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# Some critical remarks on Landau's (macroscopic) phase transition theory\*

Giuseppe Iurato<sup>†</sup>

**Abstract.** *This paper explains the existence of a particular formal model (drawn from Theoretical Astrophysics) whose thermodynamical phenomenology shows a possible second order phase transition (according to Landau's Thermodynamical Theory) that seems does not verify the (Birman-Goldrich-Jarić) "chain subduction criterion" and the (Ascher's) "maximality criterion" of Landau's Phenomenological Theory. Therefore, it follows that Landau's Phenomenological Theory is more restrictive than the Landau's Thermodynamical Theory.*

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## 1 Introduction

Lev D. Landau ([La], [LL5]) given two theoretical formulations of the (macroscopic) second order phase transition theory, the first called (*Landau's macroscopic Thermodynamical Theory*), and the second called (*Landau's macroscopic Phenomenological Theory*). These theories are based (especially the second) on some symmetry principles and elementary methods of the representation theory of groups.

In this context, there exist criteria<sup>1</sup> that give necessary and/or sufficient conditions for a second order phase transition (according to Landau).

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<sup>1</sup>See [As1], [As2], [AK], [Bi2], [Bo], [CLP], [GL], [Ja1], [IT O], [LL5], [Ly], [Mi1], [Mi3], [SH], [Th], [To1].

Among these, we recall the (*Birman-Goldrich-Jarić*) *chain subduction criterion*<sup>2</sup> and the (*Ascher's*) *maximality criterion*<sup>3</sup>.

In this paper, we compare these two criteria with a well-known formal model of theoretical astrophysics – already analyzed in [BR], and called the *Maclaurin-Jacobi pattern* – which seems to exhibit a second order phase transition (in the sense of Landau), according to Landau's Thermodynamical Theory.

In the articles [CMR], [Ho], [In], [Ks], [Mi2], [Mi3], [MM], [Si], and in the textbooks [Ly], [SH], we find other formal confirmations to the theoretical result achieved in [BR].

Nevertheless, for this latter pattern seem does not subsist these two criteria (see next § 5).

## 2 The Maclaurin-Jacobi pattern

The Maclaurin-Jacobi pattern ([Le1]) is a simplified mathematical model of the Self-Gravitating Systems Theory (cf. [BF], [Ch], [CMR], [FP], [Jar], [Ko], [Lam], [Le1], [Le2], [Lyt], [Pa], [St1], [Ta1], [Ta2]), and of the Physics of Compact Objects (cf. [KW], [Pa], [ST]).

In Astrophysics, there exists the problem of determining the (closed) stationary equilibrium configurations of a self-gravitating rotating fluid. If we suppose that the fluid is subjected to a rigid rotation around one of its symmetry axis, under the further simplified hypotheses of homogeneity, incompressibility and non-viscosity of the rotating fluid, we then obtain a first linear solution to this problem, i.e. the Maclaurin-Jacobi pattern.

Applied to polytropic fluid configurations with index  $n \leq 0.808$  ([Va], [St2]), to the theory of elliptic galaxies (cf. [OP], [St2]) and to the Black Holes thermodynamics (cf. [Da1], [Da2], [Wa]), this model provides theoretical predictions in good agreement with the corresponding experimental data, so that the Maclaurin-Jacobi pattern is a valid model of Theoretical Astrophysics<sup>4</sup>.

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<sup>2</sup>See [Bi1], [CLP], [GB], [Ja1], [Ja2], [Ja3], [Ja4], [Ja7], [JB], [SA], [To1].

<sup>3</sup>See [As1], [As2], [AK], [Bi2], [CLP], [Ja1], [Ja6], [To2].

<sup>4</sup>Furthermore, there exist satisfactory applications of this model to the Theory of *Collective Models* of atomic nuclei (see [ALLMNRA], [BK], [CPS1], [CPS2], [IA], [RS], [Ro1], [Ro2]).

Following [CMR], let us consider a homogeneous, incompressible and non-viscous fluid in rigid rotation around one of its symmetry axes, for example the  $Ox_3$  axis of a Cartesian coordinates system  $(x_1, x_2, x_3)$ . We choose a normalized orthogonal reference frame  $\{O, \hat{i}, \hat{j}, \hat{k}\}$ , whose origin  $O$  is the center of mass of the rotating fluid. We suppose that a reference frame is rotating with the fluid (with respect to which it is at rest<sup>5</sup>).

If  $\vec{\omega}$  is the angular velocity of rotation, let us suppose that  $\vec{\omega} = \omega \hat{k}$ , in order that the angular momentum is  $\vec{J} = M((\vec{\omega} \times \vec{r}) \times \vec{r})$ , where  $\vec{r}$  is the position vector (measured in the rest reference frame co-rotating with the fluid) of the generic material point of the system, and  $M$  is the total mass of the fluid.  $\vec{J}$  and  $\vec{\omega}$  are both parallel to the  $Ox_3$  axis.

Historically, C. Maclaurin ([Mc]), C.G.J. Jacobi ([Jc]) and H. Poincaré ([Po]) dealt with the dynamical problem of determining the equilibrium configurations of the fluid system. The results so obtained by the first two authors, forms the so-called "*Maclaurin-Jacobi model*" (or *M-J pattern*).

For such a fluid system, in the above mentioned rest reference frame co-rotating with the fluid, the (Euler's) hydrostatic equilibrium equation is

$$(1) \quad \nabla(p(\vec{r}) - \varphi(\vec{r})) = 0,$$

where  $p$  is the hydrostatic pressure and  $\varphi$  the potential of the fluid system. In the rest reference frame, let  $S(\vec{r}; J^2) = 0$  be the surface equation of the unknown equilibrium configuration. If we suppose  $S(\vec{r}; J^2) > 0$  inside the fluid, then the region  $V_S \subset \mathbb{R}^3$  is bounded, connected and take up by the fluid, and represented by  $S(\vec{r}; J^2) \geq 0$ . This surface is parametrical dependent by  $J^2$ , where  $J$  is the modulus of the angular momentum. The potential  $\varphi$  of this well-defined self-gravitating fluid, is given by the sum of the gravitational potential  $\varphi_g$  and of the centrifugal potential  $\varphi_c$ , so that

$$(2) \quad \varphi[S](\vec{r}; J^2) = \varphi_g[S](\vec{r}) + \varphi_c[S](\vec{r}; J^2),$$

where

$$\varphi_g[S](\vec{r}) = \frac{1}{4\pi} \int_{S(\vec{s}) \geq 0} \frac{d\vec{s}}{|\vec{r} - \vec{s}|}$$

is the gravitational potential, and

$$\varphi_c[S](\vec{r}; J^2) = \frac{J^2}{2I^2}(x_1^2 + x_2^2), \quad I[S] = \frac{15}{8\pi} \int_{S(\vec{r}) \geq 0} (x_1^2 + x_2^2) d\vec{r}$$

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<sup>5</sup>In this case, such a reference is properly called *co-rotating rest reference frame* of the given fluid system.

are, respectively, the centrifugal potential and the inertial moment of the fluid system, computed with respect to the axis  $Ox_3$ , with  $\vec{r} = (x_1, x_2, x_3)$ . The gravitational potential satisfies the Poisson's equation

$$\Delta\varphi_g[S](\vec{r}) = -\rho, \quad \rho(\vec{r}) = 1 \quad \text{if } S(\vec{r}) \geq 0, \quad \rho(\vec{r}) = 0 \quad \text{if } S(\vec{r}) < 0$$

where  $\rho$  is the (constant) density of the fluid. Moreover, the associated boundary conditions are the following

$$p(\vec{r})|_{S(\vec{r})=0} = 0, \quad \lim_{r \rightarrow \infty} \varphi_g(\vec{r}) = 0, \quad \varphi[S](\vec{r}; J^2)|_{S(\vec{r})=0} = \text{const.}, \quad r = |\vec{r}|.$$

Through the bifurcation method (see [CMR], [Pe1], [Pe2], [Dy]), and neglecting, for simplicity, the various stability questions (see [CMR], [Lyt]), a first (linear) approximated solution to the non-linear differential system

$$(3) \quad \begin{cases} \nabla(p - \varphi) = 0 \\ \Delta\varphi_g = -\rho \\ p(\vec{r})|_{S(\vec{r})=0} = \lim_{r \rightarrow \infty} \varphi_g[S](\vec{r}) = 0 \\ \varphi[S](\vec{r}; J^2)|_{S(\vec{r})=0} = \text{const.} \end{cases}$$

gives the so-called *Maclaurin-Jacobi pattern*, that consists of the first two sequences of (infinite and stationary<sup>6</sup>) solutions  $S(\vec{r}; J^2)$  of (3), for suitable values of the parameter<sup>7</sup>  $J^2$ . Precisely, we have a first sequence of (infinite and stationary<sup>8</sup>) solutions, called *Maclaurin's sequence*

$$\mathcal{S}_{Mac} = \{S(\vec{r}; J^2); 0 < J^2 < 0,384436\},$$

and a second sequence of (infinite and stationary<sup>9</sup>) solutions, called *Jacobi's sequence*

$$\mathcal{S}_{Jac} = \{S(\vec{r}; J^2); 0,384436 \leq J^2 < 0,632243\}.$$

The first sequence of solutions is geometrically formed by two-axial ellipsoids (or spheroids) whose symmetry axis of rotation is  $Ox_3$ , whereas the second sequence of solutions is geometrically formed by three-axes ellipsoids. The figures of the first sequence are called *Maclaurin's ellipsoids* (or *spheroids*),

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<sup>6</sup>But no static.

<sup>7</sup>With respects to suitable natural units, relative to the physical and geometrical parameters of an ellipsoid, the parameter  $J^2$  takes well-defined values (see [CMR]).

<sup>8</sup>But no static.

<sup>9</sup>But no static.

whereas the figures of the second sequence are called *Jacobi's ellipsoids*. When  $\omega \rightarrow 0$ , by a corollary of an important theorem of L. Lichtenstein (cf. [Le1], [Li]), the only stationary and static equilibrium configurations, compatible with (3), are those spherics (corresponding to  $J^2 = \omega = 0$ ), so that the relative symmetry group is  $O(3)$ . Little by little that  $\omega$  increases (and hence  $J^2$  increases as well), the fluid mass takes, at first, a two-axial ellipsoidal shape, with symmetry axis the rotation axis and symmetry group<sup>10</sup>  $D_{\infty h}$ , whereas, as soon as  $J^2$  has values not lower than 0,384436, the fluid mass takes the shape of a three-axes ellipsoid, with relative symmetry group  $D_{2h}$ .

Increasing  $J^2$  beyond the value 0,632243, we obtain another sequence of (infinite and stationary<sup>11</sup>) solutions (of the system (3)), represented by new geometrical figures, called *pearlike configurations* of Poincaré, whose symmetry group is  $C_{2v}$ . If the latter sequence, called *Poincaré's sequence*, is

$$\mathcal{S}_{Poin} = \{S(\vec{r}; J^2); 0,632243 \leq J^2 < 1,346350\},$$

then  $\{\mathcal{S}_{Macl}, \mathcal{S}_{Jac}, \mathcal{S}_{Poin}\}$  will be called *Maclaurin-Jacobi-Poincaré pattern*.

The main interest of this paper is restricted at discussing the Maclaurin-Jacobi pattern, that includes the first two sequences  $\{\mathcal{S}_{Macl}, \mathcal{S}_{Jac}\}$  and their temporal evolution  $\mathcal{S}_{Macl} \rightarrow \mathcal{S}_{Jac}$ .

### 3 The second order phase transitions

Let us briefly recall the main definitions of Landau's phase transition theories.

• **Landau's Thermodynamical Theory.** A first semi-empirical theory of phase transitions, was given by P. Ehrenfest (1933) on the basis of the previous thermodynamical results achieved by J.W. Gibbs and F. Van der Waals. It is based on the discontinuity (of various orders) of the free energy<sup>12</sup>  $F(P, T) = U(P, T) - T \cdot S(P, T)$  (with  $U$  internal energy,  $S$  entropy,  $P$  pressure), so that a phase transition is of order  $n \geq 1$  if there exists, at least, a partial derivative of  $F$ , of order  $n$ , that exhibits a discontinuity, while all previous partial derivatives of order  $m < n$ , are continuous. However, this classification was incomplete and not physically consistent.

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<sup>10</sup>As regard the various notions and group notations, let us follows [CMR], [LL3], Chap. XII., and [Ga], [MZ].

<sup>11</sup>But no static.

<sup>12</sup>Among the thermodynamical variables  $(P, T, V)$ , we choose  $(P, T)$  as independent.

In 1937, L.D. Landau ([La]), starting from the results of the predecessors (above mentioned), gave a first version of a continuum phase transition theory, called *Landau's thermodynamic theory* (see [LL5], Chap. XIV, §143). This theory, before all, distinguishes between phase transitions 'with order parameter' and phase transitions 'without order parameter'. The transitions of the first type are those involving phases having the same symmetry group<sup>13</sup> or different symmetry groups not connected by any relation of the type group-subgroup. The transitions of the second type, instead, regard phases whose symmetry groups are different but related by some relation of the type group-subgroup. These are characterized (differently from those of the first type) by a symmetry change of the order parameter, that must be invariant with respect to each symmetry groups of the two phases. Besides the symmetry change, the thermodynamical functions, characterizing the physical state of the system subjected to such phase transition, may not assume any values because it must change according to the above mentioned symmetry change, and in dependence on such order parameter.

The latter critical analysis has been systematically developed by Landau in a second time after his first Thermodynamical Theory, and represents the so-called *Landau's Phenomenological Theory*. Is is stronger (as physical theory) than his initial Thermodynamical Theory.

The (possible) counterexample described in this paper, just confirms this non-equivalence.

In the framework of the Thermodynamical Theory, for phase transitions with order parameter, Landau first assumes that the free energy must be function of the order parameter, as an independent and extensive variable  $\eta$ , typical of the thermodynamical system under examination<sup>14</sup>. In addition, he assumes (*Landau's hypothesis*) that  $F(P, T, \eta)$  must be analytic<sup>15</sup> in a neighborhood of the critical point  $(P_c, T_c)$  (where such a transition occurs), that must be  $\eta = 0$  in the initial phase ( $T < T_c$ ) and that must be  $\eta \neq 0$  in the final phase ( $T > T_c$ ). Hence, under such conditions, in a neighborhood of the critical point  $(P_c, T_c)$ , we must have

$$(4) \quad F(P, T, \eta) = F_0 + \alpha(P, T)\eta + A(P, T)\eta^2 + B(P, T)\eta^3 + C(P, T)\eta^4 + \dots$$

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<sup>13</sup>For instance, the two phases of a liquid-gas transition, has  $O(3)$  as a symmetry group.

<sup>14</sup>Examples of order parameter are the spontaneous polarization in a ferroelectric transition, or the magnetic susceptibility in a ferromagnetic transition.

<sup>15</sup>This is a weak-point of the Landau's theory, since  $(P_c, T_c)$  is a simple singularity where, formally, there is no functional analyticity.

where  $F_0 = F(P, T, 0)$  and  $\alpha, A, B, C, \dots$  are independent of  $\eta$ . Landau considers only a fourth order expansion of the type (4) (called *Landau's polynomial*), and assumes<sup>16</sup> the symmetry  $\eta \rightarrow -\eta$ , so that, in the expansion (4), the odd degree terms are zero. Moreover, since we consider equilibrium phase transitions, thermodynamically  $F$  should be a minimum in  $(P_c, T_c)$  for  $\eta = 0$ , so that the expansion (4) reduces to the following Landau's polynomial (of fourth degree)

$$(5) \quad F(P, T, \eta) = F_0 + A(P, T)\eta^2 + C(P, T)\eta^4.$$

In particular, by the minimum necessary conditions for  $F$  in  $(P_c, T_c, 0)$ , we must have

$$(\partial F / \partial \eta)_{(P_c, T_c, 0)} = 0, \quad (\partial^2 F / \partial \eta^2)_{(P_c, T_c, 0)} > 0,$$

that is (via (4))

$$(6) \quad A(P, T) = a(P)(T - T_c), \quad \eta = \eta_0(T - T_c)^{1/2}, \quad C(P_c, T_c) > 0,$$

with  $\eta_0$  constant. Landau puts (6) as necessary and sufficient conditions for classifying, as continuum, a phase transition.

• **Landau's Phenomenological Theory** (see [LL5], Chap. XIV, §145). In Landau's Thermodynamical Theory (as well as in the previous theories), many important theoretical questions about continuum phase transitions with order parameter remained unsolved; in this theory, subsisted only some vague and unfounded assumptions. The main question concerns the determination of the functional dependence laws of the values of the thermodynamical functions on the typical symmetry change of a continuum phase transition with order parameter. For this purpose, if  $G_0 \rightsquigarrow G_1$  denotes such a generic phase transition, from the initial phase with symmetry group  $G_0$  to the final phase with symmetry group  $G_1$ , Landau puts  $\eta = 0$  (and  $G_0$ -invariance) in the initial phase, and  $\eta \neq 0$  (and  $G_1$ -invariance) in the final phase. Moreover, he assumes that the symmetry  $G_1$  of the final phase (of the transition), must be the isotropy group of the order parameter  $\eta \neq 0$ , with respect to a well-defined irreducible representation<sup>17</sup>  $\mathcal{D}_{G_0}$  of  $G_0$ , satisfying further conditions, respectively called *Landau's condition* and *Lifšits's condition* (see next §4). Hence, as regard what has been said above (and that forms the essence of the so-called *Landau's Phenomenological Theory*), it is correct to denote this transition in the compact notation<sup>17</sup>  $G_0 \xrightarrow{\mathcal{D}} G_1$ .

<sup>16</sup>This symmetry is motivated by theoretical physics reasons (since we must have the same solution considering both  $\eta$  and  $-\eta$ ), and by stability reasons.

<sup>17</sup>From now on, for simplicity sake's, if not otherwise specified, any irreducible representation  $\mathcal{D}_{G_0}$  of  $G_0$ , will be denoted only by  $\mathcal{D}$ , without any other indices. Furthermore, we suppose that it denotes always an irreducible representation of  $G_0$ .



## 4 The "chain subduction criterion" and the "maximality criterion"

Let  $G_0 \xrightarrow{\mathcal{D}} G_1$  be a generic continuum phase transition from an initial phase with symmetry group  $G_0$  (where  $\eta = 0$  in the pre-existed phase) to the final phase with symmetry group  $G_1$  (where  $\eta \neq 0$  in the resulting phase), associated to the irreducible representation  $\mathcal{D}$  of  $G_0$ , and having order parameter  $\eta \in \mathbb{R}^+ \cup \{0\}$ . In this case, we say (with Landau) that  $G_1$  is a *allowed* subgroup<sup>18</sup> with respect to  $\mathcal{D}$ .

Subsequently (see [CLP]), according to Landau and other authors, if this phase transition is of second order, then the following necessary or sufficient conditions must hold<sup>19</sup>:

1.  $G_1$  is a subgroup of  $G_0$ ,
2. (*Landau's condition*)  $([\mathcal{D}]^3|I(G_0)) = 0$ , that is the symmetric cube of  $\mathcal{D}$  does not contain the identity representation  $I$  of  $G_0$ ,
3. (*Lifšits's condition*)  $(\{\mathcal{D}\}^2|V(G_0)) = 0$ , that is the anti-symmetric square of  $\mathcal{D}$  does not contain any arbitrary vectorial representation  $V$  of  $G_0$ ,
4. (*Birman's subduction criterion*)  $\mathcal{D}$  subduces the identity representation of  $G_1$ ,
5. (*Ascher's maximality criterion*)  $G_1$  is a maximal subgroup of  $G_0$ .

Successively, F.E. Goldrich and J.L. Birman ([GB]) introduce, for finite (or countable) groups, a necessary stronger condition than condition 4. of above

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<sup>18</sup>This is coherent with Landau's definition of continuum phase transition with order parameter, and in agreement with the fact that  $G_0$  and  $G_1$  must be in some relation of the type group-subgroup. See, also, the condition 1. that follows.

<sup>19</sup>The original sources are: for Landau's condition, see [La], for Lifšits's condition, see [Lif], for Birman's condition, see [Bir], and, for Ascher's condition, see [As1].

([Bi1]). The integer number<sup>20</sup>

$$(7) \quad i_{\mathcal{D}}(G_1) = \frac{1}{|G_1|} \sum_{g \in G_1} \chi_{\mathcal{D}}(g),$$

that gives the number of times that  $\mathcal{D}|_{G_1}$  (=restriction of  $\mathcal{D}$  to  $G_1$ ) contains the identity representation of  $G_1$ , is called *subduction index* (or *frequency*) of  $\mathcal{D}$ , with respect to  $G_1$ . Therefore, the (*Birman-Goldrich*) *chain subduction criterion* ([GB]) states that, if  $\mathcal{D}$  is an irreducible representation of  $G_0$ ,  $G_1$  and  $G'_1$  are subgroups of  $G_0$  such that  $G'_1$  is a subgroup of  $G_1$  (that is  $G'_1 \subseteq G_1 \subseteq G_0$ ) and  $i_{\mathcal{D}}(G_1) = i_{\mathcal{D}}(G'_1) = 1$ , then  $G'_1$  is not a permitted subgroup (in accordance with Landau) with respect<sup>21</sup> to  $\mathcal{D}$ .

In consequence of the criticisms moved in [LPC], M.V. Jarić ([Ja2]) extends this chain subduction criterion to the more general context including the case where  $i_{\mathcal{D}}(G_1) = i_{\mathcal{D}}(G'_1)$  is any non-negative integer, reaching the following, more general (*Birman-Goldrich-Jarić*) *chain subduction criterion*. If  $\mathcal{D}$  is an irreducible representation of  $G_0$ , and  $G'_1, G_1$  are subgroups of  $G_0$  such that  $G'_1 \subset G_1 \subseteq G_0$ , with  $G'_1$  subgroup of  $G_1$  and  $i_{\mathcal{D}}(G'_1) = i_{\mathcal{D}}(G_1) \in \mathbb{N} \cup \{0\}$ , then  $G'_1$  is not a (Landau's) allowed subgroup with respect to  $\mathcal{D}$ .

In the following section, these results will be applied to the Maclaurin-Jacobi pattern.

## 5 Applications to Maclaurin-Jacobi pattern

We recall that G. Bertin and L.A. Radicati (see [BR]) prove the symmetry change  $D_{\infty h} \rightarrow D_{2h}$ , associated to the dynamical evolution<sup>22</sup>  $\mathcal{S}_{Macl} \rightarrow \mathcal{S}_{Jac}$  of the Maclaurin-Jacobi pattern, may be formally interpreted as a second order phase transition in the Landau's Thermodynamical Theory.

Therefore, this transition should be, also, the same according to Landau's Phenomenological Theory (if we assume valid the equivalence between these

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<sup>20</sup>With  $\chi_{\mathcal{D}}(g)$ , we will denote the character of the element  $g \in G_1$ , computed with respect to the irreducible representation  $\mathcal{D}$  of  $G_0$ , whereas  $|G_1|$  denotes the order of  $G_1$ . For not finite groups, if we denote the usual Haar's measure on  $G_1$  with  $\mu$ , then we have  $i_{\mathcal{D}}(G_1) = \int_{G_1} \chi_{\mathcal{D}}(g) d\mu(g)$ , whereas, for countable groups, by means of the so-called *Born-Von Kàrmàn cyclicity (boundary) conditions*, a particular expression of the type (7) is possible (for further details, see [Ja3]).

<sup>21</sup>In the context of Landau's theory, this means that  $G'_1$  cannot be, with respect to the irreducible representation  $\mathcal{D}$ , the symmetry group of the final phase of a continuum phase transition of the type  $G_0 \xrightarrow{\mathcal{D}} G'_1$ .

<sup>22</sup>See the end of § 2.

theories), so that it may be classified as a transition of the type  $D_{\infty h} \xrightarrow{\mathcal{D}} D_{2h}$  with respect to some irreducible representation  $\mathcal{D}$  of  $D_{\infty h}$ , with  $D_{\infty h} = G_0$ ,  $D_{2h} = G_1$  and  $D_{2h}$  allowed subgroup (of  $D_{\infty h}$ ) with respect to  $\mathcal{D}$ . Nevertheless, it is possible to prove that such transition does not verify chain subduction criterion of Birman-Goldrich-Jarić, for any given irreducible representation  $\mathcal{D}$  of  $D_{\infty h}$ . In fact, if we consider the groups  $G_1 = D_{4h}$ ,  $G_1 = D_{6h}$ , with  $G'_1 = D_{2h}$  proper subgroup of both, we have  $D_{2h} \subset D_{4h} \subset D_{\infty h} = G_0$ ,  $D_{2h} \subset D_{6h} \subset D_{\infty h}$ , and since we'll see that  $i_{\mathcal{D}}(D_{2h}) = i_{\mathcal{D}}(D_{4h})$ ,  $i_{\mathcal{D}}(D_{2h}) = i_{\mathcal{D}}(D_{6h})$  for any given irreducible representation  $\mathcal{D}$  of  $D_{\infty h}$ , by the above mentioned Birman-Goldrich-Jarić criterion, it follows that<sup>23</sup>  $D_{2h}$  is not a (Landau's) allowed subgroup, that is  $D_{\infty h} \xrightarrow{\mathcal{D}} D_{2h}$  is not a continuum phase transition (in the sense of Landau's Phenomenological Theory), for any given irreducible representation  $\mathcal{D}$ .

In the Maclaurin-Jacobi pattern, we assume the  $Oz$  axis as a rotation axis  $h$  of the first order. Moreover, let us denote with  $\theta(t) = \omega t$  the angle of the rotation  $C_\theta$ , with  $E$  the identity, with  $I$  the inversion and with  $C_{2\phi}$  the rotations of  $\phi/2$  radians around to the rotation axis of the second order, placed in the  $Oxy$  plane.

Hence, with these notations and hypotheses, the table of characters of  $D_{\infty h}$  is the follows (see [Co], vol. I; [ITO], App. A,B; [Ja5]):

irr. repr.	$E$	$C_\theta, C_{-\theta}$	$C_{2\phi}$	$I$	$IC_\theta, IC_{-\theta}$	$IC_{2\phi}$
$A_{1g}$	1	1	1	1	1	1
$A_{1u}$	1	1	1	-1	-1	-1
$A_{2g}$	1	1	-1	1	1	-1
$A_{2u}$	1	1	-1	-1	-1	1
$E_{ng}$	2	$2 \cos(n\theta)$	0	2	$2 \cos(n\theta)$	0
$E_{nu}$	2	$2 \cos(n\theta)$	0	-2	$-2 \cos(n\theta)$	0
$E_{(n+1/2)g}$	2	$2\cos(n + 1/2)\theta$	0	2	$2\cos(n + 1/2)\theta$	0
$E_{(n+1/2)u}$	2	$2\cos(n + 1/2)\theta$	0	-2	$-2\cos(n + 1/2)\theta$	0

with  $A_{ig}, A_{iu}$   $i = 1, 2$ , real one-dimensional irreducible representations and  $E_{ng}, E_{nu}, E_{(n+\frac{1}{2})g}, E_{(n+\frac{1}{2})u}$   $n \in \mathbb{N}$ , real two-dimensional irreducible representations. Therefore, considering the group operations  $D_{2h}, D_{4h}, D_{6h}$  (see [Co], vol. I), by (7) it is immediate to verify that

$$i_{A_{1u}}(D_{6h}) = i_{A_{2g}}(D_{6h}) = i_{A_{2u}}(D_{6h}) = i_{E_{nu}}(D_{6h}) = i_{E_{(n+\frac{1}{2})u}}(D_{6h}) = 0, \quad \forall n \in \mathbb{N};$$

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<sup>23</sup>This criterion is applicable to these group chains because  $D_{\infty h}$  is countable and  $D_{4h}, D_{6h}, C_{2v}$  are finite.

$$\begin{aligned}
i_{A_{1u}}(D_{4h}) &= i_{A_{2g}}(D_{4h}) = i_{A_{2u}}(D_{4h}) = i_{E_{nu}}(D_{4h}) = i_{E_{(n+\frac{1}{2})u}}(D_{4h}) = 0, \quad \forall n \in \mathbb{N}; \\
i_{A_{1u}}(D_{2h}) &= i_{A_{2g}}(D_{2h}) = i_{A_{2u}}(D_{2h}) = i_{E_{nu}}(D_{2h}) = i_{E_{(n+\frac{1}{2})u}}(D_{2h}) = 0, \quad \forall n \in \mathbb{N}; \\
1 \leq i_{E_{ng}}(D_{6h}) &= \frac{1}{6}(1 + 5 \cos(n\theta)) \leq i_{E_{ng}}(D_{2h}) = \frac{1}{2}(1 + \cos(n\theta)), \quad \forall n \in \mathbb{N}; \\
1 \leq i_{E_{(n+1/2)g}}(D_{6h}) &= \frac{1}{6}(1 + 5 \cos(n + 1/2)\theta) \leq \\
&\leq i_{E_{(n+1/2)g}}(D_{2h}) = \frac{1}{2}(1 + \cos(n + 1/2)\theta), \quad \forall n \in \mathbb{N}; \\
1 \leq i_{E_{ng}}(D_{4h}) &= \frac{1}{4}(1 + 3 \cos(n\theta)) \leq i_{E_{ng}}(D_{2h}) = \frac{1}{2}(1 + \cos(n\theta)), \quad \forall n \in \mathbb{N}; \\
1 \leq i_{E_{(n+1/2)g}}(D_{4h}) &= \frac{1}{4}(1 + 3 \cos(n + 1/2)\theta) \leq \\
&\leq i_{E_{(n+1/2)g}}(D_{2h}) = \frac{1}{2}(1 + \cos(n + 1/2)\theta), \quad \forall n \in \mathbb{N},
\end{aligned}$$

so that

$$\left. \begin{aligned} &(i_{E_{ng}}(D_{lh}) < i_{E_{ng}}(D_{2h})) \\ &(i_{E_{(n+1/2)g}}(D_{lh}) < i_{E_{(n+1/2)g}}(D_{2h})) \end{aligned} \right\} \Leftrightarrow (0 < \theta < \frac{2k\pi}{n}) \quad \forall n \in \mathbb{N}, \quad k \in \mathbb{Z}$$

with  $l = 4, 6$ , and, fixing arbitrarily  $k$ , we have

$$\left( \lim_{n \rightarrow \infty} \frac{2k\pi}{n} = 0 \right) \Rightarrow (\theta(t) = \omega t = 0),$$

hence  $\omega = 0$  for  $t > 0$ , which is impossible because the case  $\omega = 0$  corresponds to a static self-gravitating fluid mass of spherical symmetry  $O(3)$  (see § 2, and [Le1], [Li]). Therefore

$$i_{E_{ng}}(D_{lh}) = i_{E_{ng}}(D_{2h}) \geq 1, \quad i_{E_{(n+1/2)g}}(D_{lh}) = i_{E_{(n+1/2)g}}(D_{2h}) \geq 1, \quad \forall n \in \mathbb{N}$$

with  $l = 4, 6$ . In conclusion,  $i_{\mathcal{D}}(D_{2h}) = i_{\mathcal{D}}(D_{lh})$   $l = 4, 6$ , for any given irreducible representation  $\mathcal{D}$  of  $D_{\infty h}$ , and hence, by the chain subduction criterion,  $D_{\infty h} \rightarrow D_{2h}$  is not a continuum phase transition, in contrast with the conclusion outlined in [BR]. It follows a restriction to the validity of the chain subduction criterion.

Finally, the continuum phase transition  $D_{\infty h} \rightarrow D_{2h}$  invalidates too Ascher's maximality criterion because  $D_{2h}$  is not a maximal subgroup of  $D_{\infty h}$  (hence, also of  $O(3)$ ) since  $D_{2h} \subset D_{lh} \subset D_{\infty h} (\subset O(3))$   $l = 4, 6$ , whereas, in conformity with such criterium,  $D_{2h}$  should be a maximal subgroup of  $D_{\infty h}$ .

## 6 Prospects and possible applications

The previous critical analysis, leads, among other, to a deeper critical study of the formal setting of the general (macroscopic) phase transition theory, eventually with the support of the (microscopic) statistical theory of phase transitions, also in connection with the theory of the first order phase transitions.

Indeed, further motivations for the present analysis come from possible applications of the Maclaurin-Jacobi pattern to particular phase transitions involved in some astrophysical models of stars<sup>24</sup>: for example, phase transitions may be invoked in the explanation of some aspects of the complex physical phenomenology of the solid external crust of a radio-pulsar (or of a rotating neutron star, in general), as, for instance, the (possible) formation of a superficial thin-layer of gases if we have a first order phase transition instead of a second order one.

On the other hand, a similar question is already connected with the problem of the supernova explosions (see [Gr], vol. II, Chap. XVIII, § 9, sect. a)), where (following W.A. Fowler and F. Hoyle) a particular phase transition – neutrons producing – occurs, involving a photodisintegration of  $^{56}\text{Fe}$ -atoms, forming a (solid) crystalline lattice imbedded in a degenerate (and strongly anisotropic – if there is in action an intense magnetic field<sup>25</sup>) Coulomb electron gas (see [Gr], [KW], [Pa], [ST], [Ta2]).

Again, a similar question enters into the problem of the determination of the shape left by a nova remnant from a rotating white dwarf parent (see [FH]).

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<sup>24</sup>Besides the possible applications that these arguments might have in the theory of fragmentation of atomic nuclei, and related phase transitions (see [GMS], [Mig], [RW]).

<sup>25</sup>The magnetic field configurations (see [Kc], [OPF]) of a star, are in relation with the occurrence of a phase transitions of the first order or of the second order, with different physical implications in both cases.

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