omega \mid \log \frac{|wy|}{|wx|}\frac{|xz|}{|yz|} < 2R\Big\},$$

and the asymptotic ball as

$$\mathrm{AsB}_\Omega(y, R) = \Big\{x \in \Omega \mid \log \frac{|wy|}{|wx|} < 2R\Big\}.$$

The result follows easily. □

Recall that the Löwner–John ellipsoid of Ω is the unique ellipsoid of minimal volume containing Ω. By performing affine transformations, we may assume without loss of generality that the Löwner ellipsoid of Ω is the Euclidean unit ball $\mathcal{E}$. It is known that $(1/d)\mathcal{E}$ is then contained in Ω, that is,

$$\frac{1}{d}\mathcal{E} \subset \Omega \subset \mathcal{E}.$$

Thus, in this case the assumptions of Lemma 9 are satisfied with $L = 1$ and $l = 1/d$.

2. The volume of flags

In this section we focus on flags. We present a uniform upper bound on the volume of balls in a polytopal Hilbert metric in terms of the number of flags of the polytope. This will be used to prove the identity of the volume entropy and the flag-approximability. We then show that all flag-simplices of a simplex have the same volume, asymptotically, which will be used to prove Theorem 4.

2.1. **Flags.** Recall that to a closed convex set $K \subset \mathbb{R}^d$ we can associate an equivalence relation, where two points a and b are equivalent if they are equal or if there exists an open segment $(c,d) \subset K$ containing the closed segment $[a,b]$. The equivalence classes are called *faces*. A face is called a *k-face* if the dimension of the smallest affine space containing it is k.

A 0-face is usually called an *extremal point*, or, in the case of convex polytopes, a *vertex*.

Thus defined, each face is an open set in its affine hull, that is, in the smallest affine set containing it. For instance, the segment $[a,b]$ in $\mathbb{R}$ admits three faces, namely $\{a\}$, $\{b\}$, and the open segment (a,b).

Notice that if K has non-empty interior (that is, $K \setminus \partial K \neq \emptyset$), then its d-dimensional face is its interior.

When a face f is in the relative boundary of another face F, we write $f < F$.

Definition 10 (Flag)**.** Let $\mathcal{P}$ be a closed convex d-dimensional polytope. A *maximal flag* of $\mathcal{P}$ is a $(d+1)$-tuple $(f_0, ..., f_d)$ of faces of $\mathcal{P}$ such that each f_i has dimension i, and $f_0 < \cdots < f_d$.

In this paper, a *simplex* in $\mathbb{R}^d$ is the convex hull of $d+1$ projectively independent points, that is, a triangle in $\mathbb{R}^2$, a tetrahedron in $\mathbb{R}^3$, and so forth.

Definition 11 (Flag simplex)**.** A simplex $\mathcal{S}$ is a *flag simplex* of a polytope $\mathcal{P}$ if there is a maximal flag $(f_0, ..., f_d)$ of $\mathcal{P}$ such that each f_i contains exactly one vertex of $\mathcal{S}$.

We denote by $\mathrm{Flags}(\mathcal{P})$ the set of maximal flags of a polytope $\mathcal{P}$. We use $|\cdot|$ to denote the number of elements in a finite set.

Let Σ be a simplex of dimension d. Observe that $\mathrm{Flags}(\Sigma)$ consists of $(d+1)!$ elements.

Let $\mathcal{P}$ be a convex polytope. Suppose that for each face of $\mathcal{P}$ we are given a point in the face. Then, associated to each maximal flag there is a flag simplex of $\mathcal{P}$, obtained by taking the convex hull of the corresponding points. Moreover, these flag simplices form a simplicial complex, and their union is equal to $\mathcal{P}$. We call this a *flag decomposition* of $\mathcal{P}$. If each point is the barycenter of its respective face, then the resulting flag decomposition is just the well known *barycentric decomposition*.

2.2. **Uniform upper bound on the volume of a flag.** We use $B(R)$ to denote the ball in a Hilbert geometry of radius R and centered at o, and $S(R)$ to denote the boundary of this ball. Recall that a facet is relative closure of a face of codimension 1.

Lemma 12. *For each $d \in \mathbb{N}$ and integer $D \geq d$, there exists a polynomial $p_{d,D}$ of order d such that the following holds. Let $\mathcal{P}$ be a polytope in $\mathbb{R}^d$ endowed with its Hilbert geometry, satisfying $(1/4D)\mathcal{E} \subset \mathcal{P} \subset \mathcal{E}$. Let F be a facet of $\mathcal{P}$, and let L be the cone with base F and apex o. Then,*

$$\mathrm{Vol}^{\mathrm{H}}\big(B(R) \cap L\big) \leq p_{d,D}(R)\,\mathrm{Flags}(F), \qquad \text{for all } R \geq 0.$$

Proof. We will use induction on the dimension d.

When $d = 1$, there is only one Hilbert geometry, up to isometry. In this case, $\mathrm{Vol}^{\mathrm{H}}\big(B(R) \cap L\big) = R/2$, and $\mathrm{Flags}(F) = 2$, and so the conclusion is evident.

Assume now that the conclusion is true when the dimension is $d-1$ and for any $D \geq d-1$.

We now fix some integer $D \geq d$.

Using the co-area formula in Lemma 8, we get that

$$\frac{d}{dR}\operatorname{Vol}^{\mathrm{H}}\big(B(R)\cap L\big) \le C\operatorname{Area}\big(S(R)\cap L\big),$$

for some constant C depending only on the dimension.

Denote the facets of F by $\{F_i\}_i$. So, each F_i is a face of $\mathcal{P}$ of co-dimension 2. Notice that $\sum_i \operatorname{Flags}(F_i) = \operatorname{Flags}(F)$. For each i, let L_i be the $d-1$ dimensional cone with base F_i and apex o.

Observe that, from Lemma 9, $B(R)\cap L \subset \operatorname{AsB}(R)\cap L$, for all $R \ge 0$. So, using the monotonicity of the Holmes–Thompson measure (Lemma 7), we get

$$\operatorname{Area}\big(S(R)\cap L\big) \le \operatorname{Area}\big(\operatorname{AsS}(R)\cap L\big) + \sum_i \operatorname{Area}\big(\operatorname{AsB}(R)\cap L_i\big).$$

Here $\operatorname{AsS}(R)$ is the boundary of the asymptotic ball of radius R about o.

By the minimality of flats for the Holmes–Thompson volume [11], we have that

$$\operatorname{Area}\big(\operatorname{AsS}(R)\cap L\big) \le \sum_i \operatorname{Area}\big(\operatorname{AsB}(R)\cap L_i\big).$$

From Lemma 9, we have that $\operatorname{AsB}(R) \subset B(R+c)$, where c depends only on D. Also, by the induction hypothesis, $\operatorname{Area}\big(B(R+c)\cap L_i\big) \le p_{d-1,D}(R+c)\operatorname{Flags}(F_i)$. Putting all this together, we get that

$$\frac{d}{dR}\operatorname{Vol}^{\mathrm{H}}(B(R)\cap L) \le 2Cp_{d-1,D}(R+c)\operatorname{Flags}(F).$$

The result follows upon integrating. □

The two- and three-dimensional case of the following theorem follow from Theorem 10 in first author's paper [16].

Theorem 13. *For any $d \in \mathbb{N}$, there is a polynomial p_d of degree d such that, for any $R > 0$, and for any convex polytope $\mathcal{P}$ satisfying $(1/4d)\mathcal{E} \subset \mathcal{P} \subset \mathcal{E}$, we have*

$$\operatorname{Vol^H}_{\mathcal{P}} B(o,R) \le p_d(R)\operatorname{Flags}(\mathcal{P}).$$

The same result holds for the asymptotic balls.

Proof. We will consider the metric balls. The passage from these to the asymptotic balls is done using Lemma 9.

Let p_d be the polynomial $p_{d,d}$ obtained from Lemma 12. According to that lemma, for each facet F of $\mathcal{P}$ and for each $R > 0$, we have $\operatorname{Vol^H}_{\mathcal{P}}(B(R)\cap L) \le p_d(R)\operatorname{Flags}(F)$, where L is the cone with base F and apex o. Summing over all the facets of $\mathcal{P}$, we get the result. □

2.3. **Asymptotic volume of a flag simplex.**

Lemma 14. *Let Σ be a d-dimensional simplex. Let T be a barycentric flag simplex of Σ, and let S be a flag simplex. Then, there exist projective linear maps ϕ_0 and ϕ_1 leaving Σ invariant, such that $\phi_0(T) \subset S \subset \phi_1(T)$.*

Proof. Consider Σ to be the projective space of the positive cone $P^{d+1} := (0,\infty)^{d+1}$. In this setting, after ordering the coordinates in the right way, T may be identified with the (projective space of) the set

$$T = \big\{x \in P^{d+1} \mid x_j < x_{j+1} \text{ for all } 0 \le j < d\big\}.$$

The image of T under the linear map

$$\phi\colon P^{d+1} \to P^{d+1}, \quad (x_j)_j \mapsto (\alpha_j x_j)_j,$$

where the $(\alpha_j)_j$ are positive real numbers, is

$$\phi(T)=\Big\{x\in P^{d+1}\mid \frac{x_j}{\alpha_j}<\frac{x_{j+1}}{\alpha_{j+1}}\text{ for all }0\le j<d\Big\}. \tag{2}$$

On the other hand, without loss of generality, S can be written as (the projective space of) the interior of the conical hull

$$S=\operatorname{coni}\{v^0,\dots,v^d\},$$

of a set of $d+1$ vectors of the form

$$v^i:=(0,\dots,0,v^i_i,\dots,v^i_d),\qquad 0\le i\le d,$$

where all the v^i_j are positive.

For each j, define $\rho_j>0$ so that $\rho_j<\min_i v^i_{j+1}/v^i_j$. Let $x\in S$. So, $x=\sum_i\lambda_i v^i$, for some positive coefficients $(\lambda_i)_i$. Therefore, for all $j\in\{0,\dots,n\}$,

$$x_{j+1}=\sum_i\lambda_i v^i_{j+1}>\sum_i\lambda_i\rho_j v^i_j=\rho_j x_j.$$

This shows that S is contained within a set of the form (2).

Now define, for each $j\in\{1,\dots,d+1\}$,

$$\sigma_j:=\max_{0\le i<j}\frac{v^i_j}{v^i_{j-1}}.$$

Let $x\in P^{d+1}$ be such that $x_j>\sigma_j x_{j-1}$, for all $j\in\{1,\dots,n\}$. Define coefficients (λ_i) in the following way: set $\lambda_0:=x_0/v^0_0$, and then recursively set λ_j to be such that

$$x_j=\lambda_j v^j_j+\sum_{i=0}^{j-1}\lambda_i v^i_j,\qquad\text{for all }j\in\{1,\dots,d+1\}.$$

Since, for all $j\in\{1,\dots,d+1\}$,

$$x_j>\sigma_j x_{j-1}=\sigma_j\sum_{i=0}^{j-1}\lambda_i v^i_{j-1}\ge\sum_{i=0}^{j-1}\lambda_i v^i_j,$$

we see that all the (λ_j) are positive. Thus, x can be written as a linear combination with positive coefficients of the vectors $\{v^i\}$, and is hence in S. So, we have shown that a set of the form (2) is contained in S. □

Lemma 15. *Consider the Hilbert geometry on a d-dimensional simplex Σ. Let S be a flag simplex of Σ. Then for any z in Σ,*

$$\lim_{R\to\infty}\frac{1}{R^d}\operatorname{Vol}\big(\operatorname{AsB}(z,R)\cap S\big)=\frac{1}{(d+1)!}\operatorname{Asvol}(\Sigma).$$

Proof. Because all simplices of the same dimension are affinely equivalent, we may assume that Σ is a regular simplex with the origin o as its barycenter.

Let T be a barycentric flag simplex of Σ.

A projective linear map leaving Σ invariant is an isometry of the Hilbert metric on Σ, and therefore preserves volume. Combining this with the fact that

$$B\big(x,R-d(x,y)\big)\subset B(y,R)\subset B\big(x,R+d(x,y)\big), \tag{3}$$

for any points $x,y\in\Sigma$ and $R>0$, we get

$$\lim_{R\to\infty}\frac{1}{R^d}\operatorname{Vol}\big(B(o,R)\cap\phi(T)\big)=\lim_{R\to\infty}\frac{1}{R^d}\operatorname{Vol}\big(B(o,R)\cap T\big), \tag{4}$$

for any projective linear map ϕ leaving Σ invariant.

From Lemma 14, there exist projective linear maps ϕ_0 and ϕ_1 leaving Σ invariant, such that $\phi_0(T) \subset S \subset \phi_1(T)$. Combining this with (4), we get

$$\lim_{R\to\infty} \frac{1}{R^d} \operatorname{Vol}\big(B(o,R)\cap S\big) = \lim_{R\to\infty} \frac{1}{R^d} \operatorname{Vol}\big(B(o,R)\cap T\big).$$

Denote by Π the group of permutations of vertices of Σ. Observe that Π has $(d+1)!$ elements. The group Π acts on Σ, leaving the center o of Σ fixed. We have that the sets $\{\phi(T)\}_{\phi\in\Pi}$ are pairwise distinct, and that the union of their closures is Σ. So, by symmetry,

$$\lim_{R\to\infty} \frac{1}{R^d} \operatorname{Vol}\big(B(o,R)\cap T\big) = \frac{1}{(d+1)!} \operatorname{Asvol}(\Sigma)$$

The last step is to use (3) and Lemma 9 to get that

$$\lim_{R\to\infty} \frac{1}{R^d} \operatorname{Vol}\big(\operatorname{AsB}(z,R)\cap S\big) = \lim_{R\to\infty} \frac{1}{R^d} \operatorname{Vol}\big(B(o,R)\cap S\big). \qquad \square$$

3. Asymptotic volume and Flags

We prove that the asymptotic volume of a flag simplex in a polytope is the same as in a simplex. From this we deduce Theorem 4.

Lemma 16. *Let $\mathcal{P}$ be a polytope, and let S be a flag simplex of $\mathcal{P}$. Then there exist simplices U and V satisfying $U \subset \mathcal{P} \subset V$ such that S is a flag simplex of both U and of V.*

Proof. We prove the existence of U by induction on the dimension. The one dimensional case is trivial, since here $\mathcal{P}$ is already a simplex. So, assume the result holds in dimension d, and let $\mathcal{P}$ be $d+1$-dimensional. Let p be the vertex of S that lies in the relative interior of $\mathcal{P}$. The remaining vertices of S form a flag simplex S' of a facet of $\mathcal{P}$. Applying the induction hypothesis, we get a simplex U' contained in this facet such that S' is a flag simplex of U'. It is not difficult to see that we may perturb p in such a way as to get a point $p' \in \mathcal{P}$ such that the simplex U formed from p' and U' contains p in its relative interior. It follows that $U \subset \mathcal{P}$, and that S is a flag simplex of U.

We also prove the existence of V by induction on the dimension. Again, the 1-dimensional case is trivial. As before, we assume the result holds in dimension d, and let $\mathcal{P}$ be $d+1$-dimensional. Recall that p is the vertex of S that lies in the relative interior of $\mathcal{P}$, and that the remaining vertices of S form a flag simplex S' of a facet F of $\mathcal{P}$. Applying the induction hypothesis, we get a simplex V' containing this facet such that S' is a flag simplex of V'. Denote by o the vertex of S that is also a vertex of $\mathcal{P}$. Without loss of generality we may assume that o is the origin of the vector space $\mathbb{R}^{d+1}$. Observe that if we multiply the vertices of V' by any scalar greater than 1, then S' remains a flag simplex of V'. Choose $q \in \mathbb{R}^{d+1}$ and $\alpha > 1$ such that every vertex of $\mathcal{P}$ lies in the convex hull

$$V := \operatorname{conv}\{q, \alpha V'\}.$$

Then, $\mathcal{P} \subset V$ and S is a flag simplex of V. $\square$

Proof of Theorem 4. Choose a flag decomposition of $\mathcal{P}$. Let x be the vertex that is common to all the flag simplices, which lies in the interior of $\mathcal{P}$.

Let S be any one of the flag simplices. By Lemma 16, there are simplices U and V satisfying $U \subset \mathcal{P} \subset V$ such that S is a flag simplex both of U and of V. Hence,

$$\operatorname{Vol}_U(X) \geq \operatorname{Vol}_{\mathcal{P}}(X) \geq \operatorname{Vol}_V(X), \tag{5}$$

for any measurable subset X of U. Observe that, for any $R > 0$,

$$\mathrm{AsB}_U(x,R) \cap S = \mathrm{AsB}_{\mathcal{P}}(x,R) \cap S = \mathrm{AsB}_V(x,R) \cap S. \tag{6}$$

Combining (5) and (6) with Lemma 15, we get

$$\lim_{R\to\infty} \frac{1}{R^d} \mathrm{Vol}_{\mathcal{P}}\big(\mathrm{AsB}_{\mathcal{P}}(x,R) \cap S\big) = \frac{1}{(d+1)!} \mathrm{Asvol}(\Sigma).$$

Using Lemma 9, we get

$$\lim_{R\to\infty} \frac{1}{R^d} \mathrm{Vol}_{\mathcal{P}}\big(B_{\mathcal{P}}(x,R) \cap S\big) = \frac{1}{(d+1)!} \mathrm{Asvol}(\Sigma).$$

But this holds for any flag simplex of the decomposition, and summing over all the flags we get the result. □

Proof of Corollary 5. The first author proved in [15] that the asymptotic volume of a convex body is finite if and only if it is a polytope. The result follows because the simplex has fewer flags than any other polytope of the same dimension. □

Proof of Corollary 6. When one considers the Busemann volume, the asymptotic volume of every normed space of a fixed dimension d is the same, and is equal to $\mathrm{Asvol}(\Sigma)$ since the Hilbert geometry on a simplex is isometric to a normed space. Hence $\mathrm{Asvol}(\Omega) = \mathrm{Asvol}(\Sigma)$, and the result follows from Corollary 5. □

4. A general bound on the flag complexity

Here we prove Theorem 2, that is, that the flag complexity of a d-dimensional convex body is no greater than $(d-1)/2$.

Our technique is to modify the proof of the main result of [2]. In that paper, essentially the same result was proved for the *face-approximability*, which is defined analogously to the flag-approximability, but counting the least number of faces rather than the least number of flags.

Their proof uses the witness-collector method. Assume we have a set S of points in $\mathbb{R}^d$, a set $\mathcal{W}$ of regions called *witnesses*, and a set $\mathcal{C}$ of regions called *collectors*, satisfying the following properties.

(i) each witness in $\mathcal{W}$ contains a point of S in its interior;
(ii) any halfspace H of $\mathbb{R}^d$ either contains a witness $W \in \mathcal{W}$, or $H \cap S$ is contained in a collector $C \in \mathcal{C}$;
(iii) each collector $C \in \mathcal{C}$ contains some constant number of points of S.

Let P be the convex hull of S.

We strengthen Lemma 4.1 of [2]. In what follows, given a quantity D, any other quantity is said to be $O(D)$ if it is bounded from above by a multiple, depending only on the dimension, of D.

Lemma 17. *Given a set of witnesses and collectors satisfying the above properties, the number of flags of the convex hull P of S is $O(|\mathcal{C}|)$.*

Proof. Take any facet F of P, and let H be the half-space whose intersection with P is F. As in the original proof, H does not contain any witness, for otherwise, by property (i), it would contain a point of S in its interior. So, by (ii), the intersection of H and S is contained in some collector C. Therefore, by (iii), F has at most n vertices, where n is the number of points in each collector.

So, we see that each facet has at most 2^n faces, and so has at most $(2^n)^d$ flags, since each flag can be written as an increasing sequence of d faces.

Also, the number of facets is at most $2^n|\mathcal{C}|$ since each facet has a different set of vertices, and this set is a subset of some collector.

We deduce that the number of flags is at most $(2^n)^{d+1}|\mathcal{C}|$. □

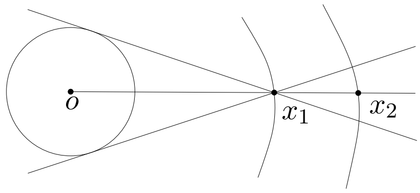

FIGURE 1. Diagram for the proof of Lemma 18.

We conclude that the main theorem of [2] holds when measuring complexity using flags instead of faces.

Proof of Theorem 2. The proof follows that of the main result of [2], but using Lemma 17 above instead of Lemma 4.1 of that paper. □

5. Upper bound on the volume entropy

We show that the volume entropy of a convex body is no greater than twice the flag approximability.

Lemma 18. *Let Ω_1 and Ω_2 be convex bodies within a Hausdorff distance $\epsilon > 0$ of each other, each containing the Euclidean ball $l{\cdot}\mathcal{E}$ of radius $l > 0$ centered at the origin. Then, $(1/\lambda)\Omega_2 \subset \Omega_1$, with $\lambda := 1 + \epsilon/l$.*

Proof. Consider a ray emanating from the origin, and let x_1 and x_2 be the intersections of this ray with the boundaries of Ω_1 and Ω_2, respectively. Let l_1 and l_2 be the distances from the origin to x_1 and x_2, respectively, and suppose that $l_2 > l_1$. Define the cone

$$F := \big\{x_1 + \alpha(x_1 - z) \mid \alpha > 0 \text{ and } z \in l{\cdot}\mathcal{E}\big\}.$$

See Figure 1. The point x of Ω_1 lying closest to x_2 can not be in the interior of the cone F. However, the distance from x to x_2 is no greater than ϵ. So, we see that $l/l_1 \le \epsilon/(l_2 - l_1)$. We deduce that

$$\frac{l_2}{l_1} = 1 + \frac{l_2 - l_1}{l_1} \le 1 + \frac{\epsilon}{l}.$$

The conclusion follows. □

Lemma 19. *Let Ω be a bounded convex domain in $\mathbb{R}^d$. The volume entropy of Ω is no greater than twice its flag approximability, that is,*

$$\mathrm{Ent}(\Omega) \le 2a_{\mathrm{f}}(\Omega).$$

Proof. Without loss of generality, we may assume that Ω is in canonical form.

Let $R > 0$, and let $\epsilon > 0$ be such that $-2R = \log \epsilon$.

Let P^* be the polytope approximating Ω within Hausdorff distance ϵ having the least number $N(\epsilon)$ of maximal flags. Write $P := (1/\lambda)P^*$, where $\lambda := 1 + d\epsilon$. From Lemma 18,

$$(1/\lambda^2)\Omega \subset P \subset \Omega. \tag{7}$$

We will henceforth assume that $\epsilon < 1/d$. Since Ω is in normal form, this implies that P satisfies the assumptions of Theorem 13. Therefore, there exists a polynomial p_d of degree d, depending only on the dimension d, such that

$$\mathrm{Vol^H}_P\big(\mathrm{AsB}_P(o, R)\big) \le N(\epsilon)p_d(R).$$

From (7),

$$\mathrm{Vol^H}_\Omega(\cdot) \le \mathrm{Vol^H}_P(\cdot).$$

Observe that $((1-\epsilon)/\lambda^2)\Omega$ is the asymptotic ball of Ω of radius R', where $-2R' = \log\epsilon'$, with $1-\epsilon' = (1-\epsilon)/\lambda^2$. Also, the asymptotic ball of P of radius R is $(1-\epsilon)P$. So, according to (7),

$$\mathrm{AsB}_\Omega(o,R') \subset \mathrm{AsB}_P(o,R).$$

Finally, Lemma 9 gives that $B(o,R') \subset \mathrm{AsB}_\Omega(o,R')$.

Putting all this together, we conclude that

$$\frac{1}{R'}\log {\mathrm{Vol}^{\mathrm{H}}}_\Omega\big(B(o,R')\big) \le 2\frac{\log\big(N(\epsilon)p_d(R)\big)}{-\log\epsilon'}.$$

We now take the limit infimum as R tends to infinity, in which case R' also tends to infinity, and ϵ and ϵ' tend to zero. A simple calculation shows that, in this limit, the ratio ϵ'/ϵ converges to $2d+1$. The result follows. □

6. Lower bound on the volume entropy

We show that the volume entropy of a convex body is no less than twice the flag approximability.

Lemma 20. *Let Ω be a bounded convex domain in $\mathbb{R}^d$. Then, $2a_{\mathrm{f}}(\Omega) \le \mathrm{Ent}(\Omega)$.*

Our proof will be a modification of the method used in [2].

We start with a lemma concerning the centroid of a convex body, otherwise known as its barycenter or center of mass.

Lemma 21. *Let D be a convex body in $\mathbb{R}^d$, having a tangent hyperplane h. Let $p \in h$ and $q \in D$ be such that the centroid x of D lies on the line segment $[pq]$. Then, $|px| \ge |pq|/(d+1)$.*

Proof. The ratio $|px|/|pq|$ is minimized when D is a simplex with a vertex at q and all the other vertices on h. □

Recall the following definitions. A *cap* C of a convex body K is a non-empty intersection of K with a halfspace H. The *base* of the cap C is the intersection of K with the hyperplane h that bounds the halfspace. An *apex* of C is a point of C of maximum distance from h. Thus, the apexes of C all lie in a hyperplane tangent to K and parallel to h. The width of the cap is the distance from any apex to h.

Let K be a convex body containing the origin o in its interior. Consider the ray emanating from o and passing through another point x. We define the ray-distance $\mathrm{ray}(x)$ to be the distance from x to the point where this ray intersects ∂K.

Lemma 22. *Let $K \subset \mathbb{R}^d$ be a convex body in canonical form. Let x be the centroid of a cap of width ϵ of K. Then, the ray-distance $\mathrm{ray}(x)$ is greater than $c\epsilon$, for some constant $c > 0$ depending on the dimension d.*

Proof. Let C be the cap of width ϵ, and let x be the centroid of its base D. Let z be an apex of C. So, z is at distance ϵ from H, the hyperplane defining the cap.

Consider the 2-plane Π containing the points o, x, and z. (If these points are collinear, then take Π to be any 2-plane containing them.)

The intersection of D with Π is a line segment. Let p and q be the endpoints of this line segment. Label them in such a way that the ray ox intersects the line segment pz at a point w. See Figure 2.

Think of D as a convex body in the H. By considering a hyperplane in H of dimension $d-2$ that supports D at p, we get from Lemma 21 that $|px| \ge |pq|/d$, since x is the centroid of D.

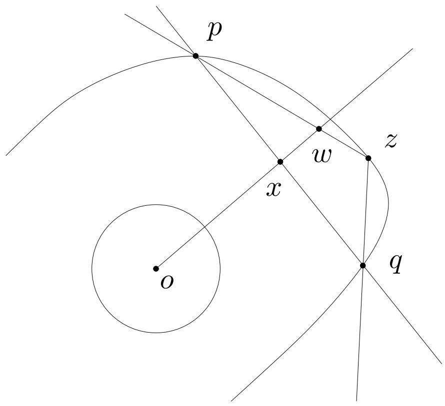

FIGURE 2. Diagram for the proof of Lemma 22.

We consider separately the cases where the angle $\angle pzq$ is acute and where it is not.

Case $\angle pzq \leq \pi/2$. Since z is at distance at most 1 from the origin, and K contains the Euclidean ball $(1/d)\mathcal{E}$, the angle $\angle pzq$ must be at least $A := 2\arcsin(1/d)$. In the present case, this implies that $\sin \angle pzq$ is at least $\sin A$.

Observe that $|zq| \geq \epsilon$.

Two applications of the sine rule give

$$|xw| = |zq| \frac{|px|}{|pq|} \frac{\sin \angle pzq}{\sin \angle pwx}.$$

We deduce that $|xw| \geq \epsilon \sin(A)/d$

Case $\angle pzq \geq \pi/2$. In this case there is a point y between p and q such that $\angle pzy = \pi/2$. Moreover, $\angle xpz \leq \pi/2$ and so there is a point w' between p and z such that $\angle xw'p = \pi/2$. Using similarity of triangles, we get

$$|xw| \geq |xw'| = \frac{|px||yz|}{|py|} \geq \frac{|px||yz|}{|pq|} \geq \epsilon/d.$$

In both cases we have shown that $|xw|$ is at least ϵ times some constant depending on the dimension. The conclusion follows since $\text{ray}(x) \geq |xw|$. $\square$

The following is part of Theorem 2 of [15].

Lemma 23. *For each dimension d, there is a constant c such that* $\text{Vol}^{\text{H}}{}_{\Omega}\big(B_\Omega(x,R)\big) \geq cR^d$, *for each convex body* Ω, *point* $x \in \Omega$, *and radius* $R > 0$.

Let K be a convex body containing a point x in its interior. The *Macbeath region* about x is defined to be

$$M'(x) := x + \Big(\frac{1}{5}(K - x) \cap \frac{1}{5}(x - K)\Big).$$

Macbeath regions are related to balls of the Hilbert geometry as follows.

Lemma 24. *Each Macbeath region* $M'(x)$ *satisfies*

$$B\Big(\frac{1}{2}\log\frac{6}{5}\Big) \subset M'(x) \subset B\Big(\frac{1}{2}\log\frac{3}{2}\Big).$$

Proof. Recall that the Funk distance between two points p and q is defined to be

$$d_F(p,q) := \log \frac{|pb|}{|qb|},$$

where b is as in the definition of the Hilbert metric in section 1. The Funk metric is not actually a metric since it is not symmetric. Its symmetrisation is the Hilbert metric: $d_\Omega(p,q) = (d_F(p,q) + d_F(q,p))/2$.

One can show that a point y is in $M'(x)$ if and only if both $d_F(x,y) \leq \log(5/4)$ and $d_F(y,x) \leq \log(6/5)$. The conclusion follows. □

The following is a modification of Lemma 3.2 of [2]. The assumptions are the same; all that has changed is the bound on the number of caps. The original bound was $O(1/\delta^{(d-1)/2})$.

Lemma 25. *Let $K \subset R^d$ be a convex body in canonical form. Let $0 < \delta \leq \Delta_0/2$, where Δ_0 is a certain constant (see [2]). Let $\mathcal{C}$ be a set of caps each of width δ, such that the Macbeath regions $M'(x)$ centered at the centroids x of the bases of these caps are disjoint. Then, $|\mathcal{C}| = O\Big(\mathrm{Vol}^{\mathrm{H}}\big(\mathrm{AsB}(R+C_0)\big)\Big)$, where $2R := -\log C\delta$, and C_0 is a constant. (Here C is the constant appearing in Lemma 22.)*

Proof. Let x be the centroid of one of the caps in $\mathcal{C}$. By Lemma 22, the ray-distance satisfies $\mathrm{ray}(x) \geq C\delta = \exp(-2R)$. Using that K is contained in the unit ball, we get that $x \in \mathrm{AsB}(R)$. So, by Lemma 24, the Macbeath region $M'(x)$ is contained within $\mathrm{AsB}(R + C_0)$, with $C_0 := (1/2)\log(3/2)$.

Combining Lemmas 23 and 24, we get that there is a constant C_1 such that each Macbeath region $M'(x)$ has volume at least C_1. A volume argument now gives that $|\mathcal{C}|C_1 \leq \mathrm{Vol}^{\mathrm{H}}(\mathrm{AsB}(R + C_0))$. □

We can now prove the lower bound on the volume entropy.

Proof of Lemma 20. We follow the method of [2], but using the bound in Lemma 25 on the number of non-intersecting Macbeath regions, rather than that in Lemma 3.2 of [2].

Given an $\epsilon > 0$, this method produces a set of points S, a set $\mathcal{W}$ of witnesses, and a set $\mathcal{C}$ of collectors satisfying the assumptions in section 4, such that the convex hull of S is an ϵ-approximation of K. Furthermore, Lemma 25 leads to the following bound on the number of collectors:

$$|\mathcal{C}| \leq \mathrm{Vol}^{\mathrm{H}}\big(\mathrm{AsB}(R + C_0)\big)/C_1,$$

where $2R := -\log C\delta$ and $\delta := c_1\epsilon/\big(\beta\log(1/\epsilon)\big)$, for some constant c_1 depending only on the dimension.

Since we are concerned with the flag-approximability, we must, just as in the proof of Theorem 2, use Lemma 17 from section 4 instead of Lemma 4.1 of [2]. We get that the number $N(\epsilon)$ of flags in the approximating polytope is at most a fixed multiple $C_3|\mathcal{C}|$ of $|\mathcal{C}|$.

Now let ϵ tend to zero. Observe that $\log\delta/\log\epsilon$ converges to 1. So,

$$\begin{aligned} a_{\mathrm{f}}(\Omega) &= \liminf_{\epsilon\to 0} \frac{\log N(\epsilon)}{-\log\epsilon} \\ &\leq \liminf_{R\to\infty} \log\Big(\frac{C_3}{C_1}\Big)\frac{\log \mathrm{Vol}^{\mathrm{H}}\big(\mathrm{AsB}(R + C_0)\big)}{2R + \log C} \\ &= \frac{1}{2}\,\mathrm{Ent}(\Omega). \end{aligned}$$ □

The proof of the main result of the paper is now complete.

Proof of Theorem 1. We combine Lemmas 19 and 20. □

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Institut Montpellierain Alexander Grothendieck, Université de Montpellier, Case Courrier 051, Place Eugène Bataillon, F–34395 Montpellier Cedex, France
E-mail address: `Constantin.Vernicos@math.univ-montp2.fr`

INRIA and CMAP, Ecole polytechnique, Palaiseau Cedex, France
E-mail address: `cormac.walsh@inria.fr`