Formulation and convergence of the finite volume method for conservation laws on spacetimes with boundary

Jan G Giesselmann, Philippe G Lefloch

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Abstract

We study nonlinear hyperbolic conservation laws posed on a differential $(n+1)$-manifold with boundary referred to as a spacetime, and in which the "flux" is defined as a flux field of n-forms depending on a parameter (the unknown variable) —a class of equations recently proposed by LeFloch and Okutmustur. Our main result is a proof of the convergence of the finite volume method for weak solutions satisfying suitable entropy inequalities. A main difference with previous work is that we allow here for slices with boundary and, in addition, introduce a new formulation of the finite volume method which involves the notion of total flux functions. Under a natural global hyperbolicity condition on the spacetime and provided the spacetime is foliated by compact slices (with boundary), we establish an existence and uniqueness theory for the initial and boundary value problem, and derive a contraction property in a geometrically natural L^1-type distance.

Contents

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1 Introduction

The mathematical study of weak solutions to hyperbolic conservation laws on curved manifolds was initiated by LeFloch and co-authors almost ten years ago [1, 2, 7, 20], and this subject extended in several directions [4, 5, 8, 6, 14, 15, 17, 19, 21]. The main motivation comes from geophysical fluid dynamics (in which the problems are posed on a curved surface such as the sphere) and general relativity (in which the Einstein-Euler equations are expressed on a manifold whose metric is one of the unknowns). The class of hyperbolic conservation laws provides a simplified, yet very challenging, model on which one can develop fundamental techniques and understand aspects of nonlinear wave propagation and shock formation on manifolds.

In the present paper, we built upon the work by LeFloch and Okusmustur [20] and consider conservation laws whose flux is expressed as a family of differential forms. Let M be an oriented, smooth $(n+1)$-dimensional manifold endowed with a smooth family of n-forms $\omega = \omega(u)$ which we refered to as a *flux field*. The conservation law of interest then reads

$$d(\omega(u)) = 0, \tag{1.1}$$

where d denotes the exterior derivative and in which the unknown $u : M \to \mathbb{R}$ is a scalar field defined on this manifold. The study of weak solutions to this type of equations started in Ben-Artzi and LeFloch [7] for the class of equations

$$\partial_t u + \operatorname{div}_g f(u) = 0, \qquad M = \mathbb{R}_+ \times N, \tag{1.2}$$

where N is a closed, oriented manifold endowed with a Riemannian metric g and f is a prescribed family of flux vector fields on N. In a local coordinate system $(t, x^1, \dots, x^n)$ associated with the basis of tangent vectors $\partial_t, \partial_{x^1}, \dots, \partial_{x^n}$, we can express any vector field f by its coefficients f_i, i.e. one sets $f(u) = \sum_i f_i(u)\partial_{x^i}$. The Riemannian metric can be expressed as $g_{ij} := g(\partial_{x^i}, \partial_{x^j})$, so that

$$\operatorname{div}_g f := \frac{1}{\sqrt{|g|}} \partial_{x^i} \big(f_i \sqrt{|g|} \big),$$

where $|g|$ denotes the absolute value of the determinant of (g_{ij}).

Observe that we recover (1.2) from (1.1) by choosing

$$\omega(u) = u\sqrt{|g|} dx^1 \wedge \cdots \wedge dx^n + \sum_{i=1}^{n} (-1)^i f_i(u) \sqrt{|g|} dt \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n.$$

Hence, the field ω combines information on the flux field f and the volume form $\sqrt{|g|}dx^1 \wedge \cdots \wedge dx^n$ induced by g. In particular, no separate information on the geometric structure of M is required and the knowledge of the flux field is sufficient for writing (1.1). As pointed out in [20], using the general form (1.1) instead of the corresponding formulations for Lorentzian or Riemannian manifolds is not only more general but also allows one to develop a mathematical theory which is conceptually and technically simpler.

Our main objective is to further extend the results in [20] and allow for time slices *with a non-empty boundary*, i.e., there is a part of the boundary of M which does not constitute 'initial data' for the Cauchy problem and on which "boundary condition" must be prescribed. Recall that imposing boundary conditions for nonlinear hyperbolic conservation laws is a delicate matter, since

the nature of the data to be prescribed depends on the (unknown) solution. In fact, the boundary condition needs to be included in the very definition of entropy solutions, in a suitably weak sense. This issue was first discussed in [3] in the class of functions of bounded total variation and later in [23] for measure-valued solutions. We follow an approach initiated in the Euclidean setting by LeFloch and Dubois [13] and later [9, 16]. An alternative approach to boundary conditions for hyperbolic conservation laws goes back to Otto [22]. We also note that hyperbolic conservation laws posed on Riemannian manifolds with boundary were studied in [17]. Recall that the theory of initial value problems for entropy solutions to scalar conservation laws in the Euclidean space goes back to Kruzkov [18] and DiPerna [12] and the present work relies on their pioneering contributions.

We will introduce here suitable notions of weak solutions and measure-valued solutions (a la DiPerna). Our definitions make sense for general flux fields ω but, in addition, an entropy condition is imposed which singles out the relevant weak solutions. To this end, we impose that the pair (M, ω) satisfies a "global hyperbolicity condition", as we call it, which in particular provides a global time orientation and allows us to distinguish between "future" and "past" directions in the spacetime.

Our main contribution is the formulation and analysis of the finite volume method based on monotone numerical flux and on a fully geometric approach to the discretization of (1.1). For technical convenience and in order to be able to formulate the method as a time-stepping scheme, we assume that M is foliated by compact slices. We built here on several earlier works. The convergence of finite volume approximations of the initial value problem for hyperbolic conservation laws was established first in the Euclidean case by Cockburn, Coquel, and LeFloch [9] and later for Riemannian manifolds in [1]. Further generalizations were then provided in [20] and [14].

We find here that in the finite volume discretisation of (1.1) it is very natural to view the approximate solution u^h as being defined via *total flux functions* along faces of the triangulation, that is, the quantities denoted below by $q_e := \int_e i^* \omega(u_e)$, where e is a face and i^* is the pull-back operator associated to the inclusion $i : e \to M$. For some faces which we call *spacelike*, we are able to go back and forth between the total flux q_e and the "physical value" u_e. One important contribution in the present paper is that we have eliminated the need to introduce (face size) averages $|e_K^+|$ which were used in LeFloch and Okumustur's earlier formulation [20]. In the present proposal, we consider it to be natural to use the flux q_e for every face e while the quantity u_e is introduced for spacelike faces only.

Interestingly enough, the formulation of the finite volume scheme requires *further structure* on the manifold, especially an n-form along its boundary. This is necessary in order to determine an averaged value of the data within each boundary cell. It is not expected that stable schemes could be design which would require only the fluxes across the boundary.

We will prove several stability results for the proposed scheme with an emphasis on discrete versions of the entropy inequality. These stability estimates are sufficient to show that the sequence of approximate solutions created by the finite volume scheme converges to an *entropy measure valued solution* in the sense of DiPerna. By extending DiPerna's uniqueness theorem we show that the sequence indeed converges to an *entropy solution*. At the same time our analysis implies a natural extension of the L^1 contraction property satisfied by hyperbolic conservation laws in Euclidean space, see (3.9).

An outline of this paper is as follows: In Section 2 we introduce the notions of entropy weak/measure-valued solutions taking into account boundary data. Then we discuss the concept of global hyperbolicity and gather our main results in Section 3. Section 4 is devoted to the presentation of the finite volume scheme and to the derivation of local stability estimates. In Section 5, we are in a position to prove global stability estimates for the finite volume scheme and to prove convergence of the approximate solutions toward the entropy solution of the Cauchy problem.

2 Conservation laws posed on a spacetime

2.1 Weak solutions

In this preliminary section we present, in a slightly modified version, the formulation proposed in LeFloch and Okutmustur [20]. We assume that M is an oriented, smooth $(n+1)$-manifold with smooth boundary ∂M, which we refer to as a *spacetime with boundary*. Given an $(n+1)$-form α, its *modulus* is defined as the $(n+1)$-form

$$|\alpha| := |\overline{\alpha}|\, dx^0 \wedge \cdots \wedge dx^n,$$

where $\alpha = \overline{\alpha}\, dx^0 \wedge \cdots \wedge dx^n$ is written in an oriented frame determined from local coordinates $x = (x^\alpha) = (x^0, \ldots, x^n)$. If H is a hypersurface, we denote by $i = i_H : H \to M$ the canonical injection map, and by $i^* = i_H^*$ the pull-back operator acting on differential forms defined on M.

We denote by $C^\infty \Lambda^k(M)$ the space of all smooth fields of differential forms of degree $k \leq n+1$, and by $\mathcal{D}\Lambda^k(M) \subset C^\infty \Lambda^k(M)$ the subset of compactly supported fields.

Definition 2.1. *1. A* flux field ω *on the $(n+1)$-manifold M is a parameterized family $\omega(\overline{u}) \in C^\infty \Lambda^n(M)$ of smooth fields of differential forms of degree n, that depends smoothly upon the real parameter $\overline{u}$.*

2. The conservation law *associated with a flux field ω and with unknown $u : M \to \mathbb{R}$ reads*

$$d\big(\omega(u)\big) = 0, \tag{2.1}$$

where d denotes the exterior derivative operator and, therefore, $d\big(\omega(u)\big)$ is a field of differential forms of degree $(n+1)$ on M.

3. A flux field $\omega = \omega(\overline{u}) \in C^\infty \Lambda^n(M)$ is said to grow at most linearly *if there exists some $\alpha \in C^\infty \Lambda^n M$ such that for every hypersurface $\mathcal{H}$ and $\bar{u} \in \mathbb{R}$*

$$-i_{\mathcal{H}}^* \alpha \leq i_{\mathcal{H}}^* \partial_u \omega(\bar{u}) \leq i_{\mathcal{H}}^* \alpha. \tag{2.2}$$

Given a *smooth* solution u of (2.1) we can apply Stokes theorem on any open subset S that is compactly included in M and has piecewise smooth boundary ∂S:

$$0 = \int_S d(\omega(u)) = \int_{\partial S} i^*(\omega(u)). \tag{2.3}$$

There is a natural orientation on S (as a subset of M) and, therefore, on ∂S such that (2.3) indeed holds. Similarly, given any smooth and compactly supported function $\psi \in \mathcal{D}(\mathring{M}) = \mathcal{D}\Lambda^0(\mathring{M})$, we can write

$$d(\psi\, \omega(u)) = d\psi \wedge \omega(u) + \psi\, d(\omega(u)),$$

where the differential $d\psi$ is a 1-form field. Thus we find

$$\int_M d(\psi\, \omega(u)) = \int_M d\psi \wedge \omega(u)$$

and, by Stokes theorem,

$$\int_M d\psi \wedge \omega(u) = \int_{\partial M} i^*(\psi \omega(u)). \tag{2.4}$$

This identity is satisfied by every smooth solution to (2.1) and this motivates us to reformulate (2.1) in a *weak* form.

Definition 2.2. *1. A function $u : M \to \mathbb{R}$ is said to be* locally integrable *(respectively* integrable*) if it is measurable and for every non-negative, $(n+1)$-form field $\alpha \in \mathcal{D}\Lambda^{n+1}(\mathring{M})$ (resp. $\alpha \in C^\infty\Lambda^{n+1}(M)$) one has*

$$\int_M |u|\,\alpha \qquad \text{is finite.}$$

The space of all such functions is denoted by $L^1_{loc}(M)$ (resp. $L^1(M)$).

2. Given a flux field ω with at most linear growth, a function $u \in L^1_{loc}(M)$ is called a weak solution *to the conservation law* (2.1) *posed on M if for every test-function $\psi \in \mathcal{D}(\mathring{M})$*

$$\int_M d\psi \wedge \omega(u) = 0.$$

The above definition makes sense, since (2.2) implies that for every 1-form field $\rho \in \mathcal{D}\Lambda^1(M)$ there exists a non-negative $(n+1)$-form field $\beta \in \mathcal{D}\Lambda^{n+1}(M)$ such that

$$\sup_{\overline{u}\in\mathbb{R}} |\rho \wedge \partial_u \omega(\overline{u})| \leq \beta, \tag{2.5}$$

so that under the conditions in Definition 2.1 the integral $\int_M d\psi \wedge \omega(u)$ is finite.

The above definition can be immediately generalized to functions defined on the boundary and we denote by $L^1(\partial M)$ and $L^1_{\rm loc}(\partial M)$ the space of all integrable (resp. locally integrable) functions defined on the boundary of the manifold. Observe that the L^1 *norm* of functions in $L^1_{\rm loc}(M)$ or $L^1(M)$ is *not* canonically defined. On the other hand we will also use the standard notation $L^1\Lambda^n(\mathcal{H})$ for the space of all integrable n-form fields defined on an (n-dimensional) hypersurface $\mathcal{H}$; it should be observed that the L^1 norm of such a field *is* uniquely defined.

2.2 Entropy solutions

As is standard for nonlinear hyperbolic problems, weak solutions must be further constrained by imposing initial, boundary, as well as entropy conditions, which we now discuss.

Definition 2.3. *A field of n-forms $\Omega = \Omega(\overline{u}) \in C^\infty\Lambda^n(M)$ depending Lipschitz continuously on $\overline{u}$ is called a* (convex) entropy flux field *for the conservation law* (2.1) *if there exists a (convex) Lipschitz continuous function $U : \mathbb{R} \to \mathbb{R}$ such that*

$$\Omega(\overline{u}) = \int_0^{\overline{u}} \partial_u U(\overline{v})\, \partial_u \omega(\overline{v})\, d\overline{v}, \qquad \overline{u} \in \mathbb{R}.$$

It is said to be admissible *if, moreover,* $\sup |\partial_u U| < \infty$*, and the pair (U, Ω) is called an (admissible, convex)* entropy pair.

For instance, if one picks up the family of functions $U(\overline{u}, \overline{v}) := |\overline{u} - \overline{v}|$, where $\overline{v}$ is a real parameter, the corresponding family of entropy flux fields reads

$$\mathbf{\Omega}(\overline{u}, \overline{v}) := \operatorname{sgn}(\overline{v} - \overline{u})\,(\omega(\overline{v}) - \omega(\overline{u})), \tag{2.6}$$

which provides us with a natural generalization to spacetimes of Kruzkov's entropy pairs.

Based on the notion of entropy flux above, we can derive entropy inequalities in the following way. Given any smooth solution u and multiplying (2.1) by $\partial_u U(u)$ we obtain the balance law

$$d(\Omega(u)) - (d\Omega)(u) + \partial_u U(u)(d\omega)(u) = \partial_u U(u) d(\omega(u)) = 0.$$

However, for discontinuous solutions this identity can not be satisfied as an equality and, instead, we impose the *entropy inequalities*

$$d(\Omega(u)) - (d\Omega)(u) + \partial_u U(u)(d\omega)(u) \leq 0 \tag{2.7}$$

in the sense of distributions for all admissible entropy pairs (U, Ω). These inequalities can be justified, for instance, via the vanishing viscosity method, that is by searching for weak solutions that are realizable as limits of smooth solutions to a parabolic regularization of (2.1).

To prescribe initial and boundary conditions, we observe that, without further assumption on the flux field (yet to be imposed shortly below), points along the boundary ∂M can not be distinguished, and it is natural to prescribe the trace of the solution along the *whole* of the boundary ∂M. This is possible provided the boundary condition

$$u|_{\partial M} = u_B \tag{2.8}$$

associated with some data $u_B : \partial M \to \mathbb{R}$, is understood in a sufficiently *weak* sense, as now defined.

Definition 2.4. *Let $\omega = \omega(\overline{u})$ be a flux field with at most linear growth and let $u_B \in L^1(\partial M)$ be a prescribed boundary function. Then, a function $u \in L^1_{loc}(M)$ is called an* entropy solution *to the boundary value problem determined by the conservation law* (2.1) *and the boundary condition* (2.8) *if there exists a field of n-forms $\gamma \in L^1_{loc}\Lambda^n(\partial M)$ such that*

$$\begin{aligned}&\int_M \Big(d\psi \wedge \Omega(u) + \psi\, (d\Omega)(u) - \psi\, \partial_u U(u)(d\omega)(u) \Big) \\ &- \int_{\partial M} \psi_{|\partial M} \left(i^*\Omega(u_B) + \partial_u U(u_B)\big(\gamma - i^*\omega(u_B)\big)\right) \geq 0\end{aligned}$$

for every admissible convex entropy pair (U, Ω) and every function $\psi \in \mathcal{D}(M)$.

In the above definition, all integrals under consideration are finite, in particular the one involving the entropy flux since $u \in L^1_{\rm loc}(M)$ and any admissible entropy flux also satisfies the condition (2.5).

The above definition can be generalized to encompass solutions within the much larger class of measure-valued mappings. Indeed, following DiPerna [12], we consider solutions that are no longer functions but *locally integrable Young measures*, i.e., weakly measurable maps $\nu : M \to \mathrm{Prob}(\mathbb{R})$ taking values within the set of probability measures $\mathrm{Prob}(\mathbb{R})$ and such that

$$\int_M \langle \nu, |\cdot| \rangle\, \alpha$$

is finite for every $(n+1)$-form field $\alpha \in \mathcal{D}\Lambda^{n+1}(\mathring{M})$.

Definition 2.5. *Given a flux field $\omega = \omega(\overline{u})$ with at most linear growth and given a boundary function $u_B \in L^1(\partial M)$, a locally integrable Young measure $\nu : M \to Prob(\mathbb{R})$ is called an* entropy measure-valued solution *to the boundary value problem* (2.1), (2.8) *if there exists a boundary field $\gamma \in L^1_{loc}\Lambda^n(\partial M)$ such that the entropy inequalities*

$$\begin{aligned}&\int_M \Big\langle \nu, d\psi \wedge \Omega(\cdot) + \psi \left((d\Omega)(\cdot) - \partial_u U(\cdot)(d\omega)(\cdot)\right)\Big\rangle \\ &- \int_{\partial M} \psi_{|\partial M} \Big(i^*\Omega(u_B) + \partial_u U(u_B)\big(\gamma - i^*\omega(u_B)\big)\Big) \geq 0\end{aligned}$$

for every admissible convex entropy pair (U, Ω) and every function $\psi \in \mathcal{D}(M)$.

Finally, we introduce a geometric compatibility condition which is quite natural (and will simplify some of the follow-up statements), since it ensures that constants are trivial solutions to the conservation law —This is a property shared by many models of fluid dynamics such as the shallow water equations on a curved manifold.

Definition 2.6. *A flux field ω is called* geometry-compatible *if it is closed for each value of the parameter,*

$$(d\omega)(\overline{u}) = 0, \qquad \overline{u} \in \mathbb{R}. \tag{2.9}$$

When (2.9) holds, then it follows from Definition 2.3 that every entropy flux field Ω also satisfies the geometric compatibility condition

$$(d\Omega)(\overline{u}) = 0, \qquad \overline{u} \in \mathbb{R}. \tag{2.10}$$

In turn, the entropy inequalities (2.7) satisfied by an entropy solution $u : M \to \mathbb{R}$ simplify drastically, and take the form

$$d(\Omega(u)) \leq 0. \tag{2.11}$$

3 Well-posedness theory

3.1 Global hyperbolicity

In general relativity, it is a standard assumption that the spacetime should be globally hyperbolic. This notion must be adapted to the present setting, since we do not have a Lorentzian structure, but solely the n-volume form structure associated with the flux field ω.

From now on a flux field $\omega = \omega(\overline{u})$ is fixed. The following definition imposes a non-degeneracy condition on that flux, which will be assumed from now.

Definition 3.1. *The flux field $\omega = \omega(\overline{u})$ of the conservation law* (2.1) *on the manifold M is said to satisfy the* global hyperbolicity condition *if there exists a 1-form field $T \in C^\infty\Lambda^1(M)$ called a* field of observers *such that*

$$T \wedge \partial_u\omega(\overline{u}) > 0 \qquad \overline{u} \in \mathbb{R}. \tag{3.1}$$

A hypersurface $\mathcal{H}$ is called spacelike *if for every normal 1-form field N*

$$N \wedge \partial_u\omega(\overline{u}) \neq 0, \qquad \overline{u} \in \mathbb{R}, \tag{3.2}$$

and one fixes the orientation on $\mathcal{H}$ so that

$$i_{\mathcal{H}}^*\partial_u\omega(\overline{u}) > 0, \qquad \overline{u} \in \mathbb{R}. \tag{3.3}$$

To every hypersurface $\mathcal{H}$ and entropy flux Ω we can associate the function

$$q_{\mathcal{H}}^{\Omega} : \quad \mathbb{R} \to \mathbb{R}, \qquad \overline{u} \mapsto \int_{\mathcal{H}} i^*\Omega(\overline{u}),$$

which represents the *total entropy flux* along the hypersurface $\mathcal{H}$. When $\Omega = \omega$, we also use the short-hand notation $q_{\mathcal{H}} := q_{\mathcal{H}}^{\omega}$.

Lemma 3.2. *If $\mathcal{H}$ is a spacelike hypersurface, then the total flux function $q_{\mathcal{H}}$ is strictly monotone and, therefore, one-to-one on its image.*

The orientation on $\mathcal{H}$ implies the following positivity property of Kruzkov's entropy flux fields

$$i_{\mathcal{H}}^*\mathbf{\Omega}(\overline{u}, \overline{v}) \geq 0, \qquad \overline{u}, \overline{v} \in \mathbb{R}. \tag{3.4}$$

Moreover, the total entropy flux functions $q_{\mathcal{H}}^{\Omega}$ satisfy the identity

$$\partial_q \left(q_{\mathcal{H}}^{\Omega} \circ q_{\mathcal{H}}^{-1} \right) = \partial_u U \circ q_{\mathcal{H}}^{-1}. \tag{3.5}$$

Proof. Since $\mathcal{H}$ is spacelike, (3.2) holds. We introduce a vector field X along $\mathcal{H}$ (not tangential to the hypersurface) such that $\langle N, X\rangle > 0$. Then, at each point of $\mathcal{H}$ we can supplement this vector with n vectors tangent to $\mathcal{H}$ so that $(X, e_1, \dots, e_n)$ is a positively-oriented (say) basis. The main point is to make a continuous selection along $\mathcal{H}$, the specific sign being irrelevant for our present argument. Hence, by (3.2) one has $(N \wedge \partial_u \omega(u))(X, e_1, \dots, e_n)$ keeps a constant sign for all u. Since N is a normal form we have

$$(N \wedge \partial_u \omega(u))(X, e_1, \dots, e_n) = \langle N, X\rangle \, (\partial_u \omega(u))(e_1, \dots, e_n),$$

so that $(\partial_u \omega(u))(e_1, \dots, e_n)$ also keeps a constant sign for all u. This shows that the function $\overline{u} \mapsto \int_{\mathcal{H}} i^* \partial_u \omega(\overline{u})$, never vanishes. Hence, the flux $q_{\mathcal{H}}$ is a strictly monotone function.

From the definitions, we can compute

$$i^*_{\mathcal{H}} \mathbf{\Omega}(\overline{u}, \overline{v}) = \operatorname{sgn}(\overline{v} - \overline{u}) \, i^*_{\mathcal{H}} \int_{\overline{u}}^{\overline{v}} \partial_u \omega(\overline{w}) \, d\overline{w} \geq 0,$$

which is (3.4). On the other hand, the proof of the last identity (3.5) is obvious from the definitions. □

3.2 Existence and uniqueness results

The given observer T canonically determines the following notion of causality.

Definition 3.3. *Given two closed hypersurfaces $\mathcal{H}, \mathcal{H}' \subset M$ such that $\partial\mathcal{H}, \partial\mathcal{H}' \subset \partial M$, one says that $\mathcal{H}'$* lies in the future of *$\mathcal{H}$ and one writes $\mathcal{H} \prec \mathcal{H}'$ if there exists a smooth and one-to-one mapping $F : \mathcal{H} \times [0,1] \to M$ such that*

$$\begin{aligned} &F(\mathcal{H} \times \{0\}) = \mathcal{H}, \qquad && F(\mathcal{H} \times \{1\}) = \mathcal{H}', \\ &\langle T, DF(\partial_s)\rangle > 0, \qquad && F(\partial\mathcal{H} \times [0,1]) \subset \partial M, \end{aligned} \tag{3.6}$$

where DF denotes the tangent map and $\partial_s = \partial/\partial s$ denotes the coordinate tangent vector corresponding to $[0,1]$. The set $\mathcal{B}(\mathcal{H}, \mathcal{H}') := F(\partial\mathcal{H} \times [0,1]) \subset \partial M$ is called the boundary of the region limited by *$\mathcal{H}$ and $\mathcal{H}'$.*

To state how the boundary conditions are assumed we need the following definition.

Definition 3.4. *Given any open subset $S \subset M$ with piecewise smooth boundary, a smooth 1-form field N on $\mathcal{H}$ defined on a smooth manifold $\mathcal{H} \subset \partial S$ is called* outward pointing with respect to S *if $\langle N, X\rangle > 0$ on $\mathcal{H}$ for all tangent vectors $X \in T_pM \setminus \{0\}$ associated with curves* leaving S*. One calls $\mathcal{H}$ an* outflow boundary of S *(respectively an* inflow boundary of S*) if the hypersurface $\mathcal{H}$ is spacelike and any outward pointing 1-form field N on $\mathcal{H}$ satisfies the sign condition $N \wedge \partial_u \omega(\overline{u}) > 0$ (resp. $N \wedge \partial_u \omega(\overline{u}) < 0$) for all $\overline{u}$.*

In the following, we require that the inflow boundary of the spacetime $(\partial M)^-$ is non-empty, as this ensures that the boundary data will be assumed in a strong sense on an open subset of the boundary, at least. We will give an example later (in the appendix) which shows that $(\partial M)^- \neq \emptyset$ need not be a consequence of the hyperbolicity condition.

From now on we assume the existence of a foliation by spacelike hypersurfaces of M, i.e.

$$M = \bigcup_{0 \leq t \leq T} \mathcal{H}_t, \tag{3.7}$$

where each slice $\mathcal{H}_t$ is a spacelike hypersurface and has the topology of a smooth n-manifold N with boundary. Furthermore we impose that $\mathcal{H}_0$ is an inflow boundary of M. By the foliation we have $M = [0,T] \times N$ topologically and a decomposition of the boundary is induced:

$$\partial M = \mathcal{H}_0 \cup \mathcal{H}_T \cup \partial^0 M, \qquad \partial^0 M = \bigcup_{0 \le t \le T} \partial \mathcal{H}_t. \tag{3.8}$$

Our main theory of existence, uniqueness, and stability is as follows. Observe that the L^1 stability property is fully geometric in nature, in that it is stated for any two hypersurfaces such that one lies in the future of the other.

Theorem 3.5 (Well-posedness theory for conservation laws on a spacetime)**.** *Let M be an $(n+1)$-dimensional spacetime with boundary and $\omega = \omega(\overline{u})$ be a geometry-compatible flux field on M growing at most linearly, and assume that the global hyperbolicity condition* (3.1) *holds, the inflow boundary $(\partial M)^-$ is non-empty and M admits a foliation. A boundary data $u_B \in L^\infty(M)$ being prescribed on ∂M, the boundary value problem determined by the conservation law* (2.1) *and the boundary condition* (2.8) *admits a unique entropy solution $u \in L^1_{loc}(M)$ which, moreover, has well-defined L^1 traces on any spacelike hypersurface. These solutions determine a (Lipschitz continuous) contracting semi-group in the sense that for any two hypersurfaces $\mathcal{H} \prec \mathcal{H}'$ and any Kruzkov entropy $\mathbf{\Omega}$ and any $A_B \in L^\infty \Lambda^n(\partial M)$ such that*

$$|\partial_u \omega|_{\partial M}| \le A_B \text{ in } \partial M \times \mathbb{R}$$

the following inequality holds

$$\int_{\mathcal{H}'} i^*_{\mathcal{H}'} \mathbf{\Omega}\big(u_{\mathcal{H}'}, v_{\mathcal{H}'}\big) \le \int_{\mathcal{H}} i^*_{\mathcal{H}} \mathbf{\Omega}\big(u_{\mathcal{H}}, v_{\mathcal{H}}\big) + \int_{\mathcal{B}} \big|u_B - v_B\big| A_B, \tag{3.9}$$

where $\mathcal{B} := \mathcal{B}(\mathcal{H}, \mathcal{H}')$ is the boundary between $\mathcal{H}$ and $\mathcal{H}'$. Moreover, the boundary data u_B along any (inflow with respect to M) spacelike parts $\mathcal{H}_B \subset \partial M$ is assumed in the strong sense

$$\lim_{\mathcal{H} \to \mathcal{H}_B} \int_{\mathcal{H}} i^*_{\mathcal{H}} \mathbf{\Omega}\big(u_{\mathcal{H}}, v_{\mathcal{H}}\big) = \int_{\mathcal{H}_B} i^*_{\mathcal{H}_B} \mathbf{\Omega}(u_B, v_B), \tag{3.10}$$

where $\mathcal{H}$ is a sequence of hypersurfaces approaching the boundary $\mathcal{H}_B$ in a sufficiently strong topology.

More generally, one can also express the contraction property within an arbitrary open subset with smooth boundary: similarly to (3.9), the total flux over the *outflow* part of the boundary of this subset is controlled by the total flux over the remaining part. We can also extend a result originally established by DiPerna [12] (for conservation laws posed on the Euclidean space) within the broad class of entropy measure-valued solutions.

Theorem 3.6 (Uniqueness of measure-valued solutions for conservation laws on spacetimes)**.** *Let ω be a geometry-compatible flux field on a spacetime M growing at most linearly and satisfying the global hyperbolicity condition* (3.1)*. Then, any locally integrable entropy measure-valued solution ν (see Definition 2.5) to the initial value problem* (2.1), (2.8) *reduces to a Dirac mass at each point, more precisely*

$$\nu = \delta_u, \tag{3.11}$$

where $u \in L^1_{loc}(M)$ is the unique entropy solution to the same problem.

4 Finite volume scheme based on total flux functions

4.1 Triangulations and numerical flux functions

To introduce the finite volume scheme, we first need to introduce a triangulation (or, rather, a family of triangulations).

Definition 4.1. • *A* triangulation $\mathcal{T}$ *of the spacetime* M *is a set of disjoint open subsets* $K \subset M$ *called cells such that* $\cup_{K\in\mathcal{T}}\overline{K} = M$ *and the boundary of each cell* K *is the union of finitely many smooth hypersurfaces* $e \subset M$*, called the faces of* K*. The set of faces of* K *is denoted* ∂K*. For any two cells* K, K' *one also requires that* $\overline{K} \cap \overline{K}'$ *is a common face of* K *and* K' *or a submanifold of* M *with co-dimension* 2*, at least.*

- *The triangulation is said to be* admissible *if every cell* K *admits one inflow face* e_K^- *and one outflow face* e_K^+ *(in the sense of Definition 3.4 and with respect to* K*). The set of remaining faces (which might also be inflow or outflow faces) is denoted by* $\partial^0 K := \partial K \setminus \{e_K^\pm\}$*. In addition, one requires that every inflow face* e_K^- *is the outflow* $e_{K'}^+$ *of some other cell* K' *or else a subset of* $\mathcal{H}_0$*, and that every* vertical face $e_K^0 \in \partial^0 K$ *is also a vertical face of some other cell or else a part of the vertical boundary* $\partial^0 M$*.*

- *A triangulation is said to be* associated with the foliation *if there exists a sequence of times* $0 = t_0 < t_1 < \cdots < t_N = T$ *such that all spacelike faces* e_K^-, e_K^+ *are subsets of the slices* $\mathcal{H}_n := \mathcal{H}_{t_n}$ *for some* $n = 0, \ldots, N$ *and determine triangulations of them.*

It will be convenient to fix the orientation as follows. Each face $e \in \partial^0 K$ is oriented such that an n-form $\eta \in C^\infty\Lambda^n(e)$ is positive if it satisfies $N \wedge \eta > 0$ for every outward pointing 1-form N. These orientations together with the orientations of $e_K^\pm$ given in definition 3.1 will yield the desired signs in the definition of the finite volume scheme as can be seen by the following observation: For every smooth solution u of (2.1) Stokes theorem yields

$$\int_{e_K^+} i^*\omega(u) = \int_{e_K^-} i^*\omega(u) - \sum_{e_K^0 \in \partial K^0} \int_{e_K^0} i^*\omega(u). \tag{4.1}$$

The finite volume scheme, as we propose to define it here, relies on *approximate total flux* along the spacelike faces $e_K^\pm$, that is if u is a solution we replace the total flux by its average for some constant state $u_{e_K^\pm}$:

$$\int_{e_K^\pm} i^*\omega(u) \approx q_{e_K^\pm}(u_{e_K^\pm}).$$

The evolution of these values is determined by the Stokes formula provided we prescribe the total flux along vertical faces $e_K^0 \in \partial^0 K$. Precisely, for each cell K and $e \in \partial^0 K$ (oriented as above) we introduce a Lipschitz continuous numerical flux $Q_{K,e} : \mathbb{R}^2 \longrightarrow \mathbb{R}$ satisfying the classical consistency, conservation and monotonicity properties

- $Q_{K,e}(\overline{u}, \overline{u}) = q_e(\overline{u})$.
- $Q_{K,e}(\overline{u}, \overline{v}) = -Q_{K_e,e}(\overline{v}, \overline{u})$.
- $\frac{\partial}{\partial \overline{u}} Q_{K,e}(\overline{u}, \overline{v}) \geq 0, \qquad \frac{\partial}{\partial \overline{v}} Q_{K,e}(\overline{u}, \overline{v}) \leq 0,$

where K_e denotes the cell sharing the face e with K. Note that in the right hand side of the first condition it is understood that e is oriented as a boundary of K.

Now, the consequence (4.1) of Stokes formula suggests to pose, for each cell K,

$$q_{e_K^+}(u_{e_K^+}) = q_{e_K^-}(u_{e_K^-}) - \sum_{e \in \partial^0 K} Q_{K,e}(u_{e_K^-}, u_{e_{K_e}^-}), \tag{4.2}$$

as long as e is not a part of the boundary of the manifold. The set $\partial^0 K$, by definition, may also include boundary faces: for such faces, the element K_e is not defined and, instead, we determine the corresponding state $u_{e_{K_e}^-}$ (still denoted with the same symbol, with now K_e being a "fictitious" cell), as follows. Along the boundary, to handle *non-inflow faces* $e^0 \subset \partial M$ we fix a positive n-form field α_B (once for all) and define

$$u_{e^0} := \Big(\int_{e^0} \alpha_B \Big)^{-1} \int_{e^0} u_B \, \alpha_B. \tag{4.3}$$

This definition is used even on the inflow parts of the boundary (where we could also inverse the total flux function so that the additional n-form field is not really necessary on inflow faces).

Finally to guarantee the stability of the scheme we impose the following version of the *CFL stability condition:* for all $K \in \mathcal{T}$

$$\sum_{e^0 \in \partial^0 K} \left| \frac{\sup_{u,v} \partial_u Q_{K,e^0}(u,v) - \partial_v Q_{K,e^0}(u,v)}{\inf_u \partial_u q_{e_K^+}(u)} \right| \leq \frac{1}{2} \tag{4.4}$$

in which the supremum is taken over range of u_B.

4.2 Convex decomposition and local entropy inequalities

Our analysis of the finite volume scheme is based on a convex decomposition of the fluxes. This technique goes back to Tadmor [24] for one-dimensional problems, Coquel and LeFloch [10, 11] for equations in several space dimensions, and Cockburn, Coquel and LeFloch [9] for finite volume schemes. Due to the geometry compatibility of the flux (2.9) and our choice of orientations of faces Stokes Theorem implies for each cell K

$$0 = \int_K d(\omega(u_{e_K^-})) = q_{e_K^+}(u_{e_K^-}) - q_{e_K^-}(u_{e_K^-}) + \sum_{e^0 \in \partial^0 K} Q_{K,e^0}(u_{e_K^-}, u_{e_K^-}).$$

Subtracting this identity from (4.2) yields

$$q_{e_K^+}(u_{e_K^+}) - q_{e_K^+}(u_{e_K^-}) + \sum_{e^0 \in \partial^0 K} \left(Q_{K,e^0}(u_{e_K^-}, u_{e_{K_{e^0}}^-}) - Q_{K,e^0}(u_{e_K^-}, u_{e_K^-}) \right) = 0.$$

Suppose we are given some real $\lambda_{K,e^0} \geq 0$ for each vertical face $e^0 \in \partial^0 K$ with $K \in \mathcal{T}$ such that $\sum_{e^0 \in \partial^0 K} \lambda_{K,e^0} = 1$ and define $\tilde{u}_{K,e^0}$ by

$$q_{e_K^+}(\tilde{u}_{K,e^0}) = q_{e_K^+}(u_{e_K^-}) - \frac{1}{\lambda_{K,e^0}} \left(Q_{K,e^0}(u_{e_K^-}, u_{e_{K_{e^0}}^-}) - Q_{K,e^0}(u_{e_K^-}, u_{e_K^-}) \right), \tag{4.5}$$

where we will have to show that the right hand side lies in the image of $q_{e_K^+}$. So we have a *convex decomposition*

$$q_{e_K^+}(u_{e_K^+}) = \sum_{e^0 \in \partial^0 K} \lambda_{K,e^0} q_{e_K^+}(\tilde{u}_{K,e^0}). \tag{4.6}$$

For the choice of $\lambda_{K,e^0} \geq 0$, at least the ratio

$$\frac{1}{\lambda_{K,e^0}} \left(Q_{K,e^0}(u_{e_K^-}, u_{e_{K_{e^0}}^-}) - Q_{K,e^0}(u_{e_K^-}, u_{e_K^-}) \right)$$

should be finite so that $\lambda_{K,e^0} q_{e_K^+}(\tilde{u}_{K,e^0})$ is well-defined. This is fulfilled with the following definition:

$$\hat{\lambda}_K := \sum_{e^0 \in \partial^0 K} \underbrace{\left| \frac{\sup_{u,v} \left(\partial_u Q_{K,e^0}(u,v) - \partial_v Q_{K,e^0}(u,v) \right)}{\inf_u \partial_u q_{e_K^+}(u)} \right|}_{=:\hat{\lambda}_{K,e^0}} \leq \frac{1}{2}, \quad \lambda_{K,e^0} := \frac{\hat{\lambda}_{K,e^0}}{\hat{\lambda}_K}.$$

We will show that the right hand side of (4.5) lies in the image of $q_{e_K^+}$ for the case $u_{e_K^-} \leq u_{e_{K_{e^0}}^-}$, the other case follows analogously. We have

$$\begin{aligned}
0 &\leq -\frac{1}{\lambda_{K,e^0}} \left(Q_{K,e^0}(u_{e_K^-}, u_{e_{K_{e^0}}^-}) - Q_{K,e^0}(u_{e_K^-}, u_{e_K^-}) \right) \\
&\leq \frac{1}{2} \frac{\inf_u \partial_u q_{e_K^+}(u)}{\sup_{u,v} |\partial_v Q_{K,e^0}(u,v)|} \sup_{u,v} |\partial_v Q_{K,e^0}(u,v)| (u_{e_{K_{e^0}}^-} - u_{e_K^-}) \\
&\leq \frac{1}{2} (q_{e_K^+}(u_{e_{K_{e^0}}^-}) - q_{e_K^+}(u_{e_K^-}))
\end{aligned}$$

due to the monotonicity of $q_{e_K^+}$. Hence the right hand side of (4.5) lies in the interval $[q_{e_K^+}(u_{e_K^-}), q_{e_K^+}(u_{e_{K_{e^0}}^-})]$ and therefore in the image of $q_{e_K^+}$. The convex decomposition (4.6) enables us to prove entropy inequalities, as follows.

Lemma 4.2. *For every convex entropy flux Ω and cell K we have*

$$q_{e_K^+}^{\Omega}(u_{e_K^+}) \leq \sum_{e^0 \in \partial^0 K} \lambda_{K,e^0} q_{e_K^+}^{\Omega}(\tilde{u}_{K,e^0}), \tag{4.7}$$

which makes sense as e_K^+ is oriented.

Proof. Using integration by parts we get

$$\begin{aligned}
&\sum_{e^0 \in \partial^0 K} \lambda_{K,e^0} \left(i^*\Omega(\tilde{u}_{K,e^0}) - i^*\Omega(u_{e_K^+}) \right) \\
&= \sum_{e^0 \in \partial^0 K} \lambda_{K,e^0} \Big[[\partial_u U(v) i^*\omega(v)]_{u_{e_K^+}}^{\tilde{u}_{K,e^0}} + \int_{\tilde{u}_{K,e^0}}^{u_{e_K^+}} \partial_{uu} U(v) i^*\omega(v)\, dv \Big] \\
&= \sum_{e^0 \in \partial^0 K} \lambda_{K,e^0} \Big[\partial_u U(u_{e_K^+}) \left[i^*\omega(\tilde{u}_{K,e^0}) - i^*\omega(u_{e_K^+}) \right] \\
&\qquad + \int_{\tilde{u}_{K,e^0}}^{u_{e_K^+}} \partial_{uu} U(v) \left(i^*\omega(v) - i^*\omega(\tilde{u}_{K,e^0}) \right) dv \Big].
\end{aligned}$$

When we integrate over e_K^+ the first term in the last line vanishes due to (4.6), while the latter term is non-negative because U is convex and $i^*\partial_u \omega$ is a positive n-form. □

The next step is the derivation of an entropy inequality for the faces. Note that here and in the sequel we do not require entropies to be admissible.

Lemma 4.3 (Entropy inequality for the faces)**.** *For every convex entropy pair (U,Ω) and each $K\in\mathcal{T}$ and $e^0\in\partial^0 K$ there exists a numerical entropy flux function $Q^{\Omega}_{K,e^0}:\mathbb{R}^2\longrightarrow\mathbb{R}$ satisfying for every $\overline{u},\overline{v}\in\mathbb{R}$:*

- *Q^{Ω}_{K,e^0} is consistent with the entropy flux Ω:*

$$Q^{\Omega}_{K,e^0}(\overline{u},\overline{u})=\int_{e^0} i^*\Omega(\overline{u}). \tag{4.8}$$

- *Conservation*

$$Q^{\Omega}_{K,e^0}(\overline{u},\overline{v})=-Q^{\Omega}_{K_{e^0},e^0}(\overline{v},\overline{u}). \tag{4.9}$$

- *Discrete entropy inequality*

$$q^{\Omega}_{e^+_K}(\tilde{u}_{K,e^0})\le q^{\Omega}_{e^+_K}(u_{e^-_K})-\frac{1}{\lambda_{K,e^0}}\left(Q^{\Omega}_{K,e^0}(u_{e^-_K},u_{e^-_{K_{e^0}}})-Q^{\Omega}_{K,e^0}(u_{e^-_K},u_{e^-_K})\right). \tag{4.10}$$

- *Discrete boundary entropy inequality*

$$q^{\Omega}_{e^+_K}(\bar{u}_{K,e^0})\le q^{\Omega}_{e^+_K}(u_{e^-_{K_{e^0}}})+\frac{1}{\lambda_{K,e^0}}\left(Q^{\Omega}_{K,e^0}(u_{e^-_K},u_{e^-_{K_{e^0}}})-Q^{\Omega}_{K,e^0}(u_{e^-_{K_{e^0}}},u_{e^-_{K_{e^0}}})\right), \tag{4.11}$$

  where $\bar{u}_{K,e^0}$ is defined by

$$q_{e^+_K}(\bar{u}_{K,e^0}):=q_{e^+_K}(u_{e^-_{K_{e^0}}})+\frac{1}{\lambda_{K,e^0}}\left(Q_{K,e^0}(u_{e^-_K},u_{e^-_{K_{e^0}}})-Q_{K,e^0}(u_{e^-_{K_{e^0}}},u_{e^-_{K_{e^0}}})\right). \tag{4.12}$$

Remark 4.4. *Similarly to the proof of the well-definedness of $\tilde{u}_{K,e^0}$ one can show that the right hand side of* (4.12) *lies between $q_{e^+_K}(u_{e^-_K})$ and $q_{e^+_K}(u_{e^-_{K_{e^0}}})$. Hence $\bar{u}_{K,e^0}$ is well-defined.*

Proof. Step 1: For $u,v\in\mathbb{R}$, $K\in\mathcal{T}$, $e^0\in\partial^0 K$ and $K'\in\{K,K_{e^0}\}$ we introduce the notation

$$H_{K,K',e^0}(u,v):=q_{e^+_K}(u)-\frac{1}{\lambda_{K,e^0}}\left(Q_{K',e^0}(u,v)-Q_{K',e^0}(u,u)\right),$$

where

$$Q_{K_{e^0},e^0}(u,v):=-Q_{K,e^0}(v,u)\quad\text{for } e^0\in\partial^0 M,$$

and observe that

$$H_{K,K',e^0}(u,u)=q_{e^+_K}(u).$$

We claim that H_{K,K',e^0} satisfies the following monotonicity properties:

$$\frac{\partial}{\partial u}H_{K,K',e^0}(u,v)\ge 0,\qquad \frac{\partial}{\partial v}H_{K,K',e^0}(u,v)\ge 0. \tag{4.13}$$

The second property is obvious because of the monotonicity properties of the numerical fluxes. For the first property the monotonicity properties and the fact that $\partial_u q_{e^+_K}$ is positive imply

$$\begin{aligned}\frac{\partial}{\partial u}H_{K,K',e^0}(u,v)&\ge\partial_u q_{e^+_K}(u)-\frac{1}{\lambda_{K,e^0}}|\partial_u Q_{K',e^0}(u,v)|-\frac{1}{\lambda_{K,e^0}}|\partial_2 Q_{K',e^0}(u,u)|\\&\ge\left(1-2\frac{\hat{\lambda}_{K,e^0}}{\lambda_{K,e^0}}\right)\partial_u q_{e^+_K}\\&\ge\left(1-2\hat{\lambda}_K\right)\partial_u q_{e^+_K},\end{aligned}$$

which is non-negative due to the CFL-condition (4.4).

Step 2: We will establish the entropy inequalities only for the family of Kruzkov's entropies $\mathbf{\Omega}$. Therefore we introduce the numerical versions of Kruzkov's entropy fluxes

$$Q^{\mathbf{\Omega}}_{K,e^0}(u,v,c) := Q_{K,e^0}(u\vee c, v\vee c) - Q_{K,e^0}(u\wedge c, v\wedge c),$$

where $a\vee b := \max(a,b)$ and $a\wedge b := \min(a,b)$. We observe that $Q^{\mathbf{\Omega}}_{K,e^0}(u,v,c)$ satisfies the first two assertions of the lemma when the entropy flux is replaced by Kruzkov's entropy flux $\Omega = \mathbf{\Omega}$. We observe that by definition of H_{K,K',e^0}

$$\begin{aligned} &H_{K,K',e^0}(u\vee c, v\vee c) - H_{K,K',e^0}(u\wedge c, v\wedge c) \\ &= q^{\mathbf{\Omega}}_{e^+_K}(u,c) - \frac{1}{\lambda_{K,e^0}}\left(Q^{\mathbf{\Omega}}_{K',e^0}(u,v,c) - Q^{\mathbf{\Omega}}_{K',e^0}(u,u,c)\right), \end{aligned} \tag{4.14}$$

where we used

$$q^{\mathbf{\Omega}}_{e^+_K}(u,c) := q_{e^+_K}(u\vee c) - q_{e^+_K}(u\wedge c).$$

Next we will check that for any $c\in\mathbb{R}$

$$H_{K,K,e^0}(u_{e^-_K}\vee c, u_{e^-_{K_{e^0}}}\vee c) - H_{K,K,e^0}(u_{e^-_K}\wedge c, u_{e^-_{K_{e^0}}}\wedge c) \geq q^{\mathbf{\Omega}}_{e^+_K}(\tilde{u}_{K,e^0}, c). \tag{4.15}$$

To this end, we first observe that due to the monotonicity of H_{K,e^0}

$$\begin{aligned} H_{K,K',e^0}(u,v)\vee H_{K,K',e^0}(c,c) &\leq H_{K,K',e^0}(u\vee c, v\vee c), \\ H_{K,K',e^0}(u,v)\wedge H_{K,K',e^0}(c,c) &\geq H_{K,K',e^0}(u\wedge c, v\wedge c). \end{aligned} \tag{4.16}$$

Inserting $K'=K$, $u = u_{e^-_K}$ and $v = u_{e^-_{K_{e^0}}}$ in (4.16) yields

$$\begin{aligned} &H_{K,K,e^0}(u_{e^-_K}\vee c, u_{e^-_{K_{e^0}}}\vee c) - H_{K,K,e^0}(u_{e^-_K}\wedge c, u_{e^-_{K_{e^0}}}\wedge c) \\ &\geq \left| H_{K,K,e^0}(u_{e^-_K}, u_{e^-_{K_{e^0}}}) - H_{K,K,e^0}(c,c)\right| \\ &\overset{(4.5)}{=} |q_{e^+_K}(\tilde{u}_{K,e^0}) - q_{e^+_K}(c)| \\ &= \operatorname{sgn}(\tilde{u}_{K,e^0}-c)(q_{e^+_K}(\tilde{u}_{K,e^0}) - q_{e^+_K}(c)) = q^{\mathbf{\Omega}}_{e^+_K}(\tilde{u}_{K,e^0}, c). \end{aligned}$$

This proves (4.15). Combining (4.15) with (4.14) (using $K'=K$, $u = u_{e^-_K}$ and $v = u_{e^-_{K_{e^0}}}$) yields the third assertion of the lemma.

Let us now prove:

$$H_{K,K_{e^0},e^0}(u_{e^-_{K_{e^0}}}\vee c, u_{e^-_K}\vee c) - H_{K,K_{e^0},e^0}(u_{e^-_{K_{e^0}}}\wedge c, u_{e^-_K}\wedge c) \geq q^{\mathbf{\Omega}}_{e^+_K}(\bar{u}_{K,e^0}, c). \tag{4.17}$$

It follows by inserting $K' = K_{e^0}$, $u = u_{e^-_{K_{e^0}}}$ and $v = u_{e^-_K}$ in (4.16) which yields

$$\begin{aligned} &H_{K,K_{e^0},e^0}(u_{e^-_{K_{e^0}}}\vee c, u_{e^-_K}\vee c) - H_{K,K_{e^0},e^0}(u_{e^-_{K_{e^0}}}\wedge c, u_{e^-_K}\wedge c) \\ &\geq |H_{K,K_{e^0},e^0}(u_{e^-_{K_{e^0}}}, u_{e^-_K}) - H_{K,K_{e^0},e^0}(c,c)| \\ &\overset{(4.12)}{=} |q_{e^+_K}(\bar{u}_{K,e^0}) - q_{e^+_K}(c)| = q^{\mathbf{\Omega}}_{e^+_K}(\bar{u}_{K,e^0}, c). \end{aligned}$$

Inequality (4.17) combined with (4.14) (setting $K' = K_{e^0}$, $u = u_{e^-_{K_{e^0}}}$ and $v = u_{e^-_K}$) and the conservation property of $Q^{\mathbf{\Omega}}_{K,e^0}$ yields the fourth assertion of the lemma. □

Combining the last two lemmas we get:

Lemma 4.5 (Entropy inequality per cell)**.** *For each cell $K \in \mathcal{T}$ we have*

$$q^{\Omega}_{e^+_K}(u_{e^+_K}) - q^{\Omega}_{e^+_K}(u_{e^-_K}) + \sum_{e^0 \in \partial^0 K} \left(Q^{\Omega}_{K,e^0}(u_{e^-_K}, u_{e^-_{K_{e^0}}}) - Q^{\Omega}_{K,e^0}(u_{e^-_K}, u_{e^-_K}) \right) \leq 0.$$

Finally, we turn our discussion to the boundary of M.

Lemma 4.6 (Discrete boundary condition)**.** *For all convex entropy pairs (U, Ω) the following discrete boundary condition holds*

$$\begin{aligned} & Q^{\Omega}_{K,e^0}\left(u_{e^-_K}, u_{e^-_{K_{e^0}}}\right) - Q^{\Omega}_{K,e^0}\left(u_{e^-_{K_{e^0}}}, u_{e^-_{K_{e^0}}}\right) \\ \geq \; & \partial_u U(u_{e^-_{K_{e^0}}}) \left[Q_{K,e^0}(u_{e^-_K}, u_{e^-_{K_{e^0}}}) - Q_{K,e^0}(u_{e^-_{K_{e^0}}}, u_{e^-_{K_{e^0}}}) \right], \end{aligned}$$

for each e^0.

Proof. According to Lemma 3.2 the function $q^{\Omega}_{e^+_K} \circ q^{-1}_{e^+_K}$ is convex so we have using (4.12)

$$\begin{aligned} & q^{\Omega}_{e^+_K}(\bar{u}_{K,e^0}) - q^{\Omega}_{e^+_K}(u_{e^-_{K_{e^0}}}) \\ & \geq \left(q^{\Omega}_{e^+_K} \circ q^{-1}_{e^+_K} \right)' (q_{e^+_K}(u_{e^-_{K_{e^0}}})) \left(q_{e^+_K}(\bar{u}_{K,e^0}) - q_{e^+_K}(u_{e^-_{K_{e^0}}}) \right) \\ & = \frac{1}{\lambda_{K,e^0}} \left(q^{\Omega}_{e^+_K} \circ q^{-1}_{e^+_K} \right)' (q_{e^+_K}(u_{e^-_{K_{e^0}}})) \left(Q_{K,e^0}(u_{e^-_K}, u_{e^-_{K_{e^0}}}) - Q_{K,e^0}(u_{e^-_{K_{e^0}}}, u_{e^-_{K_{e^0}}}) \right) \\ & = \frac{1}{\lambda_{K,e^0}} (\partial_u U)(u_{e^-_{K_{e^0}}}) \left(Q_{K,e^0}(u_{e^-_K}, u_{e^-_{K_{e^0}}}) - Q_{K,e^0}(u_{e^-_{K_{e^0}}}, u_{e^-_{K_{e^0}}}) \right). \end{aligned}$$

Combining this inequality with (4.11) yields the assertion of the lemma. □

5 Global estimates and convergence analysis

5.1 Global entropy bounds and inequalities

We now establish a global bound on the "discrete derivatives" of the approximate solutions. To state it we introduce the following notation:

$$\begin{aligned} \partial\mathcal{T}^0 &= \{e^0 \in \partial^0 K | K \in \mathcal{T}, e^0 \subset \partial^0 M\}, & \mathcal{T}_j &= \{K \in \mathcal{T} | e^-_K \subset \mathcal{H}_j\}, \\ \partial\mathcal{T}^0_j &= \{e^0 \in \partial^0 K | K \in \mathcal{T}_j, e^0 \subset \partial^0 M\}, & \partial\mathcal{T}_j &= \{e^0 \in \partial^0 K | K \in \mathcal{T}_j\}. \end{aligned} \tag{5.1}$$

Lemma 5.1 (Global entropy dissipation estimate)**.** *The following estimate for the entropy dissipation holds*

$$\begin{aligned} & \sum_{K \in \mathcal{T}_j} q^{\Omega}_{e^+_K}(u_{e^+_K}) + c \sum_{e^0 \in \partial\mathcal{T}_j} \lambda_{K,e^0} \frac{(\inf_u \partial_u q_{e^+_K})^2}{\sup_u \partial_u q_{e^+_K}} |\tilde{u}_{K,e^0} - u_{e^+_K}|^2 \\ \leq & - \sum_{e^0 \in \partial\mathcal{T}^0_j} Q^{\Omega}_{K,e^0}(u_{e^-_K}, u_{e^-_{K_{e^0}}}) + \sum_{K \in \mathcal{T}_j} q^{\Omega}_{e^-_K}(u_{e^-_K}), \end{aligned} \tag{5.2}$$

where $2c$ is a modulus of convexity of U and Ω is the associated entropy flux field. When Ω is the n-form entropy flux field associated to the entropy $U(u) = u^2$, we have

$$\begin{aligned}&\sum_{e^0\in\partial\mathcal{T}_j}\lambda_{K,e^0}\frac{(\inf_u \partial_u q_{e_K^+})^2}{\sup_u \partial_u q_{e_K^+}}|\tilde{u}_{K,e^0}-u_{e_K^+}|^2\\ &\leq -\sum_{e^0\in\partial\mathcal{T}_j^0} Q^\Omega_{K,e^0}(u_{e_K^-},u_{e_{K_{e^0}}^-})+\sum_{K\in\mathcal{T}_j} q^\Omega_{e_K^-}(u_{e_K^-}).\end{aligned}\tag{5.3}$$

Proof. We multiply (4.10) by λ_{K,e^0} and sum over all $e_K^0\in\partial^0 K$ and all $K\in\mathcal{T}_j$. Due to (4.9) this yields

$$\begin{aligned}&\sum_{e^0\in\partial\mathcal{T}_j}\lambda_{K,e^0}q^\Omega_{e_K^+}(\tilde{u}_{K,e^0})-\sum_{K\in\mathcal{T}_j} q^\Omega_{e_K^+}(u_{e_K^-})\\ &+\sum_{e^0\in\partial\mathcal{T}_j^0} Q^\Omega_{K,e^0}(u_{e_K^-},u_{e_{K_{e^0}}^-})-\sum_{e^0\in\partial\mathcal{T}_j} Q^\Omega_{K,e^0}(u_{e_K^-},u_{e_K^-})\leq 0.\end{aligned}\tag{5.4}$$

Applying $q^\Omega_{e_K^+}\circ q^{-1}_{e_K^+}$ to the convex decomposition (4.6) yields

$$q^\Omega_{e_K^+}(u_{e_K^+})+\underbrace{\beta\sum_{e^0\in\partial^0 K}\lambda_{K,e^0}|q_{e_K^+}(\tilde{u}_{K,e^0})-q_{e_K^+}(u_{e_K^+})|^2}_{=:A}\leq\sum_{e^0\in\partial^0 K}\lambda_{K,e^0}q^\Omega_{e_K^+}(\tilde{u}_{K,e^0}),\tag{5.5}$$

where 2β is a modulus of convexity of $q^\Omega_{e_K^+}\circ q^{-1}_{e_K^+}$. Using Lemma 3.2 we have on the one hand

$$\partial_{qq}(q^\Omega_{e_K^+}\circ q^{-1}_{e_K^+})=(\partial_{uu}U\circ q^{-1}_{e_K^+})\cdot\left(\frac{1}{\partial_u q_{e_K^+}\circ q^{-1}_{e_K^+}}\right),$$

which implies $\beta\geq\frac{c}{\sup_u \partial_u q_{e_K^+}}$. On the other hand

$$|q_{e_K^+}(\tilde{u}_{K,e^0})-q_{e_K^+}(u_{e_K^+})|^2\geq(\inf_u \partial_u q_{e_K^+})^2|\tilde{u}_{K,e^0}-u_{e_K^+}|^2.$$

So we have

$$A\geq c\sum_{e^0\in\partial^0 K}\lambda_{K,e^0}\frac{(\inf_u \partial_u q_{e_K^+})^2}{\sup_u \partial_u q_{e_K^+}}|\tilde{u}_{K,e^0}-u_{e_K^+}|^2.$$

Combining (5.4) and (5.5) we get

$$\begin{aligned}&\sum_{K\in\mathcal{T}_j} q^\Omega_{e_K^+}(u_{e_K^+})+c\sum_{e^0\in\partial\mathcal{T}_j}\lambda_{K,e^0}\frac{(\inf_u \partial_u q_{e_K^+})^2}{\sup_u \partial_u q_{e_K^+}}|\tilde{u}_{K,e^0}-u_{e_K^+}|^2-\sum_{K\in\mathcal{T}_j} q^\Omega_{e_K^+}(u_{e_K^-})\\ &\leq -\sum_{e^0\in\partial\mathcal{T}_j^0} Q^\Omega_{K,e^0}(u_{e_K^-},u_{e_{K_{e^0}}^-})+\sum_{e^0\in\partial\mathcal{T}_j} Q^\Omega_{K,e^0}(u_{e_K^-},u_{e_K^-}).\end{aligned}$$

Because of (2.10) Stokes Theorem implies for every $K\in\mathcal{T}$

$$-q^\Omega_{e_K^+}(u_{e_K^-})+q^\Omega_{e_K^-}(u_{e_K^-})=\sum_{e^0\in\partial^0 K} Q^\Omega_{K,e^0}(u_{e_K^-},u_{e_K^-}).$$

Inserting this in the last inequality yields

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_j} q^{\Omega}_{e^+_K}(u_{e^+_K}) + c\sum_{e^0\in\partial\mathcal{T}_j}\lambda_{K,e^0}\frac{(\inf_u \partial_u q_{e^+_K})^2}{\sup_u \partial_u q_{e^+_K}}|\tilde{u}_{K,e^0}-u_{e^+_K}|^2 \qquad (5.6)\\
\leq\ & -\sum_{e^0\in\partial\mathcal{T}_j^0} Q^{\Omega}_{K,e^0}(u_{e^-_K},u_{e^-_{K_{e^0}}}) + \sum_{K\in\mathcal{T}_j} q^{\Omega}_{e^-_K}(u_{e^-_K}).
\end{aligned}$$

This proves the first assertion of the Lemma. Noting that for $U(u)=u^2$ we have

$$q^{\Omega}_{e^+_K}(u_{e^+_K})\geq 0$$

yields the second assertion. □

Let us choose a finite number of charts covering M (once and for all, independent of h) in such a way that (for h small enough) each element of our triangulation is contained in one chart domain. These charts induce n-forms α_{e^0} corresponding to Hausdorff measures on the faces e^0. Then we define for any $\psi\in\mathcal{D}(M)$ and cell $K\in\mathcal{T}$ with vertical face $e^0\in\partial^0 K$

$$\psi_{e^0}:=\frac{\int_{e^0}\psi\alpha_{e^0}}{\int_{e^0}\alpha_{e^0}},\quad \psi_{\partial^0 K}:=\sum_{e^0\in\partial^0 K}\lambda_{K,e^0}\psi_{e^0}.$$

Lemma 5.2 (Global entropy inequalities)**.** *Let Ω be a convex entropy flux field and $\psi\in\mathcal{D}(M)$ a non-negative test-function supported in $M\setminus\mathcal{H}_T$. Then the finite volume approximation satisfies the following global entropy inequality*

$$\begin{aligned}
&-\sum_{K\in\mathcal{T}}\int_K d(\psi\Omega)(u_{e^-_K}) - \sum_{K\in\mathcal{T}_0}\int_{e^-_K}\psi i^*\Omega(u_{e^-_K}) + \sum_{e^0\in\partial\mathcal{T}^0}\psi_{e^0}Q^{\Omega}_{K,e^0}(u_{e^-_K},u_{e^-_{K_{e^0}}})\\
&\leq A(\psi)+B(\psi)+C(\psi)+D(\psi)+E(\psi),
\end{aligned}$$

with

$$\begin{aligned}
A(\psi)&:=\sum_{K\in\mathcal{T},e^0\in\partial^0 K}\lambda_{K,e^0}(\psi_{\partial^0 K}-\psi_{e^0})\left(q^{\Omega}_{e^+_K}(\tilde{u}_{K,e^0})-q^{\Omega}_{e^+_K}(u_{e^+_K})\right),\\
B(\psi)&:=\sum_{K\in\mathcal{T},e^0\in\partial^0 K}\int_{e^0}(\psi_{e^0}-\psi)i^*\Omega(u_{e^-_K}),\\
C(\psi)&:=-\sum_{K\in\mathcal{T},e^0\in\partial^0 K}\int_{e^+_K}\lambda_{K,e^0}(\psi_{\partial^0 K}-\psi)\left(i^*\Omega(\tilde{u}_{K,e^0})-i^*\Omega(u_{e^+_K})\right),\\
D(\psi)&:=-\sum_{K\in\mathcal{T},e^0\in\partial^0 K}\int_{e^+_K}\lambda_{K,e^0}\psi\partial_u U(u_{e^+_K})(i^*\omega(\tilde{u}_{K,e^0})-i^*\omega(u_{e^+_K})),\\
E(\psi)&:=-\sum_{K\in\mathcal{T}}\int_{e^+_K}(\psi_{\partial^0 K}-\psi)\left(i^*\Omega(u_{e^+_K})-i^*\Omega(u_{e^-_K})\right).
\end{aligned}$$

Proof. By Stokes theorem we have for each cell K

$$-\int_K d(\psi\Omega)(u_{e^-_K})+\int_{e^+_K}\psi i^*\Omega(u_{e^-_K})-\int_{e^-_K}\psi i^*\Omega(u_{e^-_K})+\sum_{e^0\in\partial^0 K}\int_{e^0}\psi i^*\Omega(u_{e^-_K})=0. \qquad (5.7)$$

Multiplying (4.10) by $\psi_{e^0}\lambda_{K,e^0}$ and summing over $e^0 \in \partial^0 K$ we have for each cell K

$$\begin{aligned}&\sum_{e^0\in\partial^0 K}\lambda_{K,e^0}\psi_{e^0}q^{\Omega}_{e^+_K}(\tilde{u}_{K,e^0}) - \sum_{e^0\in\partial^0 K}\lambda_{K,e^0}\psi_{e^0}q^{\Omega}_{e^+_K}(u_{e^-_K})\\ &+\sum_{e^0\in\partial^0 K}\psi_{e^0}\left(Q^{\Omega}_{K,e^0}(u_{e^-_K},u_{e^-_{K_{e^0}}}) - Q^{\Omega}_{K,e^0}(u_{e^-_K},u_{e^-_K})\right) \quad \leq 0.\end{aligned} \tag{5.8}$$

Furthermore, we have

$$\psi_{\partial^0 K}q^{\Omega}_{e^+_K}(u_{e^-_K}) - \sum_{e^0\in\partial^0 K}\psi_{\partial^0 K}\lambda_{K,e^0}q^{\Omega}_{e^+_K}(u_{e^-_K}) = 0, \tag{5.9}$$

where we used $\sum_{e^0\in\partial^0 K}\lambda_{K,e^0} = 1$. Adding (5.7), (5.8) and (5.9) and summing over all $K \in \mathcal{T}$ yields

$$\begin{aligned}&-\sum_{K\in\mathcal{T}}\int_K d(\psi\Omega)(u_{e^-_K}) + \sum_{K\in\mathcal{T}}\int_{e^+_K}\psi i^*\Omega(u_{e^-_K}) - \sum_{K\in\mathcal{T}}\int_{e^-_K}\psi i^*\Omega(u_{e^-_K})\\ &-\sum_{K\in\mathcal{T},e^0\in\partial^0 K}\lambda_{K,e^0}(\psi_{\partial^0 K}-\psi_{e^0})\left(q^{\Omega}_{e^+_K}(\tilde{u}_{K,e^0}) - q^{\Omega}_{e^+_K}(u_{e^-_K})\right)\\ &-\sum_{K\in\mathcal{T},e^0\in\partial^0 K}\int_{e^0}(\psi_{e^0}-\psi)i^*\Omega(u_{e^-_K}) + \sum_{e^0\in\partial\mathcal{T}^0}\psi_{e^0}Q^{\Omega}_{K,e^0}(u_{e^-_K},u_{e^-_{K_{e^0}}})\\ &+\sum_{K\in\mathcal{T}}\psi_{\partial^0 K}\left(\sum_{e^0\in\partial^0 K}\lambda_{K,e^0}q^{\Omega}_{e^+_K}(\tilde{u}_{K,e^0}) - q^{\Omega}_{e^+_K}(u_{e^-_K})\right) \leq 0,\end{aligned} \tag{5.10}$$

where we used $\sum_{K\in\mathcal{T},e^0\in\partial^0 K}\psi_{e^0}Q^{\Omega}_{K,e^0}(u_{e^-_K},u_{e^-_{K_{e^0}}}) = \sum_{e^0\in\partial\mathcal{T}^0}\psi_{e^0}Q^{\Omega}_{K,e^0}(u_{e^-_K},u_{e^-_{K_{e^0}}})$ due to the conservation property (4.9). We observe that

$$\begin{aligned}\sum_{K\in\mathcal{T}}\int_{e^-_K}\psi i^*\Omega(u_{e^-_K}) &= \sum_{K\in\mathcal{T}}\int_{e^+_K}\psi i^*\Omega(u_{e^+_K}) + \sum_{K\in\mathcal{T}_0}\int_{e^-_K}\psi i^*\Omega(u_{e^-_K})\\ &\leq \sum_{K\in\mathcal{T},e^0\in\partial^0 K}\int_{e^+_K}\lambda_{K,e^0}\psi i^*\Omega(\tilde{u}_{K,e^0}) + \sum_{K\in\mathcal{T}_0}\int_{e^-_K}\psi i^*\Omega(u_{e^-_K})\\ &\quad -\sum_{K\in\mathcal{T},e^0\in\partial^0 K}\int_{e^+_K}\lambda_{K,e^0}\psi\partial_u U(u_{e^+_K})(i^*\omega(\tilde{u}_{K,e^0}) - i^*\omega(u_{e^+_K}))\end{aligned} \tag{5.11}$$

similar to the derivation of (4.7). Finally inserting (5.11) in (5.10) yields

$$\begin{aligned}&-\sum_{K\in\mathcal{T}}\int_K d(\psi\Omega)(u_{e^-_K}) - \sum_{K\in\mathcal{T}_0}\int_{e^-_K}\psi i^*\Omega(u_{e^-_K})\\ &+\sum_{e^0\in\partial\mathcal{T}^0}\psi_{e^0}Q^{\Omega}_{K,e^0}(u_{e^-_K},u_{e^-_{K_{e^0}}})\\ &+\sum_{K\in\mathcal{T},e^0\in\partial^0 K}\int_{e^+_K}\lambda_{K,e^0}(\psi_{\partial^0 K}-\psi)\left(i^*\Omega(\tilde{u}_{K,e^0}) - i^*\Omega(u_{e^-_K})\right)\\ &+\sum_{K\in\mathcal{T},e^0\in\partial^0 K}\int_{e^+_K}\lambda_{K,e^0}\psi\partial_u U(u_{e^+_K})(i^*\omega(\tilde{u}_{K,e^0}) - i^*\omega(u_{e^+_K})) \leq A(\psi) + B(\psi),\end{aligned} \tag{5.12}$$

where we have used that

$$A(\psi) = \sum_{K\in\mathcal{T},e^0\in\partial^0 K}\lambda_{K,e^0}(\psi_{\partial^0 K}-\psi_{e^0})\left(q^{\Omega}_{e^+_K}(\tilde{u}_{K,e^0}) - q^{\Omega}_{e^+_K}(u_{e^-_K})\right) \tag{5.13}$$

since for all $K \in \mathcal{T}$

$$\sum_{e^0 \in \partial^0 K} \lambda_{K,e^0}(\psi_{\partial^0 K} - \psi_{e^0}) q^{\Omega}_{e^+_K}(u_{e^-_K}) = \sum_{e^0 \in \partial^0 K} \lambda_{K,e^0}(\psi_{\partial^0 K} - \psi_{e^0}) q^{\Omega}_{e^+_K}(u_{e^+_K}) = 0.$$

Using the definitions of $C(\psi), D(\psi), E(\psi)$ the assertion of the lemma follows from (5.12). □

An easy consequence of Lemmas 5.2 and 4.6 is as follows.

Lemma 5.3. *Let (U, Ω) be an admissible, convex entropy pair and $\psi \in \mathcal{D}(M)$ a non-negative test-function compactly supported in $M \setminus \mathcal{H}_T$. Then the finite volume approximation satisfies the following global entropy inequality*

$$\begin{aligned} &- \sum_{K \in \mathcal{T}} \int_K d(\psi\Omega)(u_{e^-_K}) - \sum_{K \in \mathcal{T}_0} \int_{e^-_K} \psi i^*\Omega(u_{e^-_K}) + \sum_{e^0 \in \partial \mathcal{T}^0} \int_{e^0} \psi_{e^0} i^*\Omega(u_{e^-_{K_{e^0}}}) \\ &+ \sum_{e^0 \in \partial \mathcal{T}^0} \psi_{e^0} \partial_u U(u_{e^-_{K_{e^0}}}) \left(Q_{K,e^0}(u_{e^-_K}, u_{e^-_{K_{e^0}}}) - Q_{K,e^0}(u_{e^-_{K_{e^0}}}, u_{e^-_{K_{e^0}}}) \right) \\ &\leq A(\psi) + B(\psi) + C(\psi) + D(\psi) + E(\psi), \end{aligned} \tag{5.14}$$

where $A(\psi), B(\psi), C(\psi), D(\psi), E(\psi)$ were defined in Lemma 5.2.

5.2 Convergence analysis

Recall that we fixed a positive n-form field α_B along the boundary ∂M in (4.3). We emphasize that such a structure is necessary for the analysis of the convergence of the finite volume scheme, only, and was not required in the well-posedness theory.

We assume that there exists a constant $c > 0$ such that we have for every compact subset D of a spacelike hypersurface $\mathcal{H}$

$$\frac{\sup_u \partial_u q_D(u)}{\inf_u \partial_u q_D(u)} \leq c. \tag{5.15}$$

For the convergence analysis we need to fix a metric d on M. Then let $\{\mathcal{T}^h\}_{h>0}$ be a family of triangulations of M which are admissible with respect to the foliation $\{\mathcal{H}_t\}_{t \in [0,T]}$ satisfying the following conditions: There is a constant $c > 0$ such that

$$\begin{aligned} &d(x,y) \leq ch \quad \text{for all } x, y \in K, K \in \mathcal{T}^h, \\ &\frac{h^n}{c} < \inf_u \partial_u q_{e^+_K}(u) \leq \sup_u \partial_u q_{e^+_K}(u) < ch^n \quad \text{for all } K \in \mathcal{T}^h, \\ &\int_{e^0} \alpha_B < ch^n \quad \text{for all } e^0 \in \partial^0 K, K \in \mathcal{T}^h, e^0 \subset \partial^0 M, \\ &\#\{e^0 \in \partial^0 K\} < c \quad \text{for all } K \in \mathcal{T}^h, \\ &\#\{e^0 \in \partial^0 K : K \in \mathcal{T}^h_j, e^0 \subset \partial^0 M\} < ch^{1-n} \quad \text{for all } j \in \mathbb{N}. \end{aligned} \tag{5.16}$$

We also impose that for each compact subset $D \subset M$ there is some constant $c(D) > 0$ such that

$$\begin{aligned} &\#\{K \in \mathcal{T}^h_j : K \cap D \neq \emptyset\} \leq c(D) h^{-n} \quad \text{for all integer} j, \\ &\#\{j \in \mathbb{N} : \exists K \in \mathcal{T}^h_j : K \cap D \neq \emptyset\} \leq c(D) h^{-1}. \end{aligned} \tag{5.17}$$

In addition, for any admissible entropy field Ω, any $\bar{u} \in \mathbb{R}$ and any face $e^0 \in \partial \mathcal{T}^h$ we define $\phi^{\Omega}_{e^0}(\bar{u}) : e^0 \to \mathbb{R}$ via

$$i^*_{e^0}\Omega(\bar{u}) = \phi^{\Omega}_{e^0}(\bar{u}) \alpha_{e^0}.$$

We also define $\bar{\phi}^{\Omega}_{e^0}(\bar{u})$ as the mean value of $\phi^{\Omega}_{e^0}(\bar{u})$ with respect to α_{e^0}.

We impose the following two additional conditions on the triangulation:

1. Uniformly in $\bar{u}$ for all $K \in \mathcal{T}^h$, $e^0 \in \partial^0 K$
$$\frac{\int_{e^0} |\phi^{\Omega}_{e^0}(\bar{u}) - \bar{\phi}^{\Omega}_{e^0}(\bar{u})| \alpha_{e^0}}{\int_{e^0} \alpha_{e^0}} = o(1) \quad \text{for } h \to 0. \tag{5.18}$$

2. For every $j \geq 1$, every $\psi \in C_0^\infty(M)$, every $\bar{u} \in \mathbb{R}$ and every admissible entropy field Ω
$$\sum_{K \in \mathcal{T}^h_j} \Big| \int_{e^-_K} (\psi_{\partial^0 K^-} - \psi) i^* \Omega(u_{e^-_K}) - \int_{e^+_K} (\psi_{\partial^0 K} - \psi) i^* \Omega(u_{e^-_K}) \Big| = o(h), \tag{5.19}$$
where for $K \in \mathcal{T}_j$ we denote by K^- the cell having e^-_K as outflow face.

Remark 5.4. *1. Note that* (5.18) *is not a trivial consequence of $\Omega(\bar{u})$ being a smooth field of n-forms. Derivatives of $\phi^{\Omega}_{e^0}$ do not only depend on Ω since the regularity of e^0 (i.e., curvature in some compatible Riemannian metric) enters via $i^*_{e^0}$.*

2. For hyperbolic conservation laws on Riemannian manifolds, i.e. (1.2)*, assumption* (5.18) *is satisfied provided the curvatures of the faces of the (spatial) cells are uniformly bounded under mesh refinement.*

3. A careful study of the estimate of $B^h(\psi)$ in Lemma 5.5 reveals that we may slightly weaken assumption (5.18)*, e.g., it is not a problem if it is violated for some faces e^0 as long as their number inside any compact $D \subset M$ is bounded by $c(D)h^{-n}$.*

4. Condition (5.19) *restricts temporal changes (of the geometric properties) of the triangulation. For hyperbolic conservation laws on manifolds with Lipschitz continuously evolving Riemannian metric it is automatically satisfied, if the triangulation does not change in time. The change of its geometric properties due to the evolving Riemannian metric is compatible with* (5.19)*.*

Lemma 5.5. *Let (U, Ω) be an admissible entropy pair with $U(0) = 0$ and $\psi \in \mathcal{D}M$ a non-negative test function. Let $\{\mathcal{T}^h\}_{h>0}$ be a family of triangulations of M satisfying* (5.16) *and* (5.17)*. Provided $u_B \in L^\infty(M)$ we have $A^h(\psi), B^h(\psi), C^h(\psi), D^h(\psi), E^h(\psi) \to 0$ for $h \to 0$, for the functions $A(\psi), B(\psi), C(\psi), D(\psi), E(\psi)$ defined in Lemma 5.2.*

Proof. We start by showing $A^h(\psi) \to 0$ for $h \to 0$. Note that
$$|A^h(\psi)| \leq \sum_{K \in \mathcal{T}^h, e^0 \in \partial^0 K} \lambda_{K,e^0} |\psi_{\partial^0 K} - \psi_{e^0}| \sup_u |\partial_u q^{\Omega}_{e^+_K}| |\tilde{u}_{K,e^0} - u_{e^+_K}|. \tag{5.20}$$
We have due to (5.15)
$$|\partial_u q^{\Omega}_{e^+_K}(u)| = \left| \int_{e^+_K} \partial_u U(u) i^* \partial_u \omega \right| \leq c \|\partial_u U\|_\infty \inf_u \partial_u q_{e^+_K}(u),$$
and hence
$$|A^h(\psi)| \leq \left(C \sum_{K \in \mathcal{T}^h, e^0 \in \partial^0 K} \lambda_{K,e^0} \inf_u \partial_u q_{e^+_K} |\psi_{\partial^0 K} - \psi_{e^0}|^2 \right)^{1/2} \left(C \sum_{K \in \mathcal{T}^h, e^0 \in \partial^0 K} \lambda_{K,e^0} \inf_u \partial_u q_{e^+_K} |\tilde{u}_{K,e^0} - u_{e^+_K}|^2 \right)^{1/2} =: (I_1^h)^{1/2} (I_2^h)^{1/2}. \tag{5.21}$$

Let $D := \operatorname{supp}\psi$ then (5.16) and (5.17) imply

$$I_1^h \le Cc(D)^2 h^{-1} \sup_{K\in\mathcal{T}^h, e^0\in\partial^0 K} |\psi_{\partial^0 K} - \psi_{e^0}|^2.$$

Now (5.16) and the fact that $\psi \in \mathcal{D}M$ imply

$$|\psi_{\partial^0 K} - \psi_{e^0}| \le Ch$$

and hence

$$I_1^h \to 0 \text{ for } h \to 0.$$

In order to derive an estimate for I_2^h we define

$$L^h := \{K \in \mathcal{T}^h : K \cap D \neq \emptyset\} \text{ and } \partial^0 L^h := \{e^0 \in \partial^0 K : K \in L^h, e^0 \subset \partial^0 M\},$$

and note that it is sufficient to find an estimate for

$$\tilde{I}_2^h := \sum_{K\in L^h, e^0\in\partial^0 K} \lambda_{K,e^0} \inf_u \partial_u q_{e_K^+} |\tilde{u}_{K,e^0} - u_{e_K^+}|^2$$

instead of I_2^h. Using the first assertion of Lemma 5.1 we have

$$\begin{aligned}\tilde{I}_2^h &\le C \sum_{L^h} \Big(q^\Omega_{e_K^-}(u_{e_K^-}) - q^\Omega_{e_K^+}(u_{e_K^+}) \Big) - \sum_{e^0\in\partial^0 L^h} Q^\Omega_{K,e^0}(u_{e_K^-}, u_{e^-_{K_{e^0}}}) \\ &=: C(I_3^h + I_4^h).\end{aligned}$$

In I_3^h all terms which are inflow as well as outflow faces cancel out and hence

$$\begin{aligned}I_3^h = \sum_{K\in\mathcal{T}_0^h\cap L^h} q^\Omega_{e_K^-}(u_{e_K^-}) &\le \sum_{K\in\mathcal{T}_0^h\cap L^h} \int_{e_K^-} \int_0^{u_{e_K^-}} \partial_v U(v) i^* \partial_v \omega(v)\, dv \\ &\le \|\partial_u U\|_\infty \sum_{K\in\mathcal{T}_0^h\cap L^h} \int_{e_K^-} |u_{e_K^-}|\alpha \\ &\le \|\partial_u U\|_\infty \|u_0\|_\infty \sum_{K\in\mathcal{T}_0^h\cap L^h} \int_{e_K^-} \alpha \le C\|u_0\|_\infty \int_{\mathcal{H}_0\cap D} \alpha.\end{aligned}$$

Furthermore we get using (5.16) and (5.17)

$$\begin{aligned}I_4^h &= - \sum_{e^0\in\partial^0 L^h} Q^\Omega_{K,e^0}(u_{e_K^-}, u_{e^-_{K_{e^0}}}) \\ &\overset{\text{Lemma 4.6}}{\le} \sum_{e^0\in\partial^0 L^h} \Big(\Big| Q^\Omega_{K,e^0}(u_{e^-_{K_{e^0}}}, u_{e^-_{K_{e^0}}}) \Big| \\ &\qquad + \Big| \partial_u U(u_{e^-_{K_{e^0}}}) \big[Q_{K,e^0}(u_{e_K^-}, u_{e^-_{K_{e^0}}}) - Q_{K,e^0}(u_{e^-_{K_{e^0}}}, u_{e^-_{K_{e^0}}}) \big] \Big| \Big) \\ &\overset{(4.4)}{\le} \sum_{e^0\in\partial^0 L^h} \left(\left| \int_{e^0} i^*\Omega(u_{e^-_{K_{e^0}}}) \right| + \|\partial_u U\|_\infty \inf_u \partial_u q_{e_K^+}(u) \left| u_{e_K^-} - u_{e^-_{K_{e^0}}} \right| \right) \\ &\le \sum_{e^0\in\partial^0 L^h} \left(\int_{e^0} \int_0^{u_{e^-_{K_{e^0}}}} \partial_v U(v) i^* \partial_v \omega(v)\, dv \right) + Cc(D)^2 h^{-1} h^{1-n} h^n \Big(\sup_{K\in\mathcal{T}^h} |u_{e_K^-}| + \|u_B\|_\infty \Big) \\ &\le C\|u_B\|_\infty \|\partial_u U\|_\infty \int_{\partial^0 M\cap D} \alpha + C.\end{aligned}$$

Note that $\sup_{K\in\mathcal{T}^h}|u_{e_K^-}|\le\|u_B\|_\infty$ because of the CFL condition (4.4) and since ω is geometry compatible. So finally I_2^h is bounded and hence $A^h(\psi)$ goes to zero.

Let us now turn our attention to $B^h(\psi)$

$$\begin{aligned}|B^h(\psi)| &= \Big|\sum_{K\in\mathcal{T}^h,e^0\in\partial^0K}\int_{e^0}(\psi_{e^0}-\psi)\phi^\Omega_{e^0}(u_{e_K^-})\alpha_{e^0}\Big|\\ &= \Big|\sum_{K\in\mathcal{T}^h,e^0\in\partial^0K}\int_{e^0}(\psi_{e^0}-\psi)\big(\phi^\Omega_{e^0}(u_{e_K^-})-\bar\phi^\Omega_{e^0}(u_{e_K^-})\big)\alpha_{e^0}\Big|\\ &\le \sum_{K\in L^h,e^0\in\partial^0K}\operatorname{diam}(e^0)o(1)\int_{e^0}\alpha_{e^0}=o(1),\end{aligned}$$

where we used the regularity of ψ, (5.18), and $(5.16)_2$ and (5.17) which imply $\#\{e^0\in\partial^0K : K\in L^h\}\le Ch^{-(n+1)}$.

The term $D^h(\psi)$ can be estimated as follows:

$$D^h(\psi)=-\sum_{K\in\mathcal{T}^h,e^0\in\partial^0K}\int_{e_K^+}\lambda_{K,e^0}(\psi-\psi_{e_K^+})\partial_uU(u_{e_K^+})(i^*\omega(\tilde u_{K,e^0})-i^*\omega(u_{e_K^+})),$$

where $\psi_{e_K^+}$ is any average of ψ on e_K^+, because of (4.6). Using admissibility of U and regularity of ψ we obtain

$$|D^h(\psi)|\le C\sum_{K\in\mathcal{T}^h,e^0\in\partial^0K}\operatorname{diam}(e_K^+)\sup_u(\partial_uq_{e_K^+})|\tilde u_{K,e^0}-u_{e_K^+}|.$$

This bound for $|D^h(\psi)|$ can be estimated exactly as the bound for $|A^h|$ obtained in (5.20). The term $C^h(\psi)$ can be estimated analogously to $A^h(\psi)$ and $D^h(\psi)$.

Finally we consider $E^h(\psi)$ and note that by regrouping terms we obtain

$$\begin{aligned}E^h(\psi)&=\sum_{j=1}^\infty\sum_{e^0\in\partial\mathcal{T}_j^h}\Big(\int_{e_K^-}(\psi_{\partial^0K^-}-\psi)i^*\Omega(u_{e_K^-})-\int_{e_K^+}(\psi_{\partial^0K}-\psi)i^*\Omega(u_{e_K^-})\Big)\\ &\quad-\sum_{e^0\in\partial\mathcal{T}_0^h}\int_{e_K^+}(\psi_{\partial^0K}-\psi)i^*\Omega(u_{e_K^-})\\ &=: E_1^h(\psi)+E_2^h(\psi),\end{aligned}$$

where for $K\in\mathcal{T}_j^h$ we denote by K^- the cell having e_K^- as outflow face. Due to the regularity of ψ we have $E_2^h(\psi)\to0$ for $h\to0$. Moreover, $E_1^h(\psi)\to0$ for $h\to0$ due to assumption (5.19).

□

For every triangulation $\mathcal{T}^h$ the finite volume method (4.2) generates an approximate solution of (2.1),(2.8) defined by

$$u^h(x):=u_{e_K^-}\text{ for }x\in K.\tag{5.22}$$

Theorem 5.6 (Convergence of the finite volume schemes on a spacetime)**.** *Let ω be a geometry compatible flux field with at most linear growth and satisfying the global hyperbolicity condition on a spacetime M and let $\{u^h\}_{h>0}$ be the sequence of approximate solutions generated by the finite volume method associated to a family of triangulations $\mathcal{T}^h$ satisfying* (5.16), (5.17), (5.18) *and* (5.19)*. Then u^h converges to an entropy solution of the initial boundary value problem* (2.1),(2.8) *for $h\to0$.*

The proof of Theorem 5.6 follows from Theorem 3.6 and Lemma 5.7 below. The proof of Theorem 3.6 is omitted since it follows along the same lines as in the Riemannian setting treated in [7], once charts are chosen which are compatible with the foliation.

As the CFL condition (4.4) implies L^∞ stability of the finite volume scheme there exists a Young measure $\nu : M \to \mathrm{Prob}(\mathbb{R})$, which allows us to determine all weak-$*$ limits of composite functions $a(u^h)$ for all continuous functions a, as $h \to 0$,

$$a(u^h) \overset{*}{\rightharpoonup} \langle \nu, a \rangle = \int_{\mathbb{R}} a(\lambda)\, d\nu(\lambda). \tag{5.23}$$

Lemma 5.7 (Entropy inequalities for the Young measure)**.** *Let ν be a Young measure associated with the finite volume approximations u^h. Then, for every convex entropy flux field Ω and every non-negative test-function $\psi \in \mathcal{D}(M)$ supported compactly in $M \setminus \mathcal{H}_T$, there exists a boundary field $\gamma \in L^1_{loc}\Lambda^n(\partial M)$ such that*

$$\begin{aligned} &\int_M \Big\langle \nu, d\psi \wedge \Omega(\cdot) + \psi\left((d\Omega)(\cdot) - \partial_u U(\cdot)(d\omega)(\cdot)\right)\Big\rangle \\ &- \int_{\partial M} \psi_{|\partial M}\left(i^*\Omega(u_B) + \partial_u U(u_B)\big(\gamma - i^*\omega(u_B)\big)\right) \geq 0. \end{aligned} \tag{5.24}$$

Proof. We start with the inequality from Lemma 5.3. Recalling $(d(\psi\Omega))(u) = d\psi\wedge\Omega(u)+\psi(d\Omega)(u)$ and $d\omega = 0$ we see that the first summand of (5.14) converges to

$$-\int_M \Big\langle \nu, d\psi \wedge \Omega(\cdot) + \psi\left((d\Omega)(\cdot) - \partial_u U(\cdot)(d\omega)(\cdot)\right)\Big\rangle \tag{5.25}$$

for $h \to 0$.

We define γ as the weak-$*$ limit of the sequence $\left\{Q_{K,e^0}(u_{e_K^-}, u_{e_{K_{e^0}}^-})\right\}$ on $\partial^0 M$ and by $\gamma = i^*\omega(u_B)$ on the rest of ∂M. We know that $u_{e_{K_{e^0}}^-} \to u_B$ strongly and thus using the definition of γ we see that the other summands on the left hand side of the inequality converge to

$$\int_{\partial M} \psi_{|\partial M}\left(i^*\Omega(u_B) + \partial_u U(u_B)\big(\gamma - i^*\omega(u_B)\big)\right). \tag{5.26}$$

Finally by Lemma 5.5 the right hand side of the inequality converges to zero, which finishes the proof. □

Proof of Theorem 3.5. The existence of entropy solutions follows from the convergence of the finite volume method, once we have shown that a family of triangulations on M exists which satisfies (5.18) and (5.19) for $h \to 0$. First, we fix a Riemannian metric on N. This can be achieved by fixing an atlas $\{U_i, \varphi_i\}_{i\in I}$ of N, with a finite index set I, and a partition of unity which is subordinate to the open cover $\{U_i\}_{i\in I}$. The pull-back of the Euclidean Riemannian metric under each chart φ_i induces a Riemannian metric on U_i. Using the partition of unity these Riemannian metrics can be combined to a Riemannian metric on N. This immediately induces a Riemannian metric on $M = N \times [0, T]$.

Let us now fix a triangulation on N using so-called Karcher simplexes, [25], denoted $\{T_j\}_{j\in J}$ for a finite index set J. For each of these simplexes the so-called barycentric map [25, Def. 5.4; Thrm. 6.17] is a C^2-diffeomorphism from the standard simplex to T_j. By subdividing the standard simplex using smaller simplexes we obtain subdivisions of the T_j whose faces have bounded curvature (since the second derivatives of the finitely many barycentric maps are bounded). If the subdivisions of all T_j come from the same uniform subdivision of the standard simplex, then the ratios of cell

diameters and cell volumes will be uniformly bounded. Thus, the triangulation $\mathcal{T}$ defined in (5.27) will satisfy (5.16) and (5.17).

Then, for some given (sufficiently small) $h > 0$, we define a triangulation $\mathcal{T}$ of M by fixing a triangulation $\mathcal{S}$ of N such that the maximal diameter of any element of $\mathcal{S}$ is bounded by $h/2$. Then, we fix a $0 < \bar{h} < h$ such that $T/\bar{h}$ is a natural number and the CFL condition is satisfied for the following triangulation:

$$\mathcal{T} := \Big\{ \Big[\frac{(i-1)\bar{h}}{2}, \frac{i\bar{h}}{2} \Big] \times S \,:\, 1 \leq i \leq \frac{2T}{\bar{h}},\ S \in \mathcal{S} \Big\}. \tag{5.27}$$

The temporal faces of this triangulation have bounded curvature such that (5.18) is fulfilled, compare the first two items of Remark 5.4.

In order to see that $\mathcal{T}$ also satisfies (5.19), note that due to the construction of $\mathcal{T}$ for each cell K the "previous" cell K^- is just a translation of K in time. In particular, for any K the faces e_K^+ and e_K^- are (temporal) translations of each other. Since all terms in the integrands in (5.19) depend smoothly on time we have

$$\Big| \int_{e_K^-} (\psi_{\partial^0 K^-} - \psi) i^* \Omega(u_{e_K^-}) - \int_{e_K^+} (\psi_{\partial^0 K} - \psi) i^* \Omega(u_{e_K^-}) \Big| \leq \bar{h} \cdot h^{n+1} \leq h^{n+2}.$$

Due to (5.17) we obtain

$$\sum_{K \in \mathcal{T}_j} \Big| \int_{e_K^-} (\psi_{\partial^0 K^-} - \psi) i^* \Omega(u_{e_K^-}) - \int_{e_K^+} (\psi_{\partial^0 K} - \psi) i^* \Omega(u_{e_K^-}) \Big| \leq h^2.$$

This completes the proof of existence of entropy solutions.

In order to prove the L^1 stability property (3.9) we first need to determine γ in Definition 2.5. We choose a test-function $\psi \in C_0^\infty(\partial M)$ with sufficiently smooth support and a sequence $\{\phi_\varepsilon\}_{\varepsilon > 0} \subset C_0^\infty(M)$ such that $\phi_\varepsilon \overset{\varepsilon \to 0}{\longrightarrow} \psi H_{\partial M}$ in the sense of distributions, where $H_{\partial M}$ denotes the n-dimensional Hausdorff measure on ∂M induced by the metric chosen above. Using ϕ_ε as test-function in the entropy inequality and letting ε go to zero, we obtain

$$\int_{\partial M} \psi \big(\Omega(u) - \Omega(u_B) - \partial_u U(u_B)(\gamma - \omega(u_B)) \big) \geq 0 \tag{5.28}$$

for every admissible convex entropy pair (U, Ω). By choosing $U(u) = \pm u$ and $\Omega(u) = \pm \omega(u)$ we obtain

$$\int_{\partial M} \psi \big(\pm \omega(u) \mp \gamma \big) \geq 0 \tag{5.29}$$

which allows us to identify $\gamma = \omega(u)$.

Since ψ was arbitrary we obtain

$$\int_U \big(\Omega(u) - \Omega(u_B) - \partial_u U(u_B)(\omega(u) - \omega(u_B)) \big) \geq 0 \tag{5.30}$$

for any open $U \subset \partial M$. Choosing Kruzkov's entropy pair equation (5.30) amounts to

$$\int_U \big(\operatorname{sgn}(u - \kappa) - \operatorname{sgn}(u_B - \kappa) \big) \big(\omega(u) - \omega(u_B) \big) \geq 0. \tag{5.31}$$

By checking carefully several cases we obtain that (5.31) and a similar inequality for (v, v_B) imply the following inequality for traces on ∂M :

$$\boldsymbol{\Omega}(u, v) \geq -|u_B - v_B| A_B \tag{5.32}$$

for any $A_B \in L^\infty\Lambda^n(\partial M)$ such that $|\partial_u \omega|_{\partial M}| \leq A_B$ in $\partial M \times \mathbb{R}$. The existence of one such A_B follows since ω grows at most linearly.

By the classical doubling of variables method we obtain (in the interior of M)

$$d\mathbf{\Omega}(u,v) \leq 0 \quad \text{weakly.} \tag{5.33}$$

Let us now pick hypersurfaces with $\mathcal{H}, \mathcal{H}'$ such that $\mathcal{H}'$ lies in the future of $\mathcal{H}$ with a corresponding function $F : \mathcal{H} \times [0,1] \to M$. Due to the non-degeneracy of $DF(\partial_s)$ we can associate with each $x \in \mathrm{Im}(F)$ a "time" $s = s(x)$. For each $\varepsilon > 0$ we denote by $\chi_\varepsilon : M \to [0,\infty)$ the map given by

$$\chi_\varepsilon(x) = \begin{cases} 0, & x \notin \mathrm{Im}(F), \\ \frac{1}{\varepsilon} s(x), & x \in \mathrm{Im}(F),\ s(x) \leq \varepsilon, \\ 1, & x \in \mathrm{Im}(F),\ \varepsilon < s(x) < 1-\varepsilon, \\ \frac{1}{\varepsilon}(1-s(x)), & x \in \mathrm{Im}(F),\ s(x) \geq 1-\varepsilon. \end{cases}$$

Similarly we denote for any $\delta > 0$ by $\psi_\delta \in \mathcal{D}(M)$ a function satisfying

$$\psi_\delta|_{\partial M} \equiv 0, \quad \psi_\delta(x) = 1 \quad \text{for all } x \in M \,:\, \mathrm{dist}(x, \partial M) > \delta.$$

This definition makes sense, once we have fixed a Riemannian metric on M. Using $\phi = \chi_\varepsilon \cdot \psi_\delta$ as test-function for (5.33) and letting $\delta \to 0$ we obtain

$$\frac{1}{\varepsilon}\int_{F(\mathcal{H}\times[0,\varepsilon])} \mathbf{\Omega}(u,v) \wedge F_*(ds) - \frac{1}{\varepsilon}\int_{F(\mathcal{H}\times[1-\varepsilon,1])} \mathbf{\Omega}(u,v) \wedge F_*(ds) - \int_{\mathcal{B}} \chi_\varepsilon \mathbf{\Omega}(u,v) \;\geq\; 0, \tag{5.34}$$

where F_* denotes the push-forward along F. Inserting (5.32) into (5.34) implies

$$\frac{1}{\varepsilon}\int_{F(\mathcal{H}\times[1-\varepsilon,1])} \mathbf{\Omega}(u,v) \wedge F_*(ds) \;\leq\; \frac{1}{\varepsilon}\int_{F(\mathcal{H}\times[0,\varepsilon])} \mathbf{\Omega}(u,v) \wedge F_*(ds) + \int_{\mathcal{B}} |u_B - v_B| A_B. \tag{5.35}$$

This implies (3.9) as soon as we can show that the $\varepsilon \to 0$ limit is well-defined.

To this end we define

$$a_n := 2^n \int_{F(\mathcal{H}\times[0,2^{-n}])} \mathbf{\Omega}(u,v) \wedge F_*(ds), \quad b_n := \int_{F(\mathcal{H}\times[0,2^{-n}])\cap\partial M} |u_B - v_B| A_B.$$

Then, choosing suitable test functions in (5.33) we get $a_{n+1} \geq a_n - b_n$ and $|b_n| = \mathcal{O}(2^{-n})$, which means that $\int_{\mathcal{H}} \mathbf{\Omega}(u,v) = \lim_{n\to\infty} a_n$ exists. Similarly it can be shown that

$$\frac{1}{\varepsilon}\int_{F(\mathcal{H}\times[1-\varepsilon,1])} \mathbf{\Omega}(u,v) \wedge F_*(ds) \overset{\varepsilon\to 0}{\longrightarrow} \int_{\mathcal{H}'} \mathbf{\Omega}(u,v), \tag{5.36}$$

which completes the proof of (3.9). The proof of (3.10) is analogous. □

Appendix

The purpose of this appendix is to give examples showing that $\partial^- M \neq \emptyset$ does not follow from the hyperbolicity condition and that the existence of a foliation does not follow from the hyperbolicity condition and $\partial^- M \neq \emptyset$.

Example 1 The assumption $(\partial M)^- \neq \emptyset$ cannot be dropped, as it is essential for our finite volume scheme and does not follow from the other properties of the problem in consideration, as can be seen in the following example.

We consider the following compact and bounded manifold with boundary:

$$R = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 2\}$$

with the differentiable structure as a subset of $\mathbb{R}^2$ and the orientation determined by saying that $dx \wedge dy$ is positive. Let us study the conservation law associated to

$$\omega(x, y, u) := ux\, dx + uy\, dy.$$

So, $\partial_u \omega(x, y, u) = x\, dx + y\, dy$, i.e., $\partial_u \omega$ is independent of u and ω has at most linear growth. Furthermore, ω is geometry compatible, since $(d\omega)(x, y, \bar{u}) = 0$ holds for every $\bar{u} \in \mathbb{R}$. An observer can be defined by $T(x, y) := y\, dx - x\, dy \in C^\infty \Lambda^1 R$, since

$$T \wedge \partial_u \omega = (x^2 + y^2)\, dx \wedge dy$$

is a positive 2−form. Hence, ω satisfies the global hyperbolicity property. For every $(x, y) \in \partial R$ we have

$$T_{(x,y)} \partial R = \{\lambda(-y\, \partial_x + x\, \partial_y) | \lambda \in \mathbb{R}\}.$$

Therefore normal 1-forms have the form $N = \mu(x\, dx + y\, dy)$ with $\mu \in \mathbb{R}$ and hence $N \wedge \partial_u \omega = 0$, due to the anti-symmetry of the $\wedge$-operator on 1-forms. This shows that there is no spacelike, and in particular no inflow, part of the boundary.

Example 1 The existence of a foliation is not a consequence of the other assumptions made on the problem under consideration. We consider $M = \{(x, y) \in \mathbb{R}^2 \,|\, (x, y) \in [0, 3]^2 \setminus (1, 2)^2\}$ with the differentiable structure and orientation as a subset of $\mathbb{R}^2$. Obviously M does not admit a foliation in the sense of (3.7). However, a geometry compatible flux field and an observer are given by $\omega = -u\, dx$ and $T = dy$. The inflow boundary is $\partial^- M = [0, 3] \times \{0\} \cup [1, 2] \times \{2\} \neq \emptyset$.

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