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Convergence of the gradient algorithm for linear regression models in the continuous and discrete time cases

Laurent Praly*

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Abstract

We establish convergence to zero of the solutions of

$$\dot{\tilde{\theta}}(t) = -\phi(t)\phi(t)^\top \tilde{\theta}(t) \quad \text{or} \quad \tilde{\theta}(t) = \tilde{\theta}(t-1) - \phi(t)\phi(t)^\top \tilde{\theta}(t-1)$$

under a possibly “vanishing persistent” excitation condition.

1 Continuous time case

Given a continuous vector function¹ $t \in \mathbb{R} \mapsto \phi(t) \in \mathbb{R}^n$, bounded² in norm by $\bar{\phi}$, we let $\mathfrak{T}(t, s)$ be the transition matrix associated with the non autonomous differential equation

$$\dot{\tilde{\theta}}(t) = -\phi(t)\phi(t)^\top \tilde{\theta}(t) \tag{1}$$

i.e. satisfying

$$\partial_1 \mathfrak{T}(t, s) = -\phi(t)\phi(t)^\top \mathfrak{T}(t, s) \quad , \quad \mathfrak{T}(s, s) = I \tag{2}$$

where ∂_1 denotes the partial derivative with respect to the first argument, t here. We are interested in sufficient conditions implying

$$\lim_{t \rightarrow +\infty} |\mathfrak{T}(t, s)| = \lim_{t \rightarrow +\infty} \sup_x \frac{|\mathfrak{T}(t, s)x|}{|x|} = 0 .$$

Let x be an arbitrary (constant) vector. We have

$$\partial_1 |\mathfrak{T}(t, s)x|^2 = -2|\phi(t)^\top \mathfrak{T}(t, s)x|^2$$

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¹About the smoothness of ϕ we need only the differential equation to have solutions; so, since we have boundedness, measurability is sufficient. We need also to be able to change the order of integration.

²To obtain a bounded function ϕ a pre-processing from a “raw” function say Φ may be required. Often encountered pre-processings are

$$\phi(t) = \gamma(t) \frac{\Phi(t)}{\sqrt{1 + |\Phi(t)|^2}} \quad \text{or} \quad \phi(t) = \gamma(t) \frac{\Phi(t)}{\sqrt{r(t)}} \quad , \quad \dot{r}(t) = -\lambda r(t) + |\Phi(t)|^2 .$$

By defining the function \hat{f}_s as

$$\hat{f}_s(t) = \phi(t)^\top \mathfrak{T}(t, s)x$$

and integrating, we get

$$|\mathfrak{T}(t, s)x|^2 = |x|^2 - 2 \int_s^t |\hat{f}_s(r)|^2 dr \quad (3)$$

On another hand, by integrating (2), we get

$$\mathfrak{T}(t, s) = I - \int_s^t \phi(r)\phi(r)^\top \mathfrak{T}(r, s)dr$$

Incorporating this in the definition of \hat{f}_s yields

$$\begin{aligned} \hat{f}_s(t) &= \phi(t)^\top \mathfrak{T}(t, s)x \\ &= \phi(t)^\top \left(I - \int_s^t \phi(r)\phi(r)^\top \mathfrak{T}(r, s)dr \right) x \\ &= \phi(t)^\top x - \int_s^t (\phi(t)^\top \phi(r)) (\phi(r)^\top \mathfrak{T}(r, s)x) dr \\ &= f(t) - \int_s^t (\phi(t)^\top \phi(r)) \hat{f}_s(r)dr \end{aligned} \quad (4)$$

where we have let

$$f(t) = \phi(t)^\top x .$$

It follows that we have

$$\begin{aligned} f(u)^2 &\leq 2\hat{f}_s(u)^2 + 2 \left(\int_s^u \phi(u)^\top \phi(r) \hat{f}_s(r)dr \right)^2 \\ &\leq 2\hat{f}_s(u)^2 + 2 \left[\int_s^u (\phi(u)^\top \phi(v))^2 dv \right] \left[\int_s^u \hat{f}_s(r)^2 dr \right] \\ &\leq 2\hat{f}_s(u)^2 + 2|\phi(u)|^2 \left[\int_s^u |\phi(v)|^2 dv \right] \left[\int_s^u \hat{f}_s(r)^2 dr \right] \\ \int_s^t f(u)^2 du &\leq 2 \int_s^t \hat{f}_s(u)^2 du + 2 \int_s^t |\phi(u)|^2 \left(\int_s^u |\phi(v)|^2 dv \right) \left(\int_s^u \hat{f}_s(r)^2 dr \right) du \\ &\leq 2 \int_s^t \hat{f}_s(u)^2 du + 2 \left(\int_s^t |\phi(u)|^2 du \right) \left(\int_s^t |\phi(v)|^2 dv \right) \left(\int_s^t \hat{f}_s(r)^2 dr \right) \\ &\leq 2 \left[1 + \left(\int_s^t |\phi(u)|^2 du \right)^2 \right] \left(\int_s^t \hat{f}_s(r)^2 dr \right) \end{aligned} \quad (5)$$

So we get finally

$$\begin{aligned}
|\mathfrak{T}(t, s)x|^2 &= |x|^2 - 2 \int_s^t |\hat{f}_s(r)|^2 dr \\
&\leq |x|^2 - \frac{1}{1 + \left(\int_s^t |\phi(u)|^2 du \right)^2} \int_s^t f(u)^2 du \\
&\leq |x|^2 - \frac{1}{1 + \left(\int_s^t |\phi(u)|^2 du \right)^2} \int_s^t [\phi(u)^\top x]^2 du \\
&\leq x^\top \left(I - \frac{1}{1 + \left(\int_s^t |\phi(u)|^2 du \right)^2} \int_s^t \phi(u)\phi(u)^\top du \right) x
\end{aligned}$$

In other words we have simply

$$\mathfrak{T}(t, s)^\top \mathfrak{T}(t, s) \leq I - \frac{\int_s^t \phi(u)\phi(u)^\top du}{1 + \left(\int_s^t |\phi(u)|^2 du \right)^2}$$

Remark 1

1. Up to (4), we have identities. The conservativeness we may have in this last inequality is only in the majoration obtained in (5).
2. An upperbound for $\mathfrak{T}(t, s)\mathfrak{T}(t, s)^\top$ can be obtained in a very similar way but starting from

$$\partial_2 \mathfrak{T}(t, r) = \mathfrak{T}(t, r)\phi(r)\phi(r)^\top,$$

integrating in r backwards from t to s and using

$$f_t(r) = x^\top \mathfrak{T}(t, r)\phi(r)$$

instead of $\hat{f}_s(r)$.

3. The steps used up to (4) for the differential equation (1) can also be used for :

$$\dot{\tilde{\theta}}(t) = -\psi(t)\phi(t)^\top \tilde{\theta}(t)$$

as we have for example in the least square algorithm.

2 Discrete time case

Given a sequence of vectors $\phi(t) \in \mathbb{R}^n$ bounded³ in norm by $\bar{\phi} \leq \sqrt{2}$, we let $\mathfrak{T}(t, s)$ be the transition matrix associated with the non autonomous discrete time system

$$\tilde{\theta}(t) = (I - \phi(t)\phi(t)^\top) \tilde{\theta}(t-1)$$

³See footnote 2

i.e. satisfying

$$\mathfrak{T}(t, s) = (I - \phi(t)\phi(t)^\top) \mathfrak{T}(t-1, s) \quad , \quad \mathfrak{T}(s, s) = I$$

Our problem is to find sufficient conditions implying

$$\lim_{t \rightarrow +\infty} |\mathfrak{T}(t, s)| = \lim_{t \rightarrow +\infty} \sup_x \frac{|\mathfrak{T}(t, s)x|}{|x|} = 0 .$$

Let x be an arbitrary unit vector. We have

$$\mathfrak{T}(t, s)x = \mathfrak{T}(t-1, s)x - \phi(t)\phi(t)^\top \mathfrak{T}(t-1, s)x$$

With denoting, for $t \geq s+1$,

$$\hat{f}_s(t) = \phi(t)^\top \mathfrak{T}(t-1, s)x$$

we get

$$\begin{aligned} |\mathfrak{T}(t, s)x|^2 &= \left(\mathfrak{T}(t-1, s)x - \phi(t)\hat{f}_s(t) \right)^\top \left(\mathfrak{T}(t-1, s)x - \phi(t)\hat{f}_s(t) \right) \\ &= |\mathfrak{T}(t-1, s)x|^2 - 2x^\top \mathfrak{T}(t-1, s)^\top \phi(t)\hat{f}_s(t)\phi(t)^\top \phi(t)\hat{f}_s(t)^2 \\ &= |\mathfrak{T}(t-1, s)x|^2 - (2 - \phi(t)^\top \phi(t)) \hat{f}_s(t)^2 \\ &\quad \vdots \\ |\mathfrak{T}(s+1, s)x|^2 &= |x|^2 - (2 - \phi(s+1)^\top \phi(s+1)) \hat{f}_s(s+1)^2 \end{aligned}$$

So summation gives

$$|\mathfrak{T}(t, s)x|^2 = |x|^2 - \sum_{r=s+1}^t (2 - \phi(r)^\top \phi(r)) \hat{f}_s(r)^2 \quad (6)$$

On another hand, we have

$$\begin{aligned} \mathfrak{T}(t, s) &= \mathfrak{T}(t-1, s) - \phi(t)\phi(t)^\top \mathfrak{T}(t-1, s) \\ \mathfrak{T}(t-1, s) &= \mathfrak{T}(t-2, s) - \phi(t-1)\phi(t-1)^\top \mathfrak{T}(t-2, s) \\ &\quad \vdots \\ \mathfrak{T}(s+1, s) &= I - \phi(s+1)\phi(s+1)^\top \mathfrak{T}(s, s) \end{aligned}$$

So again summation gives

$$\mathfrak{T}(t, s) = I - \sum_{r=s+1}^t \phi(r)\phi(r)^\top \mathfrak{T}(r-1, s)$$

Incorporating this in the expression of \hat{f}_s yields, for $t \geq s+2$,

$$\begin{aligned} \hat{f}_s(t) &= \phi(t)^\top \mathfrak{T}(t-1, s)x \\ &= \phi(t)^\top \left(I - \sum_{r=s+1}^{t-1} \phi(r)\phi(r)^\top \mathfrak{T}(r-1, s) \right) x \\ &= \phi(t)^\top x - \sum_{r=s+1}^{t-1} (\phi(t)^\top \phi(r)) (\phi(r)^\top \mathfrak{T}(r-1, s)x) \\ &= f(t) - \sum_{r=s+1}^{t-1} \phi(t)^\top \phi(r) \hat{f}_s(r) \quad \forall t \geq s+2 , \end{aligned}$$

where we have let

$$f(t) = \phi(t)^\top x \quad (7)$$

We have also

$$\hat{f}_s(s+1) = \phi(s+1)^\top \mathfrak{F}(s, s)x = f(s+1)$$

With the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} f(u)^2 &\leq \left(1 + \sum_{r=s+1}^{u-1} [\phi(u)^\top \phi(r)]^2\right) \left(\sum_{r=s+1}^u \hat{f}_s(r)^2\right) \\ &\leq \left(1 + |\phi(u)|^2 \sum_{v=s+1}^{u-1} |\phi(v)|^2\right) \left(\sum_{r=s+1}^u \hat{f}_s(r)^2\right) \quad \forall u \geq s+2 \\ f(s+1)^2 &= \hat{f}_s(s+1)^2 \end{aligned}$$

and therefore

$$\begin{aligned} &\sum_{u=s+1}^t f(u)^2 \\ &\leq \hat{f}_s(s+1)^2 + \sum_{u=s+2}^t \left(1 + |\phi(u)|^2 \sum_{v=s+1}^{u-1} |\phi(v)|^2\right) \left(\sum_{r=s+1}^u \hat{f}_s(r)^2\right) \\ &\leq \left(1 + \sum_{u=s+2}^t \left(1 + |\phi(u)|^2 \sum_{v=s+1}^{u-1} |\phi(v)|^2\right)\right) \hat{f}_s(s+1)^2 \\ &\quad + \sum_{r=s+2}^t \left(\sum_{u=r}^t \left(1 + |\phi(u)|^2 \sum_{v=s+1}^{u-1} |\phi(v)|^2\right)\right) \hat{f}_s(r)^2 \\ &\leq \left((t-s) + \left(\sum_{u=s+2}^t |\phi(u)|^2\right) \left(\sum_{v=s+1}^{t-1} |\phi(v)|^2\right)\right) \hat{f}_s(s+1)^2 \\ &\quad + \sum_{r=s+2}^t \left((t-r+1) + \left(\sum_{u=r}^t |\phi(u)|^2\right) \left(\sum_{v=s+1}^{t-1} |\phi(v)|^2\right)\right) \hat{f}_s(r)^2 \\ &\leq \left((t-s) + \left(\sum_{u=s+2}^t |\phi(u)|^2\right) \left(\sum_{v=s+1}^{t-1} |\phi(v)|^2\right)\right) \left(\sum_{r=s+1}^t \hat{f}_s(r)^2\right) \\ &\leq \left((t-s) + \left(\sum_{u=s+1}^t |\phi(u)|^2\right)^2\right) \left(\sum_{r=s+1}^t \hat{f}_s(r)^2\right) \quad (8) \end{aligned}$$

With the definition (7) of $f(u)$ this inequality is

$$x^\top \left(\sum_{u=s+1}^t \phi(u)\phi(u)^\top\right) x \leq \left((t-s) + \left(\sum_{u=s+1}^t |\phi(u)|^2\right)^2\right) \left(\sum_{r=s+1}^t \hat{f}_s(r)^2\right)$$

With (6), it allows us to obtain the following upperbound for $|\mathfrak{I}(t, s)x|^2$

$$\begin{aligned} |\mathfrak{I}(t, s)x|^2 &\leq |x|^2 - \min_{r \in \{s+1, \dots, t\}} \{2 - \phi(r)^\top \phi(r)\} \left(\sum_{r=s+1}^t \hat{f}_s(r)^2 \right) \\ &\leq |x|^2 - \frac{\min_{r \in \{s+1, \dots, t\}} \{2 - \phi(r)^\top \phi(r)\}}{\left((t-s) + \left(\sum_{u=s+1}^t |\phi(u)|^2 \right)^2 \right)} x^\top \left(\sum_{u=s+1}^t \phi(u)\phi(u)^\top \right) x \end{aligned}$$

In other words we have

$$\mathfrak{I}(t, s)^\top \mathfrak{I}(t, s) \leq I - \frac{\min_{r \in \{s+1, \dots, t\}} \{2 - \phi(r)^\top \phi(r)\}}{\left((t-s) + \left(\sum_{u=s+1}^t |\phi(u)|^2 \right)^2 \right)} \left(\sum_{u=s+1}^t \phi(u)\phi(u)^\top \right).$$

Remark 2 The same final remarks as for the continuous time case can be done here. In particular about the majoration (8), a less conservative bound is obtained in the proof of [1, Theorem 4.5] (or of [2, Theorem 2.2, 1st column, p. 2052], for the case where the preprocessing mentioned at the beginning of this section is

$$\phi(t) = \frac{\Phi(t)}{\sqrt{r(t)}}, \quad r(t) = r(t-1) + |\Phi(t)|^2$$

and the above analysis is carried out exploiting the assumption that the sequence r goes to infinity in some specific way.

References

- [1] H.-F. Chen, L. Guo. Identification and stochastic adaptive control. Springer Science+Business, LLC 1991.
- [2] W. Ren, P. Kumar. Stochastic adaptive prediction and model reference control. IEEE Transactions on Automatic Control, Vol. 39. No. 10. october 1994 2047

3 Convergence for the discrete and continuous time case

Let T and t_i be strictly positive real numbers satisfying

$$t_{i+1} - t_i \leq T$$

and let $t_0 = 0$. For any t , there exists $\tau(t)$ such that t is between $t_{\tau(t)}$ and $t_{1+\tau(t)}$. We have seen that both in the discrete and continuous time case, there exist a real number π_i in $[0, 1]$ satisfying

$$|\mathfrak{I}(t_i, t_{i-1})| \leq 1 - \pi_i$$

Specifically, assuming $t \mapsto |\phi(t)|$ is bounded by $\bar{\phi}$, and since $\sqrt{1-a} \leq 1 - \frac{a}{2}$

– in the continuous time case, we have

$$\pi_i = \frac{1}{2(1 + T^2\bar{\phi}^4)} \lambda_{\min} \left(\int_{t_{i-1}}^{t_i} \phi(u)\phi(u)^\top du \right)$$

– in the discrete time case, with $\bar{\phi}^2$ smaller than 2, we have

$$\pi_i = \frac{2 - \bar{\phi}^2}{2T(1 + T\bar{\phi}^4)} \lambda_{\min} \left(\sum_{1+t_{i-1}}^{t_i} \phi(u)\phi(u)^\top \right)$$

So we have

$$\begin{aligned} |\mathfrak{I}(t, 0)| &= \left| \mathfrak{I}(t, t_{\tau(t)}) \prod_{i=1}^{\tau(t)} \mathfrak{I}(t_i, t_{i-1}) \right| \\ &\leq |\mathfrak{I}(t, t_{\tau(t)})| \prod_{i=1}^{\tau(t)} |\mathfrak{I}(t_i, t_{i-1})| \\ &\leq \prod_{i=1}^{\tau(t)} (1 - \pi_i) \\ &\leq \exp \left(- \sum_{i=1}^{\tau(t)} \pi_i \right) \end{aligned}$$

where to obtain the last inequality we have used the property

$$(1 - x) \leq \exp(-x) \quad \forall x \in \mathbb{R} .$$

We conclude that $|\mathfrak{I}(t, 0)|$ tends to 0 if we can find T and the t_i 's such that we get

$$\sum_{i=1}^{\infty} \pi_i = +\infty .$$