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Scaling limits of random bipartite planar maps with a prescribed degree sequence

Cyril MARZOUK *

28th December 2016

Abstract

We study the asymptotic behaviour of uniform random maps with a prescribed face-degree sequence, in the bipartite case, as the number of faces tends to infinity. Under mild assumptions, we show that, properly rescaled, such maps converge in distribution towards the Brownian map in the Gromov–Hausdorff sense. This result encompasses the previous ones of Miermont for uniform random quadrangulations and Le Gall for uniform random q -angulations with $q \in \{4, 6, 8, \dots\}$. It applies also to random maps sampled from a Boltzmann distribution, under a second moment assumption only, conditioned to be large in either of the sense of the number of edges, vertices, or faces. The proof relies on the convergence of so-called “discrete snakes” obtained by adding spatial positions to the nodes of uniform random plane trees with a prescribed child sequence recently studied by Broutin & Marckert. This paper can alternatively be seen as a contribution to the study of the geometry of such trees.

1 Introduction

1.1 Random planar maps as metric spaces

The study of scaling limits of large random maps, viewed as metric spaces, towards a universal object called the *Brownian map* has seen numerous developments over the last decade. This paper is another step towards this universality as we show that the Brownian map appears as limit of maps with a prescribed degree sequence. This particular model is introduced in the next subsection, let us first discuss the general idea of such studies and recall some previous results.

Recall that a (planar) map is an embedding of a finite connected graph into the two-dimensional sphere, viewed up to orientation-preserving homeomorphisms. For technical reasons, the maps we consider will always be *rooted*, which means that an oriented edge is distinguished. Maps have been widely studied in combinatorics and *random* maps are of interest in theoretical physics, for which they are natural discretised version of random geometry, in particular in the theory of quantum gravity (see e.g. [7]). One can view a map as a (finite) metric space by endowing the set of vertices with the graph distance: the distance between two vertices is the minimal number of edges of a path going from one to the other; throughout this paper, if M is a map, we will denote the associated metric space, with a slight abuse of notation, by (M, d_{gr}) . The set of all compact metric spaces, considered up to isometry, can be equipped with a metric, called the Gromov–Hausdorff distance, which makes it separable and complete [15, 13]; we can then study the convergence in distribution of random maps viewed as metric spaces.

A first result in this direction has been obtained simultaneously by Le Gall [26] and Miermont [38] using different approaches. We call *faces* of a map the connected components of the complement of the

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edges; the *degree* of a face is then the number of edges incident to it, with the convention that if both sides of an edge are incident to the same face, then it is counted twice. A *quadrangulation* is a map in which all faces have degree 4. In [26] and [38], it is shown that if \mathcal{Q}_n is a uniform random rooted quadrangulation with n faces, then the convergence in distribution

$$\left(\mathcal{Q}_n, \left(\frac{9}{8n} \right)^{1/4} d_{\text{gr}} \right) \xrightarrow[n \rightarrow \infty]{(d)} (\mathcal{M}, \mathcal{D}),$$

holds in the sense of Gromov–Hausdorff, where the limit $(\mathcal{M}, \mathcal{D})$, called the *Brownian map*, is a random compact metric space, which is almost surely homeomorphic to the 2-sphere (Le Gall & Paulin [29], Miermont [37]) and has Hausdorff dimension 4 (Le Gall [25]).

Le Gall [26] gives a quite general method to prove such a limit theorem; the main result in [26] is indeed stated for q -angulations (which are maps in which each face has degree q) with n faces, for any $q \in \{3, 4, 6, 8, \dots\}$ fixed. The limit is always the Brownian map as well as the scaling factor $n^{-1/4}$, only the multiplicative constant $(9/8)^{1/4}$ above depends on q (see the precise statement below). Note that apart from the case $q = 3$ of *triangulations*, [26] only deals with maps with even face-degrees, which corresponds in the planar case to *bipartite* maps. The non bipartite case is technically more involved and we henceforth restrict ourselves to bipartite maps as well.

1.2 Main result and notation

We generalise q -angulations by considering maps with possibly faces of different degrees. For every $n \geq 1$, we are given a sequence $\mathbf{n} = (n_i; i \geq 1)$ of non-negative integers satisfying

$$\sum_{i \geq 1} n_i = n,$$

and we denote by $\mathbf{M}(\mathbf{n})$ the set of rooted planar maps with n_i faces of degree $2i$ for every $i \geq 1$. This is a finite set, its cardinal was first calculated by Tutte [39] (this paper deals with Eulerian maps with a prescribed vertex-degree sequence, which are the dual of our present maps). Let us introduce the notation that we will use throughout this paper. Set

$$N_{\mathbf{n}} := \sum_{i \geq 1} i n_i \quad \text{and} \quad n_0 := 1 + N_{\mathbf{n}} - n. \quad (1)$$

Note that every map in $\mathbf{M}(\mathbf{n})$ contains n faces and $N_{\mathbf{n}}$ edges so, according to Euler formula, it has $2 + N_{\mathbf{n}} - n = n_0 + 1$ vertices (this shift by one will simplify some statements later). We next define a probability measure and its variance by

$$p_{\mathbf{n}}(i) := \frac{n_i}{N_{\mathbf{n}} + 1} \quad \text{for } i \geq 0 \quad \text{and} \quad \sigma_{\mathbf{n}}^2 := \sum_{i \geq 1} i^2 p_{\mathbf{n}}(i) - \left(\frac{N_{\mathbf{n}}}{N_{\mathbf{n}} + 1} \right)^2.$$

The probability $p_{\mathbf{n}}$ is (up to the fact that there are $n_0 + 1$ vertices) the empirical half face-degree distribution of a map in $\mathbf{M}(\mathbf{n})$ if one sees the vertices as faces of degree 0. Last, let us denote by

$$\Delta_{\mathbf{n}} := \max\{i \geq 0 : n_i > 0\}$$

the right edge of the support of $p_{\mathbf{n}}$.

Our main assumption is the following: there exists a probability measure $p = (p(i); i \geq 0)$ with mean 1 and variance $\sigma_p^2 := \sum_{i \geq 1} i^2 p(i) - 1 \in (0, \infty)$ such that, as $|\mathbf{n}| = n \rightarrow \infty$,

$$p_{\mathbf{n}} \Longrightarrow p, \quad \sigma_{\mathbf{n}}^2 \longrightarrow \sigma_p^2 \quad \text{and} \quad n^{-1/2} \Delta_{\mathbf{n}} \longrightarrow 0, \quad (\mathbf{H})$$

where “ \Longrightarrow ” denotes the weak convergence of probability measures, which is here equivalent to $p_{\mathbf{n}}(i) \rightarrow p(i)$ for every $i \geq 0$.

Theorem 1. Under **(H)**, if \mathcal{M}_n is sampled uniformly at random in $\mathbf{M}(\mathbf{n})$ for every $n \geq 1$, then the convergence in distribution

$$\left(\mathcal{M}_n, \left(\frac{9}{4} \frac{1-p(0)}{\sigma_p^2} \frac{1}{n} \right)^{1/4} d_{\text{gr}} \right) \xrightarrow[n \rightarrow \infty]{(d)} (\mathcal{M}, \mathcal{D}),$$

holds in the sense of Gromov–Hausdorff.

Since the graph distance is defined in terms of edges, it would be natural to make the rescaling depend on $N_{\mathbf{n}}$ rather than n . Under **(H)**, we have $n/N_{\mathbf{n}} \rightarrow 1 - p(0)$ as $n \rightarrow \infty$ so the convergence in Theorem 1 is equivalent to

$$\left(\mathcal{M}_n, \left(\frac{9}{4\sigma_p^2} \frac{1}{N_{\mathbf{n}}} \right)^{1/4} d_{\text{gr}} \right) \xrightarrow[n \rightarrow \infty]{(d)} (\mathcal{M}, \mathcal{D}),$$

in the sense of Gromov–Hausdorff, which is what we shall prove.

Remark 1. The probabilities $p_{\mathbf{n}}$ and p in **(H)** are deterministic but one could also first sample a random sequence \mathbf{n} and then sample \mathcal{M}_n uniformly at random in $\mathbf{M}(\mathbf{n})$ and the theorem would still hold.

This result recovers in particular the aforementioned one of Le Gall [26] for q -angulations when $q \geq 4$ is even. Indeed, the latter correspond to $\mathbf{M}(\mathbf{n})$ where $n_i = n$ if $i = q/2$ and $n_i = 0$ otherwise. In this case $N_{\mathbf{n}} = nq/2$ and **(H)** is fulfilled with

$$p(0) = 1 - \frac{2}{q} \quad \text{and} \quad p\left(\frac{q}{2}\right) = \frac{2}{q} \quad \text{so} \quad \sigma_p^2 = \frac{q}{2} - 1 = \frac{q-2}{2};$$

Theorem 1 therefore reads as follows.

Corollary 1 (Le Gall [26]). Fix $q \geq 4$ an even integer and for every $n \geq 1$, let $\mathcal{M}_n^{(q)}$ be a uniform random rooted q -angulation with n faces. The convergence in distribution

$$\left(\mathcal{M}_n^{(q)}, \left(\frac{9}{q(q-2)} \frac{1}{n} \right)^{1/4} d_{\text{gr}} \right) \xrightarrow[n \rightarrow \infty]{(d)} (\mathcal{M}, \mathcal{D}),$$

holds in the sense of Gromov–Hausdorff.

1.3 Boltzmann random maps

Theorem 1 also applies to random maps sampled from a *Boltzmann distribution*. Relying on results of Marckert & Miermont [32], Le Gall [26] obtained under some integrability assumptions, the convergence of such maps with n vertices towards the Brownian map as $n \rightarrow \infty$. Le Gall & Miermont [28] have also considered maps with n vertices in which the distribution of the degree of a typical face is in the domain of attraction of a stable distribution with index $\alpha \in (1, 2)$ and obtained different objects at the limit (after extraction of a subsequence). Last, Janson & Stefánsson [20] have studied maps with n edges which exhibit a *condensation* phenomenon and converge, after rescaling, towards the Brownian tree: a unique giant face emerges and its boundary collapses into a tree.

Given a sequence $w = (w(k); k \geq 1)$ of non-negative real numbers, we define a measure Ω^w on the set \mathbf{M} of rooted bipartite maps by the formula

$$\Omega^w(\mathcal{M}) = \prod_{f \in \text{Faces}(\mathcal{M})} w(\deg(f)/2), \quad \mathcal{M} \in \mathbf{M},$$

where $\text{Faces}(\mathcal{M})$ is the set of faces of \mathcal{M} and $\deg(f)$ is the degree of such a face f . One could also add in the definition of $\Omega^w(\mathcal{M})$ two extra factors, say, $a^{V(\mathcal{M})}$ and $b^{E(\mathcal{M})}$ for some constants $a, b > 0$, where

$V(\mathcal{M})$ and $E(\mathcal{M})$ are the number of vertices and edges respectively of \mathcal{M} , but this measure would be nothing but $a^2\Omega^{\tilde{w}}(\mathcal{M})$, where $\tilde{w}(k) = a^{k-1}b^k w(k)$ for every $k \geq 1$.

Set $Z_w = \Omega^w(\mathbf{M})$; whenever it is finite, the formula

$$\mathbf{P}^w(\cdot) = \frac{1}{Z_w} \Omega^w(\cdot)$$

defines a probability measure on \mathbf{M} . We consider next such random maps conditioned to have a large size for several notions of size. For every integer $n \geq 2$, let $\mathbf{M}_{E=n}$, $\mathbf{M}_{V=n}$ and $\mathbf{M}_{F=n}$ be the subsets of \mathbf{M} of those maps with respectively $n-1$ edges, $n+1$ vertices (similarly to the definition of n_0 in (1), these shifts by one will simplify some statements later) and n faces; more generally, for every $A \subset \mathbf{N}$, let $\mathbf{M}_{F,A=n}$ be the subset of \mathbf{M} of those maps with n faces whose degree belongs to $2A$ (and possibly other faces, but with a degree in $2\mathbf{N} \setminus 2A$). For every $S = \{E, V, F\} \cup \bigcup_{A \subset \mathbf{N}} \{F, A\}$ and every $n \geq 2$, we define

$$\mathbf{P}_{S=n}^w(\mathcal{M}) := \mathbf{P}^w(\mathcal{M} \mid \mathcal{M} \in \mathbf{M}_{S=n}), \quad \mathcal{M} \in \mathbf{M}_{S=n},$$

the law of a Boltzmann map conditioned to have size n ; here and later, we shall always, if necessary, implicitly restrict ourselves to those values of n for which $\Omega^w(\mathbf{M}_{S=n}) \neq 0$.

Under mild integrability conditions on w , we show in Section 7 that for every $S \in \{E, V, F\} \cup \bigcup_{A \subset \mathbf{N}} \{F, A\}$ and for every $n \geq 2$ such that it makes sense, if \mathcal{M}_n is sampled from $\mathbf{P}_{S=n}^w$, then the convergence in distribution

$$\left(\mathcal{M}_n, \left(\frac{K_S^w}{n} \right)^{1/4} d_{\text{gr}} \right) \xrightarrow[n \rightarrow \infty]{(d)} (\mathcal{M}, \mathcal{D}),$$

holds in the sense of Gromov–Hausdorff, where $K_S^w > 0$ is a constant which depends only on S and w . We refer to Theorem 4 in Section 7 for a precise statement.

Observe that if \mathcal{M} is sampled from $\mathbf{P}_{S=n}^w$ for any $S \in \{E, V, F\} \cup \bigcup_{A \subset \mathbf{N}} \{F, A\}$ then, conditional on its degree sequence, say, $\nu_{\mathcal{M}} = (\nu_{\mathcal{M}}(i); i \geq 1)$, it has the uniform distribution in $\mathbf{M}(\nu_{\mathcal{M}})$. Appealing to Remark 1, the proof of the above convergence consists in showing that $\nu_{\mathcal{M}}$ satisfies **(H)** for some deterministic probability p_w .

The case $S = V$ was obtained by Le Gall [26, Theorem 9.1] when w is *regular critical*, meaning that the distribution p_w (which is roughly that of the half-degree of a typical face when we see vertices as faces of degree 0) admits small exponential moments. Here, we generalise this result (and consider other conditionings) to all *generic critical* sequences w , i.e. those for which p_w admits a second moment.

The conditioning $S = E$ by the number of edges is somewhat special since $\mathbf{M}_{E=n}$ is finite for every $n \geq 2$ (its cardinal was expressed by Walsh [40, Equation 7]) so the distribution $\mathbf{P}_{E=n}^w(\cdot) = \Omega^w(\cdot)/\Omega^w(\mathbf{M}_{E=n})$ on $\mathbf{M}_{E=n}$ makes sense even if $Z_w = \Omega^w(\mathbf{M})$ is infinite; we shall see that the above convergence still holds in this case (Theorem 5). The simplest and most important example is the constant sequence $w(k) = 1$ for every $k \geq 1$, in which case $\mathbf{P}_{E=n}^w$ corresponds to the uniform distribution in $\mathbf{M}_{E=n}$ and we calculate $K_E^w = 1/2$, which recovers a result first due to Abraham [1].

Corollary 2 (Abraham [1]). *For every $n \geq 1$, let \mathcal{M}_n be a uniform random rooted bipartite map with n edges. The convergence in distribution*

$$\left(\mathcal{M}_n, \left(\frac{1}{2n} \right)^{1/4} d_{\text{gr}} \right) \xrightarrow[n \rightarrow \infty]{(d)} (\mathcal{M}, \mathcal{D}),$$

holds in the sense of Gromov–Hausdorff.

1.4 Approach and organisation of the paper

Our approach to proving Theorem 1 follows closely the robust one of Le Gall [26]. Specifically, we code our map \mathcal{M}_n by a certain *labelled* (or *spatial*) *two-type tree* (\mathcal{T}_n, ℓ_n) via a bijection due to Bouttier, Di

Francesco & Guitter [11]: \mathcal{T}_n is a plane tree and ℓ_n is a function which associates with each vertex of \mathcal{T}_n a label (or a spatial position) in \mathbf{Z} . Such a labelled tree is itself encoded by a pair of discrete paths $(\mathcal{C}_n^\circ, \mathcal{L}_n^\circ)$; we show that under **(H)**, this pair, suitably rescaled, converges in distribution towards a pair (\mathbf{e}, Z) called in the literature the “head of the Brownian snake” (e.g. [33, 19, 31]). The construction of the Brownian map from (\mathbf{e}, Z) is analogous to the Bouttier–Di Francesco–Guitter bijection; Theorem 1 follows from this functional limit theorem as well as a certain “invariance under re-rooting” of our maps as was shown by Le Gall [26].

To prove such an invariance principle for (\mathcal{T}_n, ℓ_n) , we further rely on a more recent bijection due to Janson & Stefánsson [20] which maps two-type trees to *one-type* trees which are easier to control. As a matter of fact, if \mathcal{M}_n is uniformly distributed in $\mathbf{M}(\mathbf{n})$ and (T_n, l_n) is its corresponding labelled one-type tree, then the unlabelled tree T_n is a uniform random tree with a prescribed degree (in the sense of child) sequence as studied by Broutin & Marckert [12]. The labelled tree (T_n, l_n) is again encoded by a pair of functions (H_n, L_n) and the main result of [12] is, under the very same assumption **(H)**, the convergence of H_n suitably rescaled towards \mathbf{e} . Our main contribution consists in strengthening this result by adding the labels in their limit theorem to show that the pair (H_n, L_n) , suitably rescaled, converges towards (\mathbf{e}, Z) , and then transporting this invariance principle back to the two-type tree (\mathcal{T}_n, ℓ_n) .

The previous works on the convergence of large random labelled trees focus on the case when the tree is a size-conditioned (one or multi-type) Galton–Watson tree and a lot of effort has been put to reduce the assumptions of the labels as much as possible, maintaining quite strong assumption on the tree itself; a common assumption is indeed to consider a Galton–Watson tree whose offspring distribution admits small exponential moments; in order to reduce the assumption on the labels, Marckert [31] even supposes the offsprings to be uniformly bounded. In this paper, we take the opposite direction: we focus only on the labels given by the bijection with planar maps, which satisfy rather strong assumptions, and work under weak assumptions on the tree (essentially a first and second moments condition). Furthermore, we consider trees with a prescribed degree sequence, which are more general than Galton–Watson trees and on which the literature is limited, which explains the length of this work.

Let us mention that other convergences towards the Brownian map similar to Theorem 1 have been obtained using also other bijections with labelled trees: Beltran & Le Gall [8] studied random quadrangulations without vertices of degree one, Addario-Berry & Albenque [3] considered random triangulations and quadrangulations without loops or multiple edges and Bettinelli, Jacob & Miermont [9] uniform random maps with n edges.

This work leaves open two questions that we plan to investigate in the future. First, one can consider non-bipartite maps with a prescribed degree sequence; we restricted ourselves here to bipartite maps because (except in the notable case of triangulations), in the non-bipartite case, the Bouttier–Di Francesco–Guitter bijection yields a more complicated labelled three-type tree which is harder to analyse; moreover, the Janson–Stefánsson bijection does not apply to such trees so the method of proof should be different.

A second direction of future work would be to relax the assumption **(H)**, in particular to consider maps with large faces. A first step would be to extend the work of Broutin & Marckert [12] on plane trees; we believe that the family of so-called inhomogeneous continuum random trees introduced in [6, 14] appear at the limit; one would then construct a family of random maps from these trees, replacing the Brownian excursion \mathbf{e} by their “exploration process” studied in [5].

This paper is organised as follows. In Section 2, we first introduce the notion of labelled one-type and two-type trees and their encoding by functions, then we describe the Bouttier–Di Francesco–Guitter and Janson–Stefánsson bijections. In Section 3, we define the pair (\mathbf{e}, Z) and the Brownian map and we state our main results on the convergence of discrete paths. Section 4 is a technical section in which we extend a “backbone decomposition” of Broutin & Marckert [12]. We prove the convergence of the pairs $(\mathcal{C}_n^\circ, \mathcal{L}_n^\circ)$ and (H_n, L_n) , encoding the labelled trees (\mathcal{T}_n, ℓ_n) and (T_n, l_n) respectively, in Section 5. Then we prove Theorem 1 in section 6. Finally, we apply our results to Boltzmann random maps in Section 7.

Acknowledgments I am deeply indebted to Grégory Miermont for suggesting me to study this model, for several discussions during the preparation of this work and for pointing me out some inaccuracies in a first draft. Many thanks also to all the persons I have asked about Remark 6, and in particular Olivier Hénard who then asked me every morning if I succeeded proving it until I tried another way (Corollary 3).

2 Maps and trees

2.1 Plane trees and their encoding with paths

Let $\mathbf{N} = \{1, 2, \dots\}$ be the set of all positive integers, set $\mathbf{N}^0 = \{\emptyset\}$ and consider the set of words

$$\mathbf{U} = \bigcup_{n \geq 0} \mathbf{N}^n.$$

For every $u = (u_1, \dots, u_n) \in \mathbf{U}$, we denote by $|u| = n$ the length of u ; if $n \geq 1$, we define its *prefix* $pr(u) = (u_1, \dots, u_{n-1})$ and for $v = (v_1, \dots, v_m) \in \mathbf{U}$, we let $uv = (u_1, \dots, u_n, v_1, \dots, v_m) \in \mathbf{U}$ be the concatenation of u and v . We endow \mathbf{U} with the *lexicographical order*: given $u, v \in \mathbf{U}$, let $w \in \mathbf{U}$ be their longest common prefix, that is $u = w(u_1, \dots, u_n)$, $v = w(v_1, \dots, v_m)$ and $u_1 \neq v_1$, then $u < v$ if $u_1 < v_1$.

A *plane tree* is a non-empty, finite subset $\tau \subset \mathbf{U}$ such that:

- (i) $\emptyset \in \tau$;
- (ii) if $u \in \tau$ with $|u| \geq 1$, then $pr(u) \in \tau$;
- (iii) if $u \in \tau$, then there exists an integer $k_u \geq 0$ such that $ui \in \tau$ if and only if $1 \leq i \leq k_u$.

We shall denote the set of plane trees by \mathbf{T} . We will view each vertex u of a tree τ as an individual of a population for which τ is the genealogical tree. The vertex \emptyset is called the *root* of the tree and for every $u \in \tau$, k_u is the number of *children* of u (if $k_u = 0$, then u is called a *leaf*, otherwise, u is called an *internal vertex*) and u_1, \dots, u_{k_u} are these children from left to right, $|u|$ is its *generation*, $pr(u)$ is its *parent* and more generally, the vertices $u, pr(u), pr \circ pr(u), \dots, pr^{|u|}(u) = \emptyset$ are its *ancestors*; the longest common prefix of two elements is their *last common ancestor*. We shall denote by $\llbracket u, v \rrbracket$ the unique non-crossing path between u and v .

Fix a tree τ with N edges and let $\emptyset = u_0 < u_1 < \dots < u_N$ be its vertices, listed in lexicographical order. We describe three discrete paths which each encode τ . First, its *Lukasiewicz path* $W = (W(j); 0 \leq j \leq N + 1)$ is defined by $W(0) = 0$ and for every $0 \leq j \leq N$,

$$W(j + 1) = W(j) + k_{u_j} - 1.$$

One easily checks that $W(j) \geq 0$ for every $0 \leq j \leq N$ but $W(N + 1) = -1$. Next, we define the *height process* $H = (H(j); 0 \leq j \leq N)$ by setting for every $0 \leq j \leq N$,

$$H(j) = |u_j|.$$

Finally, define the *contour sequence* $(c_0, c_1, \dots, c_{2N})$ of τ as follows: $c_0 = \emptyset$ and for each $i \in \{0, \dots, 2N - 1\}$, c_{i+1} is either the first child of c_i which does not appear in the sequence (c_0, \dots, c_i) , or the parent of c_i if all its children already appear in this sequence. The lexicographical order on the tree corresponds to the depth-first search order, whereas the contour order corresponds to “moving around the tree in clockwise order”. The *contour process* $C = (C(j); 0 \leq j \leq 2N)$ is defined by setting for every $0 \leq j \leq 2N$,

$$C(j) = |c_j|.$$

Without further notice, throughout this work, every discrete path shall also be viewed as a continuous function after interpolating linearly between integer times.

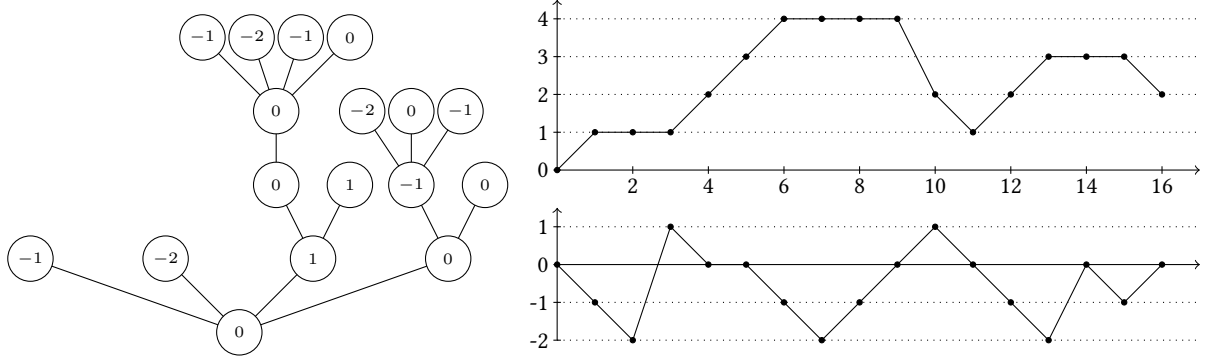


Figure 2: A one-type labelled tree, its height process on top and its label process below.

For future reference, for every $r \geq 1$, let us call a *bridge* of length r a vector (x_0, \dots, x_r) satisfying $x_0 = x_r = 0$ and $x_j - x_{j-1} \in \mathbf{Z}$ for every $1 \leq j \leq r$. Denote by

$$\mathcal{B}_r^+ := \{(x_0, \dots, x_r) : x_0 = x_r = 0 \text{ and } x_j - x_{j-1} \in \{-1, 0, 1, 2, \dots\} \text{ for } 1 \leq j \leq r\}, \quad (2)$$

the set of bridges of length r with *no negative jump*. By definition, if (T, l) is a labelled one-type tree and u is a vertex of T with $r \geq 1$ children, then the sequence $(0, l(u_1) - l(u), \dots, l(u_r) - l(u))$ belongs to \mathcal{B}_r^+ . Since the cardinal of \mathcal{B}_r^+ is $\binom{2r-1}{r-1}$, it follows that a tree T possesses

$$\prod_{u \in T: k_u \geq 1} \binom{2k_u - 1}{k_u - 1} \quad (3)$$

possible labellings; observe that this quantity depends on T only through $(k_u; u \in T)$.

2.3 The Bouttier–Di Francesco–Guitter bijection

A map is said to be *pointed* if a vertex is distinguished. Given a sequence \mathbf{n} of non-negative integers, we denote by $\mathbf{M}^*(\mathbf{n})$ the set of rooted and pointed planar maps with n_i faces with degree $2i$ for every $i \geq 1$. Let $\mathbf{T}_{\circ, \bullet}(\mathbf{n})$ denote the set of two-type trees with n_i black vertices with degree i for every $i \geq 1$; note that such a tree has n_0 white vertices and $N_{\mathbf{n}}$ edges, which are both defined in (1). Let further $\mathbf{LT}_{\circ, \bullet}(\mathbf{n})$ be the set of such labelled two-type trees.

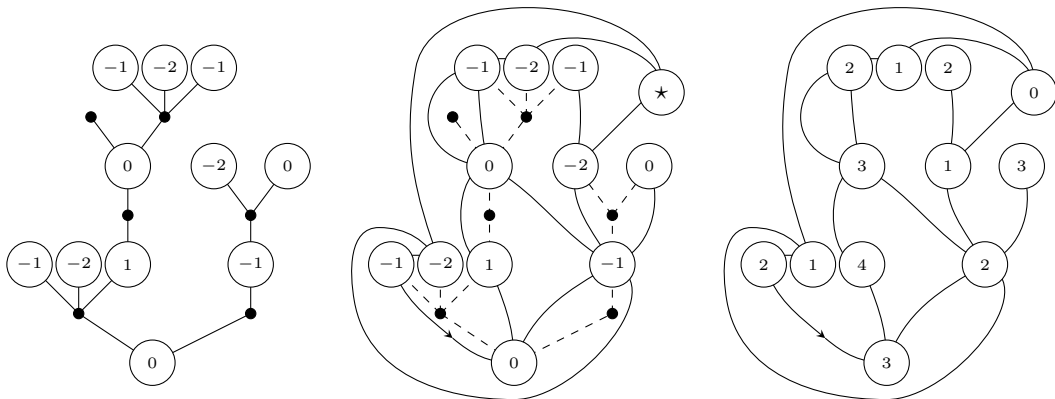


Figure 3: The Bouttier–Di Francesco–Gutter bijection.

Bouttier, Di Francesco & Guitter [11] show that $\mathbf{M}^*(\mathbf{n})$ and $\{-1, +1\} \times \mathbf{LT}_{\circ, \bullet}(\mathbf{n})$ are in bijection, we shall refer to it as the BDG bijection. Let us only recall how a map is constructed from a labelled two-type tree (\mathcal{T}, ℓ) , as depicted by Figure 3. Let N be the number of edges of \mathcal{T} , we write $(c_0^\circ, \dots, c_N^\circ)$

for its white contour sequence and we adopt the convention that $c_{N+i}^\circ = c_i^\circ$ for every $0 \leq i \leq N$. A white *corner* is a sector around a white vertex delimited by two consecutive edges; there are N white corners, corresponding to the vertices $c_0^\circ, \dots, c_{N-1}^\circ$; for every $0 \leq i \leq 2N$ we denote by e_i the corner corresponding to c_i° . We add an extra vertex \star outside of the tree \mathcal{T} and construct a map on the vertex-set of \mathcal{T} and \star by drawing edges as follows: for every $0 \leq i \leq N-1$,

- if $\ell(c_i^\circ) > \min_{0 \leq k \leq N-1} \ell(c_k^\circ)$, then we draw an edge between e_i and e_j where $j = \min\{k > i : \ell(c_k^\circ) = \ell(c_i^\circ) - 1\}$,
- if $\ell(c_i^\circ) = \min_{0 \leq k \leq N-1} \ell(c_k^\circ)$, then we draw an edge between e_i and \star .

It is shown in [11] that this procedure indeed produces a planar map \mathcal{M} , pointed at \star , and rooted at the first edge that we drew, for $i = 0$, oriented according to an external choice $\epsilon \in \{-1, +1\}$ and, further, that it is invertible. Observe that \mathcal{M} has N edges, as many as \mathcal{T} , and that the faces of \mathcal{M} correspond to the black vertices of \mathcal{T} ; one can check that the degree of a face is twice that of the corresponding black vertex, we conclude that the above procedure indeed realises a bijection between $\mathbf{M}^\star(\mathbf{n})$ and $\{-1, +1\} \times \mathbf{LT}_{\circ, \bullet}(\mathbf{n})$. One may be concerned with the fact that the vertices of \mathcal{M} different from \star are labelled, which seems at first sight to be an extra information; shift these labels by adding to each the quantity $1 - \min_{c^\circ \in \circ(\mathcal{T})} \ell(c^\circ)$ and label 0 the vertex \star , then the label of each vertex corresponds to its graph distance in \mathcal{M} to the origin \star .

2.4 The Janson–Stefánsson bijection

Let $\mathbf{T}(\mathbf{n})$ denote the set of one-type trees possessing n_i vertices with i children for every $i \geq 0$; note that such a tree has $N_{\mathbf{n}}$ edges and that $p_{\mathbf{n}}$ defined in Section 1.2 is its empirical offspring distribution. Uniform random trees in $\mathbf{T}(\mathbf{n})$ have been studied by Addario-Berry [2] who obtained sub-Gaussian tail bounds for their height and width for n fixed and Broutin & Marckert [12] who showed that, properly rescaled, under our assumption **(H)**, they converge in distribution in the sense of Gromov–Hausdorff, towards the celebrated Brownian tree, see Theorem 2 below.

Janson & Stefánsson [20] show that $\mathbf{T}(\mathbf{n})$ and $\mathbf{T}_{\circ, \bullet}(\mathbf{n})$ are in bijection, we shall refer to it as the JS bijection. In this bijection, the white vertices of the tree in $\mathbf{T}_{\circ, \bullet}(\mathbf{n})$ are mapped onto the leaves of the tree in $\mathbf{T}(\mathbf{n})$ and the black vertices in the former, with degree $k \geq 1$, are mapped onto (internal) vertices of the latter with k children. Let us recall the construction of this bijection in the two directions.

Let us start with a two-type tree \mathcal{T} ; we construct a one-type tree T with the same vertex-set as follows. First, if $\mathcal{T} = \{\emptyset\}$ is a singleton, then set $T = \{\emptyset\}$; otherwise, for every white vertex $u \in \circ(\mathcal{T})$ with $k_u \geq 1$ children, do the following: first, if $u \neq \emptyset$, draw an edge between its parent $pr(u)$ and its first child $u1$, then draw edges between any two consecutive children $u1$ and $u2$, $u2$ and $u3$, \dots , $u(k_u - 1)$ and uk_u , and finally draw an edge between uk_u and u ; if u is a leaf of \mathcal{T} , then this procedure reduces to drawing an edge between u and $pr(u)$. We root the new tree T at the first child of the root of \mathcal{T} . See Figure 4 for an illustration.

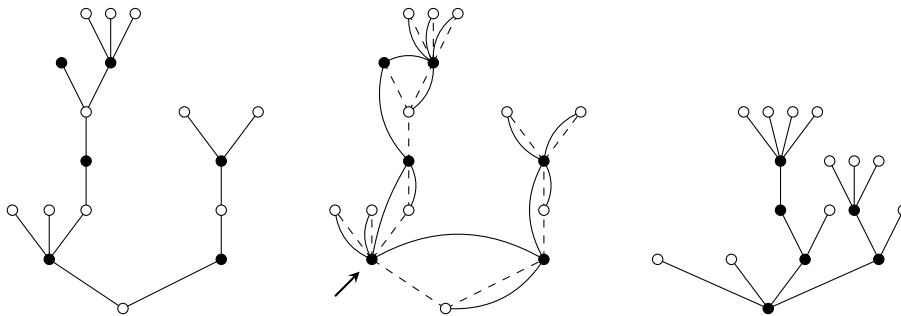


Figure 4: The Janson–Stefánsson bijection from two-type trees to one-type trees.

Conversely, given a one-type tree T , we construct a two-type tree \mathcal{T} as follows. Again, set $\mathcal{T} = \{\emptyset\}$ whenever $T = \{\emptyset\}$; otherwise, for every leaf u of T , denote by u^* its last ancestor whose last child is is

not an ancestor of u ; formally set

$$u^* := \sup \{w \in \llbracket \emptyset, u \llbracket : wk_w \notin \llbracket \emptyset, u \llbracket \}.$$

The set on the right may be empty, in which case $u^* = \emptyset$ by convention. Then draw an edge between u and every vertex $v \in \llbracket u^*, u \llbracket$. This yields a tree that we root at the last leaf of T . See Figure 5 for an illustration. One can check that the two procedures are the inverse of one another.

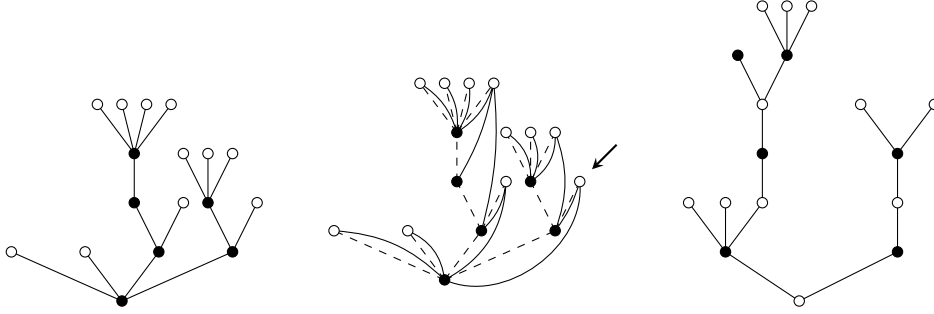


Figure 5: The Janson–Stefansson bijection from one-type trees to two-type trees.

Let further $\mathbf{LT}(\mathbf{n})$ be the set of labelled one-type trees possessing n_i vertices with i children for every $i \geq 0$, the JS bijection extends to a bijection between $\mathbf{LT}(\mathbf{n})$ and $\mathbf{LT}_{\circ, \bullet}(\mathbf{n})$ if every black vertex of a two-type tree is given the label of its white parent; see Figure 6.

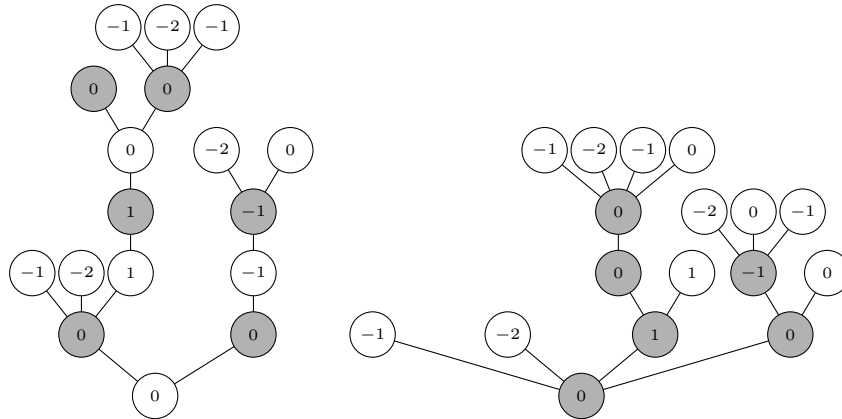


Figure 6: The Janson–Stefansson bijection with labels.

Let us explain how this bijection translates in terms of the processes encoding the labelled trees. Fix (\mathcal{T}, ℓ) a two-type labelled tree and denote by \mathcal{C}° its white contour process and \mathcal{L}° its white label process (in contour order). Fix also (T, l) a one-type labelled tree and denote by H its height process and L its label process (in lexicographical order). Finally, introduce a modified version of the height process: let N be the number of edges of T and (u_0, \dots, u_N) be its vertices listed in lexicographical order; for each integer $j \in \{0, \dots, N\}$, we let $\tilde{H}(j)$ denote the number of strict ancestors of u_j whose last child is not an ancestor of u_j , i.e.

$$\tilde{H}(j) := \# \{w \in \llbracket \emptyset, u_j \llbracket : wk_w \notin \llbracket \emptyset, u_j \llbracket \}.$$

Lemma 1. *If (T, l) and (\mathcal{T}, ℓ) are related by the JS bijection, then*

$$\mathcal{L}^\circ = L \quad \text{and} \quad \mathcal{C}^\circ = \tilde{H}.$$

Proof. Let us first prove the equality of the label processes. We use the observation from [22] that the lexicographical order on the vertices of T corresponds to the contour order of the black corners of \mathcal{T}

which, by a shift, corresponds to the contour order of the white corners of \mathcal{T} . Specifically, let N be the number of edges of both trees, fix $j \in \{0, \dots, N\}$ and consider the j -th white corner of \mathcal{T} : it is a sector around a white vertex delimited by two consecutive edges, whose other extremity is therefore black; consider the previous black corner in contour order, in the construction of the JS bijection, an edge of T starts from this corner and we claim that the other extremity of this edge is u_j the j -th vertex of T in lexicographical order. We refer to the proof of Proposition 4 and Figure 4 in [22].

It follows that if $c_j^\circ \in \circ(\mathcal{T})$ is the white vertex visited at the j -th step in the white contour sequence, then the image of u_j by the JS bijection is

- either c_j° : this is the case when c_j° is a leaf or when the white corner is the one given by the last child of c_j° and its parent;
- or a child of c_j° : precisely, its first child if the white corner is the one given by the parent of c_j° and its first child, and its k -th child if the corner is the one given by the $k - 1$ st and k -th children of c_j° .

Since a black vertex inherits the label of its white parent, we conclude that in both cases we have $L(j) = l(u_j) = \ell(c_j^\circ) = \mathcal{L}^\circ(j)$.

Next, for every $u \in T$, set

$$\tilde{H}(u) := \# \{w \in \llbracket \emptyset, u \llbracket : wk_w \notin \llbracket \emptyset, u \llbracket \};$$

if $\tilde{H}(u) \neq 0$, recall the definition

$$u^* := \sup \{w \in \llbracket \emptyset, u \llbracket : wk_w \notin \llbracket \emptyset, u \llbracket \}.$$

Fix $v \in \circ(\mathcal{T})$ a white vertex of \mathcal{T} and $w \in \bullet(\mathcal{T})$ one of its children. Denote by $\text{JS}(v), \text{JS}(w) \in T$ their image by the JS bijection, we argue that $\tilde{H}(\text{JS}(v))$ and $\tilde{H}(\text{JS}(w))$ are both equal to half the generation of v in \mathcal{T} . Denote by $u = \text{JS}(v)$; from the construction of the JS bijection, if v is different from the root of \mathcal{T} , then its parent in \mathcal{T} is mapped onto u^* and its children onto $\llbracket u^*, u \llbracket$, thus

$$\tilde{H}(\text{JS}(w)) = \tilde{H}(\text{JS}(v)) = \tilde{H}(u) = \tilde{H}(u^*) + 1 = \tilde{H}(\text{JS}(pr(v))) + 1.$$

If v is the root of \mathcal{T} , then u is the right-most leaf of T and v and its children are mapped onto the vertices of T for which $\tilde{H} = 0$. We conclude after an induction on the generation of v that indeed, $\tilde{H}(\text{JS}(w))$ and $\tilde{H}(\text{JS}(v))$ are equal, and their common value is given by half the generation of v in \mathcal{T} .

Recall the notation $c_j^\circ \in \circ(\mathcal{T})$ for the white vertex of \mathcal{T} visited at the j -th step in the white contour sequence and u_j for the j -th vertex of T in lexicographical order. Since the image of u_j by the JS bijection is either c_j° or one of its children, we conclude in both cases that $\tilde{H}(u_j)$ is half the generation of c_j° in \mathcal{T} , i.e. $\tilde{H}(j) = \mathcal{C}^\circ(j)$. \square

Recall the well-known identity between the height process H and the Łukasiewicz path W of a one-type tree (see e.g. Le Gall & Le Jan[27]):

$$H(j) = \# \left\{ i \in \{0, \dots, j-1\} : W(i) \leq \inf_{[i+1, j]} W \right\} \quad \text{for each } 0 \leq j \leq N. \quad (4)$$

Indeed, for $i < j$, we have $W(i) \leq \inf_{[i+1, j]} W$ if and only if u_i is an ancestor of u_j ; moreover, the inequality is an equality if and only if the last child of u_i is also an ancestor of u_j . A consequence of Lemma 1 is therefore the identity

$$\mathcal{C}^\circ(j) = \# \left\{ i \in \{0, \dots, i-1\} : W(i) < \inf_{[i+1, j]} W \right\} \quad \text{for each } 0 \leq j \leq N. \quad (5)$$

The latter was already observed by Abraham [1, Equation 5] without the formalism of the JS bijection, where W (which corresponds to $Y - 1$ there) was defined directly from the two-type tree.

2.5 From one-type trees to maps

The previous two subsections show that $\mathbf{M}^*(\mathbf{n})$ and $\{-1, +1\} \times \mathbf{LT}(\mathbf{n})$ are in bijection. Let us describe a direct construction of the map associated with a labelled one-type tree, depicted in Figure 7; we will not use it later but we believe it may have some interest. The construction contains two steps, the first one is close to the BDG bijection described earlier.

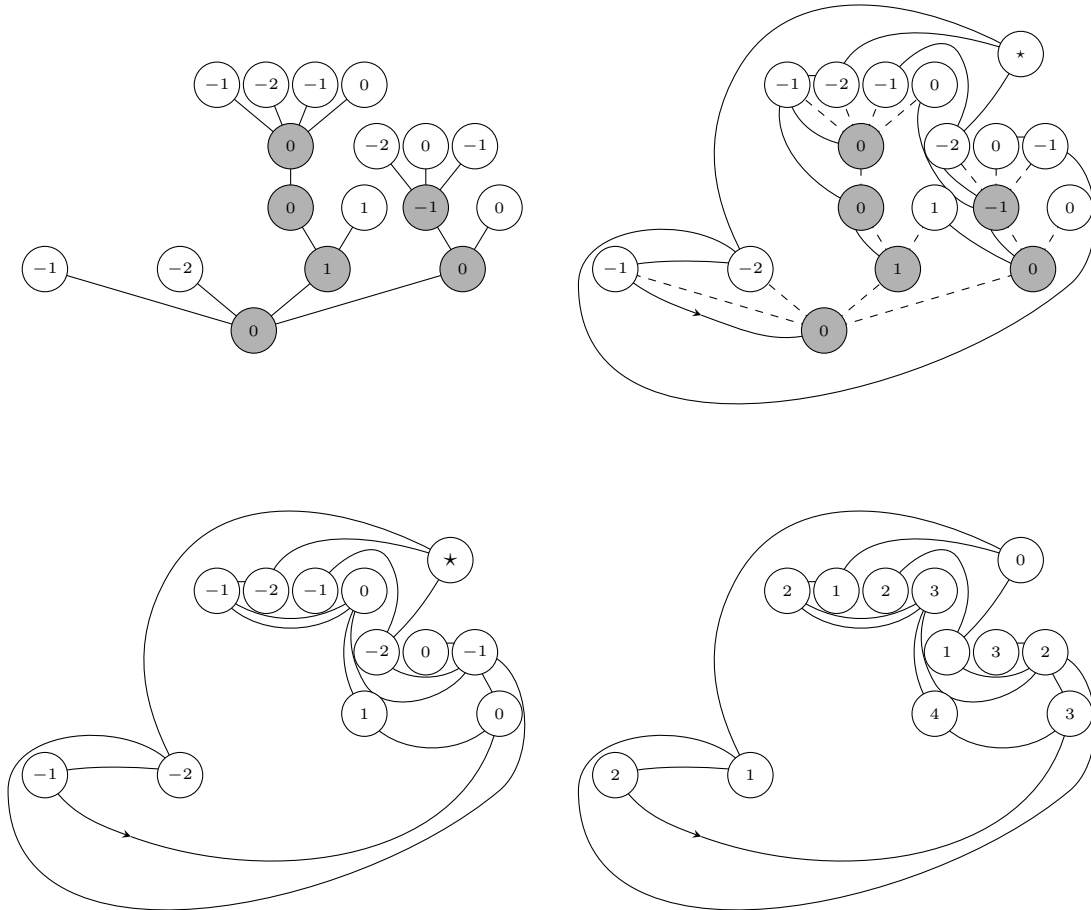


Figure 7: The composition of the Bouttier–Di Francesco–Guitter and the Janson–Stefánsson bijections; the map on the bottom right is that from Figure 3.

Let (T, l) be a labelled one-type tree with N edges and, say, n_0 leaves, and let (u_0, \dots, u_N) be the vertices of T listed in lexicographical order. For every $0 \leq i \leq N$, set $u_{N+1+i} = u_i$. We add an extra vertex \star outside of the tree T and construct a first planar graph G on the vertex-set of T and \star by drawing edges as follows: for every $0 \leq i \leq N - 1$,

- if $l(u_i) > \min_{0 \leq k \leq N} l(u_k)$, then we draw an edge between u_i and u_j where $j = \min\{k > i : l(u_k) = l(u_i) - 1\}$,
- if $l(u_i) = \min_{0 \leq k \leq N} l(u_k)$, then we draw an edge between u_i and \star .

We stress that we exclude the last vertex u_N in this construction; it indeed yields a planar graph G with N edges and $N + 2$ vertices, pointed at \star , and rooted at the first edge that we drew, for $i = 0$, the orientation of this root edge is again given by an external choice $\epsilon \in \{-1, +1\}$. In a second step, we merge every internal vertex of the tree T with its last child; then G becomes a map \mathcal{M} with N edges and $n_0 + 1$ vertices and we claim that the latter corresponds to the one obtained after the JS and then

the BDG bijection. Note again that the labels correspond, up to a shift, to the distances to the origin \star in the map \mathcal{M} .

3 The Brownian map

3.1 The Brownian snake and the Brownian map

Denote by $\mathbf{e} = (\mathbf{e}_t; t \in [0, 1])$ the standard Brownian excursion. For every $s, t \in [0, 1]$, set

$$m_{\mathbf{e}}(s, t) = \min_{r \in [s \wedge t, s \vee t]} \mathbf{e}_r \quad \text{and} \quad d_{\mathbf{e}}(s, t) = \mathbf{e}_s + \mathbf{e}_t - 2m_{\mathbf{e}}(s, t).$$

One easily checks that $d_{\mathbf{e}}$ is a random pseudo-metric on $[0, 1]$, we then define an equivalence relation on $[0, 1]$ by setting $s \sim_{\mathbf{e}} t$ whenever $d_{\mathbf{e}}(s, t) = 0$. Consider the quotient space $\mathcal{T}_{\mathbf{e}} = [0, 1] / \sim_{\mathbf{e}}$, we let $\pi_{\mathbf{e}}$ be the canonical projection $[0, 1] \rightarrow \mathcal{T}_{\mathbf{e}}$; $d_{\mathbf{e}}$ induces a metric on $\mathcal{T}_{\mathbf{e}}$ that we still denote by $d_{\mathbf{e}}$. The space $(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$ is a so-called compact real-tree, naturally rooted at $\pi_{\mathbf{e}}(0) = \pi_{\mathbf{e}}(1)$, called the *Brownian tree* coded by \mathbf{e} , introduced by Aldous [4].

We construct next another process $Z = (Z_t; t \in [0, 1])$ on the same probability space as \mathbf{e} which, conditional on \mathbf{e} , is a centred Gaussian process satisfying for every $s, t \in [0, 1]$,

$$\mathbf{E} [|Z_s - Z_t|^2 \mid \mathbf{e}] = d_{\mathbf{e}}(s, t) \quad \text{or, equivalently,} \quad \mathbf{E} [Z_s Z_t \mid \mathbf{e}] = m_{\mathbf{e}}(s, t).$$

It is known (see, e.g. Le Gall [23, Chapter IV.4]) on a more general path-valued process called the *Brownian snake* whose Z is only the “tip”) that the pair (\mathbf{e}, Z) admits a continuous version and, without further notice, we will work throughout this paper with this version. Observe that, almost surely, $Z_0 = 0$ and $Z_s = Z_t$ whenever $s \sim_{\mathbf{e}} t$ so Z can be seen as a Brownian motion indexed by $\mathcal{T}_{\mathbf{e}}$ by setting $Z_t = Z_{\pi_{\mathbf{e}}(t)}$ for every $t \in [0, 1]$. We interpret Z_x as the label of an element $x \in \mathcal{T}_{\mathbf{e}}$; the pair $(\mathcal{T}_{\mathbf{e}}, (Z_x; x \in \mathcal{T}_{\mathbf{e}}))$ is a continuous analog of labelled plane trees and the construction of the Brownian map from this pair, that we next recall, is somewhat an analog of the BDG bijection presented above.

For every $s, t \in [0, 1]$, define

$$\check{Z}(s, t) = \begin{cases} \min\{Z_r; r \in [s, t]\} & \text{if } s \leq t, \\ \min\{Z_r; r \in [s, 1] \cup [0, t]\} & \text{otherwise,} \end{cases}$$

and then

$$D_Z(s, t) = Z_s + Z_t - 2 \max\{\check{Z}(s, t); \check{Z}(t, s)\}.$$

For every $x, y \in \mathcal{T}_{\mathbf{e}}$, set

$$D_Z(x, y) = \inf \{D_Z(s, t); s, t \in [0, 1], x = \pi_{\mathbf{e}}(s) \text{ and } y = \pi_{\mathbf{e}}(t)\},$$

and finally

$$\mathcal{D}(x, y) = \inf \left\{ \sum_{i=1}^k D_Z(a_{i-1}, a_i); k \geq 1, (x = a_0, a_1, \dots, a_{k-1}, a_k = y) \in \mathcal{T}_{\mathbf{e}} \right\}.$$

Both D_Z and \mathcal{D} are pseudo-distances on $\mathcal{T}_{\mathbf{e}}$, we define an equivalence relation by setting $x \approx y$ whenever $\mathcal{D}(x, y) = 0$ for $x, y \in \mathcal{T}_{\mathbf{e}}$. The Brownian map is the quotient space $\mathcal{M} = \mathcal{T}_{\mathbf{e}} / \approx$ equipped with the metric induced by \mathcal{D} , that we still denote by \mathcal{D} . Note that \mathcal{D} can be seen as a function on $[0, 1]^2$ by setting $\mathcal{D}(s, t) = \mathcal{D}(\pi_{\mathbf{e}}(s), \pi_{\mathbf{e}}(t))$ for every $s, t \in [0, 1]$; in fact, it is the largest pseudo-distance D on $[0, 1]$ satisfying the following two properties:

$$D \leq D_Z \quad \text{and} \quad d_{\mathbf{e}}(s, t) = 0 \quad \text{implies} \quad D(s, t) = 0.$$

Thus \mathcal{M} can be seen as a quotient space of $[0, 1]$.

3.2 Functional invariance principles

Let $T_n \in \mathbf{T}(\mathbf{n})$ be a one-type tree; it has $N_{\mathbf{n}} = \sum_{i \geq 1} in_i$ edges, we denote by W_n , H_n and C_n respectively its Łukasiewicz path, its height process and its contour process. The main result of Broutin & Marckert [12] is the following.

Theorem 2 (Broutin & Marckert [12]). *Under (H), if T_n is sampled uniformly at random in $\mathbf{T}(\mathbf{n})$ for every $n \geq 1$, then the convergence in distribution*

$$\left(\frac{W_n(N_{\mathbf{n}}t)}{\sqrt{N_{\mathbf{n}}}}, \frac{H_n(N_{\mathbf{n}}t)}{\sqrt{N_{\mathbf{n}}}}, \frac{C_n(2N_{\mathbf{n}}t)}{\sqrt{N_{\mathbf{n}}}} \right)_{t \in [0,1]} \xrightarrow[n \rightarrow \infty]{(d)} \left(\sigma_p \mathbf{e}, \frac{2}{\sigma_p} \mathbf{e}, \frac{2}{\sigma_p} \mathbf{e} \right)_{t \in [0,1]}$$

holds in $\mathcal{C}([0, 1], \mathbf{R}^3)$. Consequently, the convergence in distribution

$$\left(T_n, N_{\mathbf{n}}^{-1/2} d_{\text{gr}} \right) \xrightarrow[n \rightarrow \infty]{(d)} \left(\mathcal{T}_{\mathbf{e}}, \frac{2}{\sigma_p} d_{\mathbf{e}} \right),$$

holds in the sense of Gromov–Hausdorff.

Denote by L_n the label process (in lexicographical order) of a labelled tree $(T_n, l_n) \in \mathbf{LT}(\mathbf{n})$. Consider next a labelled two-type tree $(\mathcal{T}_n, \ell_n) \in \mathbf{LT}_{\circ, \bullet}(\mathbf{n})$; it has $N_{\mathbf{n}}$ edges as well, we denote by \mathcal{C}_n° its white contour function and by \mathcal{L}_n° its label function (in contour order).

Theorem 3. *If (T_n, l_n) and (\mathcal{T}_n, ℓ_n) are related by the JS bijection and have the uniform distribution in $\mathbf{LT}(\mathbf{n})$ and $\mathbf{LT}_{\circ, \bullet}(\mathbf{n})$ respectively for every $n \geq 1$, then, under (H), the convergences in distribution*

$$\left(\left(\frac{\sigma_p^2}{4} \frac{1}{N_{\mathbf{n}}} \right)^{1/2} H_n(N_{\mathbf{n}}t), \left(\frac{9}{4\sigma_p^2} \frac{1}{N_{\mathbf{n}}} \right)^{1/4} L_n(N_{\mathbf{n}}t) \right)_{t \in [0,1]} \xrightarrow[n \rightarrow \infty]{(d)} (\mathbf{e}_t, Z_t)_{t \in [0,1]} \quad (6)$$

and

$$\left(\left(\frac{\sigma_p^2}{4p_0^2} \frac{1}{N_{\mathbf{n}}} \right)^{1/2} \mathcal{C}_n^\circ(N_{\mathbf{n}}t), \left(\frac{9}{4\sigma_p^2} \frac{1}{N_{\mathbf{n}}} \right)^{1/4} \mathcal{L}_n^\circ(N_{\mathbf{n}}t) \right)_{t \in [0,1]} \xrightarrow[n \rightarrow \infty]{(d)} (\mathbf{e}_t, Z_t)_{t \in [0,1]} \quad (7)$$

hold jointly in $\mathcal{C}([0, 1], \mathbf{R}^2)$.

Remark 3. Denote by \mathcal{C}_n the entire contour function of \mathcal{T}_n , then (7) is equivalent to the convergence in distribution in $\mathcal{C}([0, 1], \mathbf{R}^2)$

$$\left(\left(\frac{\sigma_p^2}{16p_0^2} \frac{1}{N_{\mathbf{n}}} \right)^{1/2} \mathcal{C}_n(2N_{\mathbf{n}}t), \left(\frac{9}{4\sigma_p^2} \frac{1}{N_{\mathbf{n}}} \right)^{1/4} \mathcal{L}_n^\circ(N_{\mathbf{n}}t) \right)_{t \in [0,1]} \xrightarrow[n \rightarrow \infty]{(d)} (\mathbf{e}_t, Z_t)_{t \in [0,1]}.$$

Indeed, we have already observed that $\sup_{t \in [0,1]} |\mathcal{C}_n(2N_{\mathbf{n}}t) - 2\mathcal{C}_n^\circ(N_{\mathbf{n}}t)| = 1$. In particular, we have the joint convergences

$$\left(\mathcal{T}_n, N_{\mathbf{n}}^{-1/2} d_{\text{gr}} \right) \xrightarrow[n \rightarrow \infty]{(d)} \left(\mathcal{T}_{\mathbf{e}}, \frac{4p_0}{\sigma_p} d_{\mathbf{e}} \right) \quad \text{and} \quad \left(T_n, N_{\mathbf{n}}^{-1/2} d_{\text{gr}} \right) \xrightarrow[n \rightarrow \infty]{(d)} \left(\mathcal{T}_{\mathbf{e}}, \frac{2}{\sigma_p} d_{\mathbf{e}} \right),$$

in the sense of Gromov–Hausdorff.

Remark 4. Recall from (3) that there are

$$\prod_{u \in T: k_u \geq 1} \binom{2k_u - 1}{k_u - 1}$$

possible labellings of a given plane tree T . This quantity is constant on $\mathbf{T}(\mathbf{n})$ so a uniform random unlabelled tree in $\mathbf{T}(\mathbf{n})$ which is then labelled uniformly at random (meaning that the bridges with no negative jumps, as defined by (2), at each branch-point are sampled uniformly at random and independently) has the uniform distribution in $\mathbf{LT}(\mathbf{n})$.

Let us comment on the constants in (6) and (7). The one in front of H_n is taken from Theorem 2. Next, the label of a vertex $u \in T_n$ is the sum of the increments of the labels between two consecutive ancestors; there are $|u|$ such terms, which are independent and distributed, when the ancestor has i children and the one on the path to u is the j -th one, as the j -th marginal of a uniform random bridge of length i with no negative jump, as defined in (2); the latter is a centred random variable with variance $2j(i-j)/(i+1)$. As we will see, there is typically a proportion about $p(i)$ of such ancestors so $L_n(u)$ has variance about

$$|u| \sum_{i \geq 1} \sum_{j=1}^i p(i) \frac{2j(i-j)}{i+1} = |u| \sum_{i \geq 1} p(i) \frac{i(i-1)}{3} = |u| \frac{\sigma_p^2}{3}.$$

If u is the vertex visited at time $\lfloor N_{\mathbf{n}} t \rfloor$ in lexicographical order, then $|u| \approx (4N_{\mathbf{n}}/\sigma_p^2)^{1/2} \mathbf{e}_t$ so we expect $L_n(N_{\mathbf{n}} t)$, once rescaled by $N_{\mathbf{n}}^{1/4}$, to be asymptotically Gaussian with variance

$$\sqrt{\frac{4}{\sigma_p^2} \frac{\sigma_p^2}{3}} \mathbf{e}_t = \sqrt{\frac{4\sigma_p^2}{9}} \mathbf{e}_t.$$

Regarding the two-type tree, the proof of the convergence of \mathcal{C}_n° relies on showing that, as $n \rightarrow \infty$, it is close to $p_0 H_n$ when \mathcal{T}_n and T_n are related by the JS bijection. Finally, according to Lemma 1, when \mathcal{T}_n and T_n are related by the JS bijection, then the processes \mathcal{L}_n° and L_n are equal. Note also that the previous argument relates the constant in front of \mathcal{C}_n° and that in front of \mathcal{L}_n° : typically, the proportion of black ancestors of a white vertex u which have i children and among which the j -th one is an ancestor of u is now about $p(i)/p_0$, the previous calculation yields the above variance.

We next explain how Theorem 3 will follow from several results proved in Section 5.

Proof of Theorem 3. Recall from Lemma 1 that the processes L_n and \mathcal{L}_n° are equal. Appealing to this lemma, we shall also show in Proposition 1 below the convergence in $\mathcal{C}([0, 1], \mathbf{R}^2)$

$$\left(\left(\frac{\sigma_p^2}{4} \frac{1}{N_{\mathbf{n}}} \right)^{1/2} H_n(N_{\mathbf{n}} t), \left(\frac{\sigma_p^2}{4p_0^2 N_{\mathbf{n}}} \right)^{1/2} \mathcal{C}_n^\circ(N_{\mathbf{n}} t) \right)_{t \in [0, 1]} \xrightarrow[n \rightarrow \infty]{(d)} (\mathbf{e}_t, \mathbf{e}_t)_{t \in [0, 1]}.$$

In Proposition 4, we shall show that, jointly with this convergence, for every $k \geq 1$, if (U_1, \dots, U_k) are i.i.d. uniform random variables in $[0, 1]$ independent of the trees, then the convergence

$$\left(\frac{9}{4\sigma_p^2} \frac{1}{N_{\mathbf{n}}} \right)^{1/4} (L_n(N_{\mathbf{n}} U_1), \dots, L_n(N_{\mathbf{n}} U_k)) \xrightarrow[n \rightarrow \infty]{(d)} (Z_{U_1}, \dots, Z_{U_k}) \quad (8)$$

holds in \mathbf{R}^k . Finally, in Proposition 7, we shall show that the sequence

$$\left(N_{\mathbf{n}}^{-1/4} L_n(N_{\mathbf{n}} t); t \in [0, 1] \right)_{n \geq 1}$$

is tight in $\mathcal{C}([0, 1], \mathbf{R})$. This ensures that the sequences on the left-hand side of (6) and (7) are tight in $\mathcal{C}([0, 1], \mathbf{R}^2)$. Using the equicontinuity given by this tightness, as well as the uniform continuity of the pair (\mathbf{e}, Z) , one may transpose (8) to a convergence for deterministic times, by approximating them by i.i.d. uniform random times, see Addario-Berry & Albenque [3, proof of Proposition 6.1] for a detailed argument; this characterises the sub-sequential limits of (6) and (7) in $\mathcal{C}([0, 1], \mathbf{R}^2)$ as (\mathbf{e}, Z) . \square

We prove the above intermediate results in Section 5: we show the joint convergence of H_n and \mathcal{C}_n° in Section 5.1, the convergence of random finite-dimensional marginals of L_n in Section 5.3 and the tightness of this process in Section 5.5. The proofs shall rely on a precise description of the branches from the root of T_n to i.i.d. leaves which is obtained in the next section.

4 Spinal decompositions

For a given vertex u in a plane tree T , we denote by $A_i(u)$ its number of strict ancestors with i children:

$$A_i(u) = \#\{v \in \llbracket \emptyset, u \rrbracket : k_v = i\}.$$

We write $\mathbf{A}(u) = (A_i(u); i \geq 1)$; note that $|u| = |\mathbf{A}(u)| = \sum_{i \geq 1} A_i(u)$. The quantity $\mathbf{A}(u)$ is crucial in order to control the label $l_n(u)$ of the vertex $u \in T_n$ when (T_n, l_n) is chosen uniformly at random in $\mathbf{LT}(\mathbf{n})$. Indeed, one can write

$$l_n(u) = \sum_{v \in \llbracket \emptyset, u \rrbracket} l_n(v) - l_n(\text{pr}(v)),$$

and, conditional on T_n , the random variables $l_n(v) - l_n(\text{pr}(v))$ are independent and their law depend on the number of children of $\text{pr}(v)$.

Throughout this section, the notation $\mathbf{m} = (m_i; i \geq 1)$ will stand for a sequence of non-negative integers; we set

$$LR(\mathbf{m}) = 1 + \sum_{i \geq 1} (i-1)m_i.$$

We will write $LR(u)$ for $LR(\mathbf{A}(u))$, the notation comes from the fact that removal of the path $\llbracket \emptyset, u \rrbracket$ produces a forest of $LR(u)$ trees, so, in other words, $LR(u)$ is the number of vertices lying directly on the *left* or on the *right* of this path (and the component “above”).

4.1 A one-point decomposition

The following result has been obtained by Broutin & Marckert [12]; it is not written explicitly but the arguments that we recall below can be found in Sections 3 and 5.2 there.

Lemma 2. *For every $n \geq 1$, sample T_n uniformly at random in $\mathbf{T}(\mathbf{n})$ and then sample a uniform random vertex u_n in T_n . Under **(H)**, for every $1 \leq h \leq N_n/2$ and every sequence $\mathbf{m} \in \mathbf{Z}_+^{\mathbf{N}}$ with $|\mathbf{m}| = h$, we have*

$$\mathbf{P}(\mathbf{A}(u_n) = \mathbf{m}) \leq \frac{LR(\mathbf{m})}{N_n + 1 - h} \cdot e^{h^2/N_n} \cdot \mathbf{P}(\Xi_n^{(h)} = \mathbf{m}),$$

where $\Xi_n^{(h)} = (\Xi_{n,i}^{(h)}; i \geq 1)$ is a random sequence with the multinomial distribution with parameters h and $(in_i/N_n; i \geq 1)$.

We need to generalise this decomposition to multiple i.i.d. uniform random vertices. The notation will be rather complicated so, in order to simplify the exposition, we chose to first recall the argument from [12] to prove this result before stating ours and then only point out the main differences.

Proof. Fix a plane tree T and recall that if v is a vertex of T different from its root, we denote by $\text{pr}(v)$ its parent and by $k_{\text{pr}(v)}$ the number of children of the latter; denote further by χ_v the relative position of v among the children of $\text{pr}(v)$: $\chi_v \in \{1, \dots, k_{\text{pr}(v)}\}$ satisfies $v = \text{pr}(v)\chi_v$. Define next for every vertex u the *content* of the branch $\llbracket \emptyset, u \rrbracket$ as

$$\text{Cont}(u) = ((k_{\text{pr}(v)}, \chi_v); v \in \llbracket \emptyset, u \rrbracket), \quad (9)$$

where the elements $v \in \llbracket \emptyset, u \rrbracket$ are sorted in increasing order of their height. For any sequence \mathbf{m} , denote by $\Gamma(\mathbf{m})$ the set of possible vectors $\text{Cont}(u)$ when $\mathbf{A}(u) = \mathbf{m}$ and note that

$$\#\Gamma(\mathbf{m}) = \binom{|\mathbf{m}|}{(m_i; i \geq 1)} \prod_{i \geq 1} i^{m_i}.$$

The removal of the branch $\llbracket \emptyset, u \rrbracket$ from T produces a plane forest of $LR(u)$ trees and there is a one-to-one correspondence between the pair (T, u) on the one hand and this forest and $\text{Cont}(u)$ on the

other hand. For any sequence $\mathbf{q} = (q_i; i \geq 0)$ of non-negative integers with finite sum, let $\mathbf{F}(\mathbf{q})$ be the set of plane forests having exactly q_i vertices with i children for every $i \geq 0$; such a forest possesses $r = \sum_{i \geq 0} (1-i)q_i$ roots and it is well-known that

$$\#\mathbf{F}(\mathbf{q}) = \frac{r}{|\mathbf{q}|} \binom{|\mathbf{q}|}{(q_i; i \geq 0)}.$$

Sample T_n uniformly at random in $\mathbf{T}(\mathbf{n}) = \mathbf{F}(\mathbf{n})$ and u_n uniformly at random in T_n , the previous bijection readily implies that for any sequence \mathbf{m} satisfying $m_i \leq n_i$ for every $i \geq 1$ and for any vector $C \in \Gamma(\mathbf{m})$, setting $m_0 = 0$, we have

$$\mathbf{P}(\text{Cont}(u_n) = C) = \frac{\#\mathbf{F}(\mathbf{n} - \mathbf{m})}{(N_{\mathbf{n}} + 1)\#\mathbf{F}(\mathbf{n})},$$

and so

$$\mathbf{P}(\mathbf{A}(u_n) = \mathbf{m}) = \#\Gamma(\mathbf{m}) \cdot \frac{\#\mathbf{F}(\mathbf{n} - \mathbf{m})}{(N_{\mathbf{n}} + 1)\#\mathbf{F}(\mathbf{n})}.$$

Consequently, for every sequence \mathbf{m} satisfying $m_i \leq n_i$ for every $i \geq 1$, setting $h = |\mathbf{m}|$, we have

$$\begin{aligned} \mathbf{P}(\mathbf{A}(u_n) = \mathbf{m}) &= \binom{h}{(m_i; i \geq 1)} \prod_{i \geq 1} i^{m_i} \cdot \frac{LR(\mathbf{m}) \binom{N_{\mathbf{n}}+1-h}{(n_i-m_i; i \geq 0)}}{(N_{\mathbf{n}} + 1) \frac{1}{N_{\mathbf{n}}+1} \binom{N_{\mathbf{n}}+1}{(n_i; i \geq 0)}} \\ &= \frac{LR(\mathbf{m})}{N_{\mathbf{n}} + 1 - h} \cdot \frac{h!}{\prod_{i \geq 1} m_i!} \prod_{i \geq 1} i^{m_i} \cdot \prod_{i \geq 1} \frac{n_i!}{(n_i - m_i)!} \cdot \frac{(N_{\mathbf{n}} + 1 - h)!}{(N_{\mathbf{n}} + 1)!} \\ &= \frac{LR(\mathbf{m})}{N_{\mathbf{n}} + 1 - h} \cdot \frac{h!}{\prod_{i \geq 1} m_i!} \prod_{i \geq 1} \left(\frac{in_i}{N_{\mathbf{n}}}\right)^{m_i} \cdot \prod_{i \geq 1} \frac{n_i!}{n_i^{m_i} (n_i - m_i)!} \cdot \frac{(N_{\mathbf{n}} + 1 - h)! N_{\mathbf{n}}^h}{(N_{\mathbf{n}} + 1)!}. \end{aligned}$$

Note that

$$\mathbf{P}(\Xi_{\mathbf{n}}^{(h)} = \mathbf{m}) = \frac{h!}{\prod_{i \geq 1} m_i!} \prod_{i \geq 1} \left(\frac{in_i}{N_{\mathbf{n}}}\right)^{m_i}.$$

Next, observe that $n_i! \leq n_i^{m_i} (n_i - m_i)!$ for every $i \geq 1$; finally, using the inequality $(1-x)^{-1} \leq \exp(2x)$ for $|x| \leq 1/2$, we have as soon as $h \leq N_{\mathbf{n}}/2$,

$$\frac{(N_{\mathbf{n}} + 1 - h)! N_{\mathbf{n}}^h}{(N_{\mathbf{n}} + 1)!} \leq \prod_{i=0}^{h-1} \frac{1}{1 - i/(N_{\mathbf{n}} + 1)} \leq e^{h^2/N_{\mathbf{n}}},$$

and the proof is complete. \square

Remark 5. The proof of Lemma 2 shows that, conditional on $\mathbf{A}(u_n) = \mathbf{m}$, the vector $\text{Cont}(u_n)$ has the uniform distribution in $\Gamma(\mathbf{m})$. It is then straightforward to check that, for every $h \geq 1$, if \mathbf{m} is sampled from $\Xi_{\mathbf{n}}^{(h)}$, then the unconditional law of $\text{Cont}(u_n)$ is given as follows: the entries $((k_{pr(v)}, \chi_v); v \in \llbracket \emptyset, u_n \rrbracket)$ are i.i.d., the marginal $k_{pr(v)}$ is sampled from $(in_i/N_{\mathbf{n}}; i \geq 1)$, and then, conditional on this value, the marginal χ_v has the uniform distribution in $\{1, \dots, k_{pr(v)}\}$.

Let us next present a corollary of Lemma 2 which shall be used in Section 5.5.

Corollary 3. Recall the notation $\chi_w \in \{1, \dots, k_{pr(w)}\}$ for the relative position of a vertex $w \in T_n$ among the children of its parent. Let $C = \frac{2}{p_0} - 1$ and $h_n = \frac{24}{p_0^2} \ln N_{\mathbf{n}}$ for every $n \geq 1$ and consider the event

$$\mathcal{E}_n := \left\{ \frac{\#\{w \in \llbracket u, v \rrbracket : \chi_w = 1\}}{\#\{w \in \llbracket u, v \rrbracket : \chi_w \geq 2\}} \leq C \text{ for every } u, v \in T_n \text{ with } u \in \llbracket \emptyset, v \rrbracket \text{ and } \#\llbracket u, v \rrbracket > h_n \right\}. \quad (10)$$

Under **(H)**, we have $\mathbf{P}(\mathcal{E}_n) \rightarrow 1$ as $n \rightarrow \infty$.

Proof. For every $v \in T_n$, for every $1 \leq j \leq |v|$, let us denote by $a_j(v)$ the unique element of $\llbracket \emptyset, v \rrbracket$ such that $\#\llbracket a_j(v), v \rrbracket = j$, and then set $k_j(v) = k_{a_{j+1}(v)}$ and $X_j(v) = 1$ if $\chi_{a_j(v)} = 1$ and $X_j(v) = 0$ if $\chi_{a_j(v)} \geq 2$. We may write the event \mathcal{E}_n as

$$\mathcal{E}_n = \bigcap_{v \in T_n} \bigcap_{h_n < j \leq |v|} \left\{ \frac{\#\{1 \leq i \leq j : X_i(v) = 1\}}{\#\{1 \leq i \leq j : X_i(v) = 0\}} \leq C \right\} = \bigcap_{v \in T_n} \bigcap_{h_n < j \leq |v|} \left\{ \sum_{i=1}^j X_i(v) \leq \frac{C \cdot j}{1+C} \right\}.$$

Fix $\varepsilon > 0$, from the convergence of H_n in Theorem 2, there exists $x > 0$ such that for every n large enough,

$$\mathbf{P} \left(\sup_{u \in T_n} |u| > x \sqrt{N_n} \right) < \varepsilon.$$

In fact, Addario-Berry [2] proved that the above probability is bounded above by $\exp(-cx^2/\sigma_n^2)$ for every $n \geq 1$ and $x > 0$, where $c > 0$ is a universal constant. Let v_n be a uniform random vertex of T_n , then

$$\begin{aligned} & \mathbf{P} \left(\mathcal{E}_n^c \cap \left\{ \sup_{u \in T_n} |u| \leq x \sqrt{N_n} \right\} \right) \\ & \leq \sum_{v \in T_n} \sum_{h_n < j \leq N_n} \mathbf{P} \left(\left\{ \sum_{i=1}^j X_i(v) > \frac{C \cdot j}{1+C} \right\} \cap \left\{ \sup_{u \in T_n} |u| \leq x \sqrt{N_n} \right\} \right) \\ & \leq (N_n + 1)^2 \sup_{j > h_n} \mathbf{P} \left(\left\{ \sum_{i=1}^j X_i(v_n) > \frac{C \cdot j}{1+C} \right\} \cap \left\{ \sup_{u \in T_n} |u| \leq x \sqrt{N_n} \right\} \right) \\ & \leq (N_n + 1)^2 \sup_{j > h_n} \sum_{|\mathbf{m}| \leq x \sqrt{N_n}} \mathbf{P} \left(\sum_{i=1}^j X_i(v_n) > \frac{C \cdot j}{1+C} \text{ and } \mathbf{A}(v_n) = \mathbf{m} \right), \end{aligned}$$

where in the last line, the sum is taken over all $1 \leq h \leq x \sqrt{N_n}$ and all sequences $\mathbf{m} \in \mathbf{Z}_+^N$ with $|\mathbf{m}| = h$. According to Lemma 2, for any such sequence, we have

$$\mathbf{P}(\mathbf{A}(u_n) = \mathbf{m}) \leq e^{x^2} \cdot \mathbf{P}(\Xi_n^{(h)} = \mathbf{m}) (1 + o(1)),$$

where $\Xi_n^{(h)}$ has the multinomial distribution with parameters h and $(rn_r/N_n; r \geq 1)$. Here we have used the trivial bound $LR(\mathbf{m}) \leq N_n$, which comes from the fact that if $m_i > n_i$ for some $i \geq 1$, then the left-hand side is zero.

Moreover, appealing to Remark 5, conditional on the offsprings $k_i(v_n)$'s of the ancestors $a_i(v_n)$'s, the $X_i(v_n)$'s are independent and have the Bernoulli distribution with parameter $1/k_i(v_n)$. We thus have

$$\mathbf{P} \left(\mathcal{E}_n^c \cap \left\{ \sup_{u \in T_n} |u| \leq x \sqrt{N_n} \right\} \right) \leq x e^{x^2} \cdot N_n^{5/2} \cdot \sup_{j > h_n} \mathbf{P} \left(\sum_{i=1}^j Y_{n,i} > \frac{C \cdot j}{1+C} \right) (1 + o(1)),$$

where the $Y_{n,i}$'s are independent and have the Bernoulli distribution with parameter

$$\sum_{r \geq 1} \frac{1}{r} \cdot \frac{rn_r}{N_n} = 1 - \frac{n_0 - 1}{N_n}.$$

Note that $\frac{C}{1+C} = 1 - \frac{p_0}{2}$; fix n large enough so that, according to **(H)**, $\frac{n_0 - 1}{N_n} > \frac{3p_0}{4}$ and so $\frac{C}{1+C} - (1 - \frac{n_0 - 1}{N_n}) = \frac{n_0 - 1}{N_n} - \frac{p_0}{2} > \frac{p_0}{4}$. Applying the Chernoff bound, we obtain

$$\mathbf{P} \left(\sum_{i=1}^j Y_{n,i} > \frac{C \cdot j}{1+C} \right) \leq \mathbf{P} \left(\sum_{i=1}^j (Y_{n,i} - \mathbf{E}[Y_{n,i}]) > \frac{p_0}{4} \cdot j \right) \leq \exp \left(-\frac{p_0^2}{8} \cdot j \right),$$

so finally

$$\mathbf{P} \left(\mathcal{E}_n^c \cap \left\{ \sup_{u \in T_n} |u| \leq x \sqrt{N_n} \right\} \right) \leq x e^{x^2} \cdot N_n^{5/2} \cdot \exp \left(-\frac{p_0^2}{8} \cdot h_n \right) \cdot (1 + o(1)),$$

which converges to zero as $n \rightarrow \infty$ from our choice of h_n . \square

4.2 A multi-point decomposition

We next extend the previous decomposition according to several i.i.d. uniform random vertices. Let us first introduce some notation. Fix a plane tree T and k vertices u_1, \dots, u_k of T and denote by $T(u_1, \dots, u_k)$ the tree T reduced to its root and these vertices:

$$T(u_1, \dots, u_k) = \bigcup_{1 \leq j \leq k} [\![\emptyset, u_j]\!],$$

which naturally inherits a plane tree structure from T . Denote by $k' \leq k - 1$ the number of branch-points of $T(u_1, \dots, u_k)$ and by $v_1, \dots, v_{k'}$ these branch-points. Let $F(u_1, \dots, u_k)$ be the forest obtained from $T(u_1, \dots, u_k)$ by removing the edges linking these branch-points to their children; note that $F(u_1, \dots, u_k)$ contains $k + k'$ connected components which are only single paths, i.e. each one contains one root and only one leaf and the latter is either one of the u_i 's or one of the v_i 's. Let us rank these connected components in increasing lexicographical order of their root and denote by \emptyset_j and λ_j respectively the root and the leaf of the j -th one. For every $1 \leq j \leq k + k'$ and every $i \geq 1$, we set

$$A_i^{(j)}(u_1, \dots, u_k) = \# \{z \in [\![\emptyset_j, \lambda_j]\!]: k_z = i\},$$

where k_z must be understood as the number of children in the original tree T of the vertex z . We set

$$\mathbf{A}(u_1, \dots, u_k) = \left(\mathbf{A}^{(1)}(u_1, \dots, u_k), \dots, \mathbf{A}^{(k+k')}(u_1, \dots, u_k) \right).$$

Fix $n, k \geq 1$, sample T_n uniformly at random in $\mathbf{T}(\mathbf{n})$ and then sample i.i.d. uniform random vertices $u_{n,1}, \dots, u_{n,k}$ in T_n ; denote by $\mathcal{E}_n(k)$ the following event: the reduced tree $T_n(u_{n,1}, \dots, u_{n,k})$ is binary, has k leaves and its root has only one child. It is easy to see that, under **(H)**, the probability of this event tends to 1 as $n \rightarrow \infty$. This can be seen as a consequence of Theorem 2 and the fact that such a property holds almost surely for the Brownian tree.

Lemma 3. *For every $n \geq 1$, sample T_n uniformly at random in $\mathbf{T}(\mathbf{n})$ and then sample i.i.d. uniform random vertices $u_{n,1}, \dots, u_{n,k}$ in T_n . Under **(H)**, uniformly for every $1 \leq h_1, \dots, h_{2k-1} \leq N_n/2$ and every sequences $\mathbf{m}^{(1)}, \dots, \mathbf{m}^{(2k-1)} \in \mathbf{Z}_+^{\mathbf{N}}$ with $|\mathbf{m}^{(j)}| = h_j$ for each $1 \leq j \leq 2k-1$, the quantity*

$$\mathbf{P} \left(\mathbf{A}(u_{n,1}, \dots, u_{n,k}) = (\mathbf{m}^{(1)}, \dots, \mathbf{m}^{(2k-1)}) \mid \mathcal{E}_n(k) \right)$$

is bounded above by

$$\left(\frac{\sigma_p^2}{2} \right)^{k-1} \frac{(k-1)\Delta_n + \sum_{j=1}^{2k-1} LR(\mathbf{m}^{(j)})}{N_n(N_n - h)} \exp \left(\frac{h^2 + 2h(k-2)}{N_n} \right) \cdot \prod_{j=1}^{2k-1} \mathbf{P} \left(\Xi_n^{(j)} = \mathbf{m}^{(j)} \right) (1 + o(1)),$$

where $h = h_1 + \dots + h_{2k-1}$ and $\Xi_n^{(h_j)} = (\Xi_{n,i}^{(h_j)}; i \geq 1)$ is a random sequence with the multinomial distribution with parameters h_j and $(in_i/N_n; i \geq 1)$.

Proof of Lemma 3. We treat in detail the case $k = 2$ and comment on the general case at the end. Fix $r \geq 2$ and three sequences of non-negative integers $\mathbf{m}^{(1)}, \mathbf{m}^{(2)}, \mathbf{m}^{(3)}$ with $m_0^{(1)} = m_0^{(2)} = m_0^{(3)} = 0$ and set $|\mathbf{m}_i^{(j)}| = h_j$ for each $j \in \{1, 2, 3\}$. For every $i \geq 0$, set

$$\underline{m}_i = m_i^{(1)} + m_i^{(2)} + m_i^{(3)} \quad \text{and} \quad \bar{m}_i = \underline{m}_i + \mathbf{1}_{\{i=r\}}.$$

Consider u_n and v_n independent uniform random vertices of T_n and let w_n be their most recent common ancestor. Assume that u_n is smaller in lexicographical order than v_n (otherwise exchange their names), let \hat{u}_n be the child of w_n which is an ancestor of u_n and define similarly \hat{v}_n so

$$F_n(u_n, v_n) = (\llbracket \emptyset, w_n \rrbracket, \llbracket \hat{u}_n, u_n \rrbracket, \llbracket \hat{v}_n, v_n \rrbracket).$$

Let $\text{Cont}(u_n, v_n)$ be the triplet of contents of these branches, defined in a similar way as in (9). Let $\Gamma(\mathbf{m}^{(1)}, \mathbf{m}^{(2)}, \mathbf{m}^{(3)})$ be the set of possible such triplets when $\mathbf{A}(u_n, v_n) = (\mathbf{m}^{(1)}, \mathbf{m}^{(2)}, \mathbf{m}^{(3)})$; as previously,

$$\begin{aligned} \#\Gamma(\mathbf{m}^{(1)}, \mathbf{m}^{(2)}, \mathbf{m}^{(3)}) &= \prod_{j=1}^3 \binom{h_j}{(m_i^{(j)}; i \geq 1)} \prod_{i \geq 1} i^{m_i^{(j)}} \\ &= \frac{N_{\mathbf{n}}^{h_1+h_2+h_3}}{\prod_{i \geq 1} n_i^{\overline{m}_i}} \cdot \prod_{j=1}^3 \binom{h_j}{(m_i^{(j)}; i \geq 1)} \prod_{i \geq 1} \left(\frac{in_i}{N_{\mathbf{n}}}\right)^{m_i^{(j)}} \\ &= n_r \cdot \frac{N_{\mathbf{n}}^{h_1+h_2+h_3}}{\prod_{i \geq 1} n_i^{\overline{m}_i}} \cdot \prod_{j=1}^3 \binom{h_j}{(m_i^{(j)}; i \geq 1)} \prod_{i \geq 1} \left(\frac{in_i}{N_{\mathbf{n}}}\right)^{m_i^{(j)}}. \end{aligned}$$

Observe that $LR(\overline{\mathbf{m}}) = 1 + \sum_{i \geq 1} (i-1)\overline{m}_i = 2 + (r-2) + \sum_{i \geq 1} (i-1)\overline{m}_i$ denotes the number of trees in the forest obtained from T_n by removing the reduced tree $T_n(u_n, v_n)$ when $\mathbf{A}(u_n, v_n) = (\mathbf{m}^{(1)}, \mathbf{m}^{(2)}, \mathbf{m}^{(3)})$ and $k_{w_n} = r$: there are $i-1$ components for each of the \overline{m}_i elements with i children in $\llbracket \emptyset, w_n \rrbracket \cup \llbracket \hat{u}_n, u_n \rrbracket \cup \llbracket \hat{v}_n, v_n \rrbracket$, as well as $r-2$ components corresponding to the children of w_n different from u_n and v_n , and the two components above u_n and v_n . As previously, the triplet (T_n, u_n, v_n) is characterised by the forest obtained by removing the reduced tree $T_n(u_n, v_n)$ and the content of the latter, which is $\text{Cont}(u_n, v_n)$ plus the information $(k_{w_n}, \chi_{\hat{u}_n}, \chi_{\hat{v}_n})$ about the branch-point. We therefore have for every $C \in \Gamma(\mathbf{m}^{(1)}, \mathbf{m}^{(2)}, \mathbf{m}^{(3)})$ and every $B \in \{(r, i, j); 1 \leq i < j \leq r\}$,

$$\begin{aligned} \mathbf{P}(\text{Cont}(u_n, v_n) = C \text{ and } (k_{w_n}, \chi_{\hat{u}_n}, \chi_{\hat{v}_n}) = B) &= \frac{\#\mathbf{F}(\mathbf{n} - \overline{\mathbf{m}})}{(N_{\mathbf{n}} + 1)^2 \#\mathbf{F}(\mathbf{n})} \\ &= \frac{\frac{LR(\overline{\mathbf{m}})}{|\mathbf{n} - \overline{\mathbf{m}}|} \binom{|\mathbf{n} - \overline{\mathbf{m}}|}{(n_i - \overline{m}_i; i \geq 1)}}{(N_{\mathbf{n}} + 1)^2 \frac{1}{N_{\mathbf{n}} + 1} \binom{N_{\mathbf{n}} + 1}{(n_i; i \geq 1)}} \\ &= \frac{LR(\overline{\mathbf{m}})}{(N_{\mathbf{n}} + 1)|\mathbf{n} - \overline{\mathbf{m}}|} \frac{(|\mathbf{n} - \overline{\mathbf{m}}|)!}{(N_{\mathbf{n}} + 1)!} \prod_{i \geq 1} \frac{n_i!}{(n_i - \overline{m}_i)!}. \end{aligned}$$

Since $|\mathbf{n}| = N_{\mathbf{n}} + 1$ and $|\overline{\mathbf{m}}| = h_1 + h_2 + h_3 + 1$, it follows that

$$\begin{aligned} &\mathbf{P}(\mathbf{A}(u_n, v_n) = (\mathbf{m}^{(1)}, \mathbf{m}^{(2)}, \mathbf{m}^{(3)}) \text{ and } k_{w_n} = r) \\ &= \frac{r(r-1)}{2} \cdot \#\Gamma(\mathbf{m}^{(1)}, \mathbf{m}^{(2)}, \mathbf{m}^{(3)}) \cdot \frac{LR(\overline{\mathbf{m}})}{(N_{\mathbf{n}} + 1)|\mathbf{n} - \overline{\mathbf{m}}|} \frac{(|\mathbf{n} - \overline{\mathbf{m}}|)!}{(N_{\mathbf{n}} + 1)!} \prod_{i \geq 1} \frac{n_i!}{(n_i - \overline{m}_i)!} \\ &= \frac{r(r-1)n_r}{2(N_{\mathbf{n}} + 1)} \cdot \frac{LR(\overline{\mathbf{m}})}{N_{\mathbf{n}} - (h_1 + h_2 + h_3)} \cdot \frac{(N_{\mathbf{n}} - (h_1 + h_2 + h_3))! N_{\mathbf{n}}^{h_1+h_2+h_3}}{(N_{\mathbf{n}} + 1)!} \\ &\quad \times \prod_{i \geq 1} \frac{n_i!}{n_i^{\overline{m}_i} (n_i - \overline{m}_i)!} \cdot \prod_{j=1}^3 \binom{h_j}{(m_i^{(j)}; i \geq 1)} \prod_{i \geq 1} \left(\frac{in_i}{N_{\mathbf{n}}}\right)^{m_i^{(j)}}. \end{aligned}$$

As previously, we have

$$\prod_{j=1}^3 \binom{h_j}{(m_i^{(j)}; i \geq 1)} \prod_{i \geq 1} \left(\frac{in_i}{N_{\mathbf{n}}}\right)^{m_i^{(j)}} = \prod_{j=1}^3 \mathbf{P}(\Xi_n^{(h_j)} = \mathbf{m}^{(j)}),$$

and

$$\prod_{i \geq 1} \frac{n_i!}{n_i^{\bar{m}_i} (n_i - \bar{m}_i)!} \leq 1,$$

as well as

$$\frac{(N_{\mathbf{n}} - (h_1 + h_2 + h_3))! N_{\mathbf{n}}^{h_1+h_2+h_3}}{(N_{\mathbf{n}} + 1)!} = \frac{1}{N_{\mathbf{n}} + 1} \prod_{i=0}^{h_1+h_2+h_3-1} \frac{1}{1 - i/N_{\mathbf{n}}} \leq \frac{\exp((h_1 + h_2 + h_3)^2/N_{\mathbf{n}})}{N_{\mathbf{n}} + 1},$$

as soon as $h_1 + h_2 + h_3 \leq N_{\mathbf{n}}/2$. Note that we must have $r \leq \Delta_{\mathbf{n}}$ and so

$$LR(\bar{\mathbf{m}}) = r + \sum_{i \geq 1} (i-1) \underline{m}_i = (r-3) + \sum_{j=1}^3 LR(\mathbf{m}^{(j)}) \leq \Delta_{\mathbf{n}} + \sum_{j=1}^3 LR(\mathbf{m}^{(j)}),$$

Finally, under **(H)**,

$$\sum_{r \geq 2} \frac{r(r-1)n_r}{2(N_{\mathbf{n}} + 1)} \xrightarrow{n \rightarrow \infty} \frac{\sigma_p^2}{2},$$

this concludes the case $k = 2$.

In the general case, the same argument applies: we may still decompose the tree according to its reduced subtree $T_n(u_{n,1}, \dots, u_{n,k})$ to obtain an explicit expression of the joint law of $\mathbf{A}(u_{n,1}, \dots, u_{n,k})$ and the number of children of all the branch-points of $T_n(u_{n,1}, \dots, u_{n,k})$. Specifically, denote by $v_{n,1}, \dots, v_{n,k-1}$ these branch-points, fix $\mathbf{m}^{(1)}, \dots, \mathbf{m}^{(2k-1)}$ and $r_1, \dots, r_{k-1} \leq \Delta_{\mathbf{n}}$, set $h_j = |\mathbf{m}^{(j)}|$ for $1 \leq j \leq 2k-1$ and $h = h_1 + \dots + h_{2k-1}$, as well as $\bar{m}_i = \sum_{j=1}^{2k-1} m_i^{(j)} + \sum_{j=1}^{k-1} \mathbf{1}_{\{i=r_j\}}$ for $i \geq 1$. We have then

$$\begin{aligned} & \mathbf{P} \left(\mathbf{A}(u_{n,1}, \dots, u_{n,k}) = (\mathbf{m}^{(1)}, \dots, \mathbf{m}^{(2k-1)}) \text{ and } k_{v_{n,j}} = r_j \text{ for every } 1 \leq j \leq k-1 \mid \mathcal{E}_n(k) \right) \\ &= \prod_{j=1}^{k-1} \frac{r_j(r_j-1)n_{r_j}}{2(N_{\mathbf{n}} + 1)} \cdot \frac{LR(\bar{\mathbf{m}})}{N_{\mathbf{n}} + 1 - (h + k - 1)} \cdot \frac{(N_{\mathbf{n}} + 1 - (h + k - 1))! N_{\mathbf{n}}^h}{(N_{\mathbf{n}} + 1)!} \\ & \times \prod_{i \geq 1} \frac{n_i!}{n_i^{\bar{m}_i} (n_i - \bar{m}_i)!} \cdot \prod_{j=1}^{2k-1} \binom{h_j}{(m_i^{(j)}; i \geq 1)} \prod_{i \geq 1} \left(\frac{in_i}{N_{\mathbf{n}}} \right)^{m_i^{(j)}}. \end{aligned}$$

We have

$$\sum_{r_1, \dots, r_{k-1} \geq 2} \prod_{j=1}^{k-1} \frac{r_j(r_j-1)n_{r_j}}{2(N_{\mathbf{n}} + 1)} = \left(\sum_{r \geq 2} \frac{r(r-1)n_r}{2(N_{\mathbf{n}} + 1)} \right)^{k-1} \xrightarrow{n \rightarrow \infty} \left(\frac{\sigma_p^2}{2} \right)^{k-1},$$

as well as, for $h \leq N_{\mathbf{n}}/2$,

$$\begin{aligned} \frac{(N_{\mathbf{n}} + 1 - (h + k - 1))! N_{\mathbf{n}}^h}{(N_{\mathbf{n}} + 1)!} &= \prod_{i=0}^{k-2} \frac{1}{N_{\mathbf{n}} + 1 - i} \cdot \prod_{i=0}^{h-1} \frac{1}{1 - (i + k - 2)/N_{\mathbf{n}}} \\ &\leq \frac{1 + o(1)}{N_{\mathbf{n}}^{k-1}} \cdot \exp\left(\frac{h^2 + 2h(k-2)}{N_{\mathbf{n}}}\right). \end{aligned}$$

The rest of the proof is adapted verbatim. \square

5 Functional invariance principles

We state and prove in this section the intermediate results used in the proof of Theorem 3. Let (T_n, l_n) be a uniform random labelled tree in $\mathbf{LT}(\mathbf{n})$ and let H_n and L_n denote its height and label processes.

Let also \mathcal{T}_n be its associated two-type tree, which has the uniform distribution in $\mathbf{T}_{\circ, \bullet}(\mathbf{n})$, with white contour process \mathcal{C}_n° . Our aim is to show that, under **(H)**, the convergences

$$\left(\left(\frac{\sigma_p^2}{4p_0^2 N_{\mathbf{n}}} \right)^{1/2} \mathcal{C}_n^\circ(N_{\mathbf{n}}t); t \in [0, 1] \right) \xrightarrow[n \rightarrow \infty]{(d)} (\mathbf{e}_t; t \in [0, 1]) \quad (11)$$

as well as

$$\left(\left(\frac{\sigma_p^2}{4} \frac{1}{N_{\mathbf{n}}} \right)^{1/2} H_n(N_{\mathbf{n}}t); t \in [0, 1] \right) \xrightarrow[n \rightarrow \infty]{(d)} (\mathbf{e}_t; t \in [0, 1]) \quad (12)$$

and

$$\left(\left(\frac{9}{4\sigma_p^2} \frac{1}{N_{\mathbf{n}}} \right)^{1/4} L_n(N_{\mathbf{n}}t); t \in [0, 1] \right) \xrightarrow[n \rightarrow \infty]{(d)} (Z_t; t \in [0, 1]), \quad (13)$$

hold jointly in $\mathcal{C}([0, 1], \mathbf{R})$. We prove **(11)** in the next subsection. Then we prove the convergence of random finite-dimensional marginals of $(N_{\mathbf{n}}^{-1/4} L_n(N_{\mathbf{n}} \cdot))_{n \geq 1}$ in Section 5.3 and the tightness of this sequence in Section 5.5.

5.1 Convergence of the contour

Let T_n have the uniform distribution in $\mathbf{T}(\mathbf{n})$ and let \mathcal{T}_n be its associated two-type tree, which has the uniform distribution in $\mathbf{T}_{\circ, \bullet}(\mathbf{n})$.

Proposition 1. *Under **(H)**, we have the convergence in distribution in $\mathcal{C}([0, 1], \mathbf{R}^2)$*

$$\left(\left(\frac{\sigma_p^2}{4} \frac{1}{N_{\mathbf{n}}} \right)^{1/2} H_n(N_{\mathbf{n}}t), \left(\frac{\sigma_p^2}{4p_0^2 N_{\mathbf{n}}} \right)^{1/2} \mathcal{C}_n^\circ(N_{\mathbf{n}}t) \right)_{t \in [0, 1]} \xrightarrow[n \rightarrow \infty]{(d)} (\mathbf{e}_t, \mathbf{e}_t)_{t \in [0, 1]}.$$

The key observation is the identity from Lemma 1:

$$\mathcal{C}_n^\circ = \tilde{H}_n,$$

where $\tilde{H}_n(j)$ is the number of strict ancestors of u_j , the j -th vertex of T_n in lexicographical order, whose last child is not an ancestor of u_j . We have seen in the previous section that for a “typical” vertex u of T_n , at generation $|u|$, the number of ancestors having i children for $i \geq 1$ forms approximately a multinomial sequence with parameters $|u|$ and $(in_i/N_{\mathbf{n}}; i \geq 1)$; further, for each such ancestor, there is a probability $1 - 1/i$ that its last child is not an ancestor of u and therefore contributes to \mathcal{C}_n° . Since $\sum_{i \geq 1} (1 - 1/i)(in_i/N_{\mathbf{n}}) \rightarrow 1 - (1 - p_0) = p_0$, we conclude that, at a “typical” time, $\mathcal{C}_n^\circ \approx p_0 H_n$.

Proof. The convergence of the first marginal comes from Theorem 2; since, under **(H)**, we have $p_0 = \lim_{n \rightarrow \infty} (n_0 - 1)/N_{\mathbf{n}}$ it suffices then to prove that

$$N_{\mathbf{n}}^{-1/2} \sup_{0 \leq t \leq 1} \left| \tilde{H}_n(N_{\mathbf{n}}t) - \frac{n_0 - 1}{N_{\mathbf{n}}} H_n(N_{\mathbf{n}}t) \right| \xrightarrow[n \rightarrow \infty]{\mathbf{P}} 0.$$

Note that we may restrict ourselves to times t of the form $i/N_{\mathbf{n}}$ with $1 \leq i \leq N_{\mathbf{n}}$. We proceed as in the proof of Corollary 3. Fix $\delta, \varepsilon > 0$ and choose $x > 0$ such that for every n large enough,

$$\mathbf{P} \left(\sup_{u \in T_n} |u| > x \sqrt{N_{\mathbf{n}}} \right) < \varepsilon.$$

We aim at showing

$$\limsup_{n \rightarrow \infty} \mathbf{P} \left(\sup_{1 \leq i \leq N_{\mathbf{n}}} \left| \tilde{H}_n(i) - \frac{n_0 - 1}{N_{\mathbf{n}}} H_n(i) \right| > \delta \sqrt{N_{\mathbf{n}}} \text{ and } \sup_{u \in T_n} |u| \leq x \sqrt{N_{\mathbf{n}}} \right) = 0.$$

Let i_n be a uniform random integer in $\{1, \dots, N_{\mathbf{n}}\}$ and u_n the i_n -th vertex of T_n in lexicographical order, then the previous probability is bounded above by

$$\begin{aligned} & N_{\mathbf{n}} \mathbf{P} \left(\left| \tilde{H}_n(i_n) - \frac{n_0 - 1}{N_{\mathbf{n}}} H_n(i_n) \right| > \delta \sqrt{N_{\mathbf{n}}} \text{ and } \sup_{u \in T_n} |u| \leq x \sqrt{N_{\mathbf{n}}} \right) \\ & \leq x N_{\mathbf{n}}^{3/2} \sup_{1 \leq h \leq x \sqrt{N_{\mathbf{n}}}} \sum_{|\mathbf{m}|=h} \mathbf{P}(\mathbf{A}(u_n) = \mathbf{m}) \mathbf{P} \left(\left| \tilde{H}_n(i_n) - \frac{n_0 - 1}{N_{\mathbf{n}}} h \right| > \delta \sqrt{N_{\mathbf{n}}} \mid \mathbf{A}(u_n) = \mathbf{m} \right). \end{aligned}$$

According to Lemma 2, for every n large enough, for every $1 \leq h \leq x \sqrt{N_{\mathbf{n}}}$ and every sequence $\mathbf{m} \in \mathbf{Z}_+^N$ with $|\mathbf{m}| = h$,

$$\mathbf{P}(\mathbf{A}(u_n) = \mathbf{m}) \leq e^{x^2} \cdot \mathbf{P}(\Xi_{\mathbf{n}}^{(h)} = \mathbf{m}) \cdot (1 + o(1)),$$

where $\Xi_{\mathbf{n}}^{(h)}$ is a random sequence with the multinomial distribution with parameters h and $(in_i/N_{\mathbf{n}}; i \geq 1)$. Moreover, conditional on the vector $(k_v; v \in \llbracket \emptyset, u_n \rrbracket)$, $\tilde{H}_n(i_n)$ is a sum of independent Bernoulli random variables, with respective parameter $(1 - k_v^{-1}; v \in \llbracket \emptyset, u_n \rrbracket)$. Since

$$\sum_{i \geq 1} \left(1 - \frac{1}{i}\right) \cdot \frac{in_i}{N_{\mathbf{n}}} = \frac{n_0 - 1}{N_{\mathbf{n}}},$$

we have for every n large enough, for every $1 \leq h \leq x \sqrt{N_{\mathbf{n}}}$,

$$\begin{aligned} & \sum_{|\mathbf{m}|=h} \mathbf{P}(\mathbf{A}(u_n) = \mathbf{m}) \mathbf{P} \left(\left| \tilde{H}_n(i_n) - \frac{n_0 - 1}{N_{\mathbf{n}}} h \right| > \delta \sqrt{N_{\mathbf{n}}} \mid \mathbf{A}(u_n) = \mathbf{m} \right) \\ & \leq e^{x^2} \cdot \mathbf{P} \left(\left| \sum_{i=1}^h Y_{\mathbf{n},i} - \frac{n_0 - 1}{N_{\mathbf{n}}} h \right| > \delta \sqrt{N_{\mathbf{n}}} \right) \cdot (1 + o(1)), \end{aligned}$$

where $(Y_{\mathbf{n},i}; 1 \leq i \leq h)$ are independent Bernoulli random variables with parameter $(n_0 - 1)/N_{\mathbf{n}}$. Applying the Chernoff bound, we obtain

$$\mathbf{P} \left(\left| \sum_{i=1}^h Y_{\mathbf{n},i} - \frac{n_0 - 1}{N_{\mathbf{n}}} h \right| > \delta \sqrt{N_{\mathbf{n}}} \right) \leq 2e^{-2\delta^2 N_{\mathbf{n}}/h} \leq 2e^{-2\delta^2 \sqrt{N_{\mathbf{n}}}/x}.$$

We conclude that for every $\delta, \varepsilon > 0$, there exists $x > 0$ (depending only on ε) such that for every n large enough,

$$\mathbf{P} \left(\sup_{1 \leq i \leq N_{\mathbf{n}}} \left| \tilde{H}_n(i) - \frac{n_0 - 1}{N_{\mathbf{n}}} H_n(i) \right| > \delta \sqrt{N_{\mathbf{n}}} \right) \leq \varepsilon + 2xe^{x^2} \cdot N_{\mathbf{n}}^{3/2} \cdot e^{-2\delta^2 \sqrt{N_{\mathbf{n}}}/x} \cdot (1 + o(1)),$$

which converges to ε as $n \rightarrow \infty$. □

5.2 Maximal displacement at a branch-point

Recall that for every vertex u , we denote by k_u its number of children and these children by u_1, \dots, u_{k_u} .

Proposition 2. *For every $n \geq 1$, sample (T_n, l_n) uniformly at random in $\mathbf{LT}(\mathbf{n})$. Under **(H)**, we have the convergence in probability*

$$N_{\mathbf{n}}^{-1/4} \max_{u \in T_n} \left| \max_{1 \leq j \leq k_u} l_n(u_j) - \min_{1 \leq j \leq k_u} l_n(u_j) \right| \xrightarrow[n \rightarrow \infty]{\mathbf{P}} 0.$$

To prove this result, we shall need a sub-Gaussian tail bound for the maximal gap in a random walk bridge. Recall that for $r \geq 1$, a discrete bridge of length r is a vector (B_0, \dots, B_r) satisfying $B_0 = B_r = 0$ and $B_{k+1} - B_k \in \mathbf{Z}$ for every $0 \leq k \leq r-1$. A random bridge is said to be *exchangeable* if the law of its increments $(B_1, B_2 - B_1, \dots, B_r - B_{r-1})$ is invariant under permutation.

Lemma 4. Fix $r \geq 1$ and let $B = (B_0, \dots, B_r)$ be a discrete bridge. For every $x \geq 0$ fixed, if

$$\max_{0 \leq k \leq r} B_k - \min_{0 \leq k \leq r} B_k \geq 3x,$$

then at least one of the following quantities must be smaller than or equal to $-x$:

$$\begin{aligned} \min_{0 \leq k \leq \lceil r/2 \rceil} B_k, \quad \min_{0 \leq k \leq \lceil r/2 \rceil} (B_{\lceil r/2 \rceil} - B_{\lceil r/2 \rceil - k}), \\ \min_{0 \leq k \leq \lceil r/2 \rceil} (B_{\lceil r/2 \rceil + k} - B_{\lceil r/2 \rceil}), \quad \min_{0 \leq k \leq \lceil r/2 \rceil} (B_r - B_{r-k}). \end{aligned}$$

Consequently, if B is a random exchangeable bridge, then for every $x \geq 0$, we have

$$\mathbf{P} \left(\max_{0 \leq k \leq r} B_k - \min_{0 \leq k \leq r} B_k \geq 3x \right) \leq 4 \cdot \mathbf{P} \left(\min_{0 \leq k \leq \lceil r/2 \rceil} B_k \leq -x \right).$$

Proof. Let us write $r/2$ instead of $\lceil r/2 \rceil$. Denote by

$$M_1 = \max_{0 \leq k \leq r/2} B_k, \quad m_1 = \min_{0 \leq k \leq r/2} B_k, \quad M_2 = \max_{r/2 \leq k \leq r} B_k, \quad m_2 = \min_{r/2 \leq k \leq r} B_k.$$

Suppose that the four minima in the statement are (strictly) larger than $-x$, then, since $B_r = 0$,

$$m_1 > -x, \quad B_{r/2} - M_1 > -x, \quad m_2 - B_{r/2} > -x, \quad -M_2 > -x.$$

It follows that

$$\begin{aligned} M_1 - m_1 &< (B_{r/2} + x) + x < m_2 + 3x \leq 3x, \\ M_1 - m_2 &< (B_{r/2} + x) - (B_{r/2} - x) = 2x, \\ M_2 - m_1 &< 2x, \\ M_2 - m_2 &< x - (B_{r/2} - x) \leq 2x - m_1 < 3x, \end{aligned}$$

We conclude that $\max_{0 \leq k \leq r} B_k - \min_{0 \leq k \leq r} B_k = \sup\{M_1, M_2\} - \inf\{m_1, m_2\} < 3x$.

The last claim follows after observing that if B is exchangeable, then the three processes

$$(B_{r/2} - B_{r/2-k}; 0 \leq k \leq r/2), \quad (B_{r/2+k} - B_{r/2}; 0 \leq k \leq r/2), \quad (B_r - B_{r-k}; 0 \leq k \leq r/2)$$

are distributed as $(B_k; 0 \leq k \leq r/2)$. \square

Lemma 5. Let $(S_k; k \geq 0)$ be a random walk such that $S_0 = 0$ and $(S_{k+1} - S_k; k \geq 0)$ are i.i.d. random variables, taking values in $\mathbf{Z} \cap [-b, \infty)$ for some $b \geq 0$, centred and with variance $\sigma^2 \in (0, \infty)$. There exists two constants $c, C > 0$ which only depend on b and σ such that for every $r \geq 1$ and $x \geq 0$, we have

$$\mathbf{P} \left(\max_{0 \leq k \leq r} S_k - \min_{0 \leq k \leq r} S_k \geq x \mid S_r = 0 \right) \leq C e^{-cx^2/r}.$$

Proof. First note that on the event $\{S_r = 0\}$, $\max_{0 \leq k \leq r} S_k - \min_{0 \leq k \leq r} S_k$ cannot exceed br . Moreover, on the event $\{S_r = 0\}$, the path (S_0, \dots, S_r) is an exchangeable bridge so, according to Lemma 4, it suffices to show that there exists two constants $c, C > 0$ which only depend on b and σ such that for every $r \geq 1$ and $0 \leq x \leq br$,

$$\mathbf{P} \left(\min_{0 \leq k \leq \lceil r/2 \rceil} S_k \leq -x \mid S_r = 0 \right) \leq C e^{-cx^2/r}.$$

For every $k \geq 1$ and every $x \in \mathbf{Z}$, let us set $\theta_k(x) = \mathbf{P}(S_k = -x)$. According to the local limit theorem, for every $k \geq 1$ and $x \in \mathbf{Z}$,

$$\sqrt{k}\theta_k(x) = g(x/\sqrt{k}) + \varepsilon_k(x),$$

where $g(x) = (2\pi\sigma^2)^{-1/2}e^{-x^2/(2\sigma^2)}$ and $\lim_{k \rightarrow \infty} \sup_{x \in \mathbf{Z}} |\varepsilon_k(x)| = 0$. It follows that

$$C := \sup_{r \geq 1, x \in \mathbf{Z}} \frac{\theta_{r-\lceil r/2 \rceil}(x)}{\theta_r(0)} = \sup_{r \geq 1, x \in \mathbf{Z}} \sqrt{\frac{r}{r - \lceil r/2 \rceil}} \frac{g(-x/\sqrt{r - \lceil r/2 \rceil}) + \varepsilon_{r-\lceil r/2 \rceil}(x)}{g(0) + \varepsilon_r(0)} < \infty.$$

Using the Markov property at time $\lceil r/2 \rceil$, we have thereby

$$\begin{aligned} \mathbf{P}\left(\min_{0 \leq k \leq \lceil r/2 \rceil} S_k \leq -x \mid S_r = 0\right) &= \frac{\mathbf{P}\left(\min_{0 \leq k \leq \lceil r/2 \rceil} S_k \leq -x \text{ and } S_r = 0\right)}{\mathbf{P}(S_r = 0)} \\ &= \mathbf{E}\left[\mathbf{1}_{\{\min_{0 \leq k \leq \lceil r/2 \rceil} S_k \leq -x\}} \frac{\theta_{r-\lceil r/2 \rceil}(S_{\lceil r/2 \rceil})}{\theta_r(0)}\right] \\ &\leq C \cdot \mathbf{P}\left(\min_{0 \leq k \leq \lceil r/2 \rceil} S_k \leq -x\right). \end{aligned}$$

Finally, since $-S$ is a random walk with step distribution bounded above by b , centred and with variance σ^2 , we have the following concentration inequality (see e.g. Mc Diarmid [34], Theorem 2.7 and the remark at the end of Section 2 there): for every $n \geq 1$ and every $x \geq 0$,

$$\mathbf{P}\left(\max_{0 \leq k \leq n} -S_k \geq x\right) \leq \exp\left(-\frac{x^2}{2\sigma^2 n + 2bx/3}\right).$$

We conclude that for every $r \geq 1$ and every $0 \leq x \leq br$, we have

$$\mathbf{P}\left(\min_{0 \leq k \leq \lceil r/2 \rceil} S_k \leq -x \mid S_r = 0\right) \leq C \exp\left(-\frac{x^2}{2\sigma^2 \lceil r/2 \rceil + 2bx/3}\right) \leq C \exp\left(-\frac{x^2}{(2\sigma^2 + 2b^2/3)r}\right),$$

and the proof is complete. \square

Consider the random walk $(S_i; i \geq 0)$ such that $S_0 = 0$ and $(S_{i+1} - S_i; i \geq 0)$ are i.i.d. random variables, distributed as a shifted geometric law: $\mathbf{P}(S_1 = k) = 2^{-(k+2)}$ for every $k \geq -1$. Then it is easy to check that for every $r \geq 1$, on the event $\{S_r = 0\}$, the path (S_0, \dots, S_r) has the uniform distribution in the finite set \mathcal{B}_r^+ of discrete bridges of length r with no negative jump defined in (2). Therefore, if $X^{(r)} = (X_0^{(r)}, \dots, X_r^{(r)})$ is a uniform random element of \mathcal{B}_r^+ , then we may apply Lemma 5 which reads: for every $x \geq 0$,

$$\mathbf{P}\left(\max X^{(r)} - \min X^{(r)} \geq x\right) \leq Ce^{-cx^2/r}, \quad (14)$$

where $c, C > 0$ are universal constants (which depend neither on x nor r).

Proof of Proposition 2. Recall that conditional on T_n , the sequences $(0, l_n(u_1) - l_n(u), \dots, l_n(uk_u) - l_n(u))_{u \in T_n}$ are independent and distributed respectively as a uniform random element of \mathcal{B}_r^+ when $k_u = r$ and that there are n_r such vertices in T_n . According to (14), with the same notation, for every

$\varepsilon > 0$, for every n large enough,

$$\begin{aligned}
& \mathbf{P} \left(\max_{u \in T_n} \left| \max_{1 \leq i \leq k_u} l_n(ui) - \min_{1 \leq i \leq k_u} l_n(ui) \right| \leq \varepsilon N_{\mathbf{n}}^{1/4} \right) \\
&= \prod_{r=1}^{\Delta_{\mathbf{n}}} \mathbf{P} \left(\max X^{(r)} - \min X^{(r)} \leq \varepsilon N_{\mathbf{n}}^{1/4} \right)^{n_r} \\
&\geq \prod_{r=1}^{\Delta_{\mathbf{n}}} \left(1 - C \exp \left(-c\varepsilon^2 N_{\mathbf{n}}^{1/2}/r \right) \right)^{n_r} \\
&\geq \exp \left(- \sum_{r=1}^{\Delta_{\mathbf{n}}} n_r \frac{C \exp \left(-c\varepsilon^2 N_{\mathbf{n}}^{1/2}/r \right)}{1 - C \exp \left(-c\varepsilon^2 N_{\mathbf{n}}^{1/2}/r \right)} \right) \\
&\geq \exp \left(-C \sum_{r=1}^{\Delta_{\mathbf{n}}} n_r \exp \left(-c\varepsilon^2 N_{\mathbf{n}}^{1/2}/r \right) (1 + o(1)) \right),
\end{aligned}$$

where we have used the bound $\ln(1-x) \geq -\frac{x}{1-x}$ for $x < 1$, together with the fact that, under **(H)**, we have $\sup_{1 \leq r \leq \Delta_{\mathbf{n}}} \exp(-c\varepsilon^2 N_{\mathbf{n}}^{1/2}/r) \rightarrow 0$ since $\Delta_{\mathbf{n}} = o(N_{\mathbf{n}}^{1/2})$. Recall furthermore that under **(H)**, we have $\sum_{r=1}^{\Delta_{\mathbf{n}}} r^2 n_r / N_{\mathbf{n}} \rightarrow \sigma_p^2 + 1 < \infty$, we conclude that for every n large enough, since $x \mapsto x^2 e^{-x}$ is decreasing on $[2, \infty)$,

$$\sum_{r=1}^{\Delta_{\mathbf{n}}} n_r \exp \left(-c\varepsilon^2 \frac{N_{\mathbf{n}}^{1/2}}{r} \right) \leq \sum_{r=1}^{\Delta_{\mathbf{n}}} \frac{r^2 n_r}{N_{\mathbf{n}}} \times \frac{N_{\mathbf{n}}}{\Delta_{\mathbf{n}}^2} \exp \left(-c\varepsilon^2 \frac{N_{\mathbf{n}}^{1/2}}{\Delta_{\mathbf{n}}} \right) \xrightarrow{n \rightarrow \infty} 0,$$

and the claim follows. \square

5.3 Random finite-dimensional convergence

As in Section 4, in order to make the notation easier to follow, we first treat the one-dimensional case.

Proposition 3. *For every $n \geq 1$, sample independently (T_n, l_n) uniformly at random in $\mathbf{LT}(\mathbf{n})$ and U uniformly at random in $[0, 1]$. Under **(H)**, the convergence in distribution*

$$\left(\frac{9}{4\sigma_p^2} \frac{1}{N_{\mathbf{n}}} \right)^{1/4} L_n(N_{\mathbf{n}}U) \xrightarrow[n \rightarrow \infty]{(d)} Z_U$$

holds jointly with (12).

The approach of the proof was described in Section 3.2 when explaining the constant $(9/(4\sigma_p^2))^{1/4}$ and relies on Lemma 2.

Proof. Note that the vertex u_n visited at the time $\lfloor N_{\mathbf{n}}U \rfloor$ in lexicographical order has the uniform distribution in T_n ; denote by $l_n(u_n) = L_n(\lfloor N_{\mathbf{n}}U \rfloor)$ its label and by $|u_n| = H_n(\lfloor N_{\mathbf{n}}U \rfloor)$ its height and observe that

$$\left(\frac{9}{4\sigma_p^2} \frac{1}{N_{\mathbf{n}}} \right)^{1/4} l_n(u_n) = \sqrt{\sqrt{\frac{\sigma_p^2}{4} \frac{1}{N_{\mathbf{n}}}} |u_n|} \sqrt{\frac{3}{\sigma_p^2} \frac{1}{\sqrt{|u_n|}}} l_n(u_n).$$

Since, according to (12),

$$\sqrt{\frac{\sigma_p^2}{4} \frac{1}{N_{\mathbf{n}}}} |u_n| \xrightarrow[n \rightarrow \infty]{(d)} \mathbf{e}_U,$$

it is equivalent to show that, jointly with (12), we have

$$\frac{1}{\sqrt{|u_n|}} l_n(u_n) \xrightarrow[n \rightarrow \infty]{\Rightarrow} \mathcal{N}\left(0, \frac{\sigma_p^2}{3}\right), \quad (15)$$

where $\mathcal{N}(0, \sigma_p^2/3)$ denotes the centred Gaussian distribution with variance $\sigma_p^2/3$ and “ \Rightarrow ” is a slight abuse of notation to refer to the weak convergence of the law of the random variable.

Recall that we denote by $A_i(u_n)$ the number of strict ancestors of u_n with i children:

$$A_i(u_n) = \#\{v \in \llbracket \emptyset, u_n \rrbracket : k_v = i\};$$

denote further by $A_{i,j}(u_n)$ the number of strict ancestors of u_n with i children, among which the j -th one is again an ancestor of u_n :

$$A_{i,j}(u_n) = \#\{v \in \llbracket \emptyset, u_n \rrbracket : k_v = i \text{ and } vj \in \llbracket \emptyset, u_n \rrbracket\}.$$

We have seen in Section 4 that when T_n is uniformly distributed in $\mathbf{T}(\mathbf{n})$ and u_n is uniformly distributed in T_n , then $\mathbf{A}(u_n) = (A_i(u_n); i \geq 1)$ can be compared to a multinomial sequence with parameters $|u_n|$ and $(in_i/N_n; i \geq 1)$. Observe further that given the sequence $\mathbf{A}(u_n)$, the vectors $(A_{i,j}(u_n); 1 \leq j \leq i)_{i \geq 1}$ are independent and distributed respectively according to the multinomial distribution with parameters $A_i(u_n)$ and $(\frac{1}{i}, \dots, \frac{1}{i})$.

We write

$$l_n(u_n) = \sum_{i=1}^{\Delta_n} \sum_{j=1}^i \sum_{k=1}^{A_{i,j}(u_n)} X_{i,j,k},$$

where the random variables $X_{i,j,k}$'s are all independent (and independent of $\mathbf{A}(u_n)$) and have respectively the law of the j -th marginal of a uniform random bridge in \mathcal{B}_i^+ . Each variable $X_{i,j,k}$ is centred and has finite variance, say, $\sigma_{i,j}^2$. We shall use also the notation

$$l_n^K(u_n) = \sum_{i=1}^K \sum_{j=1}^i \sum_{k=1}^{A_{i,j}(u_n)} X_{i,j,k},$$

for every $K \geq 1$. The proof of (15) is divided into two steps: we first show that for every $K \geq 1$, $l_n^K(u_n)/\sqrt{|u_n|}$ converges towards a limit which depends on K and which in turn converges towards $\mathcal{N}(0, \sigma_p^2/3)$ as $K \rightarrow \infty$, and then we show that $|l_n(u_n) - l_n^K(u_n)|/\sqrt{|u_n|}$ can be made arbitrarily small uniformly for n large enough by choosing K large enough.

Let us first prove the convergence of $l_n^K(u_n)$ as $n \rightarrow \infty$. For every $h \geq 1$, let $\Xi_{\mathbf{n}}^{(h)} = (\Xi_{\mathbf{n},i}^{(h)}; i \geq 1)$ denote a random sequence with the multinomial distribution with parameters h and $(in_i/N_n; i \geq 1)$. Fix $\varepsilon > 0$ arbitrary, according to Theorem 2, we may choose $x > 0$, such that for n large enough,

$$\mathbf{P}\left(|u_n|/\sqrt{N_n} \notin [1/x, x]\right) \leq 2 \cdot \mathbf{P}(\sigma_p \mathbf{e}_U/2 \notin [1/x, x]) < \varepsilon,$$

where U is uniformly distribution in $[0, 1]$ and independent of \mathbf{e} . Furthermore, Broutin & Marckert [12, Equation 16] have shown that

$$\mathbf{P}\left(LR(u_n) > x\sqrt{N_n}\right) \leq 16 \cdot 9 \cdot \frac{\sigma_{\mathbf{n}}^2}{x^2} < \varepsilon,$$

for n and x large enough. Finally, uniformly for $1 \leq h \leq x\sqrt{N_n}$, for every sequence $\mathbf{m} \in \mathbf{Z}_+^{\mathbf{N}}$ with $|\mathbf{m}| = h$ and $LR(\mathbf{m}) \leq x\sqrt{N_n}$, we have from Lemma 2,

$$\mathbf{P}(\mathbf{A}(u_n) = \mathbf{m}) \leq x e^{x^2} \cdot N_n^{-1/2} \cdot \mathbf{P}\left(\Xi_{\mathbf{n}}^h = \mathbf{m}\right) \cdot (1 + o(1)).$$

Fix $i \geq 1$ such that $p(i) \neq 0$. Since $\Xi_{\mathbf{n},i}^{(h)}$ has the binomial distribution with parameters h and $in_i/N_{\mathbf{n}}$, Markov inequality yields for every $\delta > 0$ and every n large enough,

$$\begin{aligned} \mathbf{P} \left(\left| \frac{N_{\mathbf{n}}}{|u_n|in_i} A_i(u_n) - 1 \right| > \delta \right) &\leq 2\varepsilon + 2x^2 e^{x^2} \sup_{\sqrt{N_{\mathbf{n}}}/x \leq h \leq x\sqrt{N_{\mathbf{n}}}} \mathbf{P} \left(\left| \frac{N_{\mathbf{n}}}{hin_i} \Xi_{\mathbf{n},i}^{(h)} - 1 \right| > \delta \right) \\ &\leq 2\varepsilon + 2x^2 e^{x^2} \sup_{\sqrt{N_{\mathbf{n}}}/x \leq h \leq x\sqrt{N_{\mathbf{n}}}} h^{-1} \delta^{-2} \left(\frac{N_{\mathbf{n}}}{in_i} - 1 \right), \end{aligned}$$

which converges to 2ε as $n \rightarrow \infty$ since $in_i/N_{\mathbf{n}} \rightarrow ip(i) \in (0, 1)$. Recall that given $A_i(u_n)$, the vector $(A_{i,j}(u_n); 1 \leq j \leq i)$ has the multinomial distribution with parameters $A_i(u_n)$ and $(\frac{1}{i}, \dots, \frac{1}{i})$ so for every $1 \leq j \leq i$, we further have

$$\frac{N_{\mathbf{n}}}{|u_n|n_i} A_{i,j}(u_n) \xrightarrow[n \rightarrow \infty]{\mathbf{P}} 1.$$

Since the random variables $X_{i,j,k}$ are independent, centred and have variance $\sigma_{i,j}^2$, the central limit theorem then reads, when $p(i) \neq 0$,

$$\frac{1}{\sqrt{|u_n|}} \sum_{k=1}^{A_{i,j}(u_n)} X_{i,j,k} \xrightarrow[n \rightarrow \infty]{\Longrightarrow} \mathcal{N}(0, p(i)\sigma_{i,j}^2). \quad (16)$$

In the case $p(i) = 0$, we claim that

$$\frac{1}{\sqrt{|u_n|}} \sum_{j=1}^i \sum_{k=1}^{A_{i,j}(u_n)} X_{i,j,k} \xrightarrow[n \rightarrow \infty]{\mathbf{P}} 0. \quad (17)$$

Indeed, with the same argument as above, it suffices to show that for every $\delta > 0$, we have

$$\lim_{n \rightarrow \infty} \sup_{\sqrt{N_{\mathbf{n}}}/x \leq h \leq x\sqrt{N_{\mathbf{n}}}} \sum_{|\mathbf{m}|=h} \mathbf{P}(\Xi_{\mathbf{n}}^{(h)} = \mathbf{m}) \mathbf{P} \left(\left| \sum_{j=1}^i \sum_{k=1}^{M_{i,j}} X_{i,j,k} \right| \geq \delta\sqrt{h} \right) = 0,$$

where the vector $(M_{i,j}; 1 \leq j \leq i)$ has the multinomial distribution with parameters m_i and $(\frac{1}{i}, \dots, \frac{1}{i})$ and is independent of the $X_{i,j,k}$'s. For every sequence \mathbf{m} , we have

$$\mathbf{P} \left(\left| \sum_{j=1}^i \sum_{k=1}^{M_{i,j}} X_{i,j,k} \right| \geq \delta\sqrt{h} \right) \leq \frac{1}{\delta^2 h} \sum_{j=1}^i \mathbf{E}[M_{i,j}] \sigma_{i,j}^2 = \frac{1}{\delta^2 h} \frac{m_i}{i} \sum_{j=1}^i \sigma_{i,j}^2,$$

whence

$$\begin{aligned} \sum_{|\mathbf{m}|=h} \mathbf{P}(\Xi_{\mathbf{n}}^{(h)} = \mathbf{m}) \mathbf{P} \left(\left| \sum_{j=1}^i \sum_{k=1}^{M_{i,j}} X_{i,j,k} \right| \geq \delta\sqrt{h} \right) &\leq \sum_{|\mathbf{m}|=h} \mathbf{P}(\Xi_{\mathbf{n}}^{(h)} = \mathbf{m}) \frac{1}{\delta^2 h} \frac{m_i}{i} \sum_{j=1}^i \sigma_{i,j}^2 \\ &\leq \mathbf{E}[\Xi_{\mathbf{n},i}^{(h)}] \frac{1}{\delta^2 h} \frac{1}{i} \sum_{j=1}^i \sigma_{i,j}^2 \\ &\leq \frac{n_i}{N_{\mathbf{n}}} \frac{1}{\delta^2} \sum_{j=1}^i \sigma_{i,j}^2. \end{aligned}$$

Since, under **(H)**, we have $n_i/N_{\mathbf{n}} \rightarrow p(i) = 0$ as $n \rightarrow \infty$ and (17) follows.

We conclude using (16), (17) and the independence of the $X_{i,j,k}$'s as i and j vary that for every $K \geq 1$, the convergence

$$\frac{1}{\sqrt{|u_n|}} l_n^K(u_n) \xrightarrow[n \rightarrow \infty]{=} \mathcal{N} \left(0, \sum_{i=1}^K p(i) \sum_{j=1}^i \sigma_{i,j}^2 \right)$$

holds. Marckert & Miermont [32, page 1664] have calculated the variance of the random variables $X_{i,j,k}$:

$$\sigma_{i,j}^2 = \frac{2j(i-j)}{i+1} \quad \text{so} \quad \sum_{j=1}^i \sigma_{i,j}^2 = \frac{i(i-1)}{3}.$$

As a consequence,

$$\sum_{i=1}^K p(i) \sum_{j=1}^i \sigma_{i,j}^2 \xrightarrow[K \rightarrow \infty]{} \sum_{i=1}^{\infty} p(i) \frac{i(i-1)}{3} = \frac{\sigma_p^2}{3},$$

which implies

$$\mathcal{N} \left(0, \sum_{i=1}^K p(i) \sum_{j=1}^i \sigma_{i,j}^2 \right) \xrightarrow[K \rightarrow \infty]{=} \mathcal{N} \left(0, \frac{\sigma_p^2}{3} \right).$$

It only remains to show that for every $\delta > 0$, we have

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P} \left(|l_n(u_n) - l_n^K(u_n)| \geq \delta \sqrt{|u_n|} \right) = 0. \quad (18)$$

Again, with the same notation as above, it is enough to show that for every $x > 0$ and every $\delta > 0$, we have

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{\sqrt{N_n}/x \leq h \leq x\sqrt{N_n}} \sum_{|\mathbf{m}|=h} \mathbf{P} \left(\Xi_n^{(h)} = \mathbf{m} \right) \mathbf{P} \left(\left| \sum_{i=K}^{\Delta_n} \sum_{j=1}^i \sum_{k=1}^{M_{i,j}} X_{i,j,k} \right| \geq \delta \sqrt{h} \right) = 0.$$

By the same calculation as above,

$$\begin{aligned} \sum_{|\mathbf{m}|=h} \mathbf{P} \left(\Xi_n^{(h)} = \mathbf{m} \right) \mathbf{P} \left(\left| \sum_{i=K}^{\Delta_n} \sum_{j=1}^i \sum_{k=1}^{M_{i,j}} X_{i,j,k} \right| \geq \delta \sqrt{h} \right) &\leq \sum_{|\mathbf{m}|=h} \mathbf{P} \left(\Xi_n^{(h)} = \mathbf{m} \right) \frac{1}{\delta^2 h} \sum_{i=K}^{\Delta_n} \frac{m_i}{i} \sum_{j=1}^i \sigma_{i,j}^2 \\ &= \frac{1}{\delta^2 h} \sum_{i=K}^{\Delta_n} \frac{1}{i} \mathbf{E} \left[\Xi_{n,i}^{(h)} \right] \sum_{j=1}^i \sigma_{i,j}^2 \\ &= \frac{1}{\delta^2} \sum_{i=K}^{\Delta_n} \frac{n_i}{N_n} \frac{i(i-1)}{3}, \end{aligned}$$

Under **(H)**,

$$\sum_{i=K}^{\Delta_n} \frac{n_i}{N_n} i(i-1) \xrightarrow[n \rightarrow \infty]{} \sum_{i \geq K} p(i) i(i-1),$$

and, in turn, the right-hand side converges 0 as $K \rightarrow \infty$. This concludes the proof of (18). \square

We next give a multi-dimensional extension of Proposition 3. The proof of the latter relied on Lemma 2, the proof of its extension appeals to Lemma 3.

Proposition 4. For every $n \geq 1$, sample independently (T_n, l_n) uniformly at random in $\mathbf{LT}(\mathbf{n})$ and U_1, \dots, U_k uniformly at random in $[0, 1]$. Under **(H)**, the convergence in distribution

$$\left(\frac{9}{4\sigma_p^2} \frac{1}{N_{\mathbf{n}}} \right)^{1/4} (L_n(N_{\mathbf{n}}U_1), \dots, L_n(N_{\mathbf{n}}U_k)) \xrightarrow[n \rightarrow \infty]{(d)} (Z_{U_1}, \dots, Z_{U_k})$$

holds jointly with (12).

Proof. As for Lemma 3, we focus on the case $k = 2$ and comment on the general case at the end. Let u_n and v_n be independent uniform random vertices of T_n and w_n be their most recent common ancestor, let further \hat{u}_n and \hat{v}_n be the children of w_n which are respectively an ancestor of u_n and v_n . We write:

$$l_n(u_n) = l_n(w_n) + (l_n(\hat{u}_n) - l_n(w_n)) + (l_n(u_n) - l_n(\hat{u}_n)),$$

and we have a similar decomposition for v_n . The point is that, conditional on T_n , u_n and v_n , the random variables $l_n(w_n)$, $l_n(u_n) - l_n(\hat{u}_n)$ and $l_n(v_n) - l_n(\hat{v}_n)$ are independent. Moreover, according to Proposition 2, with high probability, $l_n(\hat{u}_n) - l_n(w_n)$ and $l_n(\hat{v}_n) - l_n(w_n)$ are both small compared to $N_{\mathbf{n}}^{1/4}$.

According to (12), we have

$$\left(\frac{\sigma_p^2}{4} \frac{1}{N_{\mathbf{n}}} \right)^{1/2} (|w_n|, |u_n| - |\hat{u}_n|, |v_n| - |\hat{v}_n|) \xrightarrow[n \rightarrow \infty]{(d)} (m_{\mathbf{e}}(U, V), \mathbf{e}_U - m_{\mathbf{e}}(U, V), \mathbf{e}_V - m_{\mathbf{e}}(U, V)),$$

where U and V are i.i.d uniform random variables on $[0, 1]$ independent of \mathbf{e} . We shall prove that, jointly with (12),

$$\sqrt{\frac{3}{\sigma_p^2}} \left(\frac{l_n(w_n)}{\sqrt{|w_n|}}, \frac{l_n(u_n) - l_n(\hat{u}_n)}{\sqrt{|u_n| - |\hat{u}_n|}}, \frac{l_n(v_n) - l_n(\hat{v}_n)}{\sqrt{|v_n| - |\hat{v}_n|}} \right) \xrightarrow[n \rightarrow \infty]{(d)} (G_1, G_2, G_3), \quad (19)$$

where G_1, G_2, G_3 are i.i.d. standard Gaussian random variables. Proposition 2 and (19) then imply that, jointly with (12), the sequence

$$\left(\left(\frac{9}{4\sigma_p^2} \frac{1}{N_{\mathbf{n}}} \right)^{1/4} (l_n(u_n), l_n(v_n)) \right)_{n \geq 1}$$

converges in distribution towards

$$\sqrt{m_{\mathbf{e}}(U, V)}G_1 + \left(\sqrt{\mathbf{e}_U - m_{\mathbf{e}}(U, V)}G_2, \sqrt{\mathbf{e}_V - m_{\mathbf{e}}(U, V)}G_3 \right).$$

This corresponds to the claim of the proposition for $k = 2$. The proof of (19) is *mutatis mutandis* the same as that of Proposition 3: consider the three branches $[\emptyset, w_n]$, $[\hat{u}_n, u_n]$ and $[\hat{v}_n, v_n]$, we use Lemma 3 to compare the number of elements in each branch which have i children and among which the j -th one belongs to the branch to independent multinomial distributions; then we may use the arguments of the proof of Proposition 3 to each branch independently which yields (19).

The general case $k \geq 2$ hides no difficulty. Sample i.i.d. uniform random vertices $u_{n,1}, \dots, u_{n,k}$ of T_n ; appealing to Proposition 2, we neglect the contribution of the branch-points of the reduced tree $T_n(u_{n,1}, \dots, u_{n,k})$ (recall the notation from Section 4) and we decompose the labels of each vertex $u_{n,i}$ as the sum of the increments of labels over all the branches of the forest $F_n(u_{n,1}, \dots, u_{n,k})$; as for $k = 2$, Lemma 3 yields the generalisation of (19). \square

5.4 Concentration results for discrete excursions

In this subsection, we prove two concentration inequalities for the Łukasiewicz path of T_n . The first one shall be used to derive the tightness of the label process in the next subsection, the second one shall be used in Section 6 in the proof of Theorem 1.

Proposition 5. *Assume that (H) holds and let W_n be the Łukasiewicz path of a tree sampled uniformly at random in $\mathbf{T}(\mathbf{n})$. There exists a constant $C > 0$ such that, uniformly for $n \in \mathbf{N}$ and $0 \leq j < k \leq N_n + 1$ with $k - j \leq N_n/2$,*

$$\mathbf{P} \left(W_n(j) - \min_{j \leq i \leq k} W_n(i) > t \right) \leq \exp \left(-\frac{t^2}{C \cdot (k - j)} \right),$$

for every $t \geq 0$. As a consequence, for every $r > 0$, the bound

$$\mathbf{E} \left[\left(W_n(j) - \min_{j \leq i \leq k} W_n(i) \right)^r \right] \leq C(r) \cdot (k - j)^{r/2},$$

holds uniformly for $n \in \mathbf{N}$ and $0 \leq j < k \leq N_n + 1$ such that $k - j \leq N_n/2$, where $C(r) = \Gamma(1 + \frac{r}{2}) \cdot C^{r/2}$.

This result follows from Section 3 of Addario-Berry [2]. Fix $\mathbf{m} = (m_0, m_1, m_2, \dots)$ a sequence of non-negative integers with finite sum satisfying

$$M = \sum_{i \geq 0} m_i, \quad \sum_{i \geq 0} (i - 1)m_i = -1 \quad \text{and} \quad \zeta^2 = \sum_{i \geq 0} (i - 1)^2 m_i,$$

and define

$$\mathbf{B}(\mathbf{m}) := \{x = (x_1, \dots, x_M) : \#\{j : x_j = i - 1\} = m_i \text{ for every } i \geq 0\}.$$

Given $x \in \mathbf{B}(\mathbf{m})$, we consider the walk S_x defined by $S_x(0) = 0$ and $S_x(k) = x_1 + \dots + x_k$ for $1 \leq k \leq M$.

Lemma 6 (Addario-Berry [2]). *If x is sampled uniformly at random in $\mathbf{B}(\mathbf{m})$, then*

$$\mathbf{P} \left(-\min_{0 \leq i \leq k} S_x(i) \geq t \right) \leq \exp \left(-\frac{t^2}{(16\frac{\zeta^2}{M} + \frac{8}{3}(1 - \frac{1}{M}))k} \right)$$

for every $1 \leq k \leq \lfloor M/2 \rfloor$ and every $t \geq 0$.

Observe that $S_x(M) = -1$ for every $x \in \mathbf{B}(\mathbf{m})$; we define further

$$\mathbf{E}(\mathbf{m}) := \{x \in \mathbf{B}(\mathbf{m}) : S_x(k) \geq 0 \text{ for every } 1 \leq k \leq M - 1\}.$$

The sets $\mathbf{E}(\mathbf{m})$ and $\mathbf{T}(\mathbf{m})$ are in one-to-one correspondence: each path S_x with x in $\mathbf{E}(\mathbf{m})$ is the Łukasiewicz path of a tree in $\mathbf{T}(\mathbf{m})$.

For $x \in \mathbf{B}(\mathbf{m})$ and $j \in \{1, \dots, M\}$, denote by $x^{(j)} \in \mathbf{B}(\mathbf{m})$ the j -th cyclic shift of x defined by

$$x_k^{(j)} = x_{k+j \bmod M}, \quad 1 \leq k \leq M.$$

It is well-known that, given $x \in \mathbf{B}(\mathbf{m})$, we have $x^{(j)} \in \mathbf{E}(\mathbf{m})$ if and only if j is the least time at which the walk S_x achieves its minimum overall value:

$$j = \inf \left\{ 1 \leq k \leq M : S_x(k) = \inf_{1 \leq l \leq M} S_x(l) \right\}. \quad (20)$$

Given $x \in \mathbf{B}(\mathbf{m})$, we let x^* be the unique cyclic shift of x in $\mathbf{E}(\mathbf{m})$. It is a standard fact that if x has the uniform distribution in $\mathbf{B}(\mathbf{m})$, then the time j satisfying (20) has the uniform distribution on $\{1, \dots, M\}$ and then that $x^* = x^{(j)}$ is uniformly distributed in $\mathbf{E}(\mathbf{m})$ and independent of j .

Proof of Proposition 5. According to the previous remark, we know that W_n is distributed as S_{x^*} where x has the uniform distribution in $\mathbf{B}(\mathbf{n})$. With the previous notation, $M = N_{\mathbf{n}} + 1$ and

$$\varsigma^2 = (N_{\mathbf{n}} + 1)\sigma_{\mathbf{n}}^2 + \frac{N_{\mathbf{n}}^2}{N_{\mathbf{n}} + 1} - N_{\mathbf{n}} + 1 = (N_{\mathbf{n}} + 1)\sigma_{\mathbf{n}}^2 + \frac{1}{N_{\mathbf{n}} + 1}.$$

Observe that, by shift-invariance, Lemma 6 applies to S_{x^*} as well: for every $t \geq 1$, for every $1 \leq k - j \leq \lfloor N_{\mathbf{n}}/2 \rfloor$,

$$\begin{aligned} \mathbf{P} \left(S_{x^*}(j) - \min_{j \leq i \leq k} S_{x^*}(i) \geq t \right) &= \mathbf{P} \left(- \min_{0 \leq i \leq k-j} S_x(i) \geq t \right) \\ &\leq \exp \left(- \frac{t^2}{(16(\sigma_{\mathbf{n}}^2 + \frac{1}{N_{\mathbf{n}}+1}) + \frac{8}{3}(1 - \frac{1}{N_{\mathbf{n}}+1}))(k-j)} \right), \end{aligned}$$

which corresponds to the first claim, with $C := \sup_{n \geq 1} \{16(\sigma_{\mathbf{n}}^2 + \frac{1}{N_{\mathbf{n}}+1}) + \frac{8}{3}(1 - \frac{1}{N_{\mathbf{n}}+1})\} < \infty$; the second claim follows by integrating this tail bound applied to $t^{1/r}$. \square

We next show that the vertices of T_n of a given type are in some sense uniformly distributed for large n . For a tree $T \in \mathbf{T}$, a set $A \subset \mathbf{Z}_+$ and an integer $i \geq 1$, let $\Lambda_{T,i}(A)$ be the number of vertices which have a number of children in A among the first i vertices of T in lexicographical order.

Proposition 6. *Assume that (H) holds and sample T_n uniformly at random in $\mathbf{T}(\mathbf{n})$ for every $n \geq 1$. Then for every $A \subset \mathbf{Z}_+$,*

$$\mathbf{P} \left(\max_{1 \leq i \leq N_{\mathbf{n}}+1} |\Lambda_{T_n,i}(A) - p_{\mathbf{n}}(A)i| > N_{\mathbf{n}}^{3/4} \right) \xrightarrow{n \rightarrow \infty} 0.$$

Proof. For every $y \in \mathbf{B}(\mathbf{n})$, every $A \subset \mathbf{Z}_+$ and every $1 \leq i \leq N_{\mathbf{n}} + 1$, set

$$\lambda_{y,i}(A) := \#\{1 \leq k \leq i : y_k \in A - 1\}.$$

Note that $\lambda_{y,N_{\mathbf{n}}+1}(A) = (N_{\mathbf{n}} + 1)p_{\mathbf{n}}(A)$. As previously discussed, the Łukasiewicz path of T_n has the law of S_x where x is uniformly distributed in $\mathbf{E}(\mathbf{n})$, so

$$\mathbf{P} \left(\max_{1 \leq i \leq N_{\mathbf{n}}+1} |\Lambda_{T_n,i}(A) - p_{\mathbf{n}}(A)i| > N_{\mathbf{n}}^{3/4} \right) = \mathbf{P} \left(\max_{1 \leq i \leq N_{\mathbf{n}}} |\lambda_{x,i}(A) - p_{\mathbf{n}}(A)i| > N_{\mathbf{n}}^{3/4} \right).$$

Let us first consider y uniformly distributed in $\mathbf{B}(\mathbf{n})$. For each $1 \leq i \leq N_{\mathbf{n}} + 1$ fixed, $\lambda_{y,i}(A) = \sum_{k=1}^i \mathbf{1}_{\{y_k \in A-1\}}$ is the sum of i dependent Bernoulli random variables, which arise from a sampling without replacement in an urn with initial configuration of $\sum_{i \in A} n_i$ “good” balls and $N_{\mathbf{n}} + 1 - \sum_{i \in A} n_i$ “bad” balls. It is well-known that the expected value of any continuous convex function of $\lambda_{y,i}(A)$ is bounded above by the corresponding quantity for the sum of i i.i.d. Bernoulli random variables with parameter $p_{\mathbf{n}}(A)$, which arise from sampling with replacement, see e.g. Hoeffding’s seminal paper [17, Theorem 4]. In particular, the Chernoff bound for binomial random variables yields

$$\begin{aligned} \mathbf{P} \left(\max_{1 \leq i \leq N_{\mathbf{n}}} |\lambda_{y,i}(A) - p_{\mathbf{n}}(A)i| > N_{\mathbf{n}}^{3/4} \right) &\leq N_{\mathbf{n}} \max_{1 \leq i \leq N_{\mathbf{n}}} \mathbf{P} \left(|\lambda_{y,i}(A) - p_{\mathbf{n}}(A)i| > N_{\mathbf{n}}^{3/4} \right) \\ &\leq 2N_{\mathbf{n}} \max_{1 \leq i \leq N_{\mathbf{n}}} \exp \left(-2N_{\mathbf{n}}^{3/2}/i \right) \\ &= 2N_{\mathbf{n}} \exp \left(-2N_{\mathbf{n}}^{1/2} \right). \end{aligned}$$

Next, let j be as in (20) and recall that j is uniformly distributed in $\{1, \dots, N_{\mathbf{n}} + 1\}$ and that $x := y^* = y^{(j)}$ is uniformly distributed in $\mathbf{E}(\mathbf{n})$ and independent of j . If $j = N_{\mathbf{n}} + 1$, then $x = y$ and our claim follows from the above bound. We then implicitly condition j to be less than $N_{\mathbf{n}} + 1$,

in which case it has the uniform distribution in $\{1, \dots, N_{\mathbf{n}}\}$ and it is independent of x . Observe that $N_{\mathbf{n}} + 1 - j$ also has the uniform distribution in $\{1, \dots, N_{\mathbf{n}}\}$ and is independent of x , so

$$\mathbf{P} \left(\max_{1 \leq i \leq N_{\mathbf{n}}} |\lambda_{x,i}(A) - p_{\mathbf{n}}(A)i| > N_{\mathbf{n}}^{3/4} \right) \leq N_{\mathbf{n}} \mathbf{P} \left(|\lambda_{x, N_{\mathbf{n}}+1-j}(A) - p_{\mathbf{n}}(A)(N_{\mathbf{n}} + 1 - j)| > N_{\mathbf{n}}^{3/4} \right).$$

Furthermore, in our coupling, $\lambda_{x, N_{\mathbf{n}}+1-j}(A) = \#\{1 \leq k \leq N_{\mathbf{n}} + 1 - j : x_k \in A - 1\}$ is also equal to $\#\{1 \leq k \leq N_{\mathbf{n}} + 1 - j : y_{N_{\mathbf{n}}+2-k} \in A - 1\}$. By time-reversal, we have the identity

$$((y_{N_{\mathbf{n}}+2-k}; 1 \leq k \leq N_{\mathbf{n}} + 1); N_{\mathbf{n}} + 1 - j) \stackrel{(d)}{=} ((y_k; 1 \leq k \leq N_{\mathbf{n}} + 1); j'),$$

where $j' = \sup\{0 \leq k \leq N_{\mathbf{n}} : S_y(k) = \max_{1 \leq l \leq N_{\mathbf{n}}+1} S_x(l)\}$. We conclude that

$$\begin{aligned} \mathbf{P} \left(\max_{1 \leq i \leq N_{\mathbf{n}}+1} |\Lambda_{T_n, i}(A) - p_{\mathbf{n}}(A)i| > N_{\mathbf{n}}^{3/4} \right) &\leq N_{\mathbf{n}} \mathbf{P} \left(|\lambda_{y, j'}(A) - p_{\mathbf{n}}(A)j'| > N_{\mathbf{n}}^{3/4} \right) + o(1) \\ &\leq 2N_{\mathbf{n}}^2 \exp \left(-2N_{\mathbf{n}}^{1/2} \right) + o(1), \end{aligned}$$

which converges to 0. □

5.5 Tightness of the label process

Let us prove the tightness of the label process; together with Proposition 4, this will end the proof of Theorem 3 and so Theorem 1.

Proposition 7. *For every $n \geq 1$, sample (T_n, l_n) uniformly at random in $\mathbf{LT}(\mathbf{n})$. Under **(H)**, the sequence*

$$\left(N_{\mathbf{n}}^{-1/4} L_n(N_{\mathbf{n}} t); t \in [0, 1] \right)_{n \geq 1}$$

is tight in $\mathcal{C}([0, 1], \mathbf{R})$.

In the remainder of this section, we shall use the notation $C(q)$ for a positive constant which depends only on a real number q and, implicitly, on the sequences \mathbf{n} , and which will often differ from a line to another.

We shall prove that, for some sequence of events \mathcal{E}_n satisfying $\mathbf{P}(\mathcal{E}_n) \rightarrow 1$ as $n \rightarrow \infty$ (those from Corollary 3), for every $q > 4$, for every $\beta \in (0, q/4 - 1)$, for every n large enough, for every $i, j \in \{0, \dots, N_{\mathbf{n}}\}$,

$$\mathbf{E} [|L_n(i) - L_n(j)|^q \mid \mathcal{E}_n] \leq C(q) \cdot N_{\mathbf{n}}^{q/4} \cdot \left| \frac{i - j}{N_{\mathbf{n}}} \right|^{1+\beta}. \quad (21)$$

Set $L_{(n)}(t) = N_{\mathbf{n}}^{-1/4} L_n(N_{\mathbf{n}} t)$ for $n \in \mathbf{N}$ and $t \in [0, 1]$, then the previous display reads

$$\mathbf{E} [|L_{(n)}(s) - L_{(n)}(t)|^q \mid \mathcal{E}_n] \leq C(q) \cdot |s - t|^{1+\beta},$$

whenever $s, t \in [0, 1]$ are such that $N_{\mathbf{n}}s$ and $N_{\mathbf{n}}t$ are both integers. Since $L_{(n)}$ is defined by linear interpolation between such times, this bound then holds for every $s, t \in [0, 1]$ (possibly with a different constant $C(q)$). The standard Kolmogorov criterion then implies the following bound for the Hölder norm of $L_{(n)}$: for every $\alpha \in (0, 1/4)$,

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P} \left(\sup_{0 \leq s \neq t \leq 1} \frac{|L_{(n)}(s) - L_{(n)}(t)|}{|s - t|^\alpha} > K \mid \mathcal{E}_n \right) = 0;$$

since $\mathbf{P}(\mathcal{E}_n) \rightarrow 1$ as $n \rightarrow \infty$, we obtain

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P} \left(\sup_{0 \leq s \neq t \leq 1} \frac{|L_{(n)}(s) - L_{(n)}(t)|}{|s - t|^\alpha} > K \right) = 0,$$

and the sequence $(L_{(n)}; n \geq 1)$ is tight in $\mathcal{C}([0, 1], \mathbf{R})$.

The proof of (21) relies on the coding of T_n by its Łukasiewicz path. The next lemma, whose proof is left as an exercise, gathers some deterministic results that we shall need (we refer to e.g. Le Gall [24] for a thorough discussion of such results). In order to simplify the notation, we identify for the remainder of this section the vertices of a one-type tree with their index in the lexicographic order: if u and v are the i -th and j -th vertices of T_n , we write $u \leq v$ if $i \leq j$, $W_n(u)$ for $W_n(i)$ and $|u - v|$ for $|i - j|$, the lexicographic distance between u and v .

Lemma 7. *Let T be a one-type plane tree and W be its Łukasiewicz path. Fix a vertex $u \in T$, then*

$$W(uk_u) = W(u), \quad W(uj') = \inf_{[uj, uj']} W \quad \text{and} \quad j' - j = W(uj) - W(uj')$$

for every $1 \leq j \leq j' \leq k_u$.

In the course of the proof of (21), we shall need the following two ingredients. First, a consequence of the so-called Marcinkiewicz–Zygmund inequality, see e.g. Gut [16, Theorem 8.1]: fix $q \geq 2$ and consider independent and centred random variables Y_1, \dots, Y_m which admit a finite q -th moment, then there exists $C(q) \in (0, \infty)$ such that

$$\frac{1}{C(q)} \cdot \mathbf{E} \left[\left(\sum_{i=1}^m |Y_i|^2 \right)^{q/2} \right] \leq \mathbf{E} \left[\left| \sum_{i=1}^m Y_i \right|^q \right] \leq C(q) \cdot \mathbf{E} \left[\left(\sum_{i=1}^m |Y_i|^2 \right)^{q/2} \right].$$

Consider the right-most term, and raise it temporarily to the power $2/q$ in order to apply the triangle inequality for the $L^{q/2}$, the second inequality thus yields the following bound:

$$\mathbf{E} \left[\left| \sum_{i=1}^m Y_i \right|^q \right] \leq C(q) \cdot \left(\sum_{i=1}^m \mathbf{E} [|Y_i|^{q/2}] \right)^{q/2}. \quad (22)$$

Second, for every $r \geq 1$, consider $X^{(r)}$ a uniform random bridge in \mathcal{B}_r^+ , defined in (2); Le Gall & Miermont [28, Lemma 1] have shown that for every $q \geq 2$ and every $i, j \in \{0, \dots, r\}$,

$$\mathbf{E} \left[|X_i^{(r)} - X_j^{(r)}|^q \right] \leq C(q) \cdot |i - j|^{q/2}. \quad (23)$$

Proof of Proposition 7. Recall that we identify the vertices of T_n with their index in the lexicographic order. Fix $q > 4$, $\beta \in (0, q/4 - 1)$, n large enough so that \mathcal{E}_n defined by (10) has probability larger than $1/2$, and two integers $0 \leq u < v \leq N_n + 1$ with $v - u \leq \lfloor N_n/2 \rfloor$; we aim at showing

$$\mathbf{E} [|l_n(u) - l_n(v)|^q \mid \mathcal{E}_n] \leq C(q) \cdot N_n^{q/4} \cdot \left| \frac{u - v}{N_n} \right|^{1+\beta}.$$

Let $u \wedge v$, be the most recent common ancestor of u and v in T_n and further \hat{u} and \hat{v} be the children of $u \wedge v$ which are respectively ancestor of u and v . We stress that u and v are deterministic times, whereas $u \wedge v$, \hat{u} and \hat{v} are random and measurable with respect to T_n . We write:

$$l_n(u) - l_n(v) = \left(\sum_{w \in \llbracket \hat{u}, u \rrbracket} l_n(w) - l_n(pr(w)) \right) + (l_n(\hat{u}) - l_n(\hat{v})) + \left(\sum_{w \in \llbracket \hat{v}, v \rrbracket} l_n(pr(w)) - l_n(w) \right).$$

Recall the notation $1 \leq \chi_{\hat{u}} \leq \chi_{\hat{v}} \leq k_{u \wedge v}$ for the relative position of \hat{u} and \hat{v} among the children of $u \wedge v$. By construction of the labels on T_n , the bound (23) reads in our context:

$$\mathbf{E} [|l_n(\hat{u}) - l_n(\hat{v})|^q \mid T_n] \leq C(q) \cdot (\chi_{\hat{v}} - \chi_{\hat{u}})^{q/2}.$$

Next, fix $w \in \llbracket \hat{u}, u \rrbracket$, since $l_n(pr(w)) = l_n(pr(w)k_{pr(w)})$, as previously, the bound (23) gives:

$$\mathbf{E} [|l_n(w) - l_n(pr(w))|^q \mid T_n] \leq C(q) \cdot (k_{pr(w)} - \chi_w)^{q/2}.$$

Similarly, for every $w \in \llbracket \hat{v}, v \rrbracket$, we have

$$\mathbf{E} [|l_n(pr(w)) - l_n(w)|^q \mid T_n] \leq C(q) \cdot \chi_w^{q/2}.$$

According to the Marcinkiewicz–Zygmund inequality (22), we thus have

$$\begin{aligned} \mathbf{E} [|l_n(u) - l_n(v)|^q \mid T_n] &\leq C(q) \cdot \left(\sum_{w \in \llbracket \hat{u}, u \rrbracket} (k_{pr(w)} - \chi_w) + (\chi_{\hat{v}} - \chi_{\hat{u}}) + \sum_{w \in \llbracket \hat{v}, v \rrbracket} \chi_w \right)^{q/2} \\ &\leq C(q) \cdot \left(\left(\sum_{w \in \llbracket \hat{u}, u \rrbracket} (k_{pr(w)} - \chi_w) + (\chi_{\hat{v}} - \chi_{\hat{u}}) \right)^{q/2} + \left(\sum_{w \in \llbracket \hat{v}, v \rrbracket} \chi_w \right)^{q/2} \right). \end{aligned}$$

Appealing to Lemma 7, we have

$$\chi_{\hat{v}} - \chi_{\hat{u}} = W_n(\hat{u}) - W_n(\hat{v}),$$

and similarly, for every $w \in \llbracket \hat{u}, u \rrbracket$,

$$k_{pr(w)} - \chi_w = W_n(w) - W_n(pr(w)k_{pr(w)}) = W_n(wk_w) - W_n(pr(w)k_{pr(w)}),$$

so

$$\sum_{w \in \llbracket \hat{u}, u \rrbracket} (k_{pr(w)} - \chi_w) + (\chi_{\hat{v}} - \chi_{\hat{u}}) = W_n(u) - W_n(\hat{v}) = W_n(u) - \inf_{[u,v]} W_n.$$

We know from Proposition 5 that

$$\mathbf{E} \left[\left(W_n(u) - \inf_{[u,v]} W_n \right)^{q/2} \right] \leq C(q) \cdot |u - v|^{q/4};$$

since $\mathbf{P}(\mathcal{E}_n) \geq 1/2$, we conclude that

$$\mathbf{E} \left[\left(\sum_{w \in \llbracket \hat{u}, u \rrbracket} (k_{pr(w)} - \chi_w) + (\chi_{\hat{v}} - \chi_{\hat{u}}) \right)^{q/2} \middle| \mathcal{E}_n \right] \leq 2C(q) \cdot |u - v|^{q/4} \leq 2C(q) \cdot N_{\mathbf{n}}^{q/4} \cdot \left| \frac{u - v}{N_{\mathbf{n}}} \right|^{1+\beta}.$$

Consider next the branch $\llbracket \hat{v}, v \rrbracket$. Let T_n^- be the “mirror image” of T_n , i.e. the tree obtained from T_n by flipping the order of the children of every vertex; let us write $w^- \in T_n^-$ for the mirror image of a vertex $w \in T_n$; make the following observations:

- T_n^- has the same law as T_n , so in particular, its Łukasiewicz path has the same law as that of T_n ;
- for every $w \in \llbracket \hat{v}, v \rrbracket$, the quantity $\chi_w - 1$ in T_n corresponds to the quantity $k_{pr(w^-)} - \chi_{w^-}$ in T_n^- ;
- the lexicographical distance between the last descendant in T_n^- of respectively \hat{v}^- and v^- is smaller than the lexicographical distance between \hat{v} and v in T_n (the elements of $\llbracket \hat{v}, v \rrbracket = \llbracket \hat{v}^-, v^- \rrbracket$ are missing).

With these observations, the previous argument used to control the branch $\llbracket \hat{u}, u \rrbracket$ shows that

$$\mathbf{E} \left[\left(\sum_{w \in \llbracket \hat{v}, v \rrbracket} (\chi_w - 1) \right)^{q/2} \right] \leq C(q) \cdot |u - v|^{q/4},$$

and so

$$\mathbf{E} \left[\left(\sum_{w \in \llbracket \hat{v}, v \rrbracket} (\chi_w - 1) \right)^{q/2} \middle| \mathcal{E}_n \right] \leq C(q) \cdot N_{\mathbf{n}}^{q/4} \cdot \left| \frac{u - v}{N_{\mathbf{n}}} \right|^{1+\beta}.$$

Since $\chi_w \leq 2(\chi_w - 1)$ whenever $\chi_w \geq 2$, it only remains to show that

$$\mathbf{E} \left[\#\{w \in \llbracket \hat{v}, v \rrbracket : \chi_w = 1\}^{q/2} \middle| \mathcal{E}_n \right] \leq C(q) \cdot N_{\mathbf{n}}^{q/4} \cdot \left| \frac{u - v}{N_{\mathbf{n}}} \right|^{1+\beta}.$$

Let C and h_n be as in Corollary 3. On the one hand, since h_n is small compared to any positive power of $N_{\mathbf{n}}$, we have for n large enough,

$$\mathbf{E} \left[\#\{w \in \llbracket \hat{v}, v \rrbracket : \chi_w = 1\}^{q/2} \mathbf{1}_{\{\#\llbracket \hat{v}, v \rrbracket \leq h_n\}} \right] \leq h_n^{q/2} \leq N_{\mathbf{n}}^{q/4} \cdot \left| \frac{u - v}{N_{\mathbf{n}}} \right|^{1+\beta}.$$

On the other hand, if $\#\llbracket \hat{v}, v \rrbracket > h_n$, then on the event \mathcal{E}_n , we know that

$$\#\{w \in \llbracket \hat{v}, v \rrbracket : \chi_w = 1\} \leq C \cdot \#\{w \in \llbracket \hat{v}, v \rrbracket : \chi_w \geq 2\} \leq C \sum_{w \in \llbracket \hat{v}, v \rrbracket} (\chi_w - 1).$$

We then conclude from the previous bound. \square

Remark 6. With the previous notation, if one could prove

$$\mathbf{E} \left[\#\{w \in \llbracket \hat{v}, v \rrbracket : \chi_w = 1 \text{ and } k_{pr(w)} \geq 2\}^{q/2} \right] \leq C(q) \cdot |u - v|^{q/4}, \quad (24)$$

uniformly for $|u - v| \leq N_{\mathbf{n}}/2$, then it would lead to the stronger bound

$$\mathbf{E} [|L_n(u) - L_n(v)|^q] \leq C(q) \cdot |u - v|^{q/4},$$

for every $q > 4$ and every $0 \leq u < v \leq N_{\mathbf{n}} + 1$. Observe that

$$\#\{w \in \llbracket \hat{v}, v \rrbracket : \chi_w = 1 \text{ and } k_{pr(w)} \geq 2\} \leq \#\left\{w \in [u, v[: W_n(w) < \inf_{|w,v]} W_n\right\},$$

which has the same distribution as the number of strict records of a uniform random bridge in $\mathbf{B}(\mathbf{n})$ (defined in Section 5.4) after $|u - v|$ steps.

Under the stronger assumption than $\Delta_{\mathbf{n}}$ is uniformly bounded (which is the case for e.g. uniform random κ -angulations with any $\kappa \geq 4$ even fixed), the argument used to prove Proposition 5 shows that for every $r > 0$,

$$\mathbf{E} \left[\left(\max_{u \leq w \leq v} W_n(w) - W_n(u) \right)^r \right] \leq C(r) \cdot |u - v|^{r/2},$$

uniformly for $n \in \mathbf{N}$ and $0 \leq u < v \leq N_{\mathbf{n}} + 1$ such that $|u - v| \leq \lfloor N_{\mathbf{n}}/2 \rfloor$, which yields (24).

On another model, Miermont [36, Proof of Proposition 8], obtained such a bound for the number of records of a centred random walk with finite variance. The argument used in the proof of Lemma 5 enables us to extend it to such a walk conditioned to be at -1 at time $N_{\mathbf{n}} + 1$. This case corresponds to Boltzmann random maps introduced in Section 7.

6 Convergence of random maps

In this short section we deduce Theorem 1 from Theorem 3, following the argument of Le Gall [26, Section 8.3] and [25, Section 3]. First, observe that every map in $\mathbf{M}(\mathbf{n})$ has $n_0 + 1$ vertices so, if \mathcal{M}_n has the uniform distribution in $\mathbf{M}(\mathbf{n})$ and \mathcal{M}_n^\star is a pointed map obtained by distinguishing a vertex of \mathcal{M}_n uniformly at random, then \mathcal{M}_n^\star has the uniform distribution in $\mathbf{M}^\star(\mathbf{n})$. It is therefore sufficient to prove Theorem 1 with \mathcal{M}_n replaced by \mathcal{M}_n^\star .

Let \mathcal{M}_n^\star be a (deterministic) pointed and rooted planar map in $\mathbf{M}^\star(\mathbf{n})$ and denote by \star its origin; let (\mathcal{T}_n, ℓ_n) be its associated two-type labelled tree via the BDG bijection and let $(c_0^\circ, \dots, c_{N_n}^\circ)$ be the white contour sequence of the latter. Recall that the vertices c_i° are identified to the vertices of \mathcal{M}_n different from \star . For every $i, j \in \{0, \dots, N_n\}$, we set

$$d_n(i, j) = d_{\text{gr}}(c_i^\circ, c_j^\circ),$$

where d_{gr} is the graph distance of \mathcal{M}_n . We then extend d_n to a continuous function on $[0, N_n]^2$ by “bilinear interpolation” on each square of the form $[i, i+1] \times [j, j+1]$ as in [26, Section 2.5]. Recall the convention $c_{N_n+i}^\circ = c_i^\circ$ for every $0 \leq i \leq N_n$ and the interpretation, at the very end of Section 2.3, of the labels as distances from \star in \mathcal{M}_n : for every $0 \leq i \leq N_n$,

$$d_{\text{gr}}(\star, c_i^\circ) = \mathcal{L}_n^\circ(i) - \min_{0 \leq j \leq N_n} \mathcal{L}_n^\circ(j) + 1. \quad (25)$$

Then, using the triangle inequality at a point where a geodesic from c_i° to \star and a geodesic from c_j° to \star merge, Le Gall [26, Equation 4] obtains the bound

$$d_n(i, j) \leq \mathcal{L}_n^\circ(i) + \mathcal{L}_n^\circ(j) - 2 \max \left\{ \min_{i \leq k \leq j} \mathcal{L}_n^\circ(k); \min_{j \leq k \leq N_n+i} \mathcal{L}_n^\circ(k) \right\} + 2. \quad (26)$$

See also Lemma 3.1 in [25] for a detailed proof in a slightly different context.

Define for every $s, t \in [0, 1]$:

$$\begin{aligned} \mathfrak{C}_{(n)}(t) &= \left(\frac{\sigma_p^2}{16p_0^2} \frac{1}{N_n} \right)^{1/2} \mathfrak{C}_n(2N_n t), \\ \mathcal{L}_{(n)}^\circ(t) &= \left(\frac{9}{4\sigma_p^2} \frac{1}{N_n} \right)^{1/4} \mathcal{L}_n^\circ(N_n t), \\ d_{(n)}(s, t) &= \left(\frac{9}{4\sigma_p^2} \frac{1}{N_n} \right)^{1/4} d_n(N_n s, N_n t), \\ D_{\mathcal{L}_{(n)}^\circ}(s, t) &= \mathcal{L}_{(n)}^\circ(s) + \mathcal{L}_{(n)}^\circ(t) - 2 \max \left\{ \check{\mathcal{L}}_{(n)}^\circ(s); \check{\mathcal{L}}_{(n)}^\circ(t) \right\}, \end{aligned}$$

where $\check{\mathcal{L}}_{(n)}^\circ$ is defined in a similar way as \check{Z} in Section 3.1.

Proposition 8. *Let (\mathcal{T}_n, ℓ_n) have the uniform distribution in $\mathbf{LT}_{\circ, \bullet}(\mathbf{n})$ for every $n \geq 1$. Under **(H)**, the convergence in distribution of continuous paths*

$$\left(\mathfrak{C}_{(n)}(t), \mathcal{L}_{(n)}^\circ(t), d_{(n)}(s, t) \right)_{s, t \in [0, 1]} \xrightarrow[n \rightarrow \infty]{(d)} \left(\mathfrak{e}_t, Z_t, \mathfrak{D}(s, t) \right)_{s, t \in [0, 1]},$$

holds, where \mathfrak{D} is defined in Section 3.1.

Proof. The convergence (7), together with Remark 3 yields the convergence in distribution

$$\left(\mathfrak{C}_{(n)}(t), \mathcal{L}_{(n)}^\circ(t), D_{\mathcal{L}_{(n)}^\circ}(s, t) \right)_{s, t \in [0, 1]} \xrightarrow[n \rightarrow \infty]{(d)} \left(\mathfrak{e}_t, Z_t, D_Z(s, t) \right)_{s, t \in [0, 1]}.$$

The bound (26) implies further the tightness of $(d_{(n)}; n \geq 1)$, see Proposition 3.2 in [25] for a proof in a similar context. Therefore, from every sequence of integers converging to ∞ , we can extract a subsequence along which we have

$$\left(\mathcal{C}_{(n)}(t), \mathcal{L}_{(n)}^\circ(t), d_{(n)}(s, t) \right)_{s, t \in [0, 1]} \xrightarrow[n \rightarrow \infty]{(d)} (\mathbf{e}_t, Z_t, D(s, t))_{s, t \in [0, 1]}, \quad (27)$$

where $(D(s, t); 0 \leq s, t \leq 1)$ depends a priori on the subsequence. We claim that

$$D = \mathfrak{D} \quad \text{almost surely.}$$

From the bound (26), D satisfies $D \leq D_Z$, also (see Proposition 3.3 in [25]), one can check that D is a pseudo-metric on $[0, 1]$ which satisfies $D(s, t) = 0$ as soon as $d_e(s, t) = 0$. It thus follows from the maximality property discussed in section 3.1 that $D \leq \mathfrak{D}$ almost surely. Our aim is to show the following: let X, Y be i.i.d. uniform random variables on $[0, 1]$ such that the pair (X, Y) is independent of everything else, then

$$D(X, Y) \stackrel{(d)}{=} D(s_\star, Y) = Z_Y - Z_{s_\star}, \quad (28)$$

where s_\star is the (a.s. unique [30]) point at which Z attains its minimum. The second equality is a continuous analog of (25) which can be obtained from the latter by letting $n \rightarrow \infty$ along the same subsequence as in (27). Le Gall [26, Corollary 7.3] has proved that (28) holds true when D is replaced by \mathfrak{D} . In particular, $D(X, Y)$ is distributed as $\mathfrak{D}(X, Y)$. Since we know that $D \leq \mathfrak{D}$ almost surely, this implies $D(X, Y) = \mathfrak{D}(X, Y)$ almost surely which, by a density argument, implies $D = \mathfrak{D}$ almost surely.

Let us prove (28). We adapt the argument of Bettinelli & Miermont [10, Lemma 32]. Recall that the white contour sequence of \mathcal{T}_n is denoted by $(c_0^\circ, \dots, c_{N_n}^\circ)$ and let v_1, \dots, v_{n_0} be its white vertices listed in the order of their last visit in the contour sequence; for example the root is v_{n_0} . For $1 \leq i \leq n_0$, let $g(i) \in \{1, \dots, N_n\}$ be the index such that $c_{g(i)}^\circ$ is the last visit of v_i . Observe that $(c_{g(1)}^\circ, \dots, c_{g(n_0)}^\circ) = (v_1, \dots, v_{n_0})$ is an enumeration of the white vertices of \mathcal{T}_n without redundancies. We then set $g(0) = 0$ and extend g linearly to a continuous function on $[0, n_0]$. Let us prove that

$$\left(\frac{g(n_0 t)}{N_n}; t \in [0, 1] \right) \xrightarrow[n \rightarrow \infty]{\mathbf{P}} (t; t \in [0, 1]). \quad (29)$$

It is equivalent to prove

$$\left(\frac{\Lambda(N_n t)}{n_0}; t \in [0, 1] \right) \xrightarrow[n \rightarrow \infty]{\mathbf{P}} (t; t \in [0, 1]),$$

where $\Lambda(0) = 0$ and for every $1 \leq j \leq N_n$,

$$\Lambda(j) := \# \{1 \leq i \leq n_0 : v_i \in \{c_0^\circ, \dots, c_j^\circ\} \text{ and } v_i \notin \{c_{j+1}^\circ, \dots, c_{N_n}^\circ\}\},$$

denotes the number of vertices fully explored at time i in the white contour exploration. Let T_n be the image of \mathcal{T}_n by the JS bijection; it can be checked along the same line as the proof of Lemma 1 that for every $1 \leq j \leq N_n$, $\Lambda(j)$ denotes the number $\Lambda_{T_n, j}(0)$ of leaves among the first j vertices of T_n in lexicographical order. The above convergence of Λ thus follows from Proposition 6.

Fix X, Y i.i.d. uniform random variables on $[0, 1]$ such that the pair (X, Y) is independent of everything else, and set $x = c_{g(\lceil n_0 X \rceil)}^\circ$ and $y = c_{g(\lceil n_0 Y \rceil)}^\circ$. Note that x and y are uniform random white vertices of \mathcal{T}_n , they can therefore be coupled with two independent uniform random vertices x' and y' of \mathcal{M}_n^\star in such a way that the conditional probability given \mathcal{M}_n^\star that $(x, y) \neq (x', y')$ is at most $2(n_0 + 1)^{-1} \rightarrow 0$ as $n \rightarrow \infty$; we implicitly assume in the sequel that $(x, y) = (x', y')$. Since \star is also a uniform random vertex of \mathcal{M}_n^\star , we obtain that

$$d_{\text{gr}}(x, y) \stackrel{(d)}{=} d_{\text{gr}}(\star, y). \quad (30)$$

By definition,

$$d_{\text{gr}}(x, y) = d_n(g(\lceil n_0 X \rceil), g(\lceil n_0 Y \rceil)),$$

and, according to (25),

$$d_{\text{gr}}(\star, y) = \mathcal{L}_n^\circ(g(\lceil n_0 Y \rceil)) - \min_{0 \leq j \leq N_n} \mathcal{L}_n^\circ(j) + 1.$$

We obtain (28) by letting $n \rightarrow \infty$ in (30) along the same subsequence as in (27), appealing also to (29). \square

The proof of Theorem 1 is then routine.

Proof of Theorem 1. We aim at showing the convergence of metric spaces

$$\left(\mathcal{M}_n^\star, \left(\frac{9}{4\sigma_p^2} \frac{1}{N_n} \right)^{1/4} d_{\text{gr}} \right) \xrightarrow[n \rightarrow \infty]{(d)} (\mathcal{M}, \mathcal{D}), \quad (31)$$

for the Gromov–Hausdorff topology. Recall (see e.g. [13, Chapter 7.3]) that a *correspondence* between two metric spaces (X, d_X) and (Y, d_Y) is a set $R \subset X \times Y$ such that for every $x \in X$, there exists $y \in Y$ such that $(x, y) \in R$ and vice-versa. The *distortion* of R is defined as

$$\text{dis}(R) = \sup \{ |d_X(x, x') - d_Y(y, y')| ; (x, y), (x', y') \in R \}.$$

Finally, the Gromov–Hausdorff distance between (X, d_X) and (Y, d_Y) is given by ([13, Theorem 7.3.25])

$$\frac{1}{2} \cdot \inf_R \text{dis}(R),$$

where the infimum is taken over all correspondences R between (X, d_X) and (Y, d_Y) .

The proof is deterministic: we show that the convergence (31) holds whenever that in Proposition 8 does. Indeed, let $(\mathcal{M}_n^\star \setminus \{\star\}, d_{\text{gr}})$ be the metric space given by the vertices of \mathcal{M}_n^\star different from \star and their graph distance in \mathcal{M}_n^\star and observe that the Gromov–Hausdorff distance between $(\mathcal{M}_n^\star, d_{\text{gr}})$ and $(\mathcal{M}_n^\star \setminus \{\star\}, d_{\text{gr}})$ is bounded by one. Let Π be the canonical projection $\mathcal{T}_e \rightarrow \mathcal{M} = \mathcal{T}_e / \approx$, then the set

$$\mathcal{R}_n = \{ (v_{\lfloor N_n t \rfloor}, \Pi(\pi_e(t))) ; t \in [0, 1] \}.$$

is a correspondence between $(\mathcal{M}_n^\star \setminus \{\star\}, (\frac{9}{4\sigma_p^2} \frac{1}{N_n})^{1/4} d_{\text{gr}})$ and $(\mathcal{M}, \mathcal{D})$ and its distortion is given by

$$\sup_{s, t \in [0, 1]} |d_{(n)}(\lfloor N_n s \rfloor / N_n, \lfloor N_n t \rfloor / N_n) - \mathcal{D}(s, t)|,$$

which tends to 0 as $n \rightarrow \infty$ whenever the convergence in Proposition 8 holds. This concludes the proof. \square

7 Boltzmann random maps

In this last section, we state and prove the results alluded in Section 1.3 on Boltzmann random maps. Let us make two preliminary remarks. First, we shall divide by real numbers which depend on an integer n , and consider conditional probabilities with respect to events which depend on n ; we shall therefore, if necessary, implicitly restrict ourselves to those values of n for which such quantities are well-defined and statements such as “as $n \rightarrow \infty$ ” should be understood along the appropriate sequence of integers. Second, in order to avoid trivialities, any sequence of non-negative real numbers, say, $w = (w(i); i \geq 0)$, shall implicitly satisfy $w(i) > 0$ for at least one $i \geq 2$.

7.1 Rooted and pointed Boltzmann maps

Fix a sequence $w = (w(k); k \geq 1)$ of non-negative real numbers. Let \mathbf{M}^* be the set of all rooted and pointed bipartite maps, that we shall view as pairs (\mathcal{M}, \star) , where $\mathcal{M} \in \mathbf{M}$ is a rooted bipartite map, and \star is a vertex of \mathcal{M} . We adapt the distributions described in Section 1.3 to such maps by setting

$$\Omega^{w,\star}((\mathcal{M}, \star)) = \Omega^w(\mathcal{M}) = \prod_{f \in \text{Faces}(\mathcal{M})} w(\deg(f)/2), \quad (\mathcal{M}, \star) \in \mathbf{M}^*,$$

where $\text{Faces}(\mathcal{M})$ is the set of faces of \mathcal{M} and $\deg(f)$ is the degree of such a face f . We set $Z_w^* = \Omega^{w,\star}(\mathbf{M}^*)$.

Definition 1. A sequence w is called *admissible* when Z_w^* is finite.

If w is admissible, we set

$$\mathbf{P}^{w,\star}(\cdot) = \frac{1}{Z_w^*} \Omega^{w,\star}(\cdot).$$

For every integer $n \geq 2$, let $\mathbf{M}_{E=n}^*$, $\mathbf{M}_{V=n}^*$ and $\mathbf{M}_{F=n}^*$ be the subsets of \mathbf{M}^* of those maps with respectively $n - 1$ edges, $n + 1$ vertices and n faces; more generally, for every $A \subset \mathbf{N}$, let $\mathbf{M}_{F,A=n}^*$ be the subset of \mathbf{M}^* of those maps with n faces whose degree belongs to $2A$. For every $S = \{E, V, F\} \cup \bigcup_{A \subset \mathbf{N}} \{F, A\}$ and every $n \geq 2$, we define

$$\mathbf{P}_{S=n}^{w,\star}((\mathcal{M}, \star)) := \mathbf{P}^{w,\star}((\mathcal{M}, \star) \mid (\mathcal{M}, \star) \in \mathbf{M}_{S=n}^*), \quad (\mathcal{M}, \star) \in \mathbf{M}_{S=n}^*,$$

the law of a rooted and pointed Boltzmann map conditioned to have size n .

Given a sequence w , set

$$\bar{w}(0) = 1 \quad \text{and} \quad \bar{w}(k) = \binom{2k-1}{k-1} w(k) \quad \text{for } k \geq 1, \quad (32)$$

and define the power series

$$g_w(x) = \sum_{k \geq 0} x^k \bar{w}(k), \quad x \geq 0. \quad (33)$$

Denote by R_w its radius of convergence, note that g_w is convex, strictly increasing and continuous on $[0, R_w]$ and $g_w(0) = 1$. In particular, it has at most two fixed points, necessarily in $(1, R_w]$; in fact, we have the following exclusive four cases:

- There are no fixed points.
- There are two fixed points $1 < x_1 < x_2 \leq R_w$, moreover $g'_w(x_1) < 1$ and $g'_w(x_2) > 1$.
- There is a unique fixed point $1 < x \leq R_w$, with $g'_w(x) < 1$.
- There is a unique fixed point $1 < x \leq R_w$, with $g'_w(x) = 1$.

Marckert & Miermont [32] have defined another power series f_w , such that $g_w(x) = 1 + x f_w(x)$ for every $x \geq 0$. Proposition 1 in [32] reads as follows with our notation.

Proposition 9 (Marckert & Miermont [32]). *A sequence w is admissible if and only if g_w has at least one fixed point. In this case, Z_w^* is the fixed point satisfying $g'_w(Z_w^*) \leq 1$.*

The proof in [32] is based on the BDG bijection, we shall give a short proof using the composition of the BDG and the JS bijections. Following [32] let us introduce some terminology.

Definition 2. An admissible sequence w is called *critical* when Z_w^* is the unique fixed point of g_w and satisfies moreover $g'_w(Z_w^*) = 1$. It is called *generic critical* when it is admissible, critical, and $g''_w(Z_w^*) < \infty$, and *regular critical* when moreover $Z_w^* < R_w$.

Note that an admissible sequence w induces a probability measure on \mathbf{Z}_+ with mean smaller than or equal to one:

$$p_w(k) = (Z_w^*)^{k-1} \bar{w}(k), \quad k \geq 0, \quad (34)$$

where \bar{w} is given by (32); indeed,

$$\sum_{k \geq 0} p_w(k) = \frac{g_w(Z_w^*)}{Z_w^*} = 1, \quad \text{and} \quad \sum_{k \geq 0} k p_w(k) = g'_w(Z_w^*) \leq 1.$$

This distribution has mean 1 if and only if w is critical, and in this case, its variance is

$$K_w := \left(\sum_{k \geq 0} k^2 p_w(k) \right) - 1 = \left(\frac{d}{dx} x g'_w(x) \right) \Big|_{x=Z_w^*} - 1 = Z_w^* g''_w(Z_w^*), \quad (35)$$

which is finite if and only if w is generic critical. In terms of the function f_w from [32], we have $K_w = (2 + (Z_w^*)^3 f''_w(Z_w^*)) / Z_w^*$. The argument of [32, Proposition 7] show that if w is regular critical, then p_w admits small exponential moments.

Theorem 4. *Suppose w is generic critical, define p_w by (34) and K_w by (35) and set*

$$C_E^w := 1, \quad C_V^w := p_w(0) = \frac{1}{Z_w^*}, \quad C_F^w := 1 - p_w(0) = 1 - \frac{1}{Z_w^*},$$

and more generally, for every subset $A \subset \mathbf{N}$,

$$C_{F,A}^w := p_w(A) = \sum_{k \in A} (Z_w^*)^{k-1} \binom{2k-1}{k-1} w(k).$$

Fix $S \in \{E, V, F\} \cup \bigcup_{A \subset \mathbf{N}} \{F, A\}$ and for every $n \geq 2$, sample \mathcal{M}_n from $\mathbf{P}_{S=n}^w$, then the convergence in distribution

$$\left(\mathcal{M}_n, \left(\frac{9 C_S^w}{4 K_w n} \right)^{1/4} d_{\text{gr}} \right) \xrightarrow[n \rightarrow \infty]{(d)} (\mathcal{M}, \mathcal{D}),$$

holds in the sense of Gromov–Hausdorff.

Note that the Boltzmann laws in the statements are *not* the pointed versions: the maps \mathcal{M}_n are rooted but not pointed. We shall prove first that it holds under the pointed version $\mathbf{P}_{S=n}^{w,*}$, relying on the composition of the BDG and JS bijections to check that **(H)** is fulfilled with the probability p_w given by (34). Then we will show that $\mathbf{P}_{S=n}^{w,*}$ and $\mathbf{P}_{S=n}^w$ are close as $n \rightarrow \infty$; the argument will closely follow that of Bettinelli & Miermont [10, Section 7.2], see also Abraham [1, Section 6], and Bettinelli, Jacob & Miermont [9, Section 3].

Remark 7. Le Gall [26, Theorem 9.1] obtained this result in the case $S = V$, when w is supposed to be regular critical, not only generic critical. Bettinelli & Miermont [10, Theorem 5] also obtained similar convergences in the three cases $S = E, V, F$ for Boltzmann maps *with a boundary*, associated with regular critical weights. Theorem 4 completes (and improve since we only assume w to be generic critical) their Remark 2.

Note that $\mathbf{M}_{E=n}$ is finite for every $n \geq 2$ so the Boltzmann distribution $\mathbf{P}_{E=n}^w$ makes sense even if $Z_w = \infty$. The proof of Theorem 4 shows that we do not need w to be admissible in this case.

Theorem 5. *Suppose there exists $x > 0$ (necessarily unique) such that*

$$g_w(x) < \infty, \quad x g'_w(x) = g_w(x), \quad \text{and} \quad x g''_w(x) < \infty.$$

Then if \mathcal{M}_n is sampled from $\mathbf{P}_{E=n}^w$ for every $n \geq 2$, the convergence in distribution

$$\left(\mathcal{M}_n, \left(\frac{9 g_w(x)}{4 x^2 g''_w(x) n} \right)^{1/4} d_{\text{gr}} \right) \xrightarrow[n \rightarrow \infty]{(d)} (\mathcal{M}, \mathcal{D}),$$

holds in the sense of Gromov–Hausdorff.

Note that if w is generic critical, then the assumptions are fulfilled with $x = Z_w^*$: we have $g_w(Z_w^*) = Z_w^*$ so $xg'_w(x) = g_w(x)$ is equivalent to $g'_w(Z_w^*) = 1$ and then

$$\frac{g_w(x)}{x^2 g''_w(x)} = \frac{1}{Z_w^* g''_w(Z_w^*)} = \frac{1}{K_w} = \frac{C_E^w}{K_w},$$

which is coherent with Theorem 4.

As an application, consider the case $w(k) = 1$ for every $k \geq 1$: $\mathbf{P}_{E=n}^w$ is the uniform distribution in $\mathbf{M}_{E=n}$ studied by Abraham [1]. In this case, g_w has a radius of convergence equal to $1/4$ and is given by

$$g_w(x) = 1 + \sum_{k \geq 1} x^k \binom{2k-1}{k-1} = \frac{1 + \sqrt{1-4x}}{2\sqrt{1-4x}}, \quad 0 < x < 1/4.$$

Furthermore,

$$xg'_w(x) = g_w(x) \quad \text{if and only if} \quad x = \frac{3}{16}, \quad \text{and} \quad \frac{g_w(3/16)}{(3/16)^2 g''_w(3/16)} = \frac{9}{2},$$

so Theorem 5 yields Corollary 2.

The proof of Theorem 4 uses the notion of *simply generated trees* that we next introduce.

7.2 Simply generated trees

Given a sequence $w = (w(k); k \geq 0)$ of non-negative real numbers, we define a measure on the set of finite one-type tree \mathbf{T} by

$$\Pi^w(T) = \prod_{u \in T} w(k_u), \quad T \in \mathbf{T}.$$

Let $\Upsilon_w = \Pi^w(\mathbf{T})$, if the latter is finite, we define a probability measure on \mathbf{T} by

$$\text{SG}^w(\cdot) = \frac{1}{\Upsilon_w} \Pi^w(\cdot).$$

A random tree sampled according to SG^w is called a *simply generated tree*. Such distributions have been introduced by Meir & Moon [35] and studied in great detail by Janson [18] on the set of trees with a given number of vertices. A particular case is when the weight sequence w is a probability measure on \mathbf{Z}_+ with mean less than or equal to one: in this case, $\Upsilon_w = 1$ and $\text{SG}^w = \Pi^w$ is the law of a *subcritical Galton–Watson tree* with offspring distribution w ; we denote it by GW^w . When the expectation of w is exactly equal to one, we say that w (as well as any random tree sampled from GW^w) is *critical*.

Note that we may define simply generated trees with n vertices even if Υ_w is infinite by rescaling the measure Π^w restricted to this finite set by its total mass.

Lemma 8. *Let $w = (w(k); k \geq 0)$ be a sequence of non-negative real numbers. Let us denote by $\#T$ the number of vertices of a tree $T \in \mathbf{T}$.*

- (i) *Fix $c > 0$ and set $\tilde{w}(k) = c^{k-1}w(k)$ for every $k \geq 0$. Then $\Upsilon_{\tilde{w}} < \infty$ if and only if $\Upsilon_w < \infty$ and in this case, the laws $\text{SG}^{\tilde{w}}$ and SG^w coincide.*
- (ii) *Fix $a, b > 0$ and set $\hat{w}(k) = ab^k w(k)$ for every $k \geq 0$. Then the conditional laws $\text{SG}^{\hat{w}}(\cdot \mid \#T = n)$ and $\text{SG}^w(\cdot \mid \#T = n)$ coincide.*

Proof. Note that for every tree $T \in \mathbf{T}$, one has $\sum_{u \in T} k_u = \#T - 1$ and so $\sum_{u \in T} (k_u - 1) = -1$; it follows that

$$\Pi^{\tilde{w}}(T) = \prod_{u \in T} c^{k_u-1} w(k_u) = c^{-1} \Pi^w(T),$$

so $\Upsilon_{\tilde{w}} = c^{-1} \Upsilon_w$ and the first claim follows. Similarly,

$$\Pi^{\hat{w}}(T) = \prod_{u \in T} ab^{k_u} w(k_u) = a^{\#T} b^{\#T-1} \Pi^w(T),$$

so $\Pi^{\hat{w}}(\{T \in \mathbf{T} : \#T = n\}) = a^n b^{n-1} \Pi^w(\{T \in \mathbf{T} : \#T = n\})$ and the second claim follows. \square

We shall use Lemma 8 with sequences \tilde{w} or \hat{w} which are probability measures with mean 1 so, in the first case, $\text{SG}^{\tilde{w}} = \text{GW}^{\tilde{w}}$ is the law of a critical Galton–Watson tree, and in the second case, $\text{SG}^{\hat{w}}(\cdot \mid \#T = n) = \text{GW}^{\hat{w}}(\cdot \mid \#T = n)$ is the law of such a tree conditioned to have n vertices.

We next show that the empirical degree sequence of large critical Galton–Watson trees satisfies **(H)**. For a plane tree T and an integer $i \geq 0$, let us denote by $n_T(i) = \#\{u \in T : k_u = i\}$ the number of vertices of T with i children. For any subset $A \subset \mathbf{Z}_+$, set $n_T(A) = \sum_{i \in A} n_i(T)$; note that $n_T(\mathbf{Z}_+)$ is the total number of vertices of T , $n_T(0)$ is its number of leaves and $n_T(\mathbf{N})$ its number of internal vertices. Consider the empirical offspring distribution of T and its variance, given by

$$p_T(i) := \frac{n_T(i)}{n_T(\mathbf{Z}_+)} \quad \text{for } i \geq 0 \quad \text{and} \quad \sigma_T^2 := \sum_{i \geq 0} i^2 p_T(i) - \left(\frac{n_T(\mathbf{Z}_+) - 1}{n_T(\mathbf{Z}_+)} \right)^2,$$

and finally set $\Delta_T := \max\{i \geq 0 : n_T(i) > 0\}$.

Proposition 10. *Let μ be a critical distribution in \mathbf{Z}_+ with variance $\sigma^2 \in (0, \infty)$ and fix $A \subset \mathbf{Z}_+$; under $\text{GW}^\mu(\cdot \mid n_T(A) = n)$, the convergence*

$$\left(p_T, \sigma_T^2, n_T(\mathbf{Z}_+)^{-1/2} \Delta_T \right) \xrightarrow[n \rightarrow \infty]{\mathbf{P}} (\mu, \sigma^2, 0),$$

holds in probability.

This result was obtained by Broutin & Marckert [12, Lemma 11] in the case $A = \mathbf{Z}_+$. Their proof extends to the general case using arguments due to Kortchemski [21] (see in particular sections 6 and 7 there), written explicitly for $A = \{0\}$ but which hold true in general, *mutatis mutandis*, as explained in Section 8 there.

Proof. Fix $\varepsilon > 0$ and consider the event

$$E(\varepsilon) := \left\{ d \left(\left(\frac{n_T(\cdot)}{n_T(\mathbf{Z}_+)}, \sum_{i \geq 0} (i-1)^2 \frac{n_T(i)}{n_T(\mathbf{Z}_+)}, \frac{\Delta_T}{n_T(\mathbf{Z}_+)^{1/2}} \right), (\mu, \sigma^2, 0) \right) > \varepsilon \right\},$$

where d is a metric on the product space of probability measures on \mathbf{Z}_+ and \mathbf{R}^2 , compatible with the product topology. We aim at showing

$$\text{GW}^\mu(E(\varepsilon) \mid n_T(A) = n) \xrightarrow[n \rightarrow \infty]{} 0.$$

Let us denote by $(X_k; k \geq 1)$ a sequence of i.i.d. random variables with distribution $(\mu(i+1); i \geq -1)$ and $K_n(i) = \#\{1 \leq k \leq n : X_k = i-1\}$ for every $n \geq 1$ and $i \geq 0$. Consider the event

$$F(n, \varepsilon) := \left\{ d \left(\left(\frac{K_n(\cdot)}{n}, \sum_{i \geq 0} (i-1)^2 \frac{K_n(i)}{n}, \frac{\max\{i \geq 0 : K_n(i) > 0\}}{n^{1/2}} \right), (\mu, \sigma^2, 0) \right) > \varepsilon \right\},$$

Broutin & Marckert [12] have shown that

$$\mathbf{P}(F(n, \varepsilon)) \xrightarrow[n \rightarrow \infty]{} 0.$$

Recall from Section 5.4 that, given a path $x = (x_1, \dots, x_n) \in \mathbf{Z}^n$ such that $x_1 + \dots + x_n = -1$, we denote by $S_x(k) = x_1 + \dots + x_k$ for every $1 \leq k \leq n$ and by $x^* = (x_1^*, \dots, x_n^*)$ the unique cyclic shift of x satisfying furthermore $S_{x^*}(k) \geq 0$ for every $1 \leq k \leq n-1$. Let $\zeta_r(A) = \inf\{k \geq 1 : K_k(A) = \lfloor r \rfloor\}$ for every $r \geq 1$. Kortchemski [21, Proposition 6.5] shows that for every integer $n \geq 1$, the law of the vector $(X_1^*, \dots, X_{\zeta_n(A)}^*)$ under $\mathbf{P}(\cdot \mid S_X(\zeta_n(A)) = -1)$ coincides with the law of the vector

$(W_T(1), W_T(2) - W_T(1), \dots, W_T(n_T(\mathbf{Z}_+)) - W_T(n_T(\mathbf{Z}_+) - 1))$ under $\text{GW}^\mu(\cdot \mid n_T(A) = n)$, where W_T is the Łukasiewicz path of a tree T . Since $F(n, \varepsilon)$ is invariant under cyclic shift, it follows that

$$\text{GW}^\mu(E(\varepsilon) \mid n_T(A) = n) = \mathbf{P}(F(\zeta_n(A), \varepsilon) \mid S_X(\zeta_n(A)) = -1).$$

Using a time-reversibility property of $(X_1^*, \dots, X_{\zeta_n(A)}^*)$ under $\mathbf{P}(\cdot \mid S_X(\zeta_n(A)) = -1)$, see [21, Proposition 6.8], it suffices to show that

$$\mathbf{P}(F(\zeta_{n/2}(A), \varepsilon) \mid S_X(\zeta_n(A)) = -1) \xrightarrow{n \rightarrow \infty} 0.$$

As in the proof of [21, Theorem 7.1], for any $\alpha > 0$, we decompose the event $F(\zeta_{n/2}(A), \varepsilon)$ as

$$\begin{aligned} F(\zeta_{n/2}(A), \varepsilon) \cap \left\{ |S_X(\zeta_{n/2}(A))| \leq \alpha \sqrt{\sigma^2 n / (2\mu(A))} \right\} &\cap \left\{ \left| \zeta_{n/2}(A) - \frac{n}{\mu(A)} \right| \leq n^{3/4} \right\} \\ \cup \left\{ |S_X(\zeta_{n/2}(A))| > \alpha \sqrt{\sigma^2 n / (2\mu(A))} \right\} &\cup \left\{ \left| \zeta_{n/2}(A) - \frac{n}{\mu(A)} \right| > n^{3/4} \right\}. \end{aligned}$$

By [21, Lemmas 6.10 & 6.11] (argument similar to the one we use in the proof of Lemma 5, based on a local limit theorem), there exists a constant $C > 0$ independent of α such that for every n large enough, the conditional probability $\mathbf{P}(\cdot \mid S_X(\zeta_n(A)) = -1)$ of the event written at the first line of the last display is bounded above by

$$C \cdot \mathbf{P}\left(F(\zeta_{n/2}(A), \varepsilon) \text{ and } \left| \zeta_{n/2}(A) - \frac{n}{\mu(A)} \right| \leq n^{3/4}\right).$$

Next, according to [21, Equation 44],

$$\lim_{\alpha \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbf{P}\left(|S_X(\zeta_{n/2}(A))| > \alpha \sqrt{\sigma^2 n / (2\mu(A))} \mid S_X(\zeta_n(A)) = -1\right) = 0,$$

and, by [21, Lemma 6.2(i)],

$$\lim_{n \rightarrow \infty} \mathbf{P}\left(\left| \zeta_{n/2}(A) - \frac{n}{\mu(A)} \right| > n^{3/4} \mid S_X(\zeta_n(A)) = -1\right) = 0.$$

We conclude that there exists a constant $C > 0$ such that

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \mathbf{P}(F(\zeta_{n/2}(A), \varepsilon) \mid S_X(\zeta_n(A)) = -1) \\ &\leq C \limsup_{n \rightarrow \infty} \mathbf{P}\left(F(\zeta_{n/2}(A), \varepsilon) \text{ and } \left| \zeta_{n/2}(A) - \frac{n}{\mu(A)} \right| \leq n^{3/4}\right). \end{aligned}$$

On the event $|\zeta_{n/2}(A) - \frac{n}{\mu(A)}| \leq n^{3/4}$, we have for every $i \geq 0$,

$$\frac{K_{n/\mu(A) - n^{3/4}}(i)}{n/\mu(A) + n^{3/4}} \leq \frac{K_{\zeta_{n/2}(A)}(i)}{\zeta_{n/2}(A)} \leq \frac{K_{n/\mu(A) + n^{3/4}}(i)}{n/\mu(A) - n^{3/4}},$$

and the claim from the fact that $\mathbf{P}(F(n, \varepsilon)) \rightarrow 0$ as $n \rightarrow \infty$. \square

We shall also need the following concentration result. For a sequence $(x_n; n \geq 1)$ of non-negative real numbers and $\delta > 0$, we write $x_n = \text{oe}_\delta(n)$ if there exists $c_1, c_2 > 0$ such that for every $n \geq 1$, $x_n \leq c_1 \exp(-c_2 n^\delta)$.

Lemma 9. *Let μ be a critical distribution in \mathbf{Z}_+ with variance $\sigma^2 \in (0, \infty)$ and fix $A \subset \mathbf{Z}_+$; there exists $\delta > 0$ such that*

$$\text{GW}^\mu\left(\left|\frac{n_T(0)}{n} - \frac{\mu(0)}{\mu(A)}\right| > \varepsilon \mid n_T(A) = n\right) = \text{oe}_\delta(n).$$

Proof. We bound

$$\mathrm{GW}^\mu \left(\left| \frac{n_T(0)}{n} - \frac{\mu(0)}{\mu(A)} \right| > \varepsilon \mid n_T(A) = n \right) \leq \frac{\mathrm{GW}^\mu \left(\left| \frac{n_T(0)\mu(A)}{n_T(A)\mu(0)} - 1 \right| > \frac{\mu(A)}{\mu(0)}\varepsilon \mid n_T(\mathbf{Z}_+) \geq n \right)}{\mathrm{GW}^\mu(n_T(A) = n)}.$$

According to [21, Theorem 8.1],

$$\mathrm{GW}^\mu(n_T(A) = n) \underset{n \rightarrow \infty}{\sim} C \cdot n^{-3/2},$$

for some explicit constant $C > 0$ which depends only on μ and A (see [21, Theorem 3.1]). Moreover, from [21, Corollary 2.6],

$$\mathrm{GW}^\mu \left(\left| \frac{n_T(0)}{\mu(0)n_T(\mathbf{Z}_+)} - 1 \right| > n^{-1/4} \mid n_T(\mathbf{Z}_+) \geq n \right) = \mathrm{oe}_{1/2}(n).$$

This holds also when 0 is replaced by A ; it follows that

$$\mathrm{GW}^\mu \left(\left| \frac{n_T(0)\mu(A)}{n_T(A)\mu(0)} - 1 \right| > \frac{\mu(A)}{\mu(0)}\varepsilon \mid n_T(\mathbf{Z}_+) \geq n \right) = \mathrm{oe}_{1/2}(n),$$

and the proof is complete. \square

As a corollary, we obtain that, in a large critical Galton–Watson tree, the inverse of the number of leaves, normalised to have expectation 1, converges to 1 in L^1 .

Corollary 4. *Let μ be a critical distribution in \mathbf{Z}_+ with variance $\sigma^2 \in (0, \infty)$. For every $A \subset \mathbf{Z}_+$, we have*

$$\lim_{n \rightarrow \infty} \mathrm{GW}^\mu \left[\left| n_T(0)^{-1} \mathrm{GW}^\mu[n_T(0)^{-1} \mid n_T(A) = n]^{-1} - 1 \right| \mid n_T(A) = n \right] = 0.$$

Proof. Fix $\varepsilon \in (0, 1)$ and observe that, since $n_T(0)^{-1} \leq 1$,

$$\begin{aligned} \mathrm{GW}^\mu \left[\left| \frac{\mu(0)n}{\mu(A)n_T(0)} - 1 \right| \mid n_T(A) = n \right] \\ \leq \varepsilon + \left(\frac{\mu(0)n}{\mu(A)} + 1 \right) \mathrm{GW}^\mu \left(\left| \frac{\mu(0)n}{\mu(A)n_T(0)} - 1 \right| > \varepsilon \mid n_T(A) = n \right). \end{aligned}$$

Next, the probability in the right-hand side is bounded above by

$$\mathrm{GW}^\mu \left(\frac{n_T(0)}{n} < \frac{1}{2} \frac{\mu(0)}{\mu(A)} \mid n_T(A) = n \right) + \mathrm{GW}^\mu \left(\left| \frac{\mu(0)}{\mu(A)} - \frac{n_T(0)}{n} \right| > \frac{\varepsilon}{2} \frac{\mu(0)}{\mu(A)} \mid n_T(A) = n \right),$$

which is $\mathrm{oe}_\delta(n)$ for some $\delta > 0$ according to Lemma 9. This yields

$$\lim_{n \rightarrow \infty} \mathrm{GW}^\mu \left[\left| \frac{\mu(0)n}{\mu(A)n_T(0)} - 1 \right| \mid n_T(A) = n \right] = 0.$$

In particular,

$$\lim_{n \rightarrow \infty} \frac{\mu(0)n}{\mu(A)} \mathrm{GW}^\mu \left[n_T(0)^{-1} \mid n_T(A) = n \right] = 1,$$

and the claim follows from these two limits. \square

7.3 Convergence of Boltzmann random maps

We first prove the convergence of rooted and pointed Boltzmann maps, using the BDG and the JS bijections, and next compare the pointed and non pointed Boltzmann laws to deduce Theorems 4 and 5.

Proposition 11. *Theorems 4 and 5 hold under their respective assumptions when the measures $\mathbf{P}_{S=n}^w$ are replaced by their pointed version $\mathbf{P}_{S=n}^{w,\star}$.*

The main idea is to observe that for every $n \geq 2$ and $S \in \{E, V, F\} \cup \bigcup_{A \subset \mathbf{N}} \{F, A\}$, the composition of the BDG and the JS bijections maps the set $\mathbf{M}_{S=n}^*$ onto the subset of \mathbf{T} of those trees T satisfying $n_T(B_S) = n$, where for every $A \subset \mathbf{N}$,

$$B_E = \mathbf{Z}_+, \quad B_V = \{0\}, \quad B_F = \mathbf{N} \quad \text{and} \quad B_{F,A} = A. \quad (36)$$

Proof. Fix a rooted and pointed map $(\mathcal{M}, \star) \in \mathbf{M}^*$ and let (T, l) be its associated labelled one-type tree after the BDG and then the JS bijections. Recall that the faces of \mathcal{M} are in bijection with the internal vertices of T , whereas the vertices of \mathcal{M} different from \star are in bijection with the leaves of T ; in particular, with the notation of the previous subsection, for every $i \geq 1$, the number of faces of \mathcal{M} of degree $2i$ is given by $n_T(i)$, and $V(\mathcal{M}) - 1 = n_T(0)$. Thereby,

$$\Omega^{w,\star}((\mathcal{M}, \star)) = \prod_{f \in \text{Faces}(\mathcal{M})} w(\deg(f)/2) = \prod_{u \in T: k_u \geq 1} w(k_u).$$

Recall also from (3) the number of possible labellings of a given plane tree. The measure $\Omega^{w,\star}$ on \mathbf{M}^* thus induces a measure on \mathbf{T} , where each $T \in \mathbf{T}$ is given the weight

$$\prod_{u \in T: k_u \geq 1} \binom{2k_u - 1}{k_u - 1} w(k_u) = \Pi^{\bar{w}}(T),$$

where \bar{w} is given by (32). This shows that if (\mathcal{M}, \star) has the law $\mathbf{P}^{w,\star}$, then its associated labelled one-type tree (T, l) after the BDG and then the JS bijections is as follows: T has the law $\text{SG}^{\bar{w}}$ and then, conditional on the tree T , the labelling l is uniformly distributed amongst all possibilities. Similarly, for every $n \geq 2$ and $S \in \{E, V, F\} \cup \bigcup_{A \subset \mathbf{N}} \{F, A\}$, if (\mathcal{M}, \star) has the law $\mathbf{P}_{S=n}^{w,\star}$, then T has the law $\text{SG}^{\bar{w}}(\cdot \mid n_T(B_S) = n)$, where B_S is given by (36); furthermore, conditional on the tree T , the labelling l is uniformly distributed amongst all possibilities.

Let us now prove that Theorem 5 holds for the pointed maps sampled from $\mathbf{P}_{E=n}^{w,\star}$. Let us suppose that $x > 0$ is such that

$$g_w(x) < \infty, \quad xg'_w(x) = g_w(x), \quad \text{and} \quad xg''_w(x) < \infty.$$

Define a probability measure on \mathbf{Z}_+ similar to (34) where Z_w^* is replaced by x :

$$\mu_w(k) = \frac{x^k \bar{w}(k)}{g_w(x)}, \quad k \geq 0. \quad (37)$$

Note that

$$\sum_{k \geq 0} k x^k \bar{w}(k) = xg'_w(x), \quad \text{and} \quad \sum_{k \geq 0} k^2 x^k \bar{w}(k) = xg'_w(x) + x^2 g''_w(x),$$

the distribution μ_w has thus expectation

$$\sum_{k \geq 0} k \mu_w(k) = \frac{xg'_w(x)}{g_w(x)} = 1,$$

and variance

$$\sum_{k \geq 0} k^2 \mu_w(k) - 1 = \frac{x g'_w(x) + x^2 g''_w(x)}{g_w(x)} - 1 = \frac{x^2 g''_w(x)}{g_w(x)} \in (0, \infty).$$

According to Lemma 8(ii), the tree T has the law $\text{GW}^{\mu_w}(\cdot \mid n_T(\mathbf{Z}_+) = n)$, Proposition 10 ensures then that (H) is fulfilled with $p = \mu_w$ and we conclude from Theorem 1.

The proof of the fact that Theorem 4 holds for the pointed maps sampled from $\mathbf{P}_{S=n}^{w, \star}$ is similar. If w is generic critical, then Z_w^* satisfies the above assumptions on x and furthermore $g_w(Z_w^*) = Z_w^*$ so μ_w is the probability p_w given by (34):

$$\mu_w(k) = p_w(k) = (Z_w^*)^{k-1} \bar{w}(k), \quad k \geq 0.$$

According to Lemma 8(i), the tree T has the law $\text{GW}^{p_w}(\cdot \mid n_T(B_S) = n)$. Again, Proposition 10 ensures then that (H) is fulfilled with $p = p_w$ and we conclude from Theorem 1. \square

We have seen all the ingredients to prove Proposition 9. The proof is inspired from [32].

Proof of Proposition 9. According to the previous proof, we have

$$Z_w^* = \sum_{(\mathcal{M}, \star) \in \mathbf{M}^*} \Omega^{w, \star}((\mathcal{M}, \star)) = \sum_{T \in \mathbf{T}} \Pi^{\bar{w}}(T) = \Upsilon_{\bar{w}},$$

where \bar{w} is given by (32). Suppose that this quantity is finite, we next decompose the last term according to the degree of the root of T . If the latter is $k \geq 1$, then T is made of k trees, say T_1, \dots, T_k , attached to a root; this leads to the following equation:

$$\sum_{T \in \mathbf{T}} \Pi^{\bar{w}}(T) = \sum_{k \geq 0} \bar{w}(k) \sum_{T_1, \dots, T_k \in \mathbf{T}} \prod_{i=1}^k \Pi^{\bar{w}}(T_i) = \sum_{k \geq 0} \bar{w}(k) \left(\sum_{T \in \mathbf{T}} \Pi^{\bar{w}}(T) \right)^k,$$

in other words $Z_w^* = g_w(Z_w^*)$. Let us prove furthermore that $g'_w(Z_w^*) \leq 1$. Since, $Z_w^* = g_w(Z_w^*)$, the sequence p_w defined by $p_w(k) = (Z_w^*)^{k-1} \bar{w}(k)$ for every $k \geq 0$ is a probability and $g'_w(Z_w^*)$ is its mean. According to Lemma 8(i), the law $\text{SG}^{\bar{w}}$ coincides with SG^{p_w} so

$$\sum_{T \in \mathbf{T}} \text{SG}^{p_w}(T) = \frac{1}{\Upsilon_{\bar{w}}} \sum_{T \in \mathbf{T}} \Pi^{\bar{w}}(T) = 1.$$

We conclude that $\text{SG}^{p_w} = \text{GW}^{p_w}$ is the law of a sub-critical Galton–Watson tree with offspring distribution p_w , which has therefore mean $g'_w(Z_w^*) \leq 1$.

Conversely, suppose that g_w has at least one fixed point and let us prove that Z_w^* is finite. Recall that one of the fixed points, say $x > 0$, must satisfy $g'_w(x) \leq 1$; we set $\mu_w(k) = x^{k-1} \bar{w}(k)$ for every $k \geq 0$, the previous calculations show that μ_w is a probability measure with mean $g'_w(x) \leq 1$. According to (the proof of) Lemma 8(i), we have

$$\frac{1}{x} Z_w^* = \frac{1}{x} \sum_{(\mathcal{M}, \star) \in \mathbf{M}^*} \Omega^{w, \star}((\mathcal{M}, \star)) = \frac{1}{x} \sum_{T \in \mathbf{T}} \Pi^{\bar{w}}(T) = \sum_{T \in \mathbf{T}} \Pi^{\mu_w}(T) = 1.$$

We conclude that $Z_w^* = x$ is indeed finite. \square

Finally, we show that the pointed and non pointed Boltzmann laws are close to each other. Theorems 4 and 5 follow from Propositions 11 and 12.

Proposition 12. Fix $S \in \{E, V, F\} \cup \bigcup_{ACN} \{F, A\}$ and let $w \in \mathbf{Z}_+^{\mathbf{N}}$ satisfy the assumptions of Theorem 4 or Theorem 5 if $S = E$. Let $\phi : \mathbf{M}^* \rightarrow \mathbf{M} : (M, \star) \mapsto M$, then

$$\left\| \mathbf{P}_{S=n}^w - \phi_* \mathbf{P}_{S=n}^{w, \star} \right\|_{TV} \xrightarrow{n \rightarrow \infty} 0,$$

where $\|\cdot\|_{TV}$ refers to the total variation norm.

Proof. Notice that $\mathbf{P}_{S=n}^{w,\star}$ is absolutely continuous with respect to $\mathbf{P}_{S=n}^w$: for every measurable and bounded function $f : \mathbf{M} \rightarrow \mathbf{R}$,

$$\mathbf{E}_{S=n}^w [f(\mathcal{M})] = \mathbf{E}_{S=n}^{w,\star} [V(\mathcal{M})^{-1}]^{-1} \mathbf{E}_{S=n}^{w,\star} [V(\mathcal{M})^{-1} f(\mathcal{M})],$$

where in the right-hand side, $f(\mathcal{M})$, and similarly $V(\mathcal{M})$, is a slight abuse of notation for $f \circ \phi((\mathcal{M}, \star))$. Thus

$$\begin{aligned} & \left\| \mathbf{P}_{S=n}^w - \phi_* \mathbf{P}_{S=n}^{w,\star} \right\|_{TV} \\ &= \frac{1}{2} \sup_{-1 \leq f \leq 1} \left| \mathbf{E}_{S=n}^w [f(\mathcal{M})] - \mathbf{E}_{S=n}^{w,\star} [f(\mathcal{M})] \right| \\ &\leq \frac{1}{2} \sup_{-1 \leq f \leq 1} \mathbf{E}_{S=n}^{w,\star} \left[\left| f(\mathcal{M}) \left(\mathbf{E}_{S=n}^{w,\star} [V(\mathcal{M})^{-1}]^{-1} V(\mathcal{M})^{-1} - 1 \right) \right| \right] \\ &\leq \mathbf{E}_{S=n}^{w,\star} \left[\left| \mathbf{E}_{S=n}^{w,\star} [V(\mathcal{M})^{-1}]^{-1} V(\mathcal{M})^{-1} - 1 \right| \right] \\ &= \text{GW}^{p_w} \left[\left| \text{GW}^{p_w} [(n_T(0) - 1)^{-1} \mid n_T(B_s) = n]^{-1} (n_T(0) - 1)^{-1} - 1 \right| \mid n_T(B_s) = n \right], \end{aligned}$$

where p_w is given by (34) or (37) in the case $S = E$ and B_S is given by (36). Corollary 4 states that the last quantity above tends to zero as $n \rightarrow \infty$, which concludes the proof. \square

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